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# BOSTON UNIVERSITY SCHOOL OF MANAGEMENT

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## Math Notes

### **BU Note # 222-1**

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### **I. Introduction**

This note discusses the mathematics needed for SM222 – Modeling Business Decisions and Market Outcomes. Managers are concerned with quantitative decisions that the firm makes (such as price and output). It is hard to imagine principles that lead to decisions like “set price equal to \$10.50” and “produce 1.25 million units” that do not have some sort of mathematical basis. Most sections below contain not only a description of some aspect of math used in the course but also an explanation for why we use it.

First, we introduce some mathematical notation that we will use throughout the course.

- To signify that  $x$  is multiplied by  $y$ , we can write it in different forms:

$$x y \quad \text{or} \quad x * y \quad \text{or} \quad x (y) \quad \text{or} \quad x \cdot y$$

- When we write  $y = f(x)$ , we are saying that  $y$  is some function of  $x$ . In other words, the right hand side of this equation includes the variable  $x$ , in some unspecified form and with some unspecified numbers. For instance,  $y$  is a function of  $x$  in  $y = 11 + 3.2 x$ .
- A “variable” is a term that varies, and is represented by a name or letter, such as “ $X$ ” or “*population*.”

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- We often want to add several different expressions together. For instance, let's say that we have measured the height of twelve different people. If we denote each person by the index (or counter)  $i$  and denote the height of each person by  $h_i$ , then the average height of the 12 people is:

$$\text{average height} = \frac{1}{12} \sum_{i=1}^{12} h_i$$

This says, "Multiply 1/12 times the sum of all of the  $h_i$  's, beginning with the first person ( $i=1$ ) and ending with the 12<sup>th</sup> person ( $i=12$ )."

## II. Reviewing Some Basics: Order of Operations, Weighted Averages, Unit Conversion, Rounding, and Significant Digits

### A. Order of operations

Whether writing mathematical equations on paper or writing equations in computer programs, there are accepted rules about what gets calculated first. Otherwise, we wouldn't know in the equation  $X = 5 * 5 + 2$  whether  $X = 25+2=27$  or  $X = 5*10=50$ .

The accepted order of operations can be summarized by the acronym PEMDAS. We do the operations in this order:

P : Parentheses.

E : Exponents (computers use ^ to signify "to the power of").

M, D: Multiplication and Division

A, S: Addition and Subtraction

If there a string of multiplication/division operations, solve them from the left to the right.

Here are a few examples.

$$X = 5 * 5 + 2 = 25 + 2 = 27 \quad X = 2 + 5 * 5 = 2 + 25 = 27$$

$$X = 5 - 3 ^ 4 * 6 = 5 - ( 3^4 * 6) = - 481$$

$$X = 5 / 6 / 4 ^ 2 * 3 = [ (5/6) / 4^2 ] * 3 = 5/32 = .15625$$

### ***B. Weighted Averages***

The weighted average over N items is given by

$$\text{weighted average} = \sum_{i=1}^N w_i X_i$$

where  $w_i$  is the “weight” of each item and these weights sum to 1. For instance, let’s say that ten people take a test. Three of them answer 90% of the questions correctly, six answer 80% correctly, and one answers 70% correct. The average of these ten scores could be calculated as  $(90+90+90+80+80+80+80+80+80+70) / 10$ . However, it is simpler to calculate it with a weighted average:

$$\text{weighted average} = .3 \cdot 90 + .6 \cdot 80 + .1 \cdot 70 = 82$$

Thus, 90 has a weight of .3 because it occurs 3 of the 10 times. Note that the weights add up to 1.

### ***C. Unit Conversion***

To convert from one unit of measurement to another, one easy way is to put the units that you are starting with on the left of the paper and the units that you want to end with on the right. Then, in between, put the equivalencies as ratios. For instance, let’s say that you want to know what 10 gallons is in pints. First write:

$$10 \text{ gallons} \qquad \qquad \qquad = \qquad \qquad \text{pints}$$

Then, find the equivalency ratio for gallons and pints. There are 8 pints per gallon, so :

$$10 \text{ gallons} \quad * \quad \frac{8 \text{ pints}}{1 \text{ gallon}} \quad = \quad 80 \text{ pints}$$

Note that on the left side of the equation, the “gallons” cancel out since there are gallons in the numerator and in the denominator.

Here’s a more complicated example. To put 10 gallons into your car in the US, it costs \$29.50 (i.e. each gallon costs \$2.95). If you take your car to France, how many dollars will it cost to fill the car with the same amount of gasoline? Note: In Europe, gasoline is sold by the liter and costs €1.25 (where € is the symbol for euros). You can exchange your euros into dollars at the rate of 0.78973 euros per dollar. There are 3.7854 liters in a gallon.

