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IMPROVING POISSON APPROXIMATIONS

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Let $X_1, \ldots, X_n$, be indicator random variables, and set $W = \sum_{i=1}^{n} X_i$. We present a method for estimating the distribution of W in settings where W has an approximately Poisson distribution. Our method is shown to yield estimates significantly better than straight Poisson estimates when applied to Bernoulli convolutions, urn models, the circular k of n:F system, and a matching problem. Error bounds are given.

1. INTRODUCTION AND SUMMARY

There has been much in the recent literature (see Barbour, Holst, and Janson [2] and the references therein) on approximating the distribution of the sum of dependent indicator variables using a suitably chosen Poisson distribution. In this paper we take a new approach by combining several Poisson estimates into an estimator that, in the applications discussed, is seen to yield estimates significantly better than straight Poisson estimates. Our results show that this method generally performs better than the straight Poisson approximation when the indicators are negatively related (see Definition 1, in Section 2) and either the mean of their sum is less than one or when estimating the probabilities of sets sufficiently above this mean. In the applications discussed we present bounds on the approximation error that are roughly the factor of the mean of the sum

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times the bounds given elsewhere for straight Poisson approximations, thus yielding an improvement when this mean is less than one.

In Section 2 we present the new method (with theorems bounding its error), and in Sections 3–7 we apply it to approximate the distribution of Bernoulli convolutions, the distributions of the number of empty urns, and the number of urns with more than a given number of balls when balls are placed independently into urns, the reliability of a circular consecutive-k-of-n:F system, and a matching probability. Section 8 describes a setting that would not be expected to be well suited for our method, Section 9 gives a brief summary of our conclusions, and the Appendix contains additional data demonstrating the efficacy of the new method.

2. MAIN RESULTS

Let $X_1, \ldots, X_n$ be (not necessarily independent) Bernoulli random variables with $\lambda_i = P(X_i = 1)$, and set $W = \sum_{i=1}^n X_i$. Let $1 + V_i$ have the distribution of W given $X_i = 1$. First we present our method for approximating the distribution of W. Let $\lambda = \sum_i \lambda_i = E[W]$.

LEMMA 1:

(a)

$$P(W > 0) = \sum_i \lambda_i E\left[\frac{1}{1 + V_i}\right].$$

(b) *For* $k > 0$,

$$P(W = k) = \frac{1}{k} \sum_i \lambda_i P(V_i = k - 1).$$

PROOF: For an arbitrary random variable R, $E[WR] = \sum_i E[X_i R] = \sum_i \lambda_i \times E[R \mid X_i = 1]$. Part (a) follows by letting

$$R = \begin{cases} 1/W & \text{if } W > 0 \\ 0 & \text{otherwise} \end{cases}$$

and Part (b) with

$$R = \begin{cases} 1/W & \text{if } W = k \\ 0 & \text{otherwise.} \end{cases}$$

■

In situations where W has an approximately Poisson distribution, it is reasonable to expect that V_i also has approximately a Poisson distribution. Using a Poisson approximation in Lemma 1(b), we obtain the following with $a_i = E[V_i]$ and $k > 0$:

$$P(W = k) \approx \frac{1}{k} \sum_i \lambda_i \frac{e^{-a_i} a_i^{k-1}}{(k-1)!}.$$

To estimate $P(W \in A)$ for any set A, we sum the preceding over $k \in A$. This gives the new estimator $\mu(A)$ as follows: for any set with $0 \notin A$, we define

$$\mu(A) \equiv \sum_{k \in A} \frac{1}{k} \sum_i \lambda_i \frac{e^{-a_i} a_i^{k-1}}{(k-1)!} = \sum_i \frac{\lambda_i}{a_i} \mathrm{Po}(a_i)\{A\},$$

where

$$\mathrm{Po}(a)\{A\} \equiv \sum_{k \in A} e^{-a} a^k / k!.$$

For any set with $0 \in A$, we define

$$\mu(A) \equiv 1 - \mu(A^c).$$

Note that the new estimator of $P(W > 0)$ is

$$\mu([1,\infty)) = \sum_i \frac{\lambda_i}{a_i} (1 - e^{-a_i}).$$

Remark: Lemma 1(a) is a variation of an identity used by Stein [6] to bound errors of Poisson approximations. For some other applications of the general identity, see Aldous [1] and Ross [5].

