

Dear Editor,

*A note on reliability bounds via conditional inequalities*

**1. Introduction**

In a recent paper, Xie and Lai [3] study the approximation

$$P\left(\bigcap_{i=1}^n E_i\right) \approx P(E_1) \prod_{i=2}^n P(E_i | E_{i-1})$$

for the intersection of the events  $E_i$  and give some conditions for when the approximation is an upper bound. An example is given there where the conditions are not satisfied. Here we give an argument using conditions which are satisfied in the example, and in some new settings.

In Section 2 we prove the main result, and in Section 3 we apply it to the linear and circular consecutive  $k$ -of- $n : F$  systems, and to the distribution of time until a non-overlapping pattern first appears in a sequence of coin flips.

**2. Main result**

Here we prove the following theorem and give a useful consequence.

**Theorem 2.1.** *Given events  $E_1, E_2, \dots, E_i$  and an integer  $k > 1$ , if*

$$E_i \cup E_{i-j} \supseteq E_{i-1} \quad \text{for all } j : 2 \leq j < i, j \leq k \tag{1}$$

and

$$P\left(E_i | E_{i-1} \cap \bigcap_{m=1}^{i-k-1} E_m\right) \leq P(E_i | E_{i-1}), \tag{2}$$

then

$$P(E_i | E_{i-1} \cap \dots \cap E_1) \leq P(E_i | E_{i-1}), \tag{3}$$

provided that

$$P(E_{i-1} \cap \dots \cap E_1) > 0. \tag{4}$$

An immediate useful consequence is that when inequality (3) holds for all  $i$  we obtain

$$P\left(\bigcap_{i=1}^n E_i\right) \leq P(E_1) \prod_{i=2}^n P(E_i | E_{i-1}). \tag{5}$$

To prove the theorem we need the following lemma from [3]. Below we write  $BC$  for  $B \cap C$ .

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**Lemma 2.1.** *If  $B \subseteq A \cup C$  and  $P(BC) \neq 0$  then  $P(A | BC) \leq P(A | B)$ .*

*Proof.* With  $a = P(ABC^c)$ ,  $b = P(ABC)$ ,  $c = P(A^c BC)$

$$P(A | BC) = \frac{b}{b + c} \leq \frac{b + a}{b + c + a} = \frac{P(AB)}{P(B)} = P(A | B),$$

where the inequality follows since  $f(x) = (b + x)/(b + c + x)$  is an increasing function.

*Proof of theorem.* For a given value  $j$  such that  $2 \leq j \leq k$  let  $A = E_i$ ,  $C = E_{i-j}$ , and  $B = E_{i-1}E_{i-j-1}E_{i-j-2} \cdots E_1$ . Note that by assumption (1) of the theorem

$$B \subseteq E_{i-1} \subseteq E_i \cup E_{i-j} = A \cup C,$$

and we can apply the lemma (noting that (4) gives  $P(BC) > 0$ ) to obtain

$$P(E_i | E_{i-1}E_{i-j}E_{i-j-1} \cdots E_1) \leq P(E_i | E_{i-1}E_{i-j-1}E_{i-j-2} \cdots E_1).$$

Applying this argument for  $j = 2, 3, \dots, k$  gives a chain of inequalities leading to

$$P(E_i | E_{i-1}E_{i-2}E_{i-3} \cdots E_1) \leq P(E_i | E_{i-1}E_{i-k-1}E_{i-k-2} \cdots E_1) \leq P(E_i | E_{i-1}),$$

where the second inequality follows from assumption (2) of the theorem.

### 3. Some examples

#### 3.1. The consecutive- $k$ -out-of- $n$ : $F$ system

In this section we consider the reliability of the circular and linear consecutive- $k$ -out-of- $n$  system. Such a system has  $n$  independent components arranged in either a circle or a row and fails if there are at least  $k$  consecutive failed components. The reliability of such systems has been extensively studied (see for example [1] and [2]). Here we apply inequality (5) to get a bound.