You are starting with something in the units of gallons and changing into the units of dollars.

10 gallons = \$

To do this, write:

$$10 \text{ gallons} \quad \frac{3.7854 \text{ liter}}{\text{gallon}} \quad \frac{\text{€}1.25}{\text{liter}} \quad \frac{\text{\$}1}{\text{€}0.78973} = \$ 59.92$$

Note that the gallons cancel out, the liters cancel out and the €s cancel out so that you are left with dollars.

#### ***D. Significant Digits***

Significant digits tell us how accurately we know a number. Thus, when given the number 22.55, we know how accurate it is up to two decimal places. The rules of significant digits are:

- Non-zero digits are always significant. Thus, 22 has two significant digits, and 22.3 has three significant digits.

With zeroes, the situation is more complicated:

- Zeroes placed before other digits are not significant; 0.046 has two significant digits.
- Zeroes placed between other digits are always significant; 4009 has four significant digits.
- Zeroes placed after other digits but behind a decimal point are significant; 7.90 has three significant digits.
- It is impossible to tell whether zeroes at the end of a number, but before the decimal point, are significant without additional information. For instance, in the number 25,000, it is not clear whether this has two significant digits, or whether it has as many as 5 significant digits. With 5 significant digits, the number is exactly 25,000 and not, for instance, 25,001 or 24,999. However, if we knew there were only 2 significant digits in 25,000, it means that we know that the number might be rounded from anything between 24,501 and 25,499 (including the end points. To avoid uncertainty, you can use words like “thousands” instead of zeroes, or use scientific notation like  $25 \times 10^3$  (in which case it has 2 significant digits) or  $2.500 \times 10^3$  (in which case it has 4 significant digits.)

- Integers have essentially an unlimited number of significant digits. If I know there are two people in the room, I also know that there are 2.00 and 2.00000 people in the room. People don't come in fractions.

When doing arithmetic, it is important to understand how accurately you can identify the solution, i.e. how many significant digits it has. The rules are:

- Significant digits are In a calculation involving multiplication or division, the number of significant digits in the answer should equal the smallest number of significant digits in any one of the numbers being multiplied or divided. For instance,  $.44931 * .11 = .049$  (2 significant digits only).
- When numbers are being added or subtracted, the number of *decimal places* (not significant digits) in the answer should be the same as the smallest number of decimal places in any of the numbers being added or subtracted. Here is an example:

$$\begin{array}{r} 5.67 \text{ (two decimal places)} \\ +1.1 \text{ (one decimal place)} \\ \hline 7.7 \text{ (one decimal place)} \end{array}$$

- When doing multi-step calculations, *keep at least one more significant digit in intermediate results* than needed in your final answer. For instance, if a final answer requires two significant digits and you are multiplying 3 numbers which have 3, 5 and 4 significant digits respectively, carry at least three significant digits in all calculations. If you were instead to round-off all your intermediate answers to only two digits, you are discarding the information contained in the third digit, and as a result the *second* digit in your final answer might be incorrect.

### III. Straight lines (or, linear functions)

A major part of modeling is describing how some variable depends on another. For example, a demand curve relates the quantity demanded to price. Another example is the consumption function, which relates aggregate consumption to such underlying determinants as aggregate income and interest rates. Both in applying principles of economics and in using examples to illustrate points, it is useful to give a precise functional form to these relationships. The simplest such form is a straight line.

Any straight line can be written in the form:

$$(1) \quad y = mx + b$$

Here,  $m$  is the slope of the line, which you may have learned to define as “rise over run”.

In more mathematical terms, the slope  $m$  equals:

$$m = \frac{\Delta y}{\Delta x}$$

where  $\Delta$  denotes "the change in". In equation (1),  $b$  is the "y-intercept," or the value of  $y$  when  $x = 0$ . You should verify for yourself that the "x-intercept," the value of  $x$  when  $y = 0$ , equals  $-b/m$ .

The above quotation marks are to emphasize that the axes in our models are not always labeled "x" and "y." In many economic applications, the y axis represents price and the x axis represents quantity. An example of a linear demand curve is:

$$(2) \quad P = 100 - 5Q$$

This demand curve hits the price axis at 100, which means that for a price of 100 or greater, demand is 0.<sup>2</sup> Its slope is -5. As the equation is written here, the interpretation of this slope is that to sell one extra unit, the price must be lowered by 5.