It is also possible to obtain the following lower bound.

Corollary 1:

$$P(W > 0) \geq \sum_i \frac{\lambda_i}{1 + a_i}.$$

Proof: By Jensen's inequality we have for a random variable Y, $E[1/Y] \geq 1/E[Y]$. The result follows by applying this to the expectation in Lemma 1(a). ∎

Remark: In Ross [4] the inequality of Corollary 1 is shown to be better than the second moment inequality $P(W > 0) \geq (E[W])^2/E[W^2]$. It is seen from examples that simply using this bound as an approximation of $P(W > 0)$ can be better than the straight Poisson approximation.

In the following theorem we establish bounds on the error of the new estimator.

Theorem 1:

$$d(W,\mu) \equiv \sup_A |P(W \in A) - \mu(A)| \leq \sum_i \lambda_i d(V_i, \mathit{Poisson}(a_i)).$$

Proof: For any set A, since $|P(W \in A) - \mu(A)| = |P(W \in A^c) - \mu(A^c)|$, we can without loss of generality assume that $0 \notin A$. Letting $f_i(k) = P(V_i = k-1) - e^{-a_i}(a_i)^{k-1}/(k-1)!$ and summing over k in Lemma 1(b), we obtain

$$|P(W \in A) - \mu(A)| = \left| \sum_{k \in A} \sum_i \frac{\lambda_i}{k} f_i(k) \right| \leq \sum_i \lambda_i \left| \sum_{k \in A} \frac{f_i(k)}{k} \right|.$$

Next, with $B_i = \{k \in A : f_i(k) > 0\}$, we know that

$$\begin{aligned} 0 &\leq \sum_{k \in B_i} \frac{f_i(k)}{k} \leq \sum_{k \in B_i} f_i(k) \\ &= P(1 + V_i \in B_i) - P(1 + \text{Poisson}(a_i) \in B_i) \\ &\leq d(V_i, \text{Poisson}(a_i)) \end{aligned}$$

and similarly

$$0 \geq \sum_{k \in A \setminus B_i} \frac{f_i(k)}{k} \geq -d(V_i, \text{Poisson}(a_i)),$$

which when combined yield

$$\left| \sum_{k \in A} \frac{f_i(k)}{k} \right| \leq d(V_i, \text{Poisson}(a_i))$$

and the result. ■

Remark: In cases where the X_i are negatively correlated, Poisson approximations to V_i are generally better than approximations to W, for the indicators comprising the former indicate rarer events. In these cases we thus see that the supremum error of the new estimator is roughly λ times the supremum error of the straight Poisson approximation. In the applications considered below, upper bounds on $d(V_i, \text{Poisson}(a_i))$ were roughly the same as upper bounds on $d(W, \text{Poisson}(\lambda))$.

We now present cases for which we can prove that this new approximation is better than the straight Poisson approximation. First we need a definition and lemma.

Definition 1: *The random variables X_i, $1 \leq i \leq n$ are negatively related if there exist random variables X_{ij} so that for each i, $1 \leq i \leq n$, the random variables $(X_{ij};\ 1 \leq j \leq n, j \neq i)$ have the same joint distribution as $(X_j;\ 1 \leq j \leq n, j \neq i \mid X_i = 1)$ and $X_{ij} \leq X_j$ a.s. for $1 \leq j \leq n, j \neq i$.*

Note that being negatively related is a stronger condition than being negatively correlated. Correlation is a pairwise property, whereas being negatively related depends on the entire joint distribution.

