# STATIONARY EQUILIBRIA OF ECONOMIES WITH A CONTINUUM OF HETEROGENEOUS CONSUMERS

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**Abstract**

This paper studies stationary equilibria of a production economy with a continuum of heterogeneous consumers subject to borrowing constraints and individual Markov endowment shocks. The existence of a stationary equilibrium is established. The equilibrium interest rate is less than the rate of time preference of the most patient consumer provided a constraint that restricts the borrowing limit is satisfied. Properties of stationary equilibria including wealth distribution, precautionary savings and comparative statics are analyzed.

**Key words:** stationary equilibria, heterogeneity, precautionary savings, lattice theory, aggregation, comparative statics, borrowing constraint, wealth distribution, law of large numbers

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# 1 INTRODUCTION

## 1.1 Outline

A class of incomplete markets models, referred to as the *Bewley-style model*, has been extensively studied [9, 11, 15, 24, 1, 2, 20, 25, 26]. These studies are motivated by the fact that the standard representative-agent model (or complete markets model) fails to explain many phenomena observed in the data, e.g., the equity premium puzzle, the low risk-free rate puzzle, inequality, dispersion and skewness in wealth distribution, high concentration of stock ownership, large volumes of trade, and the high volatility of individual consumption relative to aggregate consumption. This suggests that models with heterogeneity and incomplete markets may be useful. The Bewley-style model is the workhorse of this class of models.

The typical environment can be described as follows. There is a continuum of consumers who make consumption and savings decisions subject to borrowing constraints and labor endowment shocks. There is one asset (capital) and aggregate shocks are absent.[1] Finally, a single competitive firm hires workers and rents capital to produce output for consumption and saving. A *stationary (competitive) equilibrium* is defined by a system of *constant* prices (interest rate and wage) and allocations such that individuals optimize and markets clear.

This paper studies the existence and properties of stationary equilibria when (i) consumers differ in preferences and endowment shocks (including both their distributions and realizations), and (ii) endowment shocks follow Markov processes with a compact state space.

The key to the analysis is to reformulate the Bewley-style model along the lines of [22] and [21]. In particular, the dynamic economy is described in terms of a sequence of deterministic aggregate distributions over consumers' characteristics (individual asset holdings and the realization of endowment shocks) across the population.[2] The associated long-run invariant distribution is the principal object of study. The main results of the paper are Theorems 4.1-4.3.

Theorem 4.1 establishes that a stationary equilibrium exists in which the interest rate is less than the rate of time preference of the most patient consumer provided a constraint that restricts the borrowing limit is satisfied. Moreover, in any stationary equilibrium, if this constraint on the borrowing limit is satisfied, then the interest rate cannot exceed or equal the rate of time preference of the most patient consumer. This contrasts sharply with the deterministic case where the equilibrium interest rate is equal to the rate of time preference of the most patient consumer (e.g., [6]). Thus, aggregate savings in the Bewley-style model exceed that in the deterministic case. These extra savings are often called

[1] See [30] for a numerical analysis and [36] for a theoretical analysis of models with aggregate shocks.

[2] Similar formulation is adopted in models of anonymous games [33, 27, 7, 29, 18].

precautionary savings.

Theorem 4.2 characterizes the wealth distribution and the mass of consumers who are borrowing constrained in any stationary equilibrium. It shows that the wealth distribution depends crucially on differences in population, discount factors, borrowing constraints, risk aversion, and shock distributions of different types of consumers. In particular, these differences can help to explain skewness in wealth distribution. For instance, consider differences in discount factors and population.[3] Theorem 4.2 shows that (i) aggregate savings and aggregate wealth for the more patient consumers are greater than for the less patient; (ii) the fraction of the borrowing constrained consumers among the more patient consumers is smaller than that among the less patient; (iii) only more patient consumers can hold very large assets. Therefore, if the mass of more patient consumers is sufficiently small, then a small fraction of rich consumers can hold very large assets and wealth, and a large fraction of poor consumers will be borrowing constrained.

Finally, Theorem 4.3 establishes the following comparative statics result: If the common borrowing constraint for each consumer is tightened, or if almost every consumer's discount factor is increased, then there is a corresponding equilibrium such that the interest rate falls and aggregate savings increase. This relation between the borrowing constraint and the interest rate (aggregate savings) has been numerically demonstrated by a number of studies surveyed in [20]. However, it has not been established formally in the literature.

The analysis must surmount two major difficulties. First, since markets are incomplete, the usual social planning approach cannot be applied. This leads me to analyze each individual agent's optimization problem using dynamic programming. Note that each individual's decision depends on his own individual state (asset holdings and the realization of shock), and the aggregate distribution over individual states in the population. The effect of the aggregate distribution on an individual's decision problem is transmitted through market prices — the interest rate and the wage. If the aggregate distribution is random, then it makes each consumer's decision problem and analysis of the model complicated. In fact, a stationary equilibrium requires that aggregate (economy-wide) variables be constant. In particular, the aggregate distribution is nonrandom and it enters each consumer's decision problem as a parameter.

Second, in order to make aggregate distributions nonrandom, one must rely on some law of large numbers for a continuum of random variables. Judd [28] points out that the sample path of these random variables may be nonmeasurable and the usual law of large numbers cannot hold even when there is no measurability problem. There are several approaches in the literature to dealing with these problems [19, 47, 4, 44]. In this paper, I apply the construction of Feldman and Gilles [19, Proposition 2] that dispenses with the

[3] In a model with aggregate shocks, Krusell and Smith [30] show numerically that differences in discount factors can match skewness.

cross sectional independence condition for a continuum of random variables.[4]

## 1.2 Related Literature

I now review briefly the related literature. In a model with no production, no borrowing, and Markov shocks with a finite state space, Bewley [10, 11] proves the existence of stationary equilibrium. A similar model in which borrowing is allowed is analyzed in [15] and [24]. Clarida [15] assumes i.i.d. shocks. Huggett [24] considers Markov shocks, but relies on numerical methods. For a production economy, Aiyagari [1] informally analyzes the existence and properties of stationary equilibrium for i.i.d. shocks and bounded utility functions. He then provides numerical results for Markov shocks. Subsequently, Huggett [25] and Huggett and Ospina [26] analyze some properties of stationary equilibria. However, they do not study the existence and comparative statics.

The above cited papers often ignore the technical issues surrounding measurability and the law of large numbers. Moreover, most of those papers focus on ex ante identical consumers. Then, taking some law of large numbers for granted, the aggregate distribution over individual states across consumers equals the individual distribution of any consumer's states so that aggregate asset holdings, consumption and wealth equal the corresponding individual expected values. This greatly simplifies analysis because the steady-state value of any aggregate variable equals the expected value of the corresponding individual variable with respect to its stationary distribution.

Recently, Barut [5] informally invokes the law of large numbers described in [44] and establishes the existence of stationary equilibrium. He assumes no borrowing, finitely many types of consumers and bounded utility functions. Moreover, he does not analyze behavior of the equilibrium interest rate and aggregate savings.

My analysis permits unbounded (time-additive expected) utility functions and Markov shocks. This is motivated by (i) most widely-used utility functions such as power and logarithmic utility are unbounded; and (ii) individual earnings data are best fitted by Markov processes (see, e.g., [1]). My proof of the existence of stationary equilibrium follows the idea in [10, 15, 1]. Specifically, given the Feldman-Gilles construction, I focus on the individual consumption and savings problem taking the interest rate and wage as given. I establish conditions such that there is an ergodic measure over the joint process of asset holdings and shocks. Then I show that the invariant aggregate distribution is given by the mean of the individual distribution taken with respect to the Lebesgue measure over an index set of consumers. Thus, the cases of ex ante identical consumers, finitely and countably many types of consumers, and uncountably many types of consumers can all be dealt with. Finally, after establishing properties of aggregate capital supply, I construct a stationary equilibrium by finding an interest rate such that the capital market clears. To my knowl-

[4]This approach is applied to models of strategic market games with i.i.d. shocks in [29].

edge, the existence theorem 4.1 has not been established formally in the literature under the general assumptions made here.

Importantly, I use lattice theory [46, 23, 37] in order to provide more thorough characterizations of stationary equilibria. This is achieved mainly via comparative statics analysis on the optimal policy functions, and the ergodic set and ergodic distribution of the joint process of asset holdings and endowment shocks. Such an analysis also generalizes several results and insights provided by the literature on one-person consumption and saving models in a partial equilibrium framework [40, 41, 8, 31, 34, 32, 42, 14, 16, 13].

Finally, note that all my analysis extends to the case of a pure exchange economy where the single asset is in zero net supply. In particular, an existence theorem similar to Theorem 4.1 can be obtained.

The remainder of the paper is organized as follows. Section 2 sets up the model. Section 3 studies the one-person decision problem. Section 4 proves the existence of stationary equilibrium and analyzes the properties of stationary equilibria. Proofs are relegated to appendices.

# 2 THE MODEL

Consider an economy with a large number of infinitely-lived consumers and a single firm. Time is discrete and denoted by $t = 0, 1, 2, \ldots$. Uncertainty is represented by a probability space $(\Omega, \mathcal{F}, P)$ on which all stochastic processes are defined.

*Notation.* For any subspace $\mathbb{D}$ in some $d$-dimensional Euclidean space $\mathbb{R}^d$, denote by $\mathcal{B}(\mathbb{D})$ the Borel $\sigma$-algebra of $\mathbb{D}$, by $\mathcal{P}(\mathbb{D})$ the space of probability measures on $\mathcal{B}(\mathbb{D})$ endowed with the weak convergence topology, by $C(\mathbb{D})$ the set of real-valued continuous functions on $\mathbb{D}$. Any product topological space is endowed with the product topology. Finally, for any sets $\mathbb{D}$ and $\mathbb{E}$ in some Euclidean space, $\mathcal{B}(\mathbb{D}) \otimes \mathcal{B}(\mathbb{E})$ denotes the product $\sigma$-algebra.

## 2.1 Consumers

There is a continuum of consumers distributed on the interval $I = [0, 1]$ according to the Lebesgue measure $\phi$. Consumers may differ in preferences and endowment shock processes.

*Information structure and endowments.* Consumer $i \in I$ is endowed with one unit of labor at each date $t$ and a deterministic asset level $a_0^i \in (0, \infty)$ at the beginning of time 0. Labor endowment is subject to random shocks represented by a stochastic process $(s_t^i)_{t\geq 0}$ where $s_0^i$ is a deterministic constant.

At the beginning of date $t$, consumer $i$ observes his labor endowment shock $s_t^i$. His information is represented by a $\sigma$-algebra $\mathcal{F}_t^i$ generated by past and current shocks $\{s_n^i\}_{n=0}^t$.

Assume that $(s_t^i)$ satisfies:

**Assumption 1** *For $\phi$-a.e. $i$:*

*(a) $(s_t^i)$ is a Markov process with the stationary transition function $Q^i : \mathbb{S} \times \mathcal{B}(\mathbb{S}) \to [0,1]$, where $\mathbb{S} \equiv [\underline{s}, \overline{s}] \subset \mathbb{R}_{++}$.*

*(b) $Q^i$ has a positive density $Q^i(s, ds')/ds'$ and satisfies that for all $s^1, s^2 \in \mathbb{S}$, $\int_{\mathbb{S}} |Q^i(s^1, ds')/ds' - Q^i(s^2, ds')/ds'| ds' < \kappa |s^1 - s^2|$, where $\kappa > 0$ is small enough as detailed in Remark 2.4.*

*(c) $Q^i$ is monotone in the sense of first-order stochastic dominance: For any bounded, increasing and measurable function $h : \mathbb{S} \to \mathbb{R}$, $\int_{\mathbb{S}} h(s') Q^i(s, ds')$ is increasing in $s$.*

**Remark 2.1** *Part (b), adapted from [17], imposes a smoothness condition on $Q^i$ or its density. It is stronger than the Feller property which requires that $\int_{\mathbb{S}} h(s') Q^i(\cdot, ds')$ be continuous if $h$ is a bounded and continuous function on $\mathbb{S}$. It is important for monotonicity of the optimal savings policy in the realization of shocks. Part (c) captures persistence in earnings. It is key to establishing monotonicity of the value and policy functions in the realization of shocks. Note that the continuous state space assumption is not essential. Most results to follow are still valid for a countable state space.*

*Consumption Space.* There is a single good. A *consumption plan* $c^i \equiv (c_t^i)_{t=0}^{\infty}$ for consumer $i$ is a nonnegative real-valued process such that $c_t^i$ is $\mathcal{F}_t^i$-measurable. Denote by $\mathcal{C}^i$ the space of all consumption plans for consumer $i$.

*Budget and borrowing constraints.* An *asset accumulation plan* $(a_{t+1}^i)_{t \geq 0}$ for consumer $i$ is a real-valued process such that $a_{t+1}^i$ is $\mathcal{F}_t^i$-measurable.

In each period $t$, consumer $i$ consumes $c_t^i$ and accumulates assets $a_{t+1}^i$ subject to the familiar budget constraint:

$$c_t^i + a_{t+1}^i = (1 + r_t) a_t^i + w_t s_t^i, \ a_0^i \text{ given}, \tag{1}$$

where $r_t$ is the interest rate and $w_t$ is the wage. Each consumer $i$ can borrow, but there is a common lower bound on assets $\underline{a}_t \leq 0$ at each date $t$ for all consumers.[5] Thus the borrowing constraint is given by:

$$a_{t+1}^i \geq \underline{a}_t. \tag{2}$$

To ensure that debt is eventually repaid, $\underline{a}_t$ must be specified further. One specification provided by Aiyagari [1] is:

$$\underline{a}_t = \max\{-b, -\sum_{s=1}^{\infty} w_{t+s} \underline{s} / R_{t,s}\}.$$

[5] The analysis in the sequel extends to the case where different types of consumers face different borrowing limits.

where $b \geq 0$ and $R_{t,s} = (1 + r_{t+1}) \cdots (1 + r_{t+s})$. Thus, the consumer's indebtedness is limited by the lesser of (i) an exogenously specified borrowing limit $b$, and (ii) the present value of his lowest level of labor income, $\sum_{s=1}^{\infty} w_{t+s}\underline{s}/R_{t,s}$.

If $r_t = r$ and $w_t = w$ are constants for all $t$, then the above specification becomes

$$\underline{a}_t = \underline{a} \equiv \max\{-b, -w\underline{s}/r\} \text{ if } r > 0;\ \underline{a}_t = -b \text{ if } r \leq 0. \tag{3}$$

Figure 1 plots $\underline{a}$ as a function of $r$. Note that the analysis to follow permits other forms of borrowing limits.

Finally, let $\mathbb{A} = [\underline{a}, \infty)$, and denote by $\mathcal{A}^i$ the set of all asset accumulation plans of consumer $i$ that satisfy the budget constraint (1) and the borrowing constraint (2). A consumption plan $c \in \mathcal{C}^i$ corresponding to an asset accumulation plan $a \in \mathcal{A}^i$ is called (budget) *feasible*.

*Preferences.* Consumer $i$'s preferences are represented by an expected utility function defined on $\mathcal{C}^i$ :

$$U^i(c) = E\left[\sum_{t=0}^{\infty} (\beta^i)^t u^i(c_t)\right],\ (c_t) \in \mathcal{C}^i,$$

where $\beta^i$ is the discount factor satisfying $0 < \beta_{\min} \leq \beta^i < 1$, and $u^i : \mathbb{R}_+ \to \mathbb{R}$ is the felicity function satisfying:

**Assumption 2** *For $\phi$-a.e. $i$,*

*(a) $u^i$ is strictly increasing, strictly concave, twice continuously differentiable and satisfies $u^i(0) = 0$ and $(u^i)'(0) = \infty$;*[6]

*(b) $1 < \gamma^i = -\lim_{c\to\infty} \log(u^i)'(c)/\log(c) < \infty$;*

*(c) $-(u^i)''(c) \geq \eta > 0$ for any bounded set in $\mathbb{R}_{++}$.*

**Assumption 3** *For $\phi$-a.e. $i$, there is a $\theta > 0$ such that*

$$Var\left(\sum_{\tau=0}^{\infty} (\beta^i)^\tau s^i_{t+\tau} \,\middle|\, \mathcal{F}^i_t\right) \geq \theta,\ t \geq 0.$$

**Remark 2.2** *(i) The assumption $u^i(0) = 0$ is a convenient normalization. Its content is boundedness below. It is not crucial as illustrated by the examples in Section 3.1. The Inada condition $(u^i)'(0) = \infty$ ensures that optimal consumption is always positive. In this case, the borrowing limit $\underline{a}_t$ must satisfy $\underline{a}_{t+1} < (1+r_t)\underline{a}_t + w_t s_t$, for all $t$, so that a positive consumption plan is always feasible.*

[6] I write $\lim_{c\to 0+}(u^i)'(c)$ as $(u^i)'(0)$. Similar notation applies to any right or left derivative. When I say a function is differentiable on a closed (or half closed) interval, I mean that this function is differentiable on its interior and has a finite left or a right derivative.