While a line can always be written in the general form above, it does not have to be. The above form is convenient for graphing. Particularly because placing price on the y-axis and quantity on the x-axis seems to get the causality backwards, it might be desirable to solve the above equation for quantity instead of price:

$$(3) \quad Q = 20 - \frac{1}{5}P$$

(You should be able to verify that this is true.)<sup>3</sup>

As we will see in "Simultaneous Equations" below, it may also be useful to write the equation as:

$$(4) \quad P + 5Q = 100$$

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<sup>2</sup> Of course, the equation says that if the price is greater than 100, demand is negative. In most cases, though, negative demand simply does not make sense. The equation represents the demand curve for prices of 100 or lower. Similarly, the equation is assumed to represent the demand curve for prices that are 0 or greater, since a negative price usually does not make economic sense.

<sup>3</sup> Sometimes, the forms of equations (2) and (3) are distinguished by calling the latter the "demand curve" and the former the "inverse demand curve." Equation (3) is the more intuitive way of writing the relationship. Equation (2) is used frequently, however, because it simplifies graphing and the solution of certain problems (like profit-maximization for a monopolist).

#### IV. Simultaneous Equations

Most economic variables are determined by the interaction of several variables and effects. The most common example of this point is that prices and quantities are determined by the interaction of supply and demand. Most people readily accept the graphical treatment of supply and demand. In the supply-demand graph, the market price and output are determined by the intersection of the supply and demand curve. Algebraically, the price and quantity at this intersection point can be determined by the simultaneous solution of the supply and demand curves.

There are several ways to solve simultaneous (linear) equations. Probably the easiest is known as elimination. Consider the system:

$$(5) \quad \begin{aligned} 4y &= 3x - 2 \\ 5x &= 3y + 7 \end{aligned}$$

Rewrite both equations in the general form of equation (4) so that they can be stacked as follows:

$$(6) \quad \begin{aligned} 4y - 3x &= -2 \\ -3y + 5x &= 7 \end{aligned}$$

Now, the trick is to multiply each equation by a number so that when you add the two equations together, one of the variables is eliminated.

Suppose we choose to eliminate  $y$ . To do this, follow the following steps:

1. Find the coefficient on  $y$  in the second equation. In the above example, it is  $-3$ .
2. Multiply both sides of the first equation by this number.
3. Find the coefficient on  $y$  in the first equation (before it was multiplied by  $-3$ ). In the above example, it is  $4$ .
4. Multiply both sides of the second equation by the **negative** of that number. That is, multiply the second equation by  $-4$ .

These steps yield:

$$(7) \quad \begin{aligned} -12y + 9x &= 6 \\ 12y - 20x &= -28 \end{aligned}$$

We can now add the two equations together, eliminating  $y$ :

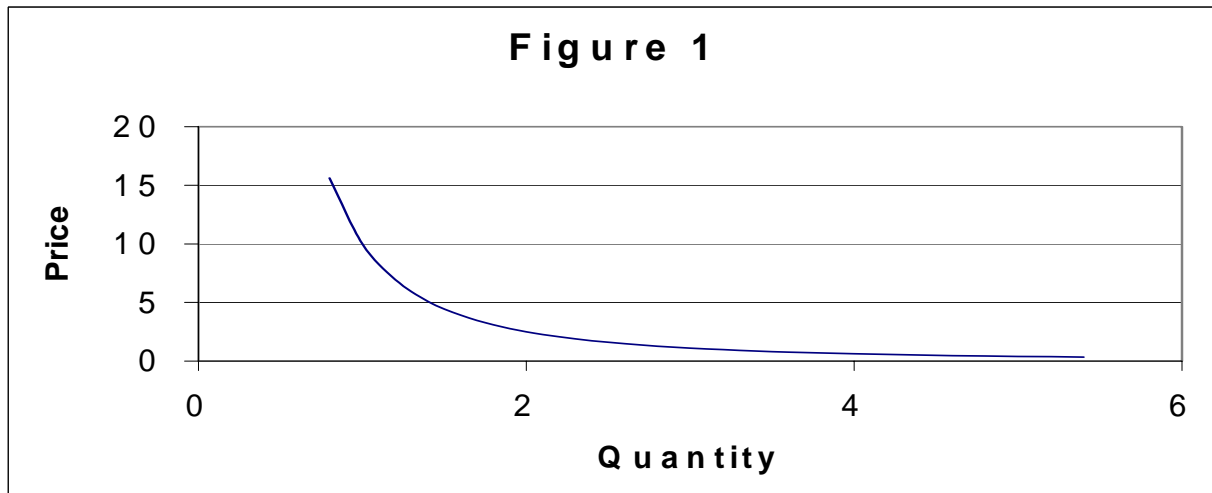
$$(8) \quad -11x = -22$$

Solving this equation for  $x$  gives  $x=2$ , which can then be plugged back into either of the original equations to get  $y=1$ . (As an exercise, solve by eliminating  $x$  instead of  $y$ .)