Lemma 2: *If the X_i are negatively related and for each i, given $X_i = 1$, the random variables $X_j, j \neq i$, are conditionally negatively related, then for $k \geq 3\lambda + 1$*

$$P(W \geq k) \leq 2 \sum_i \frac{\lambda_i}{k} Po(a_i)\{[k-1, \infty)\}.$$

PROOF: Summing over k in Lemma 1(b), we have

$$P(W \geq k) \leq \frac{1}{k} \sum_i \lambda_i P(V_i \geq k-1).$$

Next, we need Theorem 2.R from Barbour et al. [2], which states that when Y is comprised of negatively related indicators, with $k \geq 3\lambda$,

$$P(Y \geq k) \leq 2\,\mathrm{Po}(\lambda)\{[k,\infty)\}.$$

Applying this inequality to the V_i, the result is obtained. ∎

THEOREM 2: *Under the conditions of the previous lemma, there exists an integer*

$$k^* \leq \max(1 + (\log 6 - a + \lambda)/\log(\lambda/a), 3\lambda + 1),$$

where $a = \max_i a_i$, *so that given any set* A *where* $\forall k \in A$, $k \geq k^*$

$$|P(W \in A) - \mu(A)| < |P(W \in A) - Po(\lambda)\{A\}|.$$

PROOF: Let $B(k)$ be the bound given by Lemma 2. Note that it suffices to find a k^* so that for all $k \geq k^*$

$$\frac{\mathrm{Po}(\lambda)\{k\}}{B(k)} > 2,$$

which implies that

$$B(k) < \tfrac{1}{2}\mathrm{Po}(\lambda)\{A\}$$

when $k = \min_{i \in A} i$ and $k \geq k^*$. Since $P(W \in A) \leq B(k)$ and $\mu(A) \leq \mu([k,\infty)) < B(k)$, we then obtain the result.

To find such a k^*, note that when $k \geq 3\lambda + 1$ we have (see Barbour et al. [2, Property A.2.3(ii)])

$$\frac{k}{2\lambda} B(k) \leq \mathrm{Po}(a)\{[k-1,\infty)\} \leq \frac{k}{k-a}\,\mathrm{Po}(a)\{k-1\} < \frac{3}{2} e^{-a} a^{k-1}/(k-1)!;$$

thus,

$$\frac{\mathrm{Po}(\lambda)\{k\}}{B(k)} > \frac{1}{3} e^{-(\lambda - a)} (\lambda/a)^{k-1},$$

which is above 2 when $k \geq 1 + (\log 6 - a + \lambda)/\log(\lambda/a)$. Note that X_i negatively related implies $a < \lambda$. ∎

As an application of this theorem, note that in the case where 100 balls are put randomly into 20 urns and W counts the number of empty urns, $k^* \leq 7$. This means that the new approximation is strictly better when estimating the probabilities of sets whose elements are all at least 7.

3. BERNOULLI CONVOLUTIONS

Here we consider the case where the X_i are independent Bernoulli random variables with $\lambda_i = P(X_i = 1)$. In the next theorem we give an error bound for the new approximation, which is the factor $\lambda = E[W]$ times the bound given in Barbour et al. [2] for the straight approximation. We also show that the supremum error of the new approximation is less than the supremum error of the straight approximation when λ is small. The new approximation can thus be viewed as an improvement when λ is small.

THEOREM 3: *For* $\lambda < 1$,

(a)

$$d(W, \mathit{Poisson}(\lambda)) \leq \sum_i \lambda_i^2.$$

(b)

$$d(W, \mu) \leq \lambda \sum_i \lambda_i^2.$$

(c)

$$d(W, \mu) \leq 14\lambda d(W, \mathit{Poisson}(\lambda)).$$

PROOF: Part (a) is given in Barbour et al. [2, p. 8]. To obtain part (b), note that, because V_i also is a convolution of Bernoulli random variables, part (a) gives

