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CHARACTERIZING LOSSES DURING BUSY PERIODS IN FINITE BUFFER SYSTEMS

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Abstract

For multiple-server finite-buffer systems with batch Poisson arrivals, we explore how the distribution of the number of losses during a busy period changes with the buffer size and the initial number of customers. We show that when the arrival rate equals the maximal service rate ($\rho = 1$), as the buffer size increases the number of losses in a busy period increases in the convex sense, and when $\rho > 1$, as the buffer size increases the number of busy period losses increases in the increasing convex sense. Also, the number of busy period losses is stochastically increasing in the initial number of customers. A consequence of our results is that, when $\rho = 1$, the mean number of busy period losses equals the mean batch size of arrivals regardless of the buffer size. We show that this invariance does not extend to general arrival processes.

Keywords: Finite buffer system; busy period losses; M^X/GI/1/*n* queue

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1. Introduction

Consider an M^X/GI/*c*/*n* system with *c* parallel servers and with batches of arrivals occurring according to a Poisson process with rate λ, and where a random batch size, X, has mean β. Service times are independent and identically distributed with mean $1/\mu$ (and are exponentially distributed if $c > 1$). The system can hold at most *n* customers, including the one in service. Let $L_n(i)$ be the number of losses during a busy period assuming that the busy period starts with i customers, and let L_n be the number of losses during a standard busy period that starts with a batch arrival with random size X. We develop a recursion relating $L_{n+1}(i)$ and $L_n(i)$. This allows us to explore the way in which the distribution of the number of losses during a busy period changes with the buffer size. We show that when the arrival rate equals the maximal service rate ($\rho = \lambda\beta/c\mu = 1$), the number of losses increases in the convex sense (defined later) in the buffer size, and when $\rho > 1$, the number of losses increases in the increasing convex sense in the buffer size. We also consider how the distribution of $L_n(i)$ changes as a function of i. In the single server case, i.e. for M^X/GI/1/*n* systems, we can extend our results to permit server vacations.

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A consequence of our results for distributions of losses are invariance results for the mean number of losses when $\rho = 1$. In particular, for all n, $\mathrm{E}\, L_n = 1$ for the M/GI/1/n queue [1], [9], $\mathrm{E}\, L_n(i) = i$ for the M^X/GI/1/n queue [3], $\mathrm{E}\, L_n = \beta$ for the M^X/GI/1/n queue, and $\mathrm{E}\, L_n = \mathrm{E}\, A$ for the M^X/GI/1/n queue with vacations, where A is the number of arrivals during a vacation. Also, $\mathrm{E}\, L_n(i)$ is independent of n for the M^X/M/c/n queue (and $\mathrm{E}\, L_n(i) = i + 1$ for the M^X/M/2/n queue). We show that this invariance for mean losses when $\rho = 1$ does not extend to general arrival processes. Indeed, even with a single exponential server (a GI/M/1/n queue), if $\mathrm{E}\, L_n(1) = 1$ for all n, then the arrival process must be a Poisson process with $\lambda = \mu$.

Some of our work generalizes Righter [8], who considered the M/GI/1/n system. Peköz [7] studied the moments for the number of losses during a busy period for the M/GI/1/n and the GI/M/1/n systems. Abramov [2], [4], [5] also studied losses in the GI/M/1/n system. Wolff [9] gave conditions under which $\mathrm{E}\, L_n \le 1$ in the GI/GI/1/n system and slightly more general queueing systems.

2. The M^X/GI/1/n queue

We first consider the single-server queue with batch arrivals and general services. Many of the results of this section are similar to those in [8] so their proofs are omitted. We also strengthen and correct Corollary 3.2 of [8], which should have stated that $\mathrm{E}\, L_n \le \beta$ rather than $\mathrm{E}\, L_n \le 1$.

Let $L_n(i)$ be the number of losses in an M^X/GI/1/n busy period that starts with i customers, and let us consider the relationship between $L_{n+1}(i)$ and $L_n(i)$. Throughout, we permit $i > n$, in which case $L_n(i) = i - n + L_n(n)$. For convenience, we refer to the n system and the $n + 1$ system. In the following notation, a subscript of n indicates that we are considering an M^X/GI/1/n system, and an argument of i means that there are i customers initially present. Let $I_n(i)$ be an indicator that is 1 if there is a loss during the busy period and 0 otherwise, let $\tilde{L}_n(i)$ be distributed, independent of all else, as the number of losses during a busy period given that at least one loss occurs, and let $L'_n(i)$ be an independent and identically distributed version of $L_n(i)$. A coupling argument gives us the following distributional relationship between $L_{n+1}(i)$ and $L_n(i)$.