## V. Multiplicative and Other Nonlinear Functions

Linear relationships between two variables indicate that every time one variable goes up by 1, the other goes up (or down) by some constant amount. For instance, equation (3) says that every time price rises by \$1, Q falls by .2.

However, often the relationship between two economic variables is not linear. For instance, a demand curve might look like:



The equation behind this demand curve is not linear, but instead:

$$(9) \quad P = \frac{100}{Q^2}$$

Recalling that in general,  $1/Y^x = Y^{-x}$ , equation (9) is the same as:

$$(10) \quad P = 100Q^{-2}$$

Alternatively, equation (10) can be solved for Q. First divide both sides by  $P \cdot Q^2$ :

$$1/Q^2 = 100/P \quad \text{which is just} \quad Q^2 = 100/P \quad \text{so that}$$

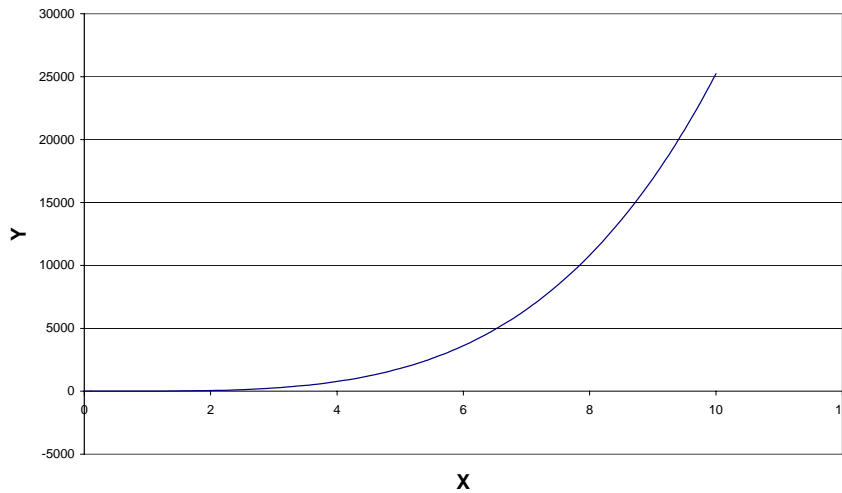
$$(11) \quad Q = \frac{10}{\sqrt{P}} = 10 P^{-0.5} \quad (\text{Recall that the square root of } P \text{ equals } P^{0.5})$$

These equations are known as “power laws” because Q varies according to a power (in this case, -0.5) of P.

Of course, the exponent in a power law might be any number, positive or negative. For instance

(12)  $Y = 4X^{3.8}$  would look like:

**Figure 2**

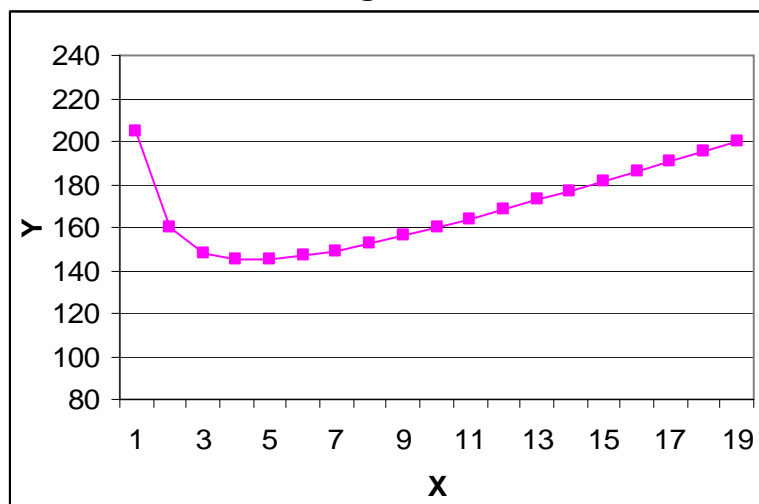


A variable may also enter into a nonlinear function more than once. Consider the following equation:

$$Y = 100 + 5X + 100/X$$

Its graph would look like this:

**Figure 3**



Often several factors simultaneously affect an economic variable. For instance, both income and price may affect the quantity demanded. The relationship might be linear, such as:

$$(13) \quad Q = 10 - \frac{1}{5}P - \frac{1}{10}I$$

where  $I$  is the average income per capital level in thousands. However, the relationship might instead be multiplicative, such as:

$$(14) \quad Q = 4.4 \frac{I}{P} = 4.4 I P^{-1}$$

Note how all terms in (13) are added together, while all terms in (14) are multiplied (or divided) by each other. The exponent of  $I$  in (14) is implicitly "1". The exponent on  $P$  is negative because higher  $P$  decreases  $Q$ . Exponents in multiplicative functions are not limited to 1 and -1, but can take any values.