$$d(V_i, \text{Poisson}(a_i)) \leq \sum_{j:j \neq i} \lambda_j^2 \leq \sum_j \lambda_j^2,$$

where $a_i = \lambda - \lambda_i$. Applying Theorem 1 gives part (b). To obtain part (c), Barbour et al. [2, Remark 3.2.2] give

$$d(W, \text{Poisson}(\lambda)) \geq \frac{1}{14} \sum_i \lambda_i^2,$$

which when combined with parts (a) and (b) yields the result. ■

A program was written to compare exact values of $P(W = k)$ with our estimates (see Table 1). Our estimates are seen to be better than straight Poisson estimates. In Table 1 $\lambda_i = P(X_i = 1) = i/1000, i = 1, \ldots, 10$, and $a_i = \lambda - \lambda_i$.

Using the inequality from Corollary 1, the bound $P(W = 0) \leq .947751$ is obtained. By the previous theorem, the straight approximation's error is always less than .0004 while the new approximation's error is less than .000021.

4. EMPTY URNS

Consider m urns and n balls. Each ball is independently placed into urn i with probability p_i. Letting X_i be the indicator of the event that urn i is empty, W

TABLE 1. Comparison of Estimates for Bernoulli Convolutions

k	$P(W=k)$	New Approximation	Straight Approximation
0	.946302	.946299	.946485
1	.052413	.052422	.052056
2	.001266	.001257	.001431
3	.000018	.000020	.000026

counts the number of empty urns. In cases when n is large compared with m, it is reasonable to believe that W has approximately a Poisson distribution.

Here

$$\lambda_i = P(X_i = 1) = (1 - p_i)^n$$

and

$$a_i = E[V_i] = \sum_{j:j\neq i} \left(1 - \frac{p_j}{1 - p_i}\right)^n.$$

In the next theorem we give an error bound for the new approximation, which is the factor $\lambda = E[W]$ times the bound given in Barbour et al. [2] for the straight approximation.

THEOREM 4: *For* $\lambda < 1$,

(a)

$$d(W, \mathit{Poisson}(\lambda)) \leq \lambda \max_i \lambda_i + n\left(\frac{\log n}{n - \log n} + \frac{4}{n}\right)^2.$$

(b)

$$d(W, \mu) \leq \lambda\left(\lambda \max_i \lambda_i + n\left(\frac{\log n}{n - \log n} + \frac{4}{n}\right)^2\right).$$

PROOF: Part (a) is a slight modification of Theorem 6.D from Barbour et al. [2]. To show part (b), note that V_i also counts the number of empty urns in a problem with urn i removed, so we can apply part (a) to bound $d(V_i$, Poisson$(a_i))$. Since the X_i are negatively related (see Barbour et al. [2]), $a_i \leq \lambda$ and $a_i \max_{j:j\neq i} E[X_j \mid X_i = 1] \leq \lambda \max_j \lambda_j$ for each i. This with part (a) implies

$$d(V_i, \text{Poisson}(a_i)) \leq \lambda \max_j \lambda_j + n\left(\frac{\log n}{n - \log n} + \frac{4}{n}\right)^2,$$

which with Theorem 1 implies the result. ∎

Table 2. Comparison of Approximations for the Empty Urns Problem

k	$P(W=k)$	New Approximation	Straight Approximation
0	.86690	.86689	.87464
1	.13227	.13230	.11715
2	.00084	.00080	.00785

A comparison of our approximations, Poisson approximations, and exact values is shown in Table 2, where we have 4 urns and 20 balls and $p_i = P(\text{ball goes into urn } i) = i/10, i = 1, \ldots, 4$.

The inequality in Corollary 1 yields $P(W=0) \leq .86767$, which is below the straight Poisson estimate.