For the linear system, number the components from left to right and let  $E_i$  be the event that the  $k$  consecutive components starting with component  $i$  are not all failed. Note that

$$P(\text{system works}) = P\left(\bigcap_{i=1}^{n-k} E_i\right). \tag{6}$$

The following result mentioned in [3] holds:

**Proposition 3.1.** *If each component fails independently with probability  $q$ ,*

$$P(\text{linear system works}) \leq P(E_1) \prod_{i=2}^{n-k+1} P(E_i | E_{i-1}) = (1 - q^k) \left(\frac{1 - 2q^k + q^{k+1}}{1 - q^k}\right)^{n-k}.$$

*Proof.* Consider two adjacent or overlapping runs of length  $k$ , and a third run starting between the two. Since the components of the third run are completely contained in the other two runs, if the components in the third run are not all failed it ensures that the components in at least one of the other two runs are not all failed. This is exactly condition (1) of the theorem. Also, since non-overlapping runs are independent, condition (2) holds. The result

follows from inequality (5) applied to Equation (6) since  $P(E_1) = 1 - q^k$  and calculation gives  $P(E_i | E_{i-1}) = (1 - 2q^k + q^{k+1})/(1 - q^k)$ .

*Note.* In [3, p. 106] the theorem presented had as its assumption that

$$E_{i-1} \subseteq E_i \cup E_{i-j} \text{ for all } j \geq 2 \text{ and for all } i \geq 1.$$

Though in [3, p. 111] it was asserted to hold, in the example here this condition does not hold since for a 4-of-13 system if the first four and last four components are the only failed components, then  $E_9$  occurs but neither  $E_{10}$  nor  $E_1$  occur.

### 3.2. Non-overlapping patterns

In this section we consider the problem of waiting for a non-overlapping pattern of heads and tails of length  $k$  in a sequence of coin flips. If  $E_i$  is the event that the pattern does not appear as a run in flips  $i, i + 1, \dots, i + k - 1$ , a non-overlapping pattern is one where  $E_i^c \cap E_{i-j}^c = \emptyset$  if  $0 < j < k$ .

**Proposition 3.2.** *Letting  $T$  be the number of flips required until the pattern first appears as a run,*

$$P(T > n) \leq \prod_{i=1}^{n-k+1} P(E_i) = (P(E_1))^{n-k+1}.$$

*Proof.* Since non-overlapping implies  $(E_i \cup E_{i-j})^c = \emptyset, j < k$ , we immediately have  $\Omega \subseteq E_i \cup E_{i-j}$ , where  $\Omega$  is the whole sample space. Thus condition (1) holds for the events  $(E_1, \dots, E_{i-1}, \Omega, E_i)$ , and (2) also holds since events separated by at least  $k$  flips are independent. Applying Theorem 2.1 we get

$$P(E_i | E_1 \cdots E_{i-1}) \leq P(E_i | \Omega) = P(E_i).$$

The proposition follows using inequality (5) and noting that events  $E_{n-k+2}, \dots, E_n$  all trivially occur.

### 3.3. The circular consecutive $k$ -out-of- $n$ : $F$ system

For the circular system, number the components clockwise and, using the same definitions as the previous section, let  $E_i$  be the event that there is a clockwise run of  $k$  failed components starting with component  $i$ . The following holds:

**Proposition 3.3.**

$$P(\text{circular system works}) \leq P(E_1) \prod_{i=2}^{n-k+1} P(E_i | E_{i-1}E_1).$$

*Proof.* For a fixed value of  $i$  define events

$$(F_{i-1}, \dots, F_1) \equiv (E_i, E_{i-1} \cap E_1, E_{i-k}, E_{i-k+1}, \dots, E_{i-2}, E_2, E_3, \dots, E_{i-k-1})$$

and let  $k^* = k+n-i$ . Note that conditions (1) and (2) apply and we can then apply Theorem 2.1 using these events and the integer  $k^*$  to obtain

$$P(F_{i-1} | F_{i-2} \cdots F_1) \leq P(F_{i-1} | F_{i-2}),$$

or equivalently

$$P(E_i | E_{i-1} \cdots E_1) \leq P(E_i | E_{i-1}E_1).$$

### References

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