Linear and multiplicative equations have very different implications. An example shows this point:

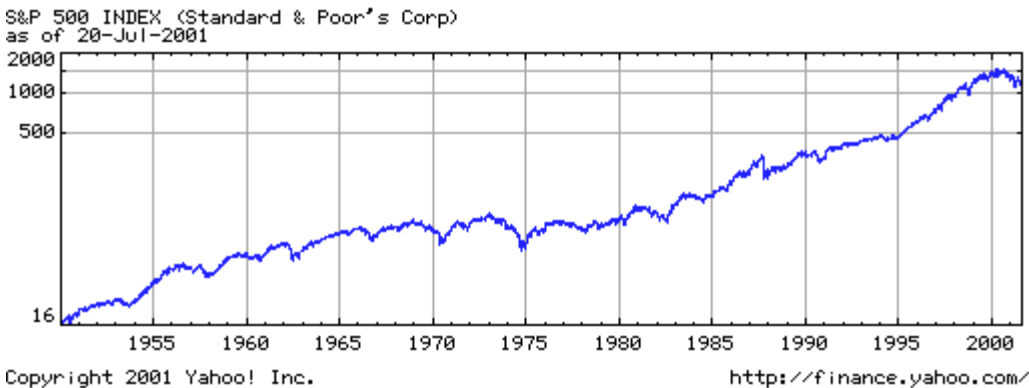
- Start at  $P=20$  and  $I=50$  (000), so  $Q = 11$  in both equations 13 and 14. (The numbers were intentionally chosen so the starting point would be the same!)
- Double the income level to 100 (000). In linear equation (13), this raises  $Q$  by  $1/10$  of 50, or 5, so  $Q$  increases to 16. In multiplicative equation (14), this doubles  $Q$  to 22, an increase of 11.
- Next start at a higher  $P$ ,  $P=40$ . (In linear equation (13),  $Q$  starts at 7 but in multiplicative equation (14)  $Q$  starts at 5.5.) If once again income doubled from 50 to 100,  $Q$  would change by 5 in linear equation (13), just as before. But in multiplicative equation (14),  $Q$  would double from 5.5 to 11.

In linear forms, an increase of 1 in one variable always changes the other variable by the same *amount*. Is this a reasonable assumption? Probably not, particularly if we are comparing very different size markets. For instance, compare demand in New York to demand in a much smaller city, Toledo. If prices rose by \$1 in each city, we wouldn't expect demand to change by exactly the same *amount* in the two different cities.

What is considerably more plausible is that a 1% increase in the price would have the same *percentage effect* on demand in the two cities. Equations (11) and (14) have this characteristic, as we will show later in this note.

## VI. Logarithms

For many students, logarithms are a dim, unpleasant memory from high school. To appreciate their use, however, look at any plot of stock prices over a reasonably long period of time. For example, if you go to Yahoo and click on "Finance/Quotes" and then do the appropriate clicks to get the S&P 500 for the maximum time span that Yahoo will give, you will see something like:



This graph is plotted on a log scale, which you can tell because the distance from 500 to 1000 equals the distance from 1000 to 2000. Note that you do not have to ask Yahoo to give you the log scale. In this case, it chooses the log scale automatically. If you really want to see the data on a linear scale, it will oblige you but it defaults to a log scale because most people who understand the difference would consider the log scale to provide the better view of the data.

Learning to use logarithms is simple and doing so will make the principles learned in managerial economics and other courses a much more practical tool for you. In comparing data, it is frequently more meaningful to do so in percentage rather than absolute terms. For example, real GNP grew by \$94.7 billion (1982 dollars) from 1949 to 1950 and by \$99.2 billion from 1985 to 1986. In absolute terms, the latter number is bigger. The percentage growth was greater during the earlier period (8.5% vs. 2.7%), however, and it is the percentage growth that is most important. The great value of logarithms arises from the property that *the difference in the log of two numbers measures their percentage difference*.

Logs can be expressed to any base. You are probably most familiar with the base of 10. When "log" is used without a subscript, we will understand that the base is 10.

Logs are exponents of the base. For instance, if:

$$y = 10^x$$

Then the log of  $y$  to the base 10 is  $x$ , which we write:  $\log_{10}y = x$  or simply  $\log y = x$

Another way to define a log to the base 10 is:

$$10^{\log y} = y$$

More commonly in this course, we will use natural logs where the base is the irrational number  $e = 2.71828\dots$ <sup>4</sup> We use the notation  $\ln$  to denote the natural log:

$$y = e^x \text{ implies } \ln y = x.$$

$$e^{\ln y} = y$$

Recall the following properties about logs, which apply no matter what base is used:

$$\log a b = \log a + \log b$$

$$\log a/b = \log a - \log b$$

$$\log a^b = b \log a$$

Combining these properties,  $\log (a^b y^x) = b \log a + x \log y$

**A. Logs and exponential growth paths (Optional, but good practice with logs)**

To see the value of using logs, consider forecasting future sales of a product with a time trend. As we will discuss in class, you would generally like to do something more sophisticated than this, but time trends are usually a good way to start.