5. VERY FULL URNS

Again, consider m urns, n balls, with each ball independently placed into urn i with probability p_i. Letting X_i be the indicator of the event that urn i has at least b balls, W counts the number of urns with at least b balls. In cases when these are rare events, it is reasonable to believe that W is roughly Poisson. Here,

$$\lambda_i = P(X_i = 1) = \sum_{j=b}^{n} \binom{n}{j} (p_i)^j (1 - p_i)^{n-j},$$

and by conditioning on the number of balls in urn i

$$a_i = \frac{1}{\lambda_i} \sum_{j:j \neq i} \sum_{k=b}^{n} \sum_{x=b}^{n-k} \frac{n!}{x!k!(n-x-k)!} p_i^x p_j^k (1 - p_i - p_j)^{n-x-k}.$$

A comparison of the approximations with exact values is shown in Table 3, where we have 6 urns and 10 balls, $b = 5$, and $p_i = 1/6, i = 1, \ldots, 6$. W counts the number of urns with at least 5 balls.

The inequality of Corollary 1 yields $P(W=0) \leq .90735$, which is again below the straight Poisson estimate.

Table 3. Comparison of the Approximations for the Full Urns Problem

k	$P(W=k)$	New Approximation	Straight Approximation
0	.9072910	.9072911	.91140
1	.0926468	.0926469	.08455
2	.0000625	.0000624	.00392

6. CIRCULAR CONSECUTIVE k OF $n{:}F$ SYSTEM

Flip n coins independently, each with $P(\text{heads}) = p = 1 - q$, and arrange them around a circle. What is the probability of seeing a run of at least k heads? Numbering the coins $1, \ldots, n$, and letting

$$X_i = \begin{cases} 1 & \text{if a clockwise run of } k \text{ heads ends with coin } i \\ 0 & \text{otherwise.} \end{cases}$$

with $W = \sum_i X_i$, we are interested in $P(W > 0)$, known in the reliability literature as the failure probability of a circular consecutive k of $n{:}F$ system. In cases where this is an unlikely event, a Poisson approximation is reasonable but we can do better by first redefining the indicators to eliminate the clustering caused by overlapping runs. Given there is a run ending with coin i, it is more likely that there will also be overlapping runs ending with coins adjacent to i. This motivates the following definition to eliminate the possibility of overlaps: let

$$Y_i = \begin{cases} 1 & \text{if a clockwise run of } k \text{ heads followed by a tail ends with coin } i \\ 0 & \text{otherwise.} \end{cases}$$

Note that the events $\{\sum_i X_i > 0\}$ and $\{\sum_i Y_i > 0\}$ differ only by the rare event that all n coins come up heads. Letting

$$\lambda = E\left[\sum_i Y_i\right] = nqp^k,$$

our approximation yields

$$P(W > 0) \approx \frac{\lambda}{a}(1 - e^{-a}),$$

where

$$a = E\left[\sum_{i:i\neq 1} Y_i \,\middle|\, Y_1 = 1\right] = E\left[\sum_{i=k+2}^{n-k} Y_i\right] = (n - 2k - 1)qp^k.$$

The first equality above follows by symmetry, and the second by the impossibility of overlaps.

In the next theorem we give a bound for the error, which again is essentially the factor λ times a bound for the straight approximation error given in Barbour et al. [2].

THEOREM 5:

(a)

$$|P(W > 0) - (1 - e^{-\lambda})| \le (2k + 1)qp^k + p^n.$$

(b)

$$\left| P(W > 0) - \frac{\lambda}{a}(1 - e^{-a}) \right| \leq \lambda(2k + 1)qp^k + p^n.$$

PROOF: Part (a) is a modification of inequality (4.4) on page 164 of Barbour et al. [2]. Though the argument there is for linearly arranged coins, the same argument can be applied to circularly arranged coins; add p^n for the case when all coins come up heads. Part (b) follows upon applying the inequality to obtain

$$d(V_i, \text{Poisson}(a_i)) \leq (2k + 1)qp^k,$$

where V_i now counts the number of times k heads is followed by a tail for a linear arrangement with $n - k - 1$ coins. The result follows by applying Theorem 1. ■

Remark: Similar bounds can be obtained for the problem where the coins are arranged in a line, though the new approximation becomes slightly more complicated to evaluate.