*(ii) $-\gamma^i$ is called the asymptotic exponent of $(u^i)'$ [12, 41, 42]. Assumption 2 (b) implies that if $c > c^0$ and $0 \le \rho_1 < \gamma^i < \rho_2$, then $(c/c^0)^{\rho_1} \le (u^i)'(c^0)/(u^i)'(c) \le (c/c^0)^{\rho_2}$ for $c^0$ large enough. It also implies that $u'(\infty) = 0$. If it is violated, then there may exist some positive $\phi-$measure of consumers whose asset holdings go to infinity and no long-run distribution for assets would exist. See [41] for an example of exponential utility that violates (b).*

*(iii) Part (c), adapted from [39], imposes a strong form of concavity on $u^i$. Its role is to establish monotonicity of the optimal asset accumulation policy in the realization of shocks.*

**Remark 2.3** *Assumption 3 is adapted from [13] and will be used only to establish long-run behavior of optimal asset holdings when $\beta^i(1+r) = 1$. It requires that the shock processes be 'sufficiently stochastic' in the sense that, for almost every consumer, the conditional variance of discounted future endowment shocks is uniformly bounded away from zero.*

*Decision problem.* Consumer $i$'s problem is given by:

$$\sup_{(c_t^i, a_{t+1}^i)_{t\ge 0} \in \mathcal{C}^i \times \mathcal{A}^i} U^i(c^i). \tag{4}$$

The *value function* $V^i(a,s)$ is defined by the above supremum when $(a_0^i, s_0^i) = (a,s) \in \mathbb{A} \times \mathbb{S}$. The plans $(c_t^i)$ and $(a_t^i)$ are optimal if $V^i(a,s)$ is achieved by $(c_t^i, a_{t+1}^i)_{t\ge 0} \in \mathcal{C}^i \times \mathcal{A}^i$.

*Allocation.* An *allocation* $((c_t^i, a_{t+1}^i)_{t\ge 0})_{i\in I}$ is a collection of consumption and asset accumulation plans $(c_t^i, a_{t+1}^i)_{t\ge 0}$, $i \in I$. An allocation $((c_t^i, a_{t+1}^i)_{t\ge 0})_{i\in I}$ is *admissible* if both $c_t^i = c_t(i,\omega)$ and $a_{t+1}^i = a_{t+1}(i,\omega)$ are $\mathcal{B}(I) \otimes \mathcal{F}_t$-measurable where $\mathcal{F}_t$ is the smallest $\sigma$-algebra containing $\mathcal{F}_t^i$ for all $i \in I$, $\mathcal{F}_t = \vee_{i\in I} \mathcal{F}_t^i$, $t \ge 0$. This measurability requirement ensures certain integrals are well defined (see [18] for further discussion if it is violated). Since both $c_t^i$ and $a_{t+1}^i$ are $\mathcal{F}_t^i$-measurable for all fixed $i \in I$, they are also $\mathcal{F}_t$-measurable. Thus, the essential content of admissibility is that $c_t^i$ and $a_{t+1}^i$ must be $\mathcal{B}(I)$-measurable for each fixed $\omega \in \Omega$. To ensure that admissible allocations exist, I assume:

**Assumption 4** *For each $t$, $s_t : I \times \Omega \to \mathbb{S}$ is $\mathcal{B}(I) \otimes \mathcal{F}_t$-measurable. Furthermore, as functions of $i$, (a) $\beta : I \to (0,1)$ is $\mathcal{B}(I)$-measurable; (b) $u(\cdot, c) : I \to \mathbb{R}$ is $\mathcal{B}(I)$-measurable for each $c \in \mathbb{R}_+$; (c) $Q(\cdot, s, B) : I \to [0,1]$ is $\mathcal{B}(I)$-measurable for each $s \in \mathbb{S}$ and $B \in \mathcal{B}(\mathbb{S})$.*

## 2.2 The Firm

There is a single firm renting capital at (net) rate $r_t$ and hiring labor at wage $w_t$ at date $t$ to produce output $Y_t$ with technology $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ :

$$Y_t = F(K_t, N_t) + (1-\delta)K_t,$$

where $F$ is homogeneous of degree one, aggregate capital $K_t$ is $\mathcal{F}_{t-1}$-measurable, aggregate labor $N_t$ is $\mathcal{F}_t$-measurable, and $\delta \in (0,1)$ is the depreciation rate. Note that capital is transformed from consumers' accumulated assets.

Normalize $N_t = 1$ and assume the following:

**Assumption 5** *$F$ is strictly increasing, strictly concave, and continuously differentiable and satisfies:* $F(0,1) = 0$, $F_{12} > 0$, $\lim_{K\to 0} F_1(K,1) = \infty$, $\lim_{K\to\infty} F_1(K,1) \leq \delta$.

**Remark 2.4** *(i) Assumption 5 implies that there is a maximal sustainable stock of capital $K_{\max}$, which is given by the unique solution to the equation $F(K,1) = \delta K$.*

*(ii) I now give bounds on $\kappa$ in Assumption 1 (c) in terms of primitives. In partial equilibrium analysis in Section 3, I assume*

$$0 < \kappa \leq \frac{\eta w}{u'(r\underline{a} + w\underline{s})}, \tag{5}$$

*since $r, w$, and $\underline{a}$ are all fixed constants. However, in general equilibrium analysis in Section 4, I assume that $r\underline{a} + w\underline{s} \geq \varepsilon$ for some $\varepsilon > 0$ and $r \in (-\delta, 1/\beta_{\min} - 1)$, where $w = w(r) \equiv F_2(F_1^{-1}(r+\delta), 1)$. Let $\underline{K}$ be the unique value such that $F_1(\underline{K}, 1) = 1/\beta_{\min} - 1 + \delta$ and let $\underline{w} \equiv F_2(\underline{K}, 1)$. Then I assume*

$$0 < \kappa \leq \frac{\eta \underline{w}}{u'(\varepsilon)}. \tag{6}$$

Finally, competitive profit maximization implies that for all $t \geq 0$,

$$r_t = F_1(K_t, 1) - \delta, \tag{7}$$

$$w_t = F_2(K_t, 1). \tag{8}$$

### 2.3 Stationary Competitive Equilibrium

If individual asset holdings and exogenous shocks at date $t \geq 0$ are $a_t^i$ and $s_t^i$, respectively, $i \in I$, then the aggregate distribution over asset accumulation and shocks across consumers, $\overline{\lambda}_t \in \mathcal{P}(\mathbb{A} \times \mathbb{S})$, is defined by:

$$\overline{\lambda}_t(A \times B) = \phi(i \in I : (a_t^i, s_t^i) \in A \times B), \ A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S}). \tag{9}$$

Thus, $\overline{\lambda}_t(A \times B)$ is the measure of consumers whose asset holdings and shocks at date $t$ lie in the set $A \times B$. Note that $\overline{\lambda}_t$ is a random measure since $a_t^i = a_t^i(\omega)$ and $s_t^i = s_t^i(\omega)$ are random variables.

The implication of this definition is that each aggregate variable at date $t$ can be written as a suitable integral with respect to the aggregate distribution $\overline{\lambda}_t$, e.g.,

$$\int_I a_t^i \phi(di) = \int_{\mathbb{A}\times\mathbb{S}} a \overline{\lambda}_t(da, ds), \ \int_I s_t^i \phi(di) = \int_{\mathbb{A}\times\mathbb{S}} s \overline{\lambda}_t(da, ds)$$

$$\int_I c_t^i \phi(di) = \int_{\mathbb{A}\times\mathbb{S}} [(1+r)a + ws] \overline{\lambda}_t(da, ds) - \int_{\mathbb{A}\times\mathbb{S}} a \overline{\lambda}_{t+1}(da, ds).$$

Thus, if prices and aggregate variables are required to be constant, a sufficient condition is that the aggregate distribution is a time invariant and *nonrandom* measure.

I now define the concept of stationary equilibrium.

**Definition 1** *A stationary (competitive) equilibrium* $(((a^i_{t+1}, c^i_t)_{t\geq 0})_{i\in I}, (r, w), \overline{\lambda})$ *consists of an admissible allocation* $((a^i_{t+1}, c^i_t)_{t\geq 0})_{i\in I}$*, a system of prices* $(r, w) \in \mathbb{R}^2$*, and a measure* $\overline{\lambda} \in \mathcal{P}(\mathbb{A} \times \mathbb{S})$ *such that: Given* $w_t = w$ *and* $r_t = r$ *for all* $t \geq 0$*, then*

*(i) For* $\phi$*-a.e.* $i$*,* $(a^i_{t+1}, c^i_t)_{t\geq 0}$ *solves problem (4).*

*(ii) The firm maximizes profits so that (7) and (8) are satisfied for* $K_t = \int_I a^i_t \phi(di),\ t \geq 0.$

*(iii) Markets clear, i.e., for all* $t \geq 0$*,*

$$\int_I s^i_t \phi(di) = 1, \tag{10}$$

$$\int_I c^i_t \phi(di) + K_{t+1} = F(K_t, 1) + (1 - \delta)K_t. \tag{11}$$

*(iv) The aggregate distribution is invariant and nonrandom, i.e.,* $\overline{\lambda}_t(\omega) = \overline{\lambda}$ *a.s., where* $\overline{\lambda}_t$ *is given by (9),* $t \geq 0$*.*

I now sketch the key idea of the construction of an equilibrium detailed in later sections. I first study the one-person decision problem in section 3 where the main result is that under suitable conditions there is a unique invariant distribution $\lambda^{*i} \in \mathcal{P}(\mathbb{A} \times \mathbb{S})$ for $\phi$-a.e. $i$.

Equilibrium requires that the aggregate distribution be a invariant and nonrandom measure. In section 4.1, I show how one can apply a 'law of large numbers' for a continuum of random variables to fulfill this requirement. Then I show that the invariant aggregate distribution is generated by:

$$\overline{\lambda}^*(\cdot) \equiv \int_I \lambda^{*i}(\cdot)\phi(di). \tag{12}$$

Finally, I show that aggregate demand and supply of capital are continuous functions of the interest rate. Thus the equilibrium interest rate can be determined from the capital market clearing condition and an equilibrium is constructed (see Figure 1).

# 3 THE ONE-PERSON DECISION PROBLEM

This section focuses on a single person's decision problem in partial equilibrium so that the agent index $i$ is suppressed. Moreover, $r$ and $w$ are assumed to be constant, as they must be in a stationary equilibrium. Some results in this section will be used in later sections. Some generalize the extant literature and are of independent interest.

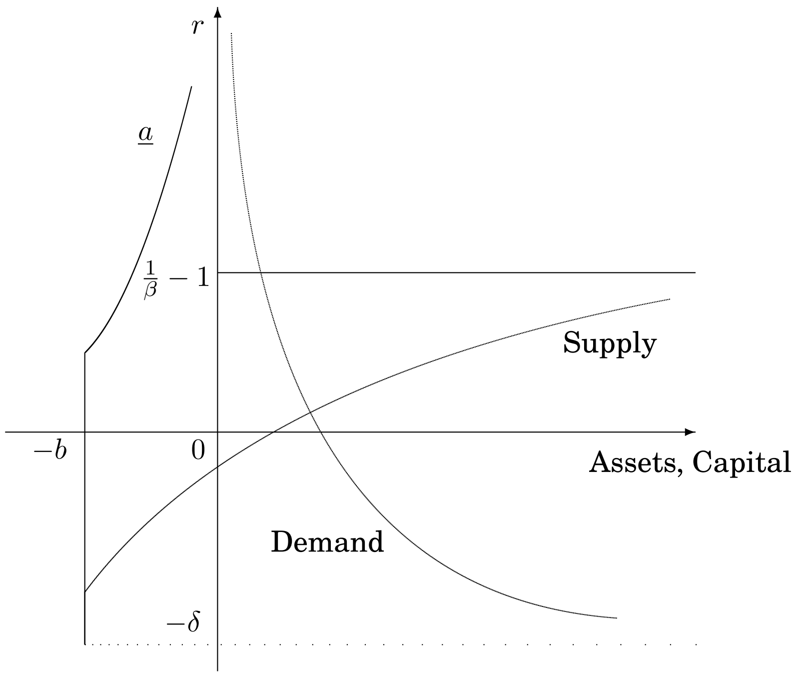


Figure 1: Determination of an equilibrium interest rate $r^*$

I first state further assumptions:

**Assumption 6** $r\underline{a} + w\underline{s} > 0.$

**Assumption 7** $1 + r > 0$ *and* $w > 0.$

Assumption 6 says that even if the consumer's wealth is at the lowest level $(1 + r)\underline{a} + w\underline{s}$ and his borrowing reaches the limit $\underline{a}$, it is still feasible to have positive consumption. Because $r$ and $w$ are endogenous in general equilibrium, the above assumptions must be consistent with general equilibrium (see Theorems 4.1). In particular, Assumption 7 follows immediately from (7)-(8). Finally, given (3), Assumption 6 implies that $\underline{a} = -b$ and $b < w\underline{s}/r$ if $r > 0$.

## 3.1 The Value Function and Policy Functions

I analyze a typical consumer's decision problem (4) by dynamic programming. Recall that $\mathbb{A} = [\underline{a}, \infty)$, $\mathbb{S} = [\underline{s}, \overline{s}]$ and that $C(\mathbb{A} \times \mathbb{S})$ denotes the set of continuous functions defined on $\mathbb{A} \times \mathbb{S}$. Define an operator $T : C(\mathbb{A} \times \mathbb{S}) \to C(\mathbb{A} \times \mathbb{S})$ by the following problem for any function $v \in C(\mathbb{A} \times \mathbb{S})$ :[7]

$$Tv(a, s) = \sup_{a' \in \Gamma(a,s)} u((1 + r)a + ws - a') + \beta \int_{\mathbb{S}} v(a', s')Q(s, ds') \tag{13}$$

where $\Gamma(a, s) = [\underline{a}, (1 + r)a + ws]$. The $n$-step operator $T^n$ can be defined in the usual fashion. The objective is to study the fixed point of the operator $T$ and the corresponding optimal policies.

**Theorem 3.1** *Suppose that Assumptions 1 (a)-(b), 2 (a)-(b), 6, and 7 are satisfied. Then:*

*(i) There are two functions* $L : \mathbb{A} \times \mathbb{S} \to \mathbb{R}$ *and* $M : \mathbb{A} \times \mathbb{S} \to \mathbb{R}$ *such that the operator* $T$ *has a unique fixed point in* $\mathcal{V} \equiv \{v \in C(\mathbb{A} \times \mathbb{S}) : L(a, s) \leq v(a, s) \leq M(a, s), \forall (a, s) \in \mathbb{A} \times \mathbb{S}\}$ *and it is the value function of problem (4). Moreover,* $\{T^n v\}$ *converges to* $V$ *pointwise and uniformly on any compact set in* $\mathbb{A} \times \mathbb{S}$ *for any* $v \in \mathcal{V}$.

*(ii) There exists a unique continuous asset accumulation policy function* $g : \mathbb{A} \times \mathbb{S} \to \mathbb{A}$ *solving the fixed point problem* $V = TV$. *The optimal consumption policy function* $f : \mathbb{A} \times \mathbb{S} \to \mathbb{R}_+$ *is given by* $f(a, s) = (1 + r)a + ws - g(a, s)$, *and it is continuous. Furthermore, the* $n$*-period optimal consumption and asset accumulation policies* $c^n : \mathbb{A} \times \mathbb{S} \to \mathbb{R}_+$ *and* $k^n : \mathbb{A} \times \mathbb{S} \to \mathbb{A}$ *corresponding to* $T^n v, v \in \mathcal{V}$, *converge respectively to* $f$ *and* $g$ *pointwise and uniformly on any compact subset of* $\mathbb{A} \times \mathbb{S}$.

[7] In the sequel, I may write $\int_{\mathbb{S}} v(a', s')Q(s, ds')$ as $E[v(a', s') \mid s]$. Note that $\Gamma(a, s) \neq \emptyset$ by Assumption 6.

Since $u$ is unbounded, standard dynamic programming techniques such as the Blackwell Theorem and the Contraction Mapping Theorem cannot be applied. Thus I analyze (13) by backward induction and establish convergence from truncated finite horizon problem to infinite horizon problem (4) by exploiting monotonicity of sequences of finite horizon value and policy functions.[8]

More specifically, in the appendix, I show that $V(a,s)$ is bounded below and above by some functions $L(a,s)$ and $M(a,s)$, respectively. Then I show that $L(a,s) \leq T^n L(a,s) \leq T^n M(a,s) \leq M(a,s)$. Further, $\{T^n L\}$ is an increasing sequence of functions and $\{T^n M\}$ is a decreasing sequence of functions. Finally, I establish that $\{T^n L\}$ and $\{T^n M\}$ converge pointwise to a common function, which is the value function $V$. The optimal policy functions are obtained from convergence of the corresponding $n-$period optimal policies.

Given $u(0) = 0$, the lower bound $L$ can be taken to be zero. The upper bound $M$ is obtained from Assumption 2 (b). Boundedness below is not necessary for the argument to work. To illustrate, consider two examples where $u(c) = c^\alpha/\alpha$, $\alpha < 0$, and $u(c) = \log(c)$.[9] Note that the latter also violates Assumption 2 (b).