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<sup>4</sup> Optional: You do not need to remember the numerical value of  $e$ . Calculators and computers have the exponential function built in to them. There is, however, a business related explanation for what  $e$  is. Suppose a bank offers you  $x$  percent annual interest. (For concreteness, let us say that  $x$  is 5%). The effective rate of interest depends on how frequently the bank compounds the interest. If the bank compounds the interest semi-annually, it gives you 2.5% interest twice a year. This will give you slightly more money at the end of the year than if the interest is compounded annually because the second 2.5% interest is applied not only to the principle but also to the 2.5% interest earned after 6 months. By the same principle, you get still more money if the interest is compounded quarterly (1.25% four times a year) and still more if it is compounded monthly ( $5\% / 12 = .4167\%$  twelve times a year). If the interest is compounded continuously, then the total amount of money after a year (principle plus interest) is  $e^x$ , (where  $x$  is now expressed as a decimal instead of a percentage). In the above example, 5% compounded continuously gives an effective annual rate of 5.1271%. ( $e^{.05} = 1.051271$ ). Now, the notion of continuous compounding sounds rather abstract, but it seems less so when you think of it as an approximation to very frequent compounding. It is not unusual for banks to offer accounts in which interest is compounded daily. Daily compounding of 5% interest gives an effective annual rate of 5.1268%, which is within 3 100ths of a basis point of the approximation above. (A basis point is a 100th of a percentage point.)

You might begin this task by simply plotting sales against time. Then, you might use a ruler to draw a line that seems to fit the data best.<sup>5</sup> The line could be expressed algebraically in the form:

$$(15) \quad S_t = a + bt$$

where  $t$  is time,  $S_t$  is sales at time  $t$ , and  $a$  and  $b$  are the values of the intercept and slope of the line. You can then make your forecast by extending the line out to future years. For instance, in 5 years, forecasted sales are  $S_5 = a + b \cdot 5$ .

The problem with the above approach is that when you look at many time plots of economic variables, they do not appear linear. Rather, many of them bend upward as in Figure 2 (page 5). There is a good reason to expect this to be the case. A linear model assumes that the absolute change will be the same from year to year. A more realistic assumption is that the percentage change is constant from year to year. Therefore, a better way to forecast sales is to estimate what  $c$  and  $g$  are in the equation:

$$(16) \quad S_t = c g^t$$

Here,  $c$  represents sales at the beginning, when  $t=0$ ;  $g$  represents the growth factor. For instance, if sales started at 100 and grew 10% a year for 5 years,

$$S_5 = 100 \cdot 1.1^5 = 161.051$$

While (16) is generally a more realistic model than (15), it has what might appear to be a major disadvantage: It is difficult to sketch accurately. Later in the course, we learn the statistical method regression to get the **line** that fits the data best, but the method of regression only fits lines, not multiplicative functions such as (16).

Here is where logs come in. Equation (16) can be made linear by taking natural logs of both sides.

$$(17) \quad \ln S_t = \ln [ c g^t ]$$

Applying the property  $\ln a b = \ln a + \ln b$  yields:

$$\ln S_t = \ln c + \ln g^t$$

Further applying the property  $\ln a^b = b \ln a$  to equation (17), we get:

$$(18) \quad \ln S_t = \ln c + t \ln g$$

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<sup>5</sup> Later in this course, we learn regression, a statistical method that more accurately estimates a linear relationship from data of this sort.

Equation (18) is a line, where  $\ln S_t$  is graphed on the vertical (y) axis and  $t$  is graphed on the horizontal (x) axis. In this line,  $\ln c$  is the y-intercept and  $\ln g$  is the slope.

For instance, again assuming that  $c$  is 100 and  $g$  is 1.1, after 5 years, we forecast:

$$\ln S_t = \ln 100 + 5 \ln 1.1 = 4.6052 + 5 \cdot 0.0953 = 5.08172$$

To find  $S_t$  itself, take  $e$  to the power of each side of this equation:

$$e^{\ln S_t} = e^{\ln c + t \ln g} = e^{5.08172} = 161.057$$

### ***B. Logs and constant elasticity demand***

The other major application of natural logs will be to estimate elasticities. Recall from microeconomics that the elasticity of quantity with respect to price equals the percentage change in quantity as price increases by 1%.