A comparison of our approximations and exact values is shown in Table 4. To compute the exact values of $P(W > 0)$, we use the formula given in Peköz and Ross [3]. The straight Poisson approximation is $P(W > 0) \approx 1 - e^{-\lambda}$.

7. A MATCHING PROBLEM

All the applications presented in Sections 2–6 involve negatively related indicators, a situation ideally suited for our method. For a Poisson approximation to be reasonable, we generally need to have a sum of indicators with small means and with weak dependence between them. When the indicators comprising W are negatively related, we are at least assured that those comprising V_i have smaller means and may therefore also be amenable to a good Poisson approximation. Here we present a situation where there is weak positive dependence

TABLE 4. Comparison of Approximations for the Consecutive k-of-n:F System

k	n	p	$P(W > 0)$	New Approximation	Straight Approximation	Corollary 1 Bound
5	20	.4	.11949	.11954	.115	≥.11644
5	20	.2	.005114102	.005114106	.005107	≥.005108
5	30	.2	.00766134	.00766135	.007650	≥.007643
5	30	.5	.402	.406	.374	≥.361
3	20	.1	.01789	.01790	.01784	≥.01779
5	80	.2	.020300020	.020300019	.02027	≥.02012
5	80	.3	.12835	.12840	.12732	≥.12179

in the indicators, but our method still performs better than the straight Poisson approximation.

The problem is as follows: n people throw their hats into a ring and then each randomly draws a hat. What is the probability that any of the first k people to draw get their own hat? Let X_i, $1 \le i \le k$, be the indicator that person i gets their own hat, and let $W = \sum_{i=1}^{k} X_i$. We are interested in estimating $P(W > 0)$. Here,

$$\lambda = E[W] = k/n$$

and

$$a = (k-1)/(n-1).$$

The straight and new approximations are

$$P(W > 0) \approx 1 - e^{-\lambda}$$

and

$$P(W > 0) \approx \lambda(1 - e^{-a})/a,$$

respectively. A comparison is shown in Table 5.

8. A CASE NOT WELL SUITED FOR OUR METHOD

Our approximation relies on the assumption that V_i is roughly Poisson when W is. In this section we look at a situation where there is much stronger positive dependency between the indicators comprising the V_i than there was in the X_i, and our approximation is not expected to perform well.

The setting is a modification of the classical birthday problem: assuming all birthdays are equally likely, what is the chance that at least three people share a birthday in a group of k people? This problem was considered in Section 5; here we give a different treatment. We number the distinct triples of people and define an indicator for each: for $i = 1, \ldots, \binom{k}{3}$,

$$X_i = \begin{cases} 1 & \text{if triple number } i \text{ shares a birthday} \\ 0 & \text{otherwise.} \end{cases}$$

TABLE 5. Comparison of Approximations for the Matching Problem

k	n	$P(W>0)$	New Approximation	Straight Approximation	Corollary 1 Bound
5	10	.4018	.4036	.393	≥.346
5	20	.2251	.2254	.221	≥.206
5	30	.15559	.15568	.154	≥.146

The straight Poisson approximation yields

$$P(W > 0) \approx 1 - e^{-\lambda},$$

where $\lambda = E[W] = (1/365^2)\binom{k}{3}$. Our approximation yields

$$P(W > 0) \approx \frac{\lambda}{a}(1 - e^{-a}),$$

where

$$a = E[V_i] = \binom{k-3}{3}\Big/365^2 + 3(k-3)/365 + 3\binom{k-3}{2}\Big/365^2.$$

Here we see positive dependencies in the indicators comprising V_i since given that triple i matches birthdays there is a 1/365 chance that another person shares the same birthday and, hence, creates three additional matching triples. The comparison of the approximations with exact values is shown in Table 6.