For $u(c) = c^\alpha/\alpha$, $\alpha < 0$, it is clear that $u(c) < 0$. Thus, take $M = 0$. Because the plan saving nothing and consuming all wealth (i.e., $c_0 = (1+r)a + ws$, $c_t = ws_t$, and $a_t = 0$, $t \geq 1$) is budget feasible,

$$-\infty < ((1+r)a + ws)^\alpha/\alpha + \frac{\beta}{1-\beta}(w\overline{s})^\alpha/\alpha \leq V(a,s) < 0,$$

for all $(a,s) \in \mathbb{A} \times \mathbb{S}$. Take $L(a,s)$ as the function on the left hand side of the above second inequality.

For $u(c) = \log(c)$, one can similarly show that[10]

$$\begin{aligned} -\infty &< \log((1+r)a + ws) + \frac{\beta}{1-\beta}\log(w\underline{s}) \leq V(a,s) \leq \sum_{t=0}^{\infty} \beta^t \log(\overline{c}_t) \\ &\leq \sum_{t=0}^{\infty} \beta^t \log\left(\frac{(1+r)^{t+2} - 1}{r}\right) + \frac{1}{1-\beta}\log\left(\max(w\overline{s} + r\underline{a}, a - \underline{a})\right) < \infty, \end{aligned}$$

for all $(a,s) \in \mathbb{A} \times \mathbb{S}$. Then take $L$ and $M$ as the functions on the left side and the right side of the above inequalities, respectively.

The following theorem states some properties of the value function $V$.

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[8]The method of successive approximations is well known in the literature, e.g., [40, 41, 34, 42, 35, 7, 29]. There is no general theory of dynamic programming for unbounded utility. My analysis is closest to [29] and [38].

[9]For simplicity, I do not consider the case $0 < \alpha < 1$, for which all analysis in Section 3 goes through under the additional assumption $\beta(1+r)^\alpha < 1$ (see, e.g., [31]).

[10]Use the following fact established in the appendix: Given any initial state $(a,s)$, any feasible consumption plan $(c_t)$ must satisfy $c_t \leq \overline{c}_t \equiv \frac{1}{r}\left[(1+r)^{t+1} - 1\right](w\overline{s} + r\underline{a}) + (1+r)^{t+1}(a - \underline{a})$.

**Theorem 3.2** *Suppose that Assumptions 1 (a)-(b), 2 (a)-(b), 6, and 7 are satisfied. Then:*

*(i) $V(a,s)$ is strictly increasing, strictly concave, and continuously differentiable in $a$ on $\mathbb{A}$ for each $s \in \mathbb{S}$. Moreover, for all $(a,s) \in \mathbb{A} \times \mathbb{S}$,*

$$V_1(a,s) = (1+r)u'(f(a,s)). \tag{14}$$

*(ii) If Assumption 1 (c) also holds, then $V(a,s)$ is strictly increasing in $s$ and $V_1(a,s)$ is strictly decreasing in $s$, for each $a \in \mathbb{A}$.*

Monotonicity and concavity of the value function in the endogenous state variable $a$ established in part (i) are standard results. Equation (14) is the *envelope condition*. Differentiability of the value function and the envelope condition are typically proved using the Benveniste and Scheinkman Theorem (e.g., [43, Theorem 9.10]). However, this theorem is valid only for the case where $a$ is in the interior of $\mathbb{A}$ and $g(a,s)$ is in the interior of $\Gamma(a,s)$. I show by backward induction that $V$ is differentiable in $a$ on the whole domain $\mathbb{A}$ and that the envelope condition holds on the whole domain $\mathbb{A} \times \mathbb{S}$.

Finally, monotonicity of the value function in shocks shown in part (ii) enables me to establish the properties of the policy functions in Theorem 3.4. It is proved by backward induction and the key condition is Assumption 1 (c) — monotonicity of $Q$.

The following theorem states the well-known necessary and sufficient conditions for optimality (see [41, 42]). The proof is omitted.

**Theorem 3.3** *Suppose that Assumptions 1 (a)-(b), 2 (a)-(b), 6, and 7 are satisfied. The policy functions $f : \mathbb{A} \times \mathbb{S} \to \mathbb{R}_+$ and $g : \mathbb{A} \times \mathbb{S} \to \mathbb{A}$ are optimal if and only if*

*(i) $V$ satisfies the envelope condition (14) and*

$$V_1(a,s) \geq \beta(1+r)E\left[V_1(g(a,s),s') \mid s\right], \text{ with equality if } g(a,s) > \underline{a}; \tag{15}$$

*(ii) for all $a_{n+1} = g(a_n, s_n)$ and $s_n \in \mathbb{S}$, $n \geq t \geq 0$,*

$$\lim_{T\to\infty} \beta^{t+T} E\left[V_1(a_{t+T}, s_{t+T})a_{t+T+1} \mid s_t\right] = 0. \tag{16}$$

Equation (15) is often called the *Euler inequality*. Using the envelope condition, it can be rewritten as

$$u'((1+r)a + ws - g(a,s)) \geq \beta E\left[V_1(g(a,s),s') \mid s\right], \text{ or}$$

$$u'((1+r)a + ws - g(a,s)) \geq \beta(1+r)E\left[u'((1+r)g(a,s) + ws' - g(g(a,s),s')) \mid s\right],$$

with equality if $g(a,s) > \underline{a}$. Equation (16) is the *transversality condition*. The Euler inequality implies that the process of 'discounted' marginal utility $\beta^t(1+r)^t u'(c_t)$, $t \geq 0$, forms a supermartingale so that the Martingale Convergence Theorem can be applied to study the long run behavior of consumption and asset holdings.

The following theorem establishes properties of the policy functions.

**Theorem 3.4** *Suppose that Assumptions 1, 2 (a)-(b), 6, and 7 are satisfied.*

*(i) If* $0 < \beta(1+r) \leq 1$*, then* $f(a, s) \geq f(\underline{a}, \underline{s}) = r\underline{a} + w\underline{s} > 0$.

*(ii)* $f$ *is strictly increasing on* $\mathbb{A} \times \mathbb{S}$*. For each* $s \in \mathbb{S}$*,* $g(\cdot, s)$ *is increasing and strictly increasing in* $a$ *when* $g(a, s) > \underline{a}$*.*

*(iii) If Assumption 2 (c) holds and if* $\beta(1+r) \leq 1$*, then* $g(a, \cdot)$ *is increasing and strictly increasing when* $g(a, \cdot) > \underline{a}$*,* $a \in \mathbb{A}$*.*

*(iv)* $\lim_{a\to\infty} f(a, s) = \infty$.

Part (i) states that given $0 < \beta(1+r) \leq 1$, if the consumer's asset holdings are at the lowest level $\underline{a}$ and he receives the smallest shock $\underline{s}$, then his borrowing is constrained by the borrowing limit, i.e., $g(\underline{a}, \underline{s}) = \underline{a}$. Furthermore, in all other states, consumption weakly exceeds $r\underline{a} + w\underline{s}$.

Monotonicity of the optimal consumption policy $f$ follows from the wealth effect. Specifically, an increase in either current asset holdings or current labor endowment shocks increases the wealth of the consumer and leads to increased consumption. Monotonicity of the optimal asset accumulation policy $g$ in $a$ follows from the following familiar argument (e.g., [43]). If $g(a, s) > \underline{a}$, the Euler inequality holds with equality. When $a$ increases, the marginal utility cost $u'((1+r)a + ws - a')$ decreases since $u$ is strictly concave, while the marginal utility benefit $\beta E[V_1(a', s') \mid s]$ does not change. Because as functions of $a'$, the marginal cost curve is upward sloping and the marginal benefit curve is downward sloping by concavity of $u$ and $V(\cdot, s)$, the optimal asset accumulation $a' = g(a, s)$ must increase.

Part (iii) is important for the latter analysis.[11] The intuition is as follows. Consider interior solutions. When $s$ increases, the marginal utility cost decreases, while the marginal utility benefit also decreases because $V_1(a', \cdot)$ is decreasing and $Q$ is monotonic in the sense of first-order stochastic dominance. Whether $g$ increases in $s$ depends on which effect dominates. Only if the decrease in marginal cost exceeds the decrease in marginal benefit, $g$ increases in $s$. Assumptions 1 (b) and 2 (c) then come into play. The former guarantees that the decrease of marginal benefit curve is small, while the latter ensures that the decrease of marginal cost curve is big enough so that it dominates the former effect. Thus the overall effects lead to the increase of $g$ in $s$.

[11]When shocks are i.i.d., Proposition 2 in [23] immediately delivers monotonicity of $g$ without the assumptions in (iii). For Markov shocks, this proposition cannot be applied because one of its condition is violated, i.e., $u((1+r)a + ws - a')$ is not supermodular in $(a, s)$.

Finally, part (iv) states that whenever assets grow without bound, consumption also grows without bound. It is applied in the proofs of Lemma 3.1 and Theorem 3.7.

### 3.2 When Does the Borrowing Constraint Bind?

Equipped with the above properties of the value function and policy functions, this subsection studies the effect of the borrowing constraint.

The following theorem shows that when $\beta(1+r) \leq 1$, there is a nontrivial set of states $(a, s)$ for which the borrowing constraint binds. Furthermore, this set is characterized explicitly. This result generalizes [14, Proposition 2.1] to the case of unbounded utility functions and Markov shocks.

**Theorem 3.5** *If Assumptions 1, 2, 6, and 7 are satisfied and if $\beta(1+r) \leq 1$, then there exists a unique $s^* \in (\underline{s}, \overline{s}]$ satisfying*

$$s^* = \max\{s \in \mathbb{S} : u'(r\underline{a} + ws) \geq \beta E\left[V_1(\underline{a}, s') \mid s\right]\}, \tag{17}$$

*and a unique function $a^* : [\underline{s}, s^*] \to (\underline{a}, \infty)$ satisfying*

$$u'((1+r)a^*(s) + ws - \underline{a}) = \beta E\left[V_1(\underline{a}, s') \mid s\right], \tag{18}$$

*such that:*

*(i) $g(a, s) = \underline{a}$ and $f(a, s) = (1+r)a + ws - \underline{a}$ for $a \in [\underline{a}, a^*(s)]$ and $s \in [\underline{s}, s^*]$.*

*(ii) $g(a, s) > \underline{a}$ and $f(a, s) = (1+r)a + ws - g(a, s)$, for all $a > a^*(s)$ and $s \in [\underline{s}, s^*]$. Moreover, $g(a, s) > \underline{a}$ for all $a \in \mathbb{A}$ and $s \in (s^*, \overline{s}]$.*

The theorem is illustrated in Figure 2. There is a critical value $s^*$ for the shock and a critical value $a^*(s)$ for the asset holdings for each $s \in [\underline{s}, s^*]$. In state $(\underline{a}, s^*)$, the consumer borrows and his borrowing reaches the limit $\underline{a}$. If the realization of shocks is better than $s^*$ ($s > s^*$), then for any current asset holdings, the consumer's asset holdings tomorrow are strictly higher than $\underline{a}$. Namely, the borrowing constraint is never binding. However, if the realization of endowment shocks $s$ is smaller than $s^*$ ($s < s^*$), then the borrowing constraint is binding if and only if current asset holdings do not exceed $a^*(s)$. Moreover, if current asset holdings do not exceed $a^*(s)$, then the consumer consumes all of his wealth including savings and borrowing, i.e., $f(a, s) = (1+r)a + ws - \underline{a}$, for $a < a^*(s)$ and $s < s^*$. If one draws optimal consumption as a function of his total wealth for a fixed realization of shocks, it will have a similar shape in Deaton [16, Figure 3] which is obtained from numerical examples.

The above theorem implies that when a consumer has small asset holdings and experiences bad labor endowment shocks, then he tends to borrow and tends to be borrowing constrained.

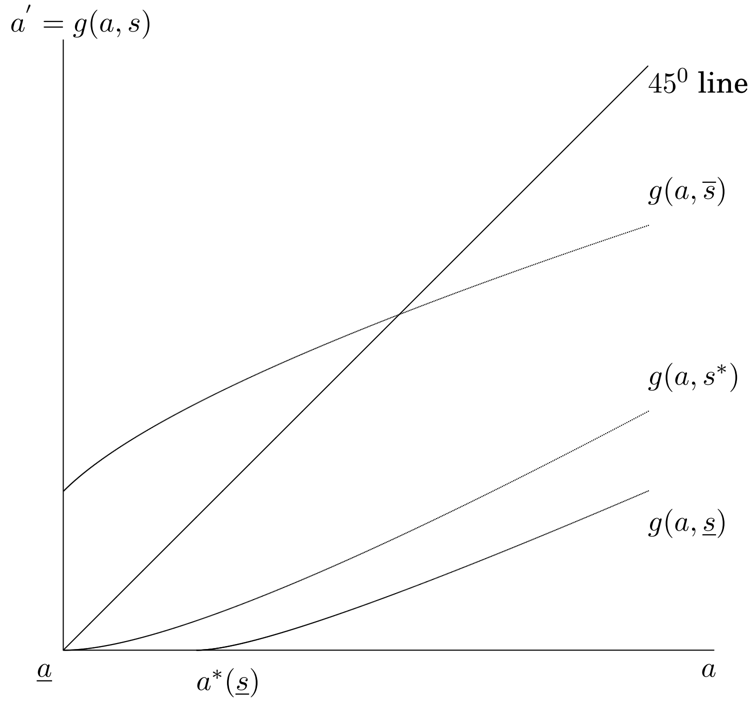


Figure 2: The effects of the borrowing constraint on the policy functions.

### 3.3 Long-run Behavior of Consumption and Asset Holdings

The optimal policy functions $f$ and $g$ generate unique optimal plans $(a_t)_{t\geq 1}$ and $(c_t)_{t\geq 0}$ for problem (4):

$$a_{t+1} = g(a_t, s_t),\ c_t = f(a_t, s_t),\ t \geq 0,\ (a_0, s_0) \text{ given.} \tag{19}$$

The objective of this subsection is to study the long-run behavior of $(a_t)$ and $(c_t)$. It is well known that if the shock process $(s_t)$ is not i.i.d., then $(a_t)$ and $(c_t)$ need not be first-order Markov processes. So I first study the joint process $(a_t, s_t)_{t\geq 0}$ which is in fact a first-order Markov process. Then the long-run behavior of $(c_t, s_t)_{t\geq 0}$ can be deduced from (19).

Consider the transition function of the Markov process $(a_t, s_t)$. By [43, Theorem 9.13], the map $\Lambda : (\mathbb{A} \times \mathbb{S}) \times \mathcal{B}(\mathbb{A} \times \mathbb{S}) \to [0, 1]$ defined by[12]

$$\Lambda(a, s; A \times B) = \mathbf{1}_A(g(a, s))Q(s, B),\ A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S}), \tag{20}$$

is the transition function generated by the shock $s$ and the policy function $a' = g(a, s)$. Define an operator $M^*_\Lambda : \mathcal{P}(\mathbb{A} \times \mathbb{S}) \to \mathcal{P}(\mathbb{A} \times \mathbb{S})$ by[13]

$$M^*_\Lambda(\lambda)(A \times B) = \int_{\mathbb{A}\times\mathbb{S}} \Lambda(a, s; A \times B)\lambda(da, ds),\ A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S}). \tag{21}$$

Thus, $M^*_\Lambda(\lambda)(A \times B)$ is the probability that the state next period lies in the set $A \times B$, given that the current state is drawn according to the distribution $\lambda$. The $n$-step transition $\Lambda^n$ and its associated operator $M^{*n}_\Lambda$ can be defined recursively in the usual fashion.

The following lemma shows that there is a unique compact ergodic set for the Markov process $(a_t, s_t)$ if $\beta(1 + r) < 1$.

**Lemma 3.1** *Under Assumptions 1, 2, 6, and 7, if* $\beta(1 + r) < 1$*, then:*

*(i) There exists a unique ergodic set* $\mathbb{K} \times \mathbb{S}$ *for the Markov process* $(a_t, s_t)$*, where* $\mathbb{K} = [\underline{a}, \overline{a}]$ *and* $\overline{a} = \min\{a \in \mathbb{A} : g(a, \overline{s}) = a\}$*. Moreover, any subset of* $(\overline{a}, \infty) \times \mathbb{S}$ *is transient.*

*(ii)* $\mathbb{K} = \{\underline{a}\}$ *if and only if* $u'(r\underline{a} + w\overline{s}) \geq \beta E\left[V_1(\underline{a}, s') \mid \overline{s}\right]$.

The intuition behind part (i) is the following: If $u'$ has an asymptotic exponent, the marginal utility of consumption is asymptotically constant for each realization of shocks when $a$ is sufficiently large. Thus, when $a$ is sufficiently large, the consumer behaves as in the deterministic case. This implies that optimal savings eventually fall over time if $\beta(1 + r) < 1$. More specifically, there is a sufficiently large $a^0$ such that $g(a, s) < a$ for all $a > a^0$ and all $s \in \mathbb{S}$. Thus there exists a fixed point of $g(\cdot, \overline{s})$. Taking the minimum of these fixed points yields $\overline{a}$.