Lemma 2.1. *For all i and n, $L_{n+1}(i) = L_n(i) - I_n(i) + I_n(i)L'_{n+1}(1)$, where $L'_{n+1}(1)$ is independent of $L_n(i)$ and $I_n(i)$.*

Let us now consider how the distribution of the number of losses during a busy period varies with the initial number of customers.

Lemma 2.2. *For all i and n, $L_n(i + 1) = L_{n-i}(1) + L_n(i)$, where $L_{n-i}(1)$ and $L_n(i)$ are independent of each other.*

Proof. For $L_n(i + 1)$, we start with $i + 1$ customers and $n - 1 - i$ empty places for new customers. The busy period will evolve (in terms of arrivals, services, and losses) until the first time, call it t, that there are i customers, which will occur before the end of the busy period. Because of the Poisson arrivals, the evolution from time t onwards is the same as an n system busy period starting with i customers. Before t, the evolution is the same as that of a busy period starting with 1 customer and with $n - 1 - i$ empty places for new customers.

When $\lambda\beta = \mu$, we can take expectations in Lemma 2.1 to show, arguing as in [8], that $\mathrm{E}\, L_n(1) = 1$ and $\mathrm{E}\, L_{n+1}(i) = \mathrm{E}\, L_n(i)$. Then, taking expectations in Lemma 2.2, we have the following, where L_n is the number of losses in a busy period starting with an arbitrary batch of size X. Part of the corollary appears in [3].

Corollary 2.1. *If* $\rho = 1$, *then* $\mathrm{E}\, L_n(i) = i$ *and* $\mathrm{E}\, L_n = \mathrm{E}\, L_n(X) = \beta$ *for all* $i, n \geq 1$.

The key relationship in Lemma 2.1 allows us to investigate how the entire distribution of L_n, not just the mean, changes with n, and we can do this even when $\rho \neq 1$. For example, Corollary 2.3 and Theorem 2.1 below tell us that when the arrival rate equals the service rate, though the mean number of busy period losses does not change with n, the variability increases with increasing buffer sizes. Recall that for two random variables, X and Y, we say that X is larger than Y in the convex sense, i.e. $X \geq_{\mathrm{cx}} Y$, if $\mathrm{E}\, f(X) \geq \mathrm{E}\, f(Y)$ for any convex function f, in the increasing convex sense, i.e. $X \geq_{\mathrm{icx}} Y$, if $\mathrm{E}\, f(X) \geq \mathrm{E}\, f(Y)$ for any increasing convex f, and in the stochastic sense, i.e. $X \geq_{\mathrm{st}} Y$, if $\mathrm{E}\, f(X) \geq \mathrm{E}\, f(Y)$ for any increasing f. In the following, $p_n(i) = \mathrm{E}\, I_n(i)$.

Corollary 2.2. (i) *If* $\rho < 1$, *then* $\mathrm{E}\, L_n(i) < i$ *and* $\mathrm{E}\, L_n < \beta$. *Moreover,* $\mathrm{E}\, L_n(i)$ *and* $\mathrm{E}\, L_n$ *are decreasing in n for all* $i, n \geq 1$.

(ii) *If* $\rho > 1$, *then* $\mathrm{E}\, L_n(i) > i$ *and* $\mathrm{E}\, L_n > \beta$. *Moreover,* $\mathrm{E}\, L_n(i)$ *and* $\mathrm{E}\, L_n$ *are increasing in n for all* $n \geq 1$, $1 \leq i \leq n$.

Corollary 2.3. *If* $\rho = 1$, *then, for all* $i, n \geq 1$,

$$\mathrm{var}(L_{n+1}(i)) = \mathrm{var}(L_n(i)) + p_n(i)\frac{\mathrm{var}(L_n(1))}{1 - p_n(1)}$$

and

$$\mathrm{var}(L_{n+1}) = \mathrm{var}(L_n) + \mathrm{E}[p_n(X)]\frac{\mathrm{var}(L_n(1))}{1 - p_n(1)}.$$

Therefore, both $\mathrm{var}(L_n(i))$ *and* $\mathrm{var}(L_n)$ *are increasing in n.*

Theorem 2.1. (i) *If* $\rho = 1$, *then* $L_{n+1}(i) \geq_{\mathrm{cx}} L_n(i)$ *and* $L_{n+1} \geq_{\mathrm{cx}} L_n$ *for all* $i, n \geq 1$.