9. CONCLUSIONS

Our results, both experimental and theoretical, show that the new approximation to the distribution of W presented here generally performs better than the

Table 6. Comparison of Approximations for the Birthday Problem

No. of People	P(a Shared Birthday)	Straight Approximation	New Approxiation
10	8.9E−4	9.0E−4	8.8E−4
20	8.2E−3	8.5E−3	8.0E−3
30	2.9E−2	3.0E−2	2.7E−2
40	6.7E−2	7.2E−2	6.2E−2
50	.13	.14	.11
60	.21	.23	.18
70	.31	.34	.26
80	.42	.46	.35
90	.53	.59	.44
100	.65	.70	.52
110	.75	.80	.60
120	.83	.88	.66
130	.90	.93	.70
140	.94	.97	.74
150	.96	.98	.77
160	.98	.99	.79
170	.99	.999	.81
180	.996	.999	.83

straight Poisson approximation when the indicators are negatively related and either $E[W] < 1$ or when estimating the probability of sets sufficiently above $E[W]$. Computing the new approximation only involves the extra information about the mean of the conditional distribution of W, information that is generally immediately available in applications. In amenable situations the bounds we obtain on the approximation error are generally much better than bounds given elsewhere for the straight Poisson approximation. The preceding criteria are met in many applications so that this new approximation is widely applicable.

References

1. Aldous, D. (1989). The harmonic mean formula for probabilities of unions: Applications to sparse random graphs. *Discrete Mathematics* 76: 167–176.
2. Barbour, A.D., Holst, L., & Janson, S. (1992). *Poisson approximation.* Oxford: Oxford University Press.
3. Peköz, E. & Ross, S.M. (1994). A simple derivation of exact reliability formulas for linear and circular consecutive k-of-n:F systems. *Journal of Applied Probability* (to appear).
4. Ross, S.M. (1994). Estimating a convolution tail probability by simulation. *Probability, Statistics, and Optimization: A Tribute to Peter Whittle.* New York: Wiley.
5. Ross, S.M. (to appear). A new simulation estimator of system reliability. *Journal of Applied Mathematics and Stochastic Analysis.*
6. Stein, C. (1986). *Approximate computation of expectations.* IMS Lecture Notes, Hayward, CA.

APPENDIX

This appendix contains additional data demonstrating the approximations.

1. With 4 urns and 14 balls and $P(\text{ball goes into urn } i) = i/10$, $i = 1, \ldots, 4$, W counts the number of empty urns.

k	$P(W = k)$	New Approximation	Straight Approximation	Bound
0	.72737	.72715	.75555	$P(W = 0) \leq .7338$
1	.26494	.26562	.21179	
2	.00768	.00702	.02968	

2. With 4 urns and 18 balls and $P(\text{ball goes into urn } i) = .2 + i/50$, $i = 1, \ldots, 4$, W counts the number of empty urns.

k	$P(W = k)$	New Approximation	Straight Approximation	Bound
0	.97432	.97433	.97462	$P(W = 0) \leq .97436$
1	.02564	.02564	.02506	
2	.00003	.00003	.00032	

3. With 4 urns and 12 balls, and $P(\text{ball goes into urn } i) = .25$, W counts the number of empty urns.

k	$P(W = k)$	New Approximation	Straight Approximation	Bound
0	.87476	.87475	.88099	$P(W = 0) \leq .87616$
1	.12378	.12381	.11163	
2	.00146	.00143	.00707	

4. Consider a Bernoulli convolution where $P(X_1 = 1) = .0016$, $P(X_2 = 1) = .0064$, $P(X_3 = 1) = .0144$, $P(X_4 = 1) = .0256$, and $P(X_5 = 1) = .04$. Letting $W = \sum_i X_i$,

k	$P(W = k)$	New Approximation	Straight Approximation	Bound
0	.91459	.91457	.91576	$P(W = 0) \leq .9169$
1	.08286	.08292	.08059	
2	.00253	.00246	.00355	
3	.00003	.00005	.00010	