Suppose you want to estimate a demand relationship in which the quantity demanded depends on price and income. For concreteness, let us suppose that you want to estimate the demand for cars. You might consider estimating the demand curve in linear form:

$$(21) \quad Q_t = b_0 + b_1 P_t + b_2 I_t$$

where  $Q_t$  is the quantity of cars demanded at time  $t$ ,  $P_t$  is the price of cars at time  $t$ , and  $I_t$  is income at time  $t$ . The problem with this equation is that because it is linear, it embodies the assumption that a \$100 increase in the price of cars would have had the same effect on the quantity demanded in, say, 1950 as it does in 1999. Because there are so many more cars sold now (and because the real price of cars has changed), this is wildly implausible. What is considerably more plausible is that a 1% increase in the price would have the same percentage effect on demand in 1950 and 1999, or in other words, that the elasticity is constant. This is also a strong assumption. (You cannot estimate a demand curve that is this simple without making strong assumptions.) But of the strong assumptions that one would like to make to facilitate estimation, constant elasticity is more likely to be nearly true than linearity.

As you will demonstrate to yourself in problem set 1, an equation that embodies the assumption of constant elasticity is the multiplicative form:

$$(22) \quad Q_t = c P_t^{-\varepsilon} I_t^{\varepsilon_I}$$

The notation is chosen to emphasize that  $\varepsilon$  is the elasticity of demand and  $\varepsilon_I$  is the income elasticity of demand.

As with the exponential time trend, this equation is not linear. However, by taking natural logs of both sides, we get a line in three dimensions (Q, P and I) of the form:

$$(23) \quad \ln Q_t = \ln c - \varepsilon \ln P_t + \varepsilon_I \ln I_t$$

For some applications, such as choosing the optimal price, the elasticity is all we want to estimate. In others, such as choosing output at that price, we need an estimate of quantity. This can be done in one of two ways. The first is to solve (23) for the natural log of Q and then find its exponential (i.e. calculate  $e^{\ln Q}$ ). The second is to take the exponential of  $\ln c$  to estimate c (i.e.  $c = e^{\ln c}$ ) and then use that estimate along with the elasticities to solve (22) directly. These are numerically equivalent. I find the first approach easier. Those of you who are still suspicious of logs might find the second way to be more intuitively appealing.

## VII. Slopes and Derivatives

### A. Slopes

The salient feature of many economic and other business relationships is the effect that a change in one variable has on another. For example, we might want to answer the question, "If price goes up by one dollar, by how much does demand drop?" Of course, we could always answer that question by substituting the current price and the current price plus a dollar into the demand curve and calculating the difference. There is an easier way, however.

In mathematical terms, the question being asked here is, "what is the slope of the demand curve (when it is written in the form of equation (3)). The slope (m) is defined as,

$$(24) \quad m = \frac{\Delta Y}{\Delta X}$$

Of course, you should already know what the slope of a line is. The key point here is that the mathematical notion of a slope is the answer to a question that is of economic interest. When a demand curve is written as it is in (3), then Q corresponds to y in equation (1), P corresponds to x, 20 corresponds to b, and -1/5 corresponds to m (slope). Rather than plugging values into the demand curve, we need only examine the coefficient on Q to know that if P goes up by 1, Q drops by 1/5. (Satisfy yourself that you get this answer by plugging values into the demand curve).

## **B. Derivatives**

The key feature that distinguishes a straight line from other functions is that its slope is the same everywhere. That is, the answer to the question, "how much does demand drop if the price goes up by 1?" does not depend on what the initial price is. Our interest in this economic question does not change, however, simply because the answer does not happen to be a constant. The generalization of the concept of a slope that applies even if the demand curve is not linear is a derivative. The derivative is a function that gives the slope of a curve at each point.<sup>6</sup> The notation for the derivative of, say, quantity (Q) with respect to price (P) is "dP/dQ."

To many students, derivatives seem much more complicated than slopes. It is important to keep in mind, though, that we are interested in derivatives for precisely the same reason that we are interested in slopes. We want to be able to quantify the change in one variable in response to a change in another. Using derivatives instead of slopes simply allows us to do so for a broader set of functions. If you find yourself resisting the concept of a derivative, you may always think of it as the slope. You will never go far wrong by mentally substituting  $\Delta P/\Delta Q$  for  $dP/dQ$ .

In addition to understanding conceptually what a derivative is, you need to know rules for taking derivatives. The first two rules are simple to memorize.

Rule 1. If  $y = a$  where  $a$  is a constant, then

$$\frac{dy}{dx} = 0$$

This rule is intuitive. If you graph a constant, you get a horizontal line i.e. a line with a slope of zero.