[12] Note that $\mathbf{1}$ is an indicator function.
[13] By [43, Theorem 8.1-8.2], this operator is well defined.

Part (ii) states that the borrowing constraint is eventually binding at all states if and only if the borrowing constraint is binding when the shock is at the highest level and the current asset holdings are at the borrowing limit.

The following theorem shows that if $\beta(1+r) < 1$, then there is a unique ergodic measure $\lambda^*$ on $\mathcal{B}(\mathbb{A} \times \mathbb{S})$ and it corresponds to the unique fixed point of $M^*_\Lambda$, i.e., $M^*_\Lambda(\lambda^*) = \lambda^*$.

**Theorem 3.6** *Under Assumptions 1, 2, 6, and 7, if* $\beta(1+r) < 1$*, then:*

*(i) There exists a unique ergodic measure* $\lambda^* \in \mathcal{P}(\mathbb{A} \times \mathbb{S})$ *for the Markov process* $(a_t, s_t)$*. Moreover,* $\lambda^*$ *is positive on any open subset of the ergodic set* $\mathbb{K} \times \mathbb{S}$*, where* $\mathbb{K} \times \mathbb{S}$ *is given in Theorem 3.1. Finally, given any* $\lambda_0 \in \mathcal{P}(\mathbb{A} \times \mathbb{S})$*,* $M^{*n}_\Lambda(\lambda_0)$ *converges weakly to* $\lambda^*$ *as* $n \to \infty$*.*

*(ii)* $\lambda^*(\{\underline{a}\} \times \mathbb{S}) = \lambda^*([\underline{a}, a^*(\underline{s})] \times [\underline{s}, s^*]) > 0$ *where* $a^*(\underline{s})$ *is given in Theorem 3.5.*

The proof applies a result stated in [35] or [17] using the following facts: (i) $\mathbb{K} \times \mathbb{S}$ is an ergodic set; (ii) $\Lambda$ has the Feller property; and (iii) $\Lambda$ is *irreducible* (See Appendix B for definition). Monotonicity of $Q$ from Assumption 1 (c) and monotonicity of $g$ established in Theorem 3.4 are key to the proof. A similar proof is given in [14] for the case of i.i.d. shocks.

Note that it is straightforward to use [43, Theorem 12.12] or [23, Theorem 2] to prove the existence of a unique invariant measure using monotonicity of $M^*_\Lambda$.[14] My result is stronger, i.e., the invariant measure is positive on any open subset of the ergodic set. Moreover, under $\lambda^*$ there is a positive probability that the borrowing constraint is binding, i.e., $\lambda^*(\{\underline{a}\} \times \mathbb{S}) > 0$.

A final theorem considers long-run properties of the optimal plans $(a_t)$ and $(c_t)$ if $\beta(1+r) \geq 1$.

**Theorem 3.7** *Under Assumptions 1, 2, 6, and 7, if* $\beta(1+r) > 1$*, or if* $\beta(1+r) = 1$ *and Assumption 3 holds, then* $\lim_{t\to\infty} a_{t+1} = \infty$ *and* $\lim_{t\to\infty} c_t = \infty$ *a.s..*

This theorem extends [42, 1] to allow for Markov shocks and extends [13] to allow for unbounded utility. The underlying intuition is as follows. If the interest rate $r$ exceeds the rate of time preference $1/\beta - 1$, then the consumer wants to postpone consumption to the future so that consumption eventually grows without bound. Because he is borrowing constrained, he has to accumulate an infinitely large amount of assets to finance an infinitely large amount of consumption in the long run. If $r$ equals $1/\beta - 1$, then consumption will also grow without bound provided that the consumer's income is sufficiently stochastic as required by Assumption 3. The key tool for the argument is the Martingale Convergence Theorem which ensures that marginal utility of consumption converges to zero if $\beta(1+r) > 1$ and to a finite random variable if $\beta(1+r) = 1$.

[14] Aiyagari [1] applies [43, Theorem 12.12] for the case of i.i.d. shocks. Huggett [24] applies [23, Theorem 2] for the case of Markov shocks with only two states.

### 3.4 Comparative Statics Analysis

This subsection studies the following question: How do the optimal policies and invariant distribution vary with changes in the discount factor $\beta$, the distribution of shocks or the transition function $Q$, the borrowing limit $\underline{a}$, or the degree of risk aversion? The main tool for the analysis below is the lattice theory developed in [46, 23, 37].

When risk aversion varies, I consider only power utility, i.e., $u(c) = c^{\alpha}/\alpha$, $\alpha < 0$. Let the exogenous parameter be $\varphi$ which may represent $\beta$, $r$, $\underline{a}$, $\alpha$, or $Q$. Then write $\varphi$ as an argument for any function that depends on any of the noted parameters. Note that $w$ can be written as a function of $r$ by (7)-(8) so that it is not treated as a parameter.

First define some partial orders.[15] For any two probability measures $\nu$ and $\nu'$ on some Borel space, say $\nu \succeq_{FSD} \nu'$ if $\int hd\nu \geq \int hd\nu'$ for any increasing and bounded function $h$. Say $\nu \succ_{FSD} \nu'$ if $\int hd\nu > \int hd\nu'$ for any strictly increasing and bounded function $h$. For any transition functions $Q$ and $Q'$, say $Q \succeq_c (\succ_c) Q'$ if $Q(s, \cdot) \succeq_{FSD} (\succ_{FSD}) Q'(s, \cdot)$ for each $s \in \mathbb{S}$. For any sets $Y$ and $Y'$ in the real line, say $Y \succeq_s Y'$ if $y \wedge y' \in Y'$ and $y \vee y' \in Y$ for any $y \in Y$ and $y' \in Y'$.[16] Finally, when the set $Y$ depends on a parameter $\varphi$, say that $Y$ is (strictly) increasing in $\varphi$ with respect to some partial order $\succeq$ if ($Y(\varphi) \neq Y(\varphi')$) $Y(\varphi) \succeq_s Y(\varphi')$ for $\varphi \succ \varphi'$.

**Theorem 3.8** *Let $(\beta^j, \underline{a}^j)$ satisfy Assumption 6, $j = 1, 2$. Under Assumptions 1, 2 (a)-(b), 6, and 7, then: For every $(a, s) \in \mathbb{A} \times \mathbb{S}$,*

*(i) $g(a, s; \beta^2) \geq g(a, s; \beta^1)$ if $\beta^2 > \beta^1$, and the strict inequality holds if $g(a, s; \beta^1) > \underline{a}$.*

*(ii) $g(a, s; Q^2) \leq g(a, s; Q^1)$ if $Q^2 \succeq_c Q^1$, and the strict inequality holds if $g(a, s; Q^2) > \underline{a}$ and $Q^2 \succ_c Q^1$.*

*(iii) $g(a, s; \underline{a}^2) \geq g(a, s; \underline{a}^1)$ if $0 \geq \underline{a}^2 > \underline{a}^1$ and $a \geq \underline{a}^2$.*

When a consumer is more patient, he is not so eager to consume now. Consequently, he saves more for future consumption. For part (ii), observe first that asset holdings and the realization of shocks are strategic substitutes because $V_1(a, s)$ is strictly decreasing in $s$ by Theorem 3.2. Then one can show that asset holdings and the conditional distribution of shocks are strategic substitutes using Assumption 1 (c). Thus, when the consumer anticipates that the distribution of shocks tomorrow is better conditional on any state today, he has less incentive to save more in order to buffer future endowment fluctuations because strategic substitution leads to a decrease in the marginal benefit of an additional unit of saving, while the marginal cost does not change. Finally, part (iii) demonstrates the numerical finding of Deaton [16] that the presence of borrowing constraint is similar to that

[15] See [46] and [23] for more details.

[16] This is the strong set order. Note that $y \wedge y' \equiv \min\{y, y'\}$ and $y \vee y' \equiv \max\{y, y'\}$.

associated with a precautionary motive for saving. The intuition is that when the consumer can borrow more, he has more freedom to use borrowing to buffer endowment fluctuations, so there is less need to accumulate assets.

**Theorem 3.9** *Under Assumptions 1, 2, 6, and 7, let $a^*(\underline{s})$ and $s^*$ be given in Theorem 3.5 and let $\mathbb{K} = [\underline{a}, \overline{a}^j]$ be given in Lemma 3.1 and $\lambda^*$ given in Theorem 3.6. If $\beta(1+r) < 1$, then:*

*(i) $\mathbb{K}$ is strictly increasing in $\beta$ and $\underline{a}$, and strictly decreasing in $Q$ and $\alpha$.*[17]

*(ii) $a^*(\underline{s})$ is strictly decreasing in $\beta$ and strictly increasing in $Q$. Moreover, $s^*$ is decreasing in $\beta$ and increasing in $Q$.*

*(iii) $\lambda^*$ is increasing in $\beta$ and $\underline{a}$.*

*(iv) $E_{\lambda^*}[a](r, \beta, \underline{a}, Q) \equiv \int_{\mathbb{A}\times\mathbb{S}} a\lambda^*(da, ds; r, \beta, \underline{a}, Q)$ is continuous in $(r, \beta, \underline{a})$ and increasing in $(\beta, \underline{a})$, where $\lambda^*$ is given in Theorem 3.6.*

Part (i) says that the ergodic set is strictly larger if either discount factor $\beta$ is increased or the borrowing constraint is tightened, and it is strictly smaller if either the conditional distribution $Q(s, \cdot)$ is increased for all $s$ or risk aversion $1-\alpha$ is decreased. Part (ii) says that the set of states at which the borrowing constraint is binding is strictly smaller if either $\beta$ is increased or $Q$ is decreased. Except for monotonicity in risk aversion, these results are intuitive given the changes of the optimal asset accumulation policy curve shown in Theorem 3.8.

For monotonicity of $\mathbb{K}$ in risk aversion, consider the equation determining $\overline{a}^1$ in Theorem 3.1, which implies that the consumer saves $\overline{a}^1$ at state $(\overline{a}^1, \overline{s})$. If his risk aversion $1-\alpha^1$ decreases to $1-\alpha^2$ ($\alpha^2 > \alpha^1$), then the returns to savings decrease so that his savings decrease. Thus $\mathbb{K}$ must decrease. The extra savings associated with high risk aversion also reflect stronger precautionary savings motive. Note that this argument is applied only locally at the point $(\overline{a}^1, \overline{s})$. It does not carry over when comparing optimal policies globally because both the marginal benefit and marginal cost vary in risk aversion and time-additive utility confounds risk aversion and intertemporal substitution. Thus the change in invariant distribution is also ambiguous.

Part (iii) says that following an increase in either $\beta$ or $\underline{a}$, the invariant distribution becomes better in the sense of the first-order stochastic dominance. This is because given any probability measure over today's states, if either $\beta$ or $\underline{a}$ increases, the transition probability that asset holdings and shocks tomorrow are at better states will be greater since asset accumulation policy is better by Theorem 3.8 and the conditional distribution of shocks is better by Assumption 1 (c) (see (20)). Thus, the distribution of states tomorrow is better and the limiting invariant distribution is also better by (21) and Theorem 3.6.[18]

[17] When $\alpha$ changes, I assume $r > 0$.

[18] The argument is roughly as follows (see [23, Corollary 3]). Let 2 index the larger value of $\beta$ or $\underline{a}$. Then

Note that although the optimal policy decreases monotonically in the transition function $Q$, the invariant distribution does not necessarily vary monotonically. This is because given today's state $(a,s)$, following an increase in $Q$, the realization of tomorrow's shocks $s'$ is more likely to be in good state by Assumption 1 (c) so that the conditional probability that the state tomorrow $(g(a,s),s')$ is at good states is not necessarily greater (see (20)). This implies that given a distribution of states today, the distribution tomorrow is not necessarily ordered and neither is the limiting distribution.

Finally, continuity of $E_{\lambda^*}[a]$ follows from [43, Theorem 12.13]. Monotonicity of $E_{\lambda^*}[a]$ in $(\beta,\underline{a})$ follows from parts (i)-(iii). Note that $E_{\lambda^*}[a]$ represents the steady-state aggregate capital supply for a given interest rate. Its properties play key roles in proving existence of stationary equilibrium and in conducting comparative statics analysis on stationary equilibrium.

# 4 EXISTENCE AND PROPERTIES OF STATIONARY EQUILIBRIA

Consider now a continuum of consumers distributed over $I=[0,1]$ according to the Lebesgue measure $\phi$ on $\mathcal{B}(I)$. As established in the preceding section, under suitable assumptions, for each consumer $i$, the value function $V^i$ is a fixed point of the operator $T^i$ defined by (13) and there exists a unique optimal asset accumulation policy $g^i$ solving the fixed point problem $V^i=T^iV^i$. This policy function and shocks $s^i$ induce a transition function $\Lambda^i:(\mathbb{A}\times\mathbb{S})\times\mathcal{B}(\mathbb{A}\times\mathbb{S})\to[0,1]$ defined by $\Lambda^i(a,s,A\times B)=\mathbf{1}_A(g^i(a,s))Q^i(s,B)$.

Define a function $g:I\times\mathbb{A}\times\mathbb{S}\to\mathbb{A}$ by $g(i,a,s)=g^i(a,s)$. This function generates a unique optimal allocation $((a^i_{t+1},c^i_t)_{t\geq 0})_{i\in I}$ as follows:

$$\begin{aligned} a^i_1 &= a_1(i,\omega)=g(i,a_0(i),s_0(i)),\ c^i_0=c_0(i,\omega)=(1+r)a_0(i)+ws_0(i)-a_1(i,\omega),\\ a^i_{t+1} &= a_{t+1}(i,\omega)=g(i,a_t(i,\omega),s_t(i,\omega)),\\ c^i_t &= c_t(i,\omega)=(1+r)a_t(i,\omega)+ws_t(i,\omega)-a_{t+1}(i,\omega),\ t\geq 0. \end{aligned} \tag{22}$$

To facilitate aggregation, the optimal allocation $((a^i_{t+1},c^i_t)_{t\geq 0})_{i\in I}$ must be admissible, i.e., as functions of $(i,\omega)$, $a^i_{t+1}=a_{t+1}(i,\omega)$ and $c^i_t=c_t(i,\omega)$ must be $\mathcal{B}(I)\otimes\mathcal{F}_t$-measurable.

**Lemma 4.1** *Under Assumptions 1, 2, 4, 6, and 7, the optimal policy function $g$ is a Caratheodory function,*[19] *and the optimal allocation $((a^i_{t+1},c^i_t)_{t\geq 0})_{i\in I}$ is admissible.*

$\Lambda^2(a,s;A\times B)\geq\Lambda^1(a,s;A\times B)\Longrightarrow M^{*2}_\Lambda(\lambda)\succeq M^{*1}_\Lambda(\lambda)\Longrightarrow\lambda^{*2}=\lim(M^{*2}_\Lambda)^n(\lambda^{*1})\succeq\lim(M^{*1}_\Lambda)^n(\lambda^{*1})=\lambda^{*1}$.

[19] Namely, as a function of $i$, $g(i,a,s)$ is $\mathcal{B}(I)$-measurable for each $(a,s)\in\mathbb{A}\times\mathbb{S}$ and as a function of $(a,s)$ it is continuous for fixed $i\in I$.

## 4.1 Aggregation and the Law of Motion for Aggregate Distributions

A stationary equilibrium requires that although each consumer faces labor endowment risk, in equilibrium risk disappears in the aggregate. In particular, the aggregate invariant distribution must be a nonrandom measure. To ensure this, one typically invokes some law of large numbers for a continuum of random variables.

When the assumption of independence of shocks across consumers is dispensed with, Feldman and Gilles [19] show that there exists a probability space $(\Omega, \mathcal{F}, P)$ and a shock process $(s_t^i)$ such that a law of large numbers for a continuum of random variables $(s_t^i)_{i\in I}$ holds. I refer to this as the *Feldman-Gilles construction* and adopt it throughout the analysis to follow.

To see how the Feldman-Gilles construction can be applied, consider the law of motion for aggregate distributions. Let $\mu^i$ be the distribution of $(s_t^i)$ on $\mathcal{B}(\mathbb{S}^\infty)$ induced by the transition function $Q^i$. Let $(a_{t+1}^i)$ be the optimal plan for consumer $i$ so that $a_{t+1}^i = g^i(a_t^i, s_t^i)$ for some $(a_t^i, s_t^i) \in \mathbb{A} \times \mathbb{S}$.