(ii) *If* $\rho > 1$, *then* $L_{n+1}(i) \geq_{\mathrm{icx}} L_n(i)$ *and* $L_{n+1} \geq_{\mathrm{icx}} L_n$ *for all* $i, n \geq 1$.

From Lemma 2.2 it is easy to see the following.

Theorem 2.2. (i) *The number of losses* $L_n(i)$ *is stochastically increasing in i for all n.*

(ii) *If* $\rho = 1$, *then* $L_n(i+1) - (i+1) \geq_{\mathrm{cx}} L_n(i) - i$ *for all* $i, n \geq 1$. *Therefore,* $\mathrm{var}(L_n(i))$ *is increasing in i.*

Now suppose that when the server empties the queue, it takes an arbitrarily distributed vacation, during which it does no service. If the queue is empty when the server returns from vacation, it takes another vacation, otherwise it serves customers until the queue is again empty. Let A be the number of arrivals during a vacation. Let L_n^{C} be the number of losses during a cycle consisting of a vacation followed by a busy period (which may be of length 0 if no arrivals occurred during the vacation). We define $L_n(i)$ as before, so $L_n(i)$ is the number of losses during a busy period of a standard $\mathrm{M}^X/\mathrm{GI}/1/n$ system that starts with i customers, where $L_n(i) = i - n + L_n(n)$ for $i \geq n$ and $L_n(0) = 0$. Then we have

$$L_n^{\mathrm{C}} = L_n(A),$$

where we think of any losses that occur during the vacation as occurring when the server returns and starts a busy period. Therefore, from Corollary 2.1, when $\rho = 1$, $\mathrm{E}\, L_n^{\mathrm{C}} = \mathrm{E}\, L_n(A) = \mathrm{E}\, A$ for all n. Corollary 2.2 and Theorem 2.1 can also be extended to L_n^{C}.

3. Multiple servers

Now we consider an $\mathrm{M}^X/\mathrm{M}/c/n$ system, where $n \geq c$ is the maximum number of customers allowed in the system, which is the queueing buffer size plus c. With multiple servers we no longer permit general service times; service times are exponential with rate μ. Again we show how the distribution of losses during a busy period varies with the initial number of customers present and with the buffer size. We also again have that the mean number of losses in a busy period is independent of the buffer size when $\rho = 1$, though the actual expression now depends on c and the distribution of batch sizes.

As before, for the $\mathrm{M}^X/\mathrm{M}/c/n$ system starting with initial state i, let $I_n(i)$ be an indicator that is 1 if there is a loss during the busy period. Now we define $\hat{L}_n(i)$ to be the number of losses before first reaching a queue length of c, given that a loss has just occurred. Then

$$L_n(i) = I_n(i)(\hat{L}_n(i) + L_n(c)).$$

Furthermore, by coupling the arrivals and services in the n system with those in the $n+1$ system we have that

$$\begin{aligned} L_{n+1}(i) &= I_n(i)(\hat{L}_n(i) - 1 + L_{n+1}(c+1)) \\ &= L_n(i) + I_n(i)(L_{n+1}(c+1) - L_n(c) - 1), \end{aligned}$$

where $L_{n+1}(c+1)$ and $L_n(c)$ are independent of $I_n(i)$. To see this, first note that if no customers are lost in the n system, then none are lost in the $n+1$ system. Otherwise, think of the first customer that is lost in the n system (call it the tagged customer) as always being at the end of the queue in the coupled $n+1$ system. After the first loss in the n system, both systems will have the same loss patterns until there are c customers in the n system. Thus, we have the following result.

Lemma 3.1. *For all $n \geq c$ and $i \geq c$, $L_{n+1}(i) = L_n(i) + I_n(i)(L_{n+1}(c+1) - L_n(c) - 1)$, where $L_{n+1}(c+1)$ and $L_n(c)$ are independent of $I_n(i)$.*

Let $\tilde{L}_n(i)$ be the number of losses in an $\mathrm{M}^X/\mathrm{M}/1/n$ queue with service rate $c\mu$. Arguing as in the proof of Lemma 2.2, we have the following result.