Rule 2. If  $y = f(x) + g(x)$  where  $f(x)$  and  $g(x)$  are two different functions of  $x$ , then

$$\frac{dy}{dx} = \frac{df}{dx} + \frac{dg}{dx}$$

In words, the derivative of the sum of two terms with respect to  $x$  is just the derivative of each term with respect to  $x$ , added together.

The final rule of derivatives we use in SM222 is less intuitive and should be memorized:

Rule 3. If  $y = a x^b$

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<sup>6</sup> The slope of a curve is the slope of the line that is tangent to the curve.

where "a" is coefficient and "b" is an exponent, the derivative of y with respect to x is given by:

$$\frac{dy}{dx} = b a x^{b-1}$$

In taking the derivative, take the original expression, multiply it by the exponent (b) and reduce the exponent by 1.

Here are some examples. If

$$(25) \quad y = 25 x^2$$

Then,

$$(26) \quad \frac{dy}{dx} = 2 \cdot 25 x^{2-1} = 50 x$$

Another example combining the three rules is:

$$(27) \quad y = 25 x^2 + 200 x + 3000$$

then, recalling that  $x^0 = 1$ ,

$$(28) \quad \frac{dy}{dx} = 2 \cdot 25 x^{2-1} + 1 \cdot 200 x^{1-1} + 0 = 50 x + 200$$

The exponent does not have to be either positive or an integer. For example, if

$$(29) \quad y = 20 x^{-2.5}$$

then:

$$(30) \quad \frac{dy}{dx} = -2.5 \cdot 20 x^{-2.5-1} = -50 x^{-3.5}$$

### VIII. Maximization

Most of the theory that we will discuss is based on the assumption that firms maximize profits. Imagine a smooth curve that goes up, has a maximum, then turns down. At the maximum, the slope is zero. Therefore, if:

- We want to maximize  $y$
- We can choose  $x$  ( $x$  is the “choice variable”) and
- We know the relationship between  $y$  and  $x$ ,

$$y = f(x)$$

- Then we can find the value of  $x$  that maximizes  $y$  by find the derivative (or slope) of  $y$  and finding the value of  $x$  where that derivative is zero.

$$\frac{dy}{dx} = 0$$

For example, suppose that a firm faces the demand curve given by equation (2) in section II of these notes, and that its unit costs (both marginal and average) are 10. Thus, its total costs are  $10Q$ . The firm's profit function is:

$$\begin{aligned} (31) \quad \pi &= R - TC \\ &= PQ - cQ \\ &= (100 - 5Q)Q - 10Q \\ &= 90Q - 5Q^2 \end{aligned}$$

where  $R$  is revenue,  $TC$  is total cost, and  $c$  is unit cost. The second equation follows because revenues are price times quantity and total costs can be written as average cost times quantity. The third equation comes from substituting the demand curve (2) for the price. The fourth equation is simply combining terms.

Given this demand curve, the assumption that the firm maximizes profits implies that the firm chooses the specific profit-maximizing output (and, in turn, price). To find what that output is, we take the derivative of the profit function and set it equal to 0. Using the derivative rules described in section above, we get:

$$(32) \quad \frac{d\pi}{dQ} = 90 - 10Q \quad 90 - 10Q = 0$$

Solving for  $Q$  gives  $Q = 9$ . (Substituting  $Q=9$  back into (2) gives  $P = 55$ ).

## IX. Some Concluding Comments

The objective of modeling business decisions is to make the principles of economics, marketing and other fields a useful tool for you in making decisions, for instance about pricing, output, advertising and investments. These decisions are inherently quantitative, and the right quantitative answer cannot be derived without mathematics.

Consider pricing decisions. Without any mathematics, it should be clear to you that, all else equal, a profit-maximizing firm will choose a higher price if the demand for its output is not particularly sensitive to price than if it is. By itself, however, that principle does not provide much of a guide about exactly what price to charge or even what the optimal price depends on. Using mathematics, however, we can show that the optimal price is given by  $P = MC/(1-1/\varepsilon)$ , where MC is marginal cost and  $\varepsilon$  is the elasticity of demand. This formula makes the general principle articulated above more precise. In addition, it indicates that the optimal price depends on two variables, the demand elasticity and marginal cost. Still, the formula is not operational if you have no idea how to determine MC and  $\varepsilon$ . Hopefully, your engineers and accountants will be able to tell you MC provided that you are careful enough to ask for marginal instead of average cost. I do not want to minimize the difficulties in getting good estimates of the elasticities of demand. Still, without the material discussed above, you haven't a prayer of getting any closer to it than a swag.<sup>7</sup>

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<sup>7</sup> Swinging Wild-A\_\_ed Guess.