By definition (9) and Bayes' Rule, for all $A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S})$,

$$\overline{\lambda}_{t+1}(A \times B) = \phi(i \in I : (a_{t+1}^i, s_{t+1}^i) \in A \times B)$$

$$= \int_{\mathbb{A}\times\mathbb{S}} \phi(i \in I : (a_{t+1}^i, s_{t+1}^i) \in A \times B \mid (a_t^m, s_t^m) = (a, s), m \in I)\phi(m \in I : (a_t^m, s_t^m) \in da \times ds).$$

Applying definition (9) again,

$$\overline{\lambda}_{t+1}(A \times B) = \int_{\mathbb{A}\times\mathbb{S}} \phi(i \in I : (a_{t+1}^i, s_{t+1}^i) \in A \times B \mid (a_t^m, s_t^m) = (a, s), m \in I)\overline{\lambda}_t(da, ds)$$

Thus, the function $\overline{\Lambda}_t : \mathbb{A} \times \mathbb{S} \times \mathcal{B}(\mathbb{A} \times \mathbb{S}) \to [0, 1]$, defined by

$$\overline{\Lambda}_t(a, s, A \times B) = \phi(i \in I : (a_{t+1}^i, s_{t+1}^i) \in A \times B \mid (a_t^m, s_t^m) = (a, s), m \in I), \tag{23}$$

is the transition function for the sequence of aggregate distribution $\{\overline{\lambda}_t\}$. If $\{\overline{\Lambda}_t\}$ is nonrandom, then the sequence of aggregate distributions $\{\overline{\lambda}_t\}$ evolves deterministically provided that the initial aggregate distribution $\overline{\lambda}_0$ is nonrandom.

*Ex ante identical consumers.* Consider first the case where each consumer has the same preferences and the same distribution of shocks, i.e., $u^i = u$, $\beta^i = \beta$, and $\mu^i = \mu$, for all $i \in I$.

Proposition 2 in [19] implies that there exists a probability space $(\Omega, \mathcal{F}, P)$ and a process $s^i = (s_t^i)$, where $s_t^i \equiv s_t(i, \omega)$ is $\mathcal{B}(I) \otimes \mathcal{F}_t$-measurable, $t \geq 0$, such that:

(i) For all $i \in I$, $s(i, \cdot)$ has common distribution $\mu$: $P(\omega \in \Omega : s^i(\omega) \in A) = \mu(A)$, $A \in \mathcal{B}(\mathbb{S}^\infty)$.

(ii) For all $\omega \in \Omega$, $\phi(i \in I : s(i, \omega) \in A) = \mu(A)$, $A \in \mathcal{B}(\mathbb{S}^\infty)$.

The above two equations say that the empirical distribution of a sample of random variables is the same as the theoretical distribution from which these random variables are drawn. Thus the transition function for $(s_t^i)$, $Q^i$, satisfies that for all $B \in \mathcal{B}(\mathbb{S})$,

$$Q^i(s,B) = Q(s,B) = P(\omega \in \Omega : s_{t+1}^i(\omega) \in B \mid s_t^i(\omega) = s) = \phi(i \in I : s_{t+1}^i \in B \mid s_t^m = s, m \in I).$$

Since all consumers are ex ante identical, they choose the same optimal policy, i.e., $g^i = g$ for all $i \in I$. Thus, by (23),

$$\begin{aligned} \overline{\Lambda}_t(a,s,A\times B) &= \phi(i \in I : (g(a,s), s_{t+1}^i) \in A \times B \mid s_t^m = s, m \in I) \\ &= \mathbf{1}_A(g(a,s))\phi(i \in I : s_{t+1}^i \in B \mid s_t^m = s, m \in I) \\ &= \mathbf{1}_A(g(a,s))Q(s,B) = \int_I \mathbf{1}_A(g^i(a,s))Q^i(s,B)\phi(di). \end{aligned}$$

That is, the aggregate transition function is equal to the transition function of the process $(a_t^i, s_t^i)$ for any individual $i \in I$.

*Countably many types of ex ante identical consumers.* Let $I$ be partitioned into countably many disjoint measurable sets $I_j$ such that $I = \cup_{j=1}^{\infty} I_j$, $q_j = \phi(I_j) > 0$, $\sum_{j=1}^{\infty} q_j = 1$, and let all consumers in the set $I_j$ be endowed with utility function $u^j$ and discount factor $\beta^j$. Furthermore, all consumers in the set $I_j$ have the same distribution of labor endowment shocks $\mu^j$.

In this case, I apply Feldman-Gilles construction as follows: Let the probability space $(\Omega, \mathcal{F}, P)$ and the process $(s_t^i)_{i\in I}$ be such that $s_t : I_j \times \Omega \to \mathbb{S}$ is $\mathcal{B}(I_j) \otimes \mathcal{F}_t$-measurable and

$$\begin{aligned} P(\omega &\in A : s^i(\omega) \in A) = \mu^j(A),\ \forall i \in I_j, \\ \phi(i &\in I_j : s^i(\omega) \in A) = \mu^j(A),\ \forall \omega \in \Omega,\ A \in \mathcal{B}(\mathbb{S}^\infty). \end{aligned}$$

Then the transition function of $(s_t^i)$ for all $i \in I_j$, $Q^j$, satisfies:

$$Q^j(s,B) \equiv P(\omega \in \Omega : s_{t+1}^i(\omega) \in B \mid s_t^i = s) = \phi(i \in I_j : s_{t+1}^i \in B \mid s_t^m = s, m \in I_j).$$

Since all consumers in $I_j$ are ex ante identical and choose the same policy function $g^j$, it follows from the previous case that the transition function for the aggregate distribution across consumers in $I_j$ is given by:

$$\begin{aligned} \overline{\Lambda}_t^j(a,s,A\times B) &\equiv \phi(i \in I_j : (a_{t+1}^i, s_{t+1}^i) \in A \times B \mid s_t^m = s, m \in I_j) \\ &= \phi(i \in I_j : (g^j(a,s), s_{t+1}^i) \in A \times B \mid s_t^m = s, m \in I_j) \\ &= \mathbf{1}_A(g^j(a,s))\phi(i \in I_j : s_{t+1}^i \in B \mid s_t^m = s, m \in I_j) \\ &= \mathbf{1}_A(g^j(a,s))Q^j(s,B). \end{aligned}$$

Then, by (23), the aggregate transition function can be written as:

$$\begin{aligned}\overline{\Lambda}_t(a,s,A\times B) &= \sum_{j=1}^{\infty}\phi(i\in I_j:(g^j(a,s),s^i_{t+1})\in A\times B\mid s^m_t=s,m\in I)\\ &= \sum_{j=1}^{\infty}\mathbf{1}_A(g^j(a,s))\phi(m\in I_j)\phi(i\in I_j:s^i_{t+1}\in B\mid s^m_t=s,m\in I_j)\\ &= \sum_{j=1}^{\infty}q_j\mathbf{1}_A(g^j(a,s))Q^j(s,B)=\int_I\mathbf{1}_A(g^i(a,s))Q^i(s,B)\phi(di).\end{aligned}$$

Thus the aggregate transition function is equal to the weighted average of the transition functions across all types of consumers.

The above analysis can be summarized in the following lemma.

**Lemma 4.2** *Under Assumptions 1, 2, 4, 6 and 7, and the Feldman-Gilles construction, the aggregate transition function is time invariant and given by* $\overline{\Lambda}(a,s,A\times B)\equiv\int_I\Lambda^i(a,s,A\times B)\phi(di)=\int_I\mathbf{1}_A(g^i(a,s))Q^i(s,B)\phi(di)$, *for* $(a,s)\in\mathbb{A}\times\mathbb{S}$ *and* $A\times B\in\mathcal{B}(\mathbb{A})\times\mathcal{B}(\mathbb{S})$.

**Remark 4.1** *When there are uncountably many types of ex ante identical consumers, one can apply a method described in [29].*[20] *Specifically, let* $I$ *and* $J$ *be copies of* $[0,1]$ *and let* $\psi$ *be a Lebesgue measure on* $\mathcal{B}(I\times J)$ *with the marginal* $\phi$ *on* $\mathcal{B}(I)$ *and the marginal* $q$ *on* $\mathcal{B}(J)$. *One can think of an element* $(i,j)\in I\times J$ *indexing a consumer* $i$ *of type* $j$. *Let any variable* $x$ *associated with consumer* $i$ *of type* $j$ *be denoted by* $x^{(i,j)}$. *Consumers of the same type* $j$ *have the same felicity* $u^j$ *and the same distribution* $\mu^j$ *of labor endowment shocks, i.e.,* $u^{(i,j)}=u^j,\mu^{(i,j)}=\mu^j,\beta^{(i,j)}=\beta^j$. *Then one can apply the Feldman-Gilles construction to show that*

$$\overline{\Lambda}_t(a,s,A\times B)=\int_J\mathbf{1}_A(g^j(a,s))Q^j(s,B)q(dj)=\int_{I\times J}\mathbf{1}_A(g^{(i,j)}(a,s))Q^{(i,j)}(s,B)\psi(di,dj).$$

Note that the shocks processes $(s^i_t)$, $i\in I$, are generally correlated across consumers due to the measurability requirement. This raises two questions: (i) Does correlation matter? (ii) Can it be avoided?

With regard to the first question, correlation does not matter for all my results and interpretations. This is because an individual's decision depends only on his own individual state and the aggregate distribution of individual states across the population (or market prices). It does not depend on any other particular individual's actions, even though he may realize that his shocks are correlated with others. Furthermore, every individual knows that his own decision does not affect the aggregate distribution or market prices. This is the cornerstone hypothesis of competitive markets and anonymous games.[21]

[20]This case will not be considered in the later analysis.
[21]See [27, 7, 29] for further discussion.

Turn to the second question. It is known that if a continuum of random variables are independent and have a common distribution, then the process is not measurable and even has no measurable standard modification with respect to the relevant product measure [28, 19, 44]. Judd [28] shows that there is a natural extension of the probability space so that this measurability problem disappears. However, one has still to face the second difficulty. That is, the set of sample realizations satisfying the law of large numbers has outer measure one and inner measure zero, therefore this set is not measurable. However, Judd shows that there is an extension of the measure so that the law of large numbers is satisfied for these random variables. Sun [44] argues that both this approach and the Feldman-Gilles construction are not a theory of the law of large numbers for a continuum of random variables because the law depends critically on a particular probability space and a particular process. He further argues that one has to go beyond the usual continuum framework. In particular, he claims hyperfinite processes which are measurable with respect to Loeb product spaces constitute the right class for the exact law of large of numbers.

This paper does not intend to propose a new theory of the law of large numbers. The purpose is to demonstrate how one can apply some 'law of large numbers' such as the Feldman-Gilles construction to dynamic economies so that aggregation removes individual risks.[22]

### 4.2 Existence of Equilibrium

It is useful to restate Definition 1 in a recursive form.

**Definition 2** *A recursive stationary (competitive) equilibrium* $((V^i, g^i)_{i\in I}, (r,w), \overline{\lambda})$ *consists of a system of constant prices* $(r,w) \in \mathbb{R}^2$*, a value function* $V : I \times \mathbb{A} \times \mathbb{S} \to \mathbb{R}$*, a policy function* $g : I \times \mathbb{A} \times \mathbb{S} \to \mathbb{A}$*, and a probability distribution* $\overline{\lambda} \in \mathcal{P}(\mathbb{A} \times \mathbb{S})$ *such that:*

*(i) Given prices* $(r,w)$*,* $V^i$ *is a fixed point of* $T^i$ *in problem (13) and* $g^i$ *is the corresponding optimal policy function for* $\phi$*-a.e.* $i$*.*

*(ii) Given prices* $(r,w)$*, the firm maximizes profits, i.e.,* $r = F_1(K,1) - \delta,\ w = F_2(K,1)$*, where* $K = \int_{\mathbb{A}\times\mathbb{S}} a\overline{\lambda}(da,ds)$*.*

*(iii) Markets clear:* $\int_{\mathbb{A}\times\mathbb{S}} s\overline{\lambda}(da,ds) = 1$*.*

*(iv)* $\overline{\lambda}$ *is an invariant distribution, i.e., for all* $A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S})$*,*

$$\overline{\lambda}(A \times B) = \int_{\mathbb{A}\times\mathbb{S}} \int_I \mathbf{1}_A(g^i(a,s))Q^i(s,B)\phi(di)\overline{\lambda}(da,ds).$$

It merits emphasis that (iv) can be justified by Lemma 4.2. Moreover, the equilibrium wage rate $w$ can be written as a function of $r$ by (7)-(8). Thus, the interest rate is in fact the only price that needs to be determined in equilibrium.

[22] Sun [45] applies the theory developed in [44] to study a static pure exchange economy. It remains to be seen how one can apply his theory to dynamic economies such as the Bewley-style model studied here.

Given a recursive stationary equilibrium $((V^i, g^i)_{i\in I}, (r, w), \overline{\lambda})$, a stationary equilibrium can be constructed as follows. Let the initial aggregate distribution be $\overline{\lambda}$, i.e., $\overline{\lambda}_0(A \times B) = \phi(i \in I : (a_0^i, s_0^i) \in A \times B) = \overline{\lambda}(A \times B)$ for all $A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S})$. Let the allocation $((a_{t+1}^i, c_t^i)_{t\geq 0})_{i\in I}$ be defined by (22). Then using Lemma 4.1, it can be shown that $(((a_{t+1}^i, c_t^i)_{t\geq 0})_{i\in I}, (r, w), \overline{\lambda})$ constitutes a stationary equilibrium.

One key step of the construction of an equilibrium is to find the invariant aggregate distribution. The following lemma establishes that the measure given by (12) is an invariant aggregate distribution.

**Lemma 4.3** *Let $\lambda^{*i}$ be an invariant distribution of the joint process of asset holdings and shocks for consumer $i$. If $i \mapsto \lambda^{*i}(D)$ is $\mathcal{B}(I)$-measurable for each $D \in \mathcal{B}(\mathbb{A} \times \mathbb{S})$, then the measure $\int_I \lambda^{*i}\phi(di)$ is the invariant aggregate distribution.*

This lemma and item (iii) in Definition 4.3 put a restriction on the shock process $(s_t^i)$. Specifically, since the marginal of $\lambda^{*i}$ on $\mathbb{S}$ is the invariant distribution of the process $(s_t^i)$ for each $i \in I$, the following restriction must hold:

$$\int_{\mathbb{A}\times\mathbb{S}} s\overline{\lambda}(da, ds) = \int_I \int_{\mathbb{A}\times\mathbb{S}} s\lambda^{*i}(da, ds)\phi(di) = 1.$$

That is, the average of the mean values of all shocks with respect to their invariant distributions must be equal to the exogenously given aggregate labor supply.

Now, the existence of a stationary equilibrium can be established.

**Theorem 4.1** *Under Assumptions 1-5, there exists a stationary equilibrium $(((a_{t+1}^i, c_t^i)_{t\geq 0})_{i\in I}, (r^*, w^*), \overline{\lambda})$ such that $\beta^i(1+r^*) < 1$ for $\phi$-a.e. $i$ provided that the borrowing limit $\underline{a}$ satisfies $r\underline{a} + w(r)\underline{s} \geq \varepsilon$ for some $\varepsilon > 0$ and $r \in (-\delta, 1/\beta_{\min} - 1)$. Furthermore, under these conditions, there is no stationary equilibrium with the interest rate $r$ and the wage $w$ satisfying $r \geq 1/\beta^i - 1$ for positive $\phi$-measure of $i$.*

It has been shown that deterministic version of the Bewley-style model exhibits an extreme distribution of capital in a stationary equilibrium. That is, only the most patient consumer determines the aggregate capital stock so that the equilibrium interest rate is equal to the rate of time preference of the most patient consumer (e.g., [6]). Theorem 4.1 shows that this result does not carry over to the Bewley-style model.

In the Bewley-style model, the equilibrium interest rate is less than that in the deterministic case. Furthermore, aggregate savings are higher than that in the deterministic case. These extra precautionary savings reflect the fact that consumers' income is random and only partially insured. This result is the key implication of the Bewley-style model. It implies that heterogeneity, incomplete markets, borrowing constraint, and uncertainty play important roles in the determination of the wealth distribution, aggregate savings and the interest rate.

Because the interest rate $r^*$ and the wage rate $w^*$ in any stationary equilibrium satisfy $r^*\underline{a} + w^*\underline{s} > 0$ and $\beta^i(1 + r^*) < 1$ for $\phi$-a.e. $i$, all partial equilibrium results established in Section 3 are valid for stationary equilibria. In particular, Theorem 3.7 and Lemma 4.4 imply that it is a necessary feature that the borrowing constraint binds for a positive mass of consumers in stationary equilibrium.[23]

The intuition behind Theorem 4.1 is as follows. For both the deterministic case and uncertainty case, if $r$ is larger than the rate of time preference of the most patient consumer (denoted by $\varrho$), then consumption will grow without bound. This violates the resource constraint given Assumption 5 so that in any stationary equilibrium $r$ must not exceed $\varrho$. In the deterministic case, if $r$ equals $\varrho$, then the most patient consumer can demand any asset level bigger than $\underline{a}$ in the long run and all other consumers are borrowing constrained. Furthermore, if $r$ is less than $\varrho$, then all consumers will be borrowing constrained. In fact, the unique steady-state equilibrium for the deterministic case is characterized by $r = \varrho$. By contrast, in the uncertainty case, $r$ cannot equal $\varrho$ because if so, it follows from Theorem 3.7 that consumption of the most patient consumer would grow without bound, contradicting the resource constraint. Importantly, if $r$ is less than $\varrho$, there exists an ergodic set and an ergodic measure for the joint process of asset holdings and shocks (Theorem 3.7). Moreover, the expected asset holdings and aggregate asset demand move continuously with respect to $r$ and increase to infinity as $r$ approaches $\varrho$. Thus, Theorem 4.1 is obtained as illustrated in Figure 1.