Lemma 3.2. *For all $n \geq c$ and $i \geq c$, $L_n(i+1) = \tilde{L}_{n-i}(1) + L_n(i)$, where $\tilde{L}_{n-i}(1)$ and $L_n(i)$ and independent of each other.*

Taking expected values in Lemma 3.2 and using Corollary 2.1 for $\tilde{L}_{n-i}(1)$, we have the following result.

Corollary 3.1. *If $\lambda\beta = c\mu > 0$, then, for all $n \geq c$ and $i \geq 1$, $\mathrm{E}\, L_n(i) = \mathrm{E}\, L_n(i-1) + 1$.*

The following corollary gives the invariance result for mean losses during a busy period for the $\mathrm{M}^X/\mathrm{GI}/1/n$ system when the arrival rate equals the maximum service rate.

Corollary 3.2. *If $\lambda\beta = c\mu > 0$, then, for all $n \geq c$ and $i \geq 1$,*

$$\mathrm{E}\, L_{n+1}(i) = \mathrm{E}\, L_n(i),$$
$$\mathrm{E}\, L_{n+1} = \mathrm{E}\, L_n.$$

Proof. Taking expected values in Lemma 3.1 and using Corollary 3.1, we have

$$\mathrm{E}\, L_{n+1}(i) = \mathrm{E}\, L_n(i) + p_n(i)(\mathrm{E}\, L_{n+1}(c) - \mathrm{E}\, L_n(c)), \tag{3.1}$$

where $p_n(i) = \mathrm{E}\, I_n(i)$ is the probability of a loss during a busy period starting with i customers. Letting $i = c$, we have

$$\mathrm{E}\, L_{n+1}(c) - \mathrm{E}\, L_n(c) = p_n(c)(\mathrm{E}\, L_{n+1}(c) - \mathrm{E}\, L_n(c)).$$

Note that $p_n(c) < 1$ because the probability that there is no loss during a busy period, $1 - p_n(c)$, is bounded below by the probability that all of the initial c customers depart before the first arrival, which is greater than 0. Hence $\mathrm{E}\, L_{n+1}(c) - \mathrm{E}\, L_n(c) = 0$. From (3.1), we then have $\mathrm{E}\, L_{n+1}(i) = \mathrm{E}\, L_n(i)$ for all $i \geq 1$, and therefore, $\mathrm{E}\, L_{n+1} = \mathrm{E}\, L_{n+1}(X) = \mathrm{E}\, L_n(X) = \mathrm{E}\, L_n$.

It is not hard to show the following result.

Corollary 3.3. (i) *If $\lambda\beta > c\mu$, then*

$$\mathrm{E}\, L_n(i) > \mathrm{E}\, L_n(i-1) + 1 \quad \textit{for all } i \geq c$$

and $\mathrm{E}\, L_n(i)$ *and* $\mathrm{E}\, L_n$ *are increasing in n for all $i \geq 1$.*

(ii) *If $\lambda\beta < c\mu$, then*

$$\mathrm{E}\, L_n(i) < \mathrm{E}\, L_n(i-1) + 1 \quad \textit{for all } i \geq c$$

and $\mathrm{E}\, L_n(i)$ *and* $\mathrm{E}\, L_n$ *are decreasing in n for all $i \geq 1$.*

Theorem 3.1. (i) *If $\lambda\beta = c\mu$, then $L_{n+1}(i) \geq_{\mathrm{cx}} L_n(i)$ for all $i \geq 1$ and $n \geq c$. In particular,* $\mathrm{var}(L_n(i))$ *is increasing in n for $n \geq c$.*

(ii) *If $\lambda\beta > c\mu$, then $L_{n+1}(i) \geq_{\mathrm{icx}} L_n(i)$ for all $i \geq 1$ and $n \geq c$.*

Proof. For part (i), using Lemma 3.1, Jensen's inequality, and Corollary 3.2, we have that, for a convex function f, when $\lambda\beta = c\mu$,

$$\begin{aligned}
\mathrm{E}\, f(L_{n+1}(i)) &= \mathrm{E}\, f(L_n(i) + I_n(i)(L_{n+1}(c+1) - L_n(c) - 1)) \\
&\geq \mathrm{E}\, f\,(L_n(i) + I_n(i)(\mathrm{E}\, L_{n+1}(c+1) - \mathrm{E}\, L_n(c) - 1)) \\
&= \mathrm{E}\, f(L_n(i)).
\end{aligned}$$

Since $f(x) = x^2$ is a convex function, it follows immediately that $\mathrm{var}(L_n(i))$ is increasing in n for $n \geq c$.

For part (ii), note that, from Corollary 3.3, we have $\mathrm{E}\, L_{n+1}(c+1) - \mathrm{E}\, L_n(c) - 1 \geq 0$ when $\lambda\beta > c\mu$. The rest of the argument follows similarly to that of part (i).

This theorem, together with Corollary 3.2, shows that, for an M^X/M/*c*/*n* system with $\lambda\beta = c\mu$, as the buffer size increases the variability of the number of losses during a busy period increases, even though the mean is invariant, depending only on the initial state i.

From Lemma 3.2 and Corollary 3.2 we have the following result.

Theorem 3.2. (i) *The number of losses $L_n(i)$ is stochastically increasing in i for all n.*

(ii) *If $\lambda\beta = \mu$, then $L_n(i+1) - (i+1) \geq_{\mathrm{cx}} L_n(i) - i$ for all $i, n \geq 1$. Therefore,* $\mathrm{var}(L_n(i))$ *is increasing in i.*

Now let us attempt to determine $\mathrm{E}\, L_n(i)$ when $\lambda\beta = c\mu$, which, from Corollary 3.1, reduces to determining $\mathrm{E}\, L_n(i)$ for $i < c$. We have, for $1 \le i \le c-1$,

$$\mathrm{E}\, L_n(i) = \frac{\lambda}{\lambda + i\mu} \mathrm{E}\, L_n(i+X) + \frac{i\mu}{\lambda + i\mu} \mathrm{E}\, L_n(i-1),$$

where $\mathrm{E}\, L_n(0) = 0$, and, from Corollary 3.1, for $i \ge c-1$,

$$\mathrm{E}\, L_n(i) = \mathrm{E}\, L_n(c-1) + i - (c-1).$$

We can solve this system of equations, but in general it will depend on the distribution of X, so let us consider some special cases.

Suppose that $X \equiv 1$ (so $\beta = 1$ and $\lambda = c\mu$). Then it is not hard to show that, for $1 \le i \le c$,

$$\mathrm{E}\, L_n(i) = \frac{c^c}{c!} \sum_{k=0}^{i-1} \frac{k!}{c^k}.$$

Also, $\mathrm{E}\, L_n = \mathrm{E}\, L_n(1) = c^c/c!$.

Now suppose that $c = 2$ and let X have a general distribution with $\mathrm{E}\, X = \beta$. When $\lambda\beta = 2\mu$, we have $\mathrm{E}\, L_n(i) = \mathrm{E}\, L_n(1) + i - 1$ for all $i \ge 1$ and

$$\begin{aligned} \mathrm{E}\, L_n(1) &= \frac{\lambda}{\lambda + \mu} \mathrm{E}\, L_n(1+X) \\ &= \frac{\lambda}{\lambda + \mu} [\mathrm{E}\, L_n(1) + \mathrm{E}\, X] \\ &= \frac{\lambda}{\mu}\beta = 2. \end{aligned}$$

Therefore, $\mathrm{E}\, L_n(i) = i + 1$ for all i and for all $n \ge 2$, and $\mathrm{E}\, L_n = \beta + 1$ for all $n \ge 2$.

Corollary 3.4. *For the M/M/c/n queue with $\lambda = c\mu$, $\mathrm{E}\, L_n = c^c/c!$ for all $n \ge c$. For the M^X/M/2/N queue with $\lambda\beta = 2\mu$, $\mathrm{E}\, L_n(i) = i + 1$ and $\mathrm{E}\, L_n = \beta + 1$ for all $n \ge 2$ and all i.*

4. The GI/M/1/*n* queue

In this section, we consider the GI/M/1/*n* queue, and show how the behavior can be different from the M/GI/1/*n* queue studied above. Corollary 2.1 gives conditions under which the mean number of losses in a busy period equals 1 for all n in the M/GI/1/*n* system. Below we show that this can only happen in a GI/M/1/*n* queue when there are Poisson arrivals. This result corrects the flawed conclusion of Proposition 2.1 in [7].

Let μ be the service rate, let T be a random variable with the interarrival distribution, and let $\lambda = 1/\mathrm{E}\, T$ be the arrival rate.