### 4.3 Properties of Equilibria

*Wealth distribution.* I first consider the situation where there are two types of consumers who differ in only one of the following parameters: discount factors, the degrees of risk aversion, borrowing limits, and distributions of endowment shocks. How are aggregate capital and wealth distributed between these two types of consumers?

Let the discount factor, the borrowing limit and the transition function of endowment shocks of a type $j$ consumer be $\beta^j$, $\underline{a}^j$ and $Q^j$ respectively, $j = 1, 2$. When consumers differ in risk aversion, I restrict attention to a particular class of utility functions of the form $u^j(c) = c^{\alpha^j}/\alpha^j$, $\alpha^j < 0$, $j = 1, 2$. Let the mass of type $j$ consumers be $q_j$ ($q_1 + q_2 = 1$). The following theorem follows from Theorems 3.6, 3.9 and Lemma 4.3, the proof of which is omitted.

**Theorem 4.2** *Given the assumptions in Theorem 4.1, consider the stationary equilibrium* $(((a^i_{t+1}, c^i_t)_{t\geq 0})_{i\in I}, (r, w), \overline{\lambda})$. *Corresponding to type* $j$, *let* $a^{*j}(\underline{s})$ *and* $s^{*j}$ *be given in Theorem 3.5*

[23] A similar result is established in [26] under stronger conditions.

*and let $\mathbb{K}^j = [\underline{a}^j, \overline{a}^j]$ be given in Lemma 3.1 and $\lambda^{*j}$ given in Theorem 3.6. Then:*

*(i) The aggregate distribution is given by $\overline{\lambda} = q_1\lambda^{*1} + q_2\lambda^{*2}$. Furthermore, if $\beta^2 > \beta^1$ or $\underline{a}^2 > \underline{a}^1$, then $\lambda^{*2} \succeq_{FSD} \lambda^{*1}$.*

*(ii) Aggregate savings and aggregate wealth for type $j$ consumers are given by $q_j \int_{\mathbb{A}\times\mathbb{S}} a\lambda^{*j}(da, ds)$ and $q_j \left[(1+r)\int_{\mathbb{A}\times\mathbb{S}} a\lambda^{*j}(da, ds) + w\int_{\mathbb{A}\times\mathbb{S}} s\lambda^{*j}(da, ds)\right]$, respectively.*

*(iii) The mass of type $j$ consumers who are borrowing constrained is $q_j\lambda^{*j}([\underline{a}^j, a^{*j}(\underline{s})] \times [\underline{s}, s^{*j}])$. Furthermore, if $\beta^2 > \beta^1$ or $Q^1 \succ_c Q^2$, then $a^{*2}(\underline{s}) < a^{*1}(\underline{s})$ and $s^{*2} \leq s^{*1}$.*

*(iv) If $\beta^2 > \beta^1$, or $\underline{a}^2 > \underline{a}^1$, or $Q^1 \succ_c Q^2$ or $\alpha^2 < \alpha^1$, then $\overline{a}^2 > \overline{a}^1$ and the consumers whose assets exceed $\overline{a}^1$ are of type 2 and their mass is $q_2\lambda^{*2}((\overline{a}^1, \overline{a}^2] \times \mathbb{S})$.*

It is observed from the data that the poorest twenty percent of the population have near zero wealth on average, whereas the richest five percent of the population hold roughly half of aggregate wealth. Such skewness in wealth cannot be generated from the Bewley-style model with ex ante identical consumers as shown by the numerical results of [1] and [30]. However, Krusell and Smith [30] show by numerical example that differences in discount factors can generate large degrees of skewness. The above theorem establishes this possibility analytically and also demonstrates that other factors such as differences in population, risk aversion, borrowing constraints, and distributions of shocks can also help to explain this pattern. For instance, if the fraction of consumers ($q_2$) whose conditional distribution of shocks is stochastically dominated is sufficiently small, then the mass of consumers who hold large capital stock (more than $\overline{a}^1$), $q_2\lambda^{*2}((\overline{a}^1, \overline{a}^2] \times \mathbb{S})$, is small. Furthermore, the mass of consumers who are borrowing constrained, $\sum_{j=1}^{2} q_j\lambda^{*j}([\underline{a}, a^{*j}(\underline{s})] \times [\underline{s}, s^{*j}])$, is large.

The above theorem also implies that: (a) When the two types of consumers have equal mass, aggregate savings and wealth for consumers with higher discount factor or tighter borrowing constraint are strictly greater than for those with lower discount factor or looser borrowing constraint. (b) The mass of type 2 consumers who are borrowing constrained is less than that of type 1 consumers if every type 2 consumer has a higher discount factor or his conditional distribution shocks is stochastically dominated. (c) The consumers who hold sufficiently large assets are of the type who have higher discount factor, or tighter borrowing constraint, or greater degree of risk aversion, or worse conditional distribution of shocks.

*Comparative statics.* Note that uniqueness of equilibrium is not guaranteed because the aggregate capital supply curve is not necessarily monotonic in the interest rate so that it may cross the aggregate capital demand curve more than once. Nevertheless, one can still conduct comparative statics analysis on the set of equilibria using the partial order $\succeq_s$ defined in Section 3.4.

**Theorem 4.3** *Under Assumptions 1-5, if the borrowing constraint is tightened, or if the* *discount factor is increased for almost every consumer, then the set of equilibrium interest rates decreases provided that the constraint on borrowing limits given in Theorem 4.1 is satisfied.*

The intuition behind this theorem is that if the borrowing constraint is tightened or if the discount factors are increased, then Theorem 3.9 (iv) and Lemma 3.6 imply that aggregate capital supply increases. Since the aggregate capital demand is fixed, the result then follows (see Figure 1).

# A Appendix: Existence and Properties of The Value and Policy Functions

**Proofs of Theorems 3.1-3.2:**

Recall problem (13):

$$Tv(a,s) = \sup_{a' \in \Gamma(a,s)} u((1+r)a + ws - a') + \beta E\left[v(a',s') \mid s\right], \tag{24}$$

where $v : \mathbb{A} \times \mathbb{S} \to \mathbb{R}$ and $\Gamma(a,s) = [\underline{a}, (1+r)a + ws]$.

Assume that Assumption 2 (a) holds and that $v$ satisfies the following properties:

**Property 1.** For each $s \in \mathbb{S}$, $v(\cdot, s)$ is increasing, concave and continuously differentiable on $\mathbb{A}$.

**Property 2.** For each $a \in \mathbb{A}$, $v_1(a, \cdot)$ is decreasing on $\mathbb{S}$.

Since $v$ is continuous and concave and $u$ is continuous and strictly concave, there exists a unique maximizer of problem (24), $k$. As a function of $(a,s)$, $k$ is continuous by the Maximum Theorem. The corresponding optimal consumption policy $c : \mathbb{A} \times \mathbb{S} \to \mathbb{R}$ is given by $c(a,s) = (1+r)a + ws - k(a,s)$. Then for all $(a,s) \in \mathbb{A} \times \mathbb{S}$,

$$Tv(a,s) = u((1+r)a + ws - k(a,s)) + \beta E\left[v(k(a,s),s') \mid s\right]. \tag{25}$$

Note that $c(a,s) > 0$ since $u'(0) = \infty$ from Assumption 2 (a).

The first-order condition for problem (24) is given by

$$u'(c(a,s)) \geq \beta E\left[v_1(k(a,s),s') \mid s\right] \text{ with equality if } k(a,s) > \underline{a}. \tag{26}$$

This condition is also sufficient for optimality by concavity.

**Lemma A.1** *Suppose $u$ satisfies Assumption 2 (a) and $v$ satisfies property 1. For fixed $s \in \mathbb{S}$, if $k(a,s) = \underline{a}$ for some $a \in (\underline{a}, \overline{a}]$, then there exists a unique $a^0 \geq a$ such that $k(\widetilde{a}, s) > \underline{a}$ for all $\widetilde{a} > a^0$, and $k(\widetilde{a}, s) = \underline{a}$ and*

$$\frac{\partial Tv}{\partial a}(\widetilde{a}, s) = (1+r)u'((1+r)\widetilde{a} + ws - \underline{a}),$$

*for all $\widetilde{a} \in [\underline{a}, a^0]$.*

**Proof:** If $k(a,s) = \underline{a}$ for some $a \in (\underline{a}, \overline{a}]$, then (26) holds. By Assumption 2 (a), $u'(c(a,s)) = u'((1+r)a + ws - \underline{a})$ is strictly decreasing in $a$ and it goes to $0$ as $a \to \infty$. Moreover, Property 1 implies that $v_1(\underline{a}, s') > 0$. Thus, the Intermediate Value Theorem and (26) imply that there is a unique $a^0 \geq a$ such that it satisfies the properties in the lemma. Finally, the equality in the lemma follows from differentiation of (25) at $\widetilde{a} \in [\underline{a}, a^0]$. ■

The following proposition shows that $Tv$ inherits some properties of $v$. I first state strong versions of properties 1-2.

**Property 1$'$**. *For each $s \in \mathbb{S}$, $v(\cdot, s)$ is strictly increasing, strictly concave and continuously differentiable on $\mathbb{A}$.*

**Property 2$'$**. *For each $a \in \mathbb{A}$, $v_1(a, \cdot)$ is strictly decreasing on $\mathbb{S}$.*

**Proposition A.1** *Suppose that Assumptions 1 (a)-(c), 2 (a), 6, and 7 hold. If $v$ satisfies properties 1-2, then the function $Tv$ satisfies properties 1$'$ and 2. Furthermore, if $v(\cdot, s)$ is strictly concave for each $s \in \mathbb{S}$ and satisfied property 2, then $Tv$ satisfies property 2$'$. Finally, for all $(a, s) \in \mathbb{A} \times \mathbb{S}$,*

$$\frac{\partial Tv}{\partial a}(a, s) = (1 + r)u'(c(a, s)). \tag{27}$$

The proof uses the following lemma. Let $\widetilde{k}$ be the optimal asset holdings and $\widetilde{c}$ be optimal consumption for problem (24) when $v$ is replaced by the function $\widetilde{v} : \mathbb{A} \times \mathbb{S} \to \mathbb{R}$. Let $k^n$ be the optimal asset holdings and $c^n$ be optimal consumption for problem (24) when $v$ is replaced by the function $v^n : \mathbb{A} \times \mathbb{S} \to \mathbb{R}$.

**Lemma A.2** *Suppose that Assumption 2 (a) holds.*

*(i) If $v \leq \widetilde{v}$ for any function $\widetilde{v} : \mathbb{A} \times \mathbb{S} \to \mathbb{R}$, then $Tv \leq T\widetilde{v}$.*

*(ii) If $v$ and $\widetilde{v}$ satisfy the properties 1-2 and $v_1 \leq \widetilde{v}_1$ then $k \leq \widetilde{k}$ and $c \geq \widetilde{c}$.*

*(iii) If $v$ satisfies properties 1-3, $v^n(\underline{a}, s)$ increases to $v(\underline{a}, s)$ and $v_1^n(a, s)$ increases to $v_1(a, s)$ for all $(a, s) \in \mathbb{A} \times \mathbb{S}$ as $n \to \infty$, then $Tv^n(a, s)$ increases to $Tv(a, s)$ and $k^n(a, s)$ increases to $k(a, s)$ for each $(a, s) \in \mathbb{A} \times \mathbb{S}$.*

**Proof:** Part (i) is obvious from (24).

(ii) Since $(1 + r)a + ws > \underline{a}$ and $u'(0) = \infty$, $c(a, s) > 0$ and $\widetilde{c}(a, s) > 0$. Suppose $k > \widetilde{k}$. Then $k > \underline{a}$. By the first-order condition (26) and $v_1 \leq \widetilde{v}_1$,

$$\begin{aligned} \beta E\left[\widetilde{v}_1(k, s') \mid s\right] &\geq \beta E\left[v_1(k, s') \mid s\right] = u'((1 + r)a + ws - k) \\ &> u'((1 + r)a + ws - \widetilde{k}) \geq \beta E\left[\widetilde{v}_1(\widetilde{k}, s') \mid s\right]. \end{aligned}$$

Thus, by concavity of $\widetilde{v}$, $k < \widetilde{k}$. This is a contradiction.

(iii) If $v_1^n$ increases to $v_1$ and $v^n(\underline{a}, s)$ increases to $v(\underline{a}, s)$, then $v^n(a, s) = v^n(\underline{a}, s) + \int_{\underline{a}}^a v_1^n(x, s)dx$ increases to $v(a, s) = v(\underline{a}, s) + \int_{\underline{a}}^a v_1(x, s)dx$. By parts (i)-(ii), $Tv^n(a, s)$ increases to a limit, denoted by $\overline{v}$, and $k^n$ increases to a limit, denoted by $\overline{k}$. It remains to show that $\overline{v} = Tv$ and $\overline{k} = k$.

I first show that $\overline{k} = k$ by considering two cases.

1. $\overline{k}(a,s) = \underline{a}$. Then $k^n = \overline{k} = \underline{a}$, $c^n(a,s) > 0$ for all $n$ sufficiently large. By the first-order condition (26), $u'((1+r)a + ws - \underline{a}) \geq \beta E\left[v_1^n(\underline{a}, s') \mid s\right]$ for all $n$ sufficiently large. By the Monotone Convergence Theorem, $u'((1+r)a + ws - \underline{a}) \geq \beta E\left[v_1(\underline{a}, s') \mid s\right]$. Since the objective function of problem (24) is strictly concave by assumption, the above first-order condition is also sufficient for optimality. Thus $k = \overline{k} = \underline{a}$.

2. $\overline{k}(a,s) > \underline{a}$. Then for sufficiently large $N$, $k^n > \underline{a}$ for $n > N$. Consider two cases. Since $u'(0) = \infty$, $c^n(a,s) > 0$ for all $n > N$. Then $k^n$ satisfies the first-order condition: $u'((1+r)a+ws-k^n) = \beta E\left[v_1^n(k^n, s') \mid s\right]$. Letting $n \to \infty$, $\overline{k} = k$ from the same argument as above.

Finally, show that $\overline{v} = Tv$. This follows from the Monotone Convergence Theorem and

$$\begin{aligned} Tv(a,s) &= u((1+r)a + ws - k) + \beta E\left[v(k, s') \mid s\right] \\ &= \lim_{n\to\infty} u((1+r)a + ws - k^n) + \beta E\left[v^n(k^n, s') \mid s\right] \\ &= \lim_{n\to\infty} Tv^n(a,s) = \overline{v}, \text{ for all } (a,s), \end{aligned}$$

as desired. ■

**Proof of Proposition A.1:** The proof consists of two steps.

*Step 1. Show that $Tv$ satisfies property 1′ and (27).*

The strict monotonicity and concavity of $Tv$ follow from standard argument. I then show the remaining properties by considering two cases.

**Case 1.** $k(a,s) > \underline{a}$ for all $a > \underline{a}$ and $s \in \mathbb{S}$. Then one can apply the Benveniste and Scheinkman Theorem [43, Theorem 9.10] to deliver differentiability of $Tv$ and (27).

**Case 2.** $k(a,s) = \underline{a}$ for some $a > \underline{a}$ and for any $s \in \mathbb{S}$. Then by Lemma A.1 there is an $a^0 > a$ such that for fixed $s$ and all $\widetilde{a} < a^0$,

$$\frac{\partial Tv}{\partial a}(\widetilde{a}, s) = (1+r)u'((1+r)\widetilde{a} + ws - \underline{a}),$$

and for all $\widetilde{a} > a^0$, $k(\widetilde{a}, s) > \underline{a}$. Thus if $\widetilde{a} > a^0$, from the preceding case, $Tv(\cdot, s)$ is continuously differentiable and satisfies (27). Taking the right derivative in (27) and left derivative in the above equation at $a^0$ yields the same value. Thus $Tv(\cdot, s)$ is continuously differentiable and satisfies (27).

Combining the above two cases and using the continuity of $u'$, one obtains that $\frac{\partial Tv}{\partial a}(\underline{a}, \underline{s}) = u'(r\underline{a} + w\underline{s})$ and that $Tv$ has property 1′ and satisfies (27) on $\mathbb{A} \times \mathbb{S}$.

*Step 2. Show that $Tv$ has property 2 or 2′.*

Suppose that $v(a,s)$ has property 2. If $k(a,s) = \underline{a}$ then $c(a,s) = (1+r)a + ws - \underline{a}$ is increasing in $s$. By the envelope condition (27) and $u'' < 0$, $\partial Tv/\partial a$ is strictly decreasing in $s$.

If $k(a,s) > \underline{a}$, then $(1+r)a + ws > \underline{a}$. The first-order condition is given by

$$u'(c(a,s)) = \beta E\left[v_1((1+r)a + ws - c(a,s), s') \mid s\right],$$

Fixing $c(a,s)$, when $s$ increases, the R.H.S. of the above equation (strictly) decreases since $v_1(a,s)$ is (strictly) decreasing in $a$ and $s$ by the hypothesis and $Q$ is monotone by Assumption 1 (c). Thus, since $u'' < 0$, $c(a,s) = (1+r)a + ws - k(a,s)$ must be (strictly) increasing in $s$. Using the envelope condition (27) yields $\frac{\partial Tv}{\partial a}(a,s) = (1+r)u'(c(a,s))$. Since $u'$ is strictly decreasing and $c(a,s)$ is (strictly) increasing in $s$, $\partial Tv(a,s)/\partial a$ is (strictly) decreasing in $s$ for each $a \in \mathbb{A}$. ■

Define a sequence of functions $\{V^n\}$ recursively by $V^0 = 0$ and $V^{n+1} = TV^n$. Let $k^{n+1} : \mathbb{A} \times \mathbb{S} \to \mathbb{R}$ be the optimal asset accumulation policy function of problem (24) when $v = V^n$. The optimal consumption policy $c^{n+1} : \mathbb{A} \times \mathbb{S} \to \mathbb{R}$ is given by $c^{n+1}(a,s) = (1+r)a + ws - k^{n+1}(a,s)$. Then it is trivial that $c^1(a,s) = (1+r)a + ws - \underline{a}$ and $k^1(a,s) = \underline{a}$.

Since $u'(0) = \infty$, $c^n(a,s) > 0$ for all $n$. Then by induction and Proposition A.1,

$$V^{n+1}(a,s) = u(c^{n+1}(a,s)) + \beta E\left[V^n(k^{n+1}(a,s), s') \mid s\right], \tag{28}$$

$$V_1^{n+1}(a,s) = (1+r)u'(c^{n+1}(a,s)). \tag{29}$$

By the Maximum Theorem $c^n$ and $k^n$ are continuous functions for all $n$. Moreover, each $V^n$ satisfies other properties stated in Proposition A.1.

**Lemma A.3** *Suppose Assumptions 1 (a)-(c), 2, 6, and 7 hold. Then:*

*(i)* $\{V^n\}$, $\{V_1^n\}$ *and* $\{k^n\}$ *are increasing sequences of functions, and* $\{c^n\}$ *is a decreasing sequence of functions.*

*(ii) There exist two functions* $L : \mathbb{A} \times \mathbb{S} \to \mathbb{R}$ *and* $M : \mathbb{A} \times \mathbb{S} \to \mathbb{R}$ *such that* $-\infty < L(a,s) \le V^n(a,s) \le V(a,s) \le M(a,s) < \infty$, *for all* $n \ge 1$ *and all* $(a,s) \in \mathbb{A} \times \mathbb{S}$. *Furthermore,* $\lim_{t\to\infty} E\left[\beta^t M(a_t, s_t) \mid s_0\right] = 0$ *for all feasible plans* $(a_t)$ *and all* $(a_0, s_0) \in \mathbb{A} \times \mathbb{S}$.

**Proof:** (i) It can be shown that $V^1(a,s) = u((1+r)a + ws - \underline{a})$ and $V_1^1(a,s) = (1+r)u'((1+r)a + ws - \underline{a})$. So $V_1^1(a,s) > V_1^0(a,s) = 0$, $V^1(a,s) \ge V^0(a,s) = 0$, $k^1 \le k^2$, and $c^1 \ge c^2$ by Lemma A.2 (ii). Thus by (27), $V_1^2(a,s) = (1+r)u'(c^2(a,s)) \ge (1+r)u'(c^1(a,s)) = V_1^1(a,s)$. By induction and Lemma A.2, one can show that $\{k^n\}$, $\{V^n\}$ and $\{V_1^n\}$ are increasing sequences, and that $\{c^n\}$ is a decreasing sequence of functions.

(ii) Since $u(0) = 0$ and $u$ is increasing, one can take $L = 0$. It is also straightforward to show that $V^n(a,s) \le V(a,s)$ for all $n \ge 0$. Observe that one can rewrite the budget constraint as

$$c_t + \widehat{a}'_{t+1} = (1+r)\widehat{a}_t + z_t,$$

where $\widehat{a}_t \equiv a_t - \underline{a} \geq 0$ and $0 \leq z_t \equiv r\underline{a} + ws_t \leq r\underline{a} + w\overline{s}$ for all $t \geq 0$. Then it can be shown that any feasible consumption plan $(c_t)$ must satisfy:

$$\begin{aligned} c_t &= z_t + (1+r)\widehat{a}_t - \widehat{a}'_{t+1} \leq z_t + (1+r)\widehat{a}_t \leq z_t + (1+r)\left[(1+r)\widehat{a}_{t-1} + z_{t-1}\right] \\ &= z_t + (1+r)z_{t-1} + (1+r)^2\widehat{a}_{t-1} \leq \cdots \leq z_t + (1+r)z_{t-1} + \cdots + (1+r)^t z_0 + (1+r)^{t+1}\widehat{a}_0 \\ &\leq \frac{1}{r}\left[(1+r)^{t+1} - 1\right](w\overline{s} + r\underline{a}) + (1+r)^{t+1}(a_0 - \underline{a}) \equiv \overline{c}_t(a_0). \end{aligned}$$

If $r < 0$, then $\overline{c}_t(a) \leq a - w\overline{s}/r - 2\underline{a}$ for all $t \geq 0$. Thus,

$$V(a,s) \leq E\left[\sum_{t=0}^{\infty} \beta^t u(\overline{c}_t(a_0))\right] = u(a - w\overline{s}/r - 2\underline{a})/(1-\beta) < \infty.$$

Take $M(a,s) = u(a - w\overline{s}/r - 2\underline{a})/(1-\beta)$.

If $r \geq 0$,then $\overline{c}_t(a)$ grows without bound. By Assumption 2 (b), one can show that for any $0 < \sigma < \gamma$, there exists a $x_\sigma > 0$ large enough such that for all $c > x_\sigma$, $u'(c) \leq c^{-\sigma}$, or

$$u(c) \leq \xi_1 \frac{c^{1-\sigma} - 1}{1-\sigma} + \xi_2 \equiv \overline{u}(c),$$

where $\xi_1 > 0$ and $\xi_2$ are constants that depend on $x_\sigma$ (e.g., [42, Lemma 3.3.2]). Since $\gamma > 1$, one can take $\sigma = 1$. Because $\overline{c}_t(a)$ grows without bound, there exists a minimal time $t_0$ such that $\overline{c}_t(a) > x_\sigma$ for all $t \geq t_0$. Thus,

$$V(a,s) \leq \sum_{t=0}^{t_0-1} \beta^t u(\overline{c}_t(a)) + \frac{\beta^{t_0}\xi_2}{1-\beta^{t_0}} + \xi_1 \sum_{n=t_0}^{\infty} \beta^n \log(\overline{c}_n(a)).$$

As shown in section 3.1, $\sum_{n=t_0}^{\infty} \beta^n \log(\overline{c}_n(a))$ is finite. Thus, one can take $M(a,s)$ as the function on the R.H.S. of the above inequality. ■

**Proof of Theorem 3.1:**

(i) Since $\{V^n(a,s)\}$ is an increasing sequence and bounded above, it converges to a finite limit $V^\infty(a,s)$. To show $V^\infty$ is a fixed point of $T$, let $c^\infty$ and $a^\infty$ be the optimal consumption and asset accumulation policies for problem $TV^\infty$. Then since $(c^\infty, a^\infty)$ is also feasible for problem $V^n = TV^{n-1}$,

$$\begin{aligned} 0 &\leq TV^\infty(a,s) - V^n(a,s) \leq TV^\infty(a,s) - u(c^\infty) - \beta E\left[V^{n-1}(a^\infty, s') \mid s\right] \\ &= \beta E\left[V^\infty(a^\infty, s') \mid s\right] - \beta E\left[V^{n-1}(a^\infty, s') \mid s\right]. \end{aligned}$$

Thus letting $n \to \infty$ and using the Monotone Convergence Theorem, the expression on the R.H.S. of the above equality goes to 0. This implies that $V^\infty$ is a fixed point of $T$. Since $V^n$ is continuous for each $n$ by the Maximum Theorem, $V^\infty$ is lower semicontinuous by Dini's Theorem.

I claim that $V^\infty = V$. By [43, Theorem 9.2], it suffices to show that $\lim_{t\to\infty} \beta^t E\left[V^\infty(a_t, s_t) \mid s_0\right] = 0$ for all feasible plans $(a_t)$ and all $(a_0, s_0) \in \mathbb{A} \times \mathbb{S}$. This is true by Lemma A.3.

To show continuity of $V$, define a sequence of functions $\{V^{*n}(a,s)\}$ by $V^{*0}(a,s) = M(a,s) < \infty$ and $V^{*n}(a,s) = T^n V^{*0}(a,s)$. Then since $TV^{*0}(a,s) \leq V^{*0}(a,s)$, Lemma A.2 implies that $\{V^{*n}\}$ is a decreasing sequence of functions. Moreover, $V^{*n}$ is a continuous function by the Maximum Theorem for each $n$. Since $V^{*n} \geq 0$ by Assumption 2 (a), $\{V^{*n}\}$ converges to a finite function $V^{*\infty}$ pointwise. By Dini's Theorem, $V^{*\infty}$ is upper semicontinuous. Similar to the previous argument, one can show that $V^{*\infty}$ is a fixed point of $T$.

Because $\lim_{t\to\infty} \beta^t E\left[V^{*\infty}(a_t, s_t) \mid s_0\right] = 0$ for all feasible plans $(a_t)$ and all $(a_0, s_0) \in \mathbb{A} \times \mathbb{S}$ by Lemma A.3, it follows from [43, Theorem 9.2] that $V^{*\infty} = V$. Thus $V$ is both lower semicontinuous and upper semicontinuous so that $V$ is continuous.

To show uniqueness of fixed points of $T$ in $\mathcal{V}$, suppose that there is another fixed point of $T$, say $\overline{v} \in \mathcal{V}$. Then since $0 \leq \overline{v}(a,s) \leq M(a,s)$, Lemma A.2 implies that $V^{\infty}(a,s) = \lim_{n\to\infty} T^n 0 \leq \overline{v} = \lim_{n\to\infty} T^n \overline{v} \leq \lim_{n\to\infty} T^n M(a,s) = V^{*\infty}$. Thus $V^{*\infty} = \overline{v} = V^{\infty} = V$.

(ii) By Lemma A.3, $\{k^n(a,s)\}$ is an increasing sequence of functions. Since this sequence is bounded above by $(1+r)a + ws$, it converges to a finite limit, denoted by $g(a,s)$. It follows that the corresponding sequence of optimal consumption polices $\{c^n(a,s)\}$ converges to a finite limit, denoted by $f(a,s) = (1+r)a + ws - g(a,s)$. Letting $n \to \infty$ in (28) and using $\lim_{n\to\infty} k^n(a,s) = g(a,s)$, one obtains that $g(a,s)$ is a maximizer for problem $V(a,s) = TV(a,s)$. Because $V$ is continuous and strictly concave, the optimal policy $g : \mathbb{A} \times \mathbb{S} \to \mathbb{A}$ must be a continuous function by the Maximum Theorem. ■

**Proof of Theorem 3.2:**

(i) By Proposition A.1, it can be inductively shown that $V^n(\cdot, s)$ satisfies the desired properties of monotonicity, concavity and differentiability. Moreover, $V_1^n(\cdot, s)$ is bounded in $[\underline{a}, a]$. Thus by the Monotone Convergence Theorem and (29),

$$V(a,s) - V(\underline{a},s) = \lim_{n\to\infty} \left[V^n(a,s) - V^n(\underline{a},s)\right] = \lim_{n\to\infty} \int_{\underline{a}}^{a} V_1^n(x,s)dx$$

$$= (1+r) \lim_{n\to\infty} \int_{\underline{a}}^{a} u'(c^n(x,s))dx = (1+r) \int_{\underline{a}}^{a} u'(f(x,s))dx.$$

So $V$ is continuously differentiable in $a$ and satisfies (14). Monotonicity and concavity of $V$ follow from induction or [43, Corollary 1, p.52].

(ii) By induction and Proposition A.1, one can show that $V_1^n(a,s)$ is strictly decreasing in $s$ for each $a \in \mathbb{A}$. Thus the limit $V_1(a,s)$ as $n \to \infty$ is decreasing in $s$ for each $a \in \mathbb{A}$. Moreover, $V_1(a,s)$ must be strictly decreasing in $s$ since $V_1(a,s) = \partial TV/\partial a$ and $\partial TV/\partial a$ is strictly decreasing in $s$ by Proposition A.1. Monotonicity of $V(a,s)$ in $s$ can be similarly shown by the above induction argument. Since it is standard [43, Theorem 9.11], I omit the detail. ■

**Proof of Theorem 3.4:**

(i) Since $0 < \beta(1+r) \leq 1$ and $V_1(a, \cdot)$ is strictly decreasing by Theorem 3.2 (ii),

$$V_1(\underline{a}, \underline{s}) > E\left[V_1(\underline{a}, s') \mid \underline{s}\right] \geq \beta(1+r)E\left[V_1(\underline{a}, s') \mid \underline{s}\right]. \tag{30}$$

Thus, Theorem 3.3 implies that $g(\underline{a}, \underline{s}) = \underline{a}$. It follows that $f(\underline{a}, \underline{s}) = r\underline{a} + w\underline{s} > 0$. Then by monotonicity of $f$ shown below, $f(a, s) \geq f(\underline{a}, \underline{s}) = r\underline{a} + w\underline{s}$.

(ii) Monotonicity of $f$ follows from the envelope condition (14) and Theorem 3.2. Monotonicity of $g(\cdot, s)$ follows from standard argument (e.g., [43]).

(iii) Let $s^1 < s^2$. If $g(a, s^1) = \underline{a}$, then $g(a, s^2) \geq \underline{a}$. So suppose that $g(a, s^1) > \underline{a}$, but $g(a, s^1) \geq g(a, s^2)$. Then by the budget constraint, $f(a, s^2) = (1+r)a + ws^2 - g(a, s^2) \geq f(a, s^1) + w(s^2 - s^1)$. Thus, by the Mean Value Theorem,

$$u'(f(a, s^1)) - u'(f(a, s^2)) = u''(\widehat{c})(f(a, s^1) - f(a, s^2)) \geq \eta w(s^2 - s^1), \tag{31}$$

where $\widehat{c} \in [f(a, s^1), f(a, s^2)]$ and the inequality follows from Assumption 2 (c).

Since $\beta(1+r) \leq 1$, it follows that

$$\begin{aligned}
& \beta E\left[V_1(g(a, s^1), s') \mid s^1\right] - \beta E\left[V_1(g(a, s^2), s') \mid s^2\right] \\
\leq \; & \beta E\left[V_1(g(a, s^1), s') \mid s^1\right] - \beta E\left[V_1(g(a, s^1), s') \mid s^2\right] \\
= \; & \beta(1+r) \int_{\mathbb{S}} u'(f(g(a, s^1), s')) \left[Q(s^1, ds') - Q(s^2, ds')\right] \\
\leq \; & \beta(1+r)u'(r\underline{a} + w\underline{s}) \int_{\mathbb{S}} \left|Q(s^1, ds')/ds' - Q(s^2, ds')/ds'\right| ds' \\
\leq \; & u'(r\underline{a} + w\underline{s}) \int_{\mathbb{S}} \left|Q(s^1, ds')/ds' - Q(s^2, ds')/ds'\right| ds',
\end{aligned}$$

where the first line follows from concavity of $V(\cdot, s')$ and the supposition that $g(a, s^1) \geq g(a, s^2)$, the second line from the envelope condition, and the third line from part (i). Thus, use (31) and the Euler inequality to derive

$$\begin{aligned}
u'(f(a, s^2)) + \eta w(s^2 - s^1) \leq u'(f(a, s^1)) = \beta E\left[V_1(g(a, s^1), s') \mid s^1\right] \\
\leq \beta E\left[V_1(g(a, s^2), s') \mid s^2\right] + u'(r\underline{a} + w\underline{s}) \int_{\mathbb{S}} \left|Q(s^1, ds')/ds' - Q(s^2, ds')/ds'\right| ds'.
\end{aligned}$$

Finally, apply the Euler inequality for $g(a, s^2) > \underline{a}$ to obtain

$$\eta w(s^2 - s^1) \leq u'(r\underline{a} + w\underline{s}) \int_{\mathbb{S}} \left|Q(s^1, ds')/ds' - Q(s^2, ds')/ds'\right| ds'.$$

This is a contradiction because Assumption 1 (b) and (5) imply that

$$\int_{\mathbb{S}} \left|Q(s^1, ds')/ds' - Q(s^2, ds')/ds'\right| ds' < \kappa(s^2 - s^1) \leq \frac{\eta w}{u'(r\underline{a} + w\underline{s})}(s^2 - s^1).$$

**Remark A.1** *In the general equilibrium analysis, $w$ and $r$ are endogenous. Given (6) and the assumptions that $r\underline{a} + w\underline{s} > \varepsilon$ and $w \geq \underline{w}$, one can still establish this result using the preceding three equations.*

(iv) If $u$ is unbounded, then $V(a,s)$ is also unbounded as $a \to \infty$. This is because the plan saving nothing and consuming all wealth is budget feasible so that

$$V(a,s) \geq u((1+r)a + ws) + E\sum_{t=1}^{\infty}\beta^t u(ws_t) \to \infty, \text{ as } a \to \infty.$$

Suppose that $f(a,s)$ is bounded above by $d < \infty$. Then $V(a,s) \leq u(d)/(1-\beta)$, which is a contradiction. If $u$ is bounded, then the argument follows from [13, Lemma 2]. ■

**Proof of Theorem 3.5:**

I first show (17). If $u'(r\underline{a} + w\overline{s}) \geq \beta E\left[V_1(\underline{a}, s') \mid \overline{s}\right]$, then $g(\underline{a}, \overline{s}) = \underline{a}$. Thus $s^* = \overline{s}$ satisfies (17). Now suppose $u'(r\underline{a} + w\overline{s}) < \beta E\left[V_1(\underline{a}, s) \mid \overline{s}\right]$. By (30), $u'(r\underline{a} + w\underline{s}) > \beta E\left[V_1(\underline{a}, s') \mid \underline{s}\right]$. Noting that $u'(r\underline{a} + ws)$ and $\beta E\left[V_1(\underline{a}, s') \mid s\right]$ are continuous and strictly decreasing in $s$ by Theorem 3.2 and Assumptions 1- 2, one can apply the Intermediate Value Theorem to obtain that there exists a unique $\underline{s} < s^* < \overline{s}$ such that (17) holds as illustrated in Figure 3.