Theorem 4.1. *If the expected number of losses in a GI/M/1/n busy period equals 1 for all n, then the arrival process must be a Poisson process with rate $\lambda = \mu$.*

Proof. Let $L_n(i)$ be the number of losses in a GI/M/1/*n* busy period that starts with i customers. That is, we assume that an arrival has just occurred, and there are i customers in the system. Let μ be the service rate and let

$$p_i = \mathrm{E}\left[\mathrm{e}^{-\mu T} \frac{(\mu T)^i}{i!}\right], \qquad i = 0, 1, \ldots,$$

be the chance of i potential service completions between arrivals. We will first show that, if

$$\mathrm{E}\, L_n(1) = 1 \quad \text{for all } n,$$

then it must follow that

$$\mathrm{E}\, L_n(i) = i \quad \text{for all } i, n$$

and

$$p_i = \frac{1}{2^{i+1}} \quad \text{for all } i.$$

We use (double) induction. First note that, by conditioning on an arrival before the first service completion, we have

$$\mathrm{E}\, L_1(1) = p_0(1 + \mathrm{E}\, L_1(1))$$

and since by assumption $\mathrm{E}\, L_1(1) = 1$, this implies that $p_0 = \frac{1}{2}$. Next, as the first induction hypothesis assume that, for a given $n > 1$, $p_i = 1/2^{i+1}$ for all $i < n - 1$. We will show that $p_{n-1} = 1/2^n$. To do this we use induction again. Since by assumption $\mathrm{E}\, L_n(1) = 1$, we assume that $\mathrm{E}\, L_n(j) = j$ for $j \le i$ and we will show that $\mathrm{E}\, L_n(i+1) = i+1$. By conditioning on the number of service completions before the next arrival, for $i < n$ it holds that

$$\mathrm{E}\, L_n(i) = \sum_{m=0}^{i-1} p_m\, \mathrm{E}\, L_n(i + 1 - m),$$

which follows because the queue length at the arrival epochs is a Markov chain and, starting a busy period with $i < n$ customers in the system, if there are $m < i$ service completions before the next arrival there will be $i + 1 - m$ customers in the system after the next arrival. Using both of the induction hypotheses and re-arranging, we have

$$\begin{aligned}
\mathrm{E}\, L_n(i + 1) &= 2\bigg(\mathrm{E}\, L_n(i) - \sum_{m=1}^{i-1} \frac{\mathrm{E}\, L_n(i + 1 - m)}{2^{m+1}}\bigg) \\
&= 2\bigg(i - \sum_{m=1}^{i-1} \frac{i + 1 - m}{2^{m+1}}\bigg) \\
&= 2\bigg(i - \frac{i}{2} + \frac{1}{2}\bigg) \\
&= i + 1.
\end{aligned}$$

Thus, we can assume that $\mathrm{E}\, L_n(i) = i$ for all $i \le n$. Again conditioning on the number of service completions before the next arrival, it holds that

$$\mathrm{E}\, L_n(n) = p_0(1 + \mathrm{E}\, L_n(n)) + \sum_{m=1}^{n-1} p_m\, \mathrm{E}\, L_n(n + 1 - m),$$

which by the induction hypotheses and re-arranging gives

$$n = \frac{1 + n}{2} + \sum_{m=1}^{n-2} \frac{n + 1 - m}{2^{m+1}} + 2p_{n-1},$$

and simplifying and solving gives $p_{n-1} = 2^{-n}$. This completes the induction.

From this it will follow that T must be exponential with rate μ using the following reasoning, which we sketch. The ith derivative of the moment generating function

$$\phi(t) = \mathrm{E}[\mathrm{e}^{tT}]$$

is

$$\phi^{(i)}(t) = \mathrm{E}[T^i \mathrm{e}^{tT}],$$

and note that

$$p_i = \phi^{(i)}(-\mu)\frac{\mu^i}{i!}.$$

Using a Taylor series expansion around $-\mu$, we get that

$$\begin{aligned}\phi(t) &= \sum_{i=0}^{\infty} \frac{(t+\mu)^i \phi^{(i)}(-\mu)}{i!} \\ &= \sum_{i=0}^{\infty} \left(\frac{t+\mu}{\mu}\right)^i p_i \\ &= \sum_{i=0}^{\infty} \left(\frac{t+\mu}{\mu}\right)^i \frac{1}{2^{i+1}} \\ &= \frac{\mu}{\mu - t}\end{aligned}$$

when t is in the radius of convergence of the power series. Since a distribution is determined by its moment generating function when defined in a neighborhood of zero (see [6, p. 408]), and exponential rate μ interarrival times gives moment generating function $\mu/(\mu - t)$, T must be exponential rate μ.

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