Turn to the proof of (18). From Figure 3, one can see that if $s > s^*$, then

$$\beta E\left[V_1(\underline{a}, s') \mid s\right] > u'(r\underline{a} + ws) \geq u'((1+r)a + ws - \underline{a}).$$

Thus, there is no $a \in [\underline{a}, \infty)$ such that $\beta E\left[V_1(\underline{a}, s') \mid s\right] = u'((1+r)a + ws - \underline{a})$. If $s < s^*$, then

$$0 < \beta E\left[V_1(\underline{a}, s') \mid s\right] < u'(r\underline{a} + ws) = \max_{a \in \mathbb{A}} u'((1+r)a + ws - \underline{a}).$$

Moreover, by Assumption 2, $\lim_{a\to\infty} u'((1+r)a + ws - \underline{a}) = 0$. Thus, by the Intermediate Value Theorem, there exists a unique $a^* > \underline{a}$ satisfying (18).

(i) If $a \in [\underline{a}, a^*(s)]$ and $s \in [\underline{s}, s^*]$, then

$$u'(f(a,s)) \geq u'(f(a^*, s)) = \beta E\left[V_1(\underline{a}, s') \mid s\right],$$

where the inequality follows from monotonicity of $f(\cdot, s)$ by Theorem 3.4 (ii) and concavity of $u$, the equality from (18). Thus, it follows from Theorem 3.3 that $g(a,s) = \underline{a}$ for every $a \in [\underline{a}, a^*(s)]$ and $s \in [\underline{s}, s^*]$.

(ii) If $a > a^*(s)$ and $s \in [\underline{s}, s^*)$, then by (18) and Theorem 3.3 and strict concavity of $V(\cdot, s)$ from Theorem 3.2,

$$\beta(1+r)E\left[V_1(g(a,s), s') \mid s\right] \leq V_1(a,s) < V_1(a^*(s), s) = \beta(1+r)E\left[V_1(\underline{a}, s') \mid s\right].$$

Thus, strict concavity of $V(\cdot, s)$ implies that $g(a,s) > \underline{a}$. Finally, if $s \in (s^*, \overline{s}]$, then by (17) and concavity of $u$, $\beta E\left[V_1(\underline{a}, s') \mid s\right] > u'(r\underline{a} + ws) \geq u'((1+r)a + ws - \underline{a})$. Thus, Theorem 3.3 implies that $g(a,s) > \underline{a}$. ■

# B Appendix: Long-Run Behavior of Consumption and Asset Holdings

**Proof of Lemma 3.1:**

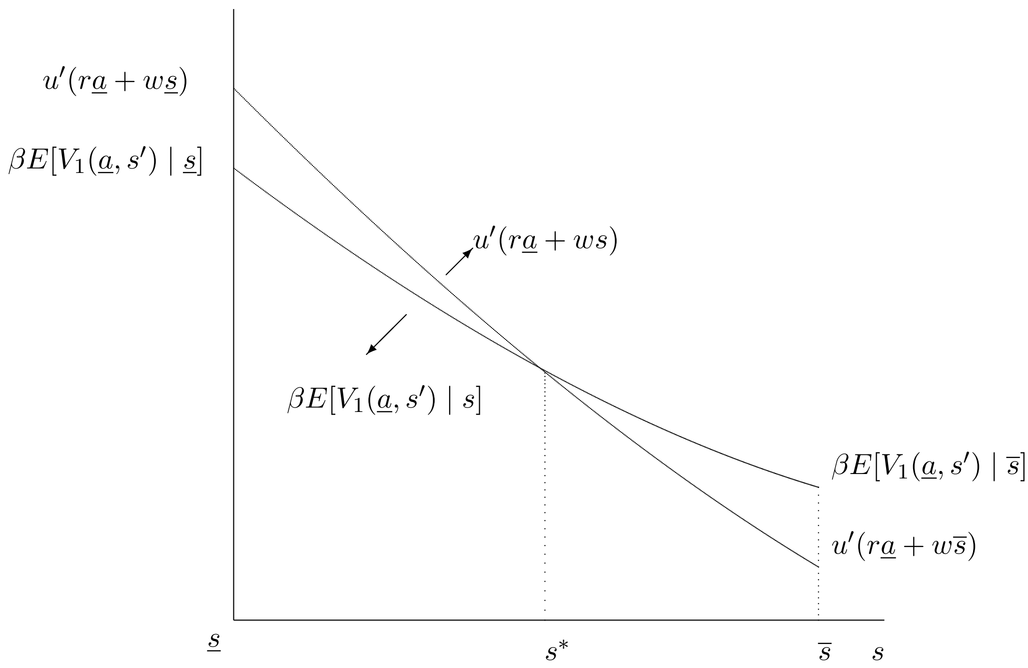


Figure 3: Determination of $s^*$

I first prove two lemmas.

**Lemma B.1** *Given Assumptions 1, 2, 6, and 7, if* $\beta(1+r) < 1$*, then* $g(a, \underline{s}) < a$ *for* $a > \underline{a}$.

**Proof:** Since $V_1(a, \cdot)$ is decreasing by Theorem 3.2 (iii) and $\beta(1+r) < 1$, using Theorem 3.3 yields:

$$V_1(a, \underline{s}) \geq E\left[V_1(a, s') \mid \underline{s}\right] > \beta(1+r)E\left[V_1(a, s') \mid \underline{s}\right],$$

for any $a > \underline{a}$. Thus, if $g(a, \underline{s}) \geq a > \underline{a}$, then by Theorem 3.3, $V_1(a, \underline{s}) = \beta(1+r)E\left[V_1(g(a, s), s') \mid \underline{s}\right]$. This contradicts the above inequality by strict concavity of $V_1(\cdot, s')$. ■

**Lemma B.2** *Given Assumptions 1, 2, 6, and 7, if* $\beta(1+r) < 1$ *and* $w \geq \underline{w}$*, then there exists a finite* $\widetilde{a} > \underline{a}$ *such that* $g(a, s) < a$ *whenever* $a > \widetilde{a}$ *for all* $s \in \mathbb{S}$.

**Proof:** Since $g(a, \overline{s}) \geq g(a, \underline{s})$ for all $a \in \mathbb{A}$ by Theorem 3.4 (iii), $(1+r)a + w\underline{s} - g(a, \overline{s}) \leq (1+r)a + w\underline{s} - g(a, \underline{s})$. This implies that

$$f(a, \overline{s}) - w(\overline{s} - \underline{s}) \leq f(a, \underline{s}) \text{ or } \frac{f(a, \overline{s})}{f(a, \underline{s})} \leq 1 + w(\overline{s} - \underline{s})/f(a, \underline{s}).$$

By the envelope condition (14), Assumption 2 (b), and Theorem 3.4 (iv), one can deduce that for $a$ large enough,

$$V_1(a, \underline{s})/V_1(a, \overline{s}) = u'(f(a, \underline{s}))/u'(f(a, \overline{s})) \leq \left[\frac{f(a, \overline{s})}{f(a, \underline{s})}\right]^{\rho_2} \leq [1 + w(\overline{s} - \underline{s})/f(a, \underline{s})]^{\rho_2},$$

where $\rho_2 > \gamma$. Thus, $\lim_{a\to\infty} V_1(a, \underline{s})/V_1(a, \overline{s}) \leq 1$. On the other hand, since $V_1$ is decreasing in $s$, $V_1(a, \underline{s}) \geq V_1(a, \overline{s})$. Thus, $\lim_{a\to\infty} V_1(a, \underline{s})/V_1(a, \overline{s}) = 1$. Since $\beta(1+r) < 1$, there exists a finite $\widetilde{a} > \underline{a}$ large enough such that for all $a > \widetilde{a}$,

$$V_1(a, \overline{s}) > \beta(1+r)V_1(a, \underline{s}) \geq \beta(1+r)E\left[V_1(a, s') \mid \overline{s}\right].$$

Finally, use a similar argument in the proof of Lemma B.1 to conclude $g(a, \overline{s}) < a$. ■

**Proof of Lemma 3.1:** Since $g(\underline{a}, \overline{s}) \geq \underline{a}$, $g(a, \overline{s}) < a$ for $a$ sufficiently large from Lemma B.2, and $g(a, \overline{s})$ is continuous from Theorem 3.4, there exists $a \in \mathbb{A}$ such that $g(a, \overline{s}) = a$. Define $\overline{a}$ as

$$\overline{a} \equiv \min\{a \in \mathbb{A} : g(a, \overline{s}) = a\}. \tag{32}$$

If $u'(r\underline{a} + w\overline{s}) < \beta E\left[V_1(\underline{a}, s) \mid \overline{s}\right]$, then $g(\underline{a}, \overline{s}) > \underline{a}$ so that $\overline{a} > \underline{a}$. Otherwise, Theorem 3.3 implies $g(\underline{a}, \overline{s}) = \underline{a}$ so that $\overline{a} = \underline{a}$.

Define $\mathbb{K} = [\underline{a}, \overline{a}]$. To show that $\mathbb{K} \times \mathbb{S}$ is the unique ergodic set, it is sufficient to show that (a) once the process $(a_t, s_t)$ has entered $\mathbb{K} \times \mathbb{S}$ there is zero probability that the process will depart from it; (b) once the process $(a_t, s_t)$ has entered any set disjoint from $\mathbb{K} \times \mathbb{S}$, there is probability 1 that the process will depart from it. That is, such a set is transient.

For part (a), if $(a_t, s_t) \in \mathbb{K} \times \mathbb{S}$, then by Theorem 3.4 and the definition of $\overline{a}$, (32), $\underline{a} \leq a_{t+1} = g(a_t, s_t) \leq g(\overline{a}, \overline{s}) = \overline{a}$. Thus one can inductively show that $(a_n, s_n) \in \mathbb{K} \times \mathbb{S}$ for any $n \geq t$.

For part (b), there are two types of sets to consider, depicted as $(\overline{a}, a^1] \times \mathbb{S}$ and $[a^1, a^2] \times \mathbb{S}$ in Figure 4. For the set $(\overline{a}, a^1] \times \mathbb{S}$ suppose that at some date $t$, $(a_t, s_t) \in (\overline{a}, a^1] \times \mathbb{S}$. Since $g(a, s) \leq g(a, \overline{s}) < a$ for any $(a, s) \in (\overline{a}, a^1] \times \mathbb{S}$ as shown in Figure A.1. By monotonicity of $g$ established in Theorem 3.4, one can inductively show that the sequence $\{a_n\}_{n \geq t}$ defined by $a_{n+1} = g(a_n, s_n)$ is strictly decreasing for any sequence of shocks $\{s_n\}_{n>t}$. Thus there is an $N$ large enough such that $a_N \leq \overline{a}$ with probability 1.

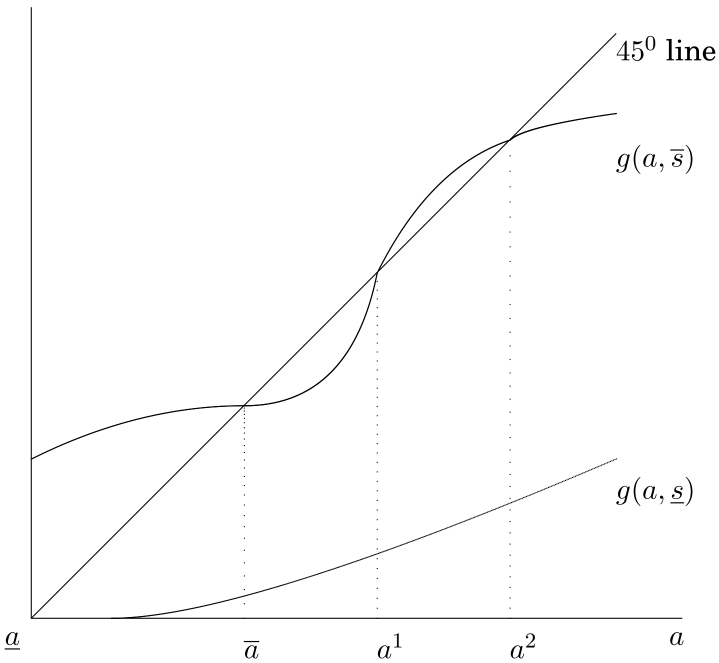


Figure 4: Ergodic set.

Turn to the set $[a^1, a^2] \times \mathbb{S}$. Suppose that $(a_t, s_t) \in [a^1, a^2] \times \mathbb{S}$, at some date $t$. Since $g(a, \underline{s}) < a$ for $a > \underline{a}$ by Lemma B.1 and $g(a, \cdot)$ is continuous by Theorem 3.2, there exists an $\widehat{s} > \underline{s}$ such that for any $s \leq \widehat{s}$, $g(a, \widehat{s}) < a$. Thus the sequence $\{a_{t+n}, s_{t+n}\}$, defined by $a_{t+1} = g(a_t, s_t)$, $a_{t+n+1} = g(a_{t+n}, s_{t+n})$ and $s_{t+n} \leq \widehat{s}$ for $n > 0$, is strictly decreasing. Thus, for any sequence of shocks $\{s_{t+n}\}_{n>1}$ such that $\underline{s} \leq s_{t+n} \leq \widehat{s}$, $a_N < a^1$ for $N$ sufficiently large. The probability of this occurrence is at least $\zeta$ given by:

$$\zeta = \int_{\underline{s}}^{\widehat{s}} \cdots \int_{\underline{s}}^{\widehat{s}} \int_{\underline{s}}^{\widehat{s}} Q(s^{N-1}, ds^N) Q(s^{N-2}, ds^{N-1}) \cdots Q(s_t, ds^1) > 0.$$

Thus, with probability at least $\zeta$, the sequence $\{(a_{t+n}, s_{t+n})\}_{n \geq 0}$ leaves $[a^1, a^2] \times \mathbb{S}$ and never returns. Consequently, the expected number of visits to $[a^1, a^2] \times \mathbb{S}$ is at most $\sum_{j=1}^{\infty}(1-\zeta)^j < \infty$. This implies that $[a^1, a^2] \times \mathbb{S}$ is a transient set. ■

**Proof of Theorem 3.6:**

If $\overline{a} = \underline{a}$, the results are trivial. Consider now $\overline{a} > \underline{a}$.

(i) The proof follows from the following facts and [17] or [35, Theorem 6.1].

*Fact 1. Feller property:* $\Lambda$ has the Feller property because the policy function $g$ is continuous and $Q$ satisfies the Feller property (see [43, Exercise 8.10]).

*Fact 2. Ergodicity:* $\mathbb{K} \times \mathbb{S}$ *is the unique ergodic set as shown in the preceding theorem.*

*Fact 3. Irreducibility:* Any open subset $A \times B$ of $\mathbb{K} \times \mathbb{S}$ can be reached by a sequence starting from any point $(a^0, s^0) \in \mathbb{K} \times \mathbb{S}$ in a finite number of steps with positive probability.

Partition $\mathbb{K}\times\mathbb{S}$ into two subsets, denoted by $(\mathbb{K}\times\mathbb{S})_1$ (the area above the $45^0$ line in Figure 4) and $(\mathbb{K} \times \mathbb{S})_2$ (the area below the $45^0$ line in Figure 4). For any point $(a, s) \in (\mathbb{K} \times \mathbb{S})_1$, $g(a, s) > a$ and for any point $(a, s) \in (\mathbb{K} \times \mathbb{S})_2$ , $g(a, s) < a$. Then the desired result can be easily established (e.g., [17, Theorem 4.2]) because the probability that the next period state $(g(a^0, s^0), s')$ starting from $(a^0, s^0)$ enters any open subset of $(\mathbb{K} \times \mathbb{S})_1$ or $(\mathbb{K} \times \mathbb{S})_1$ is positive.

(ii) Using the definition of $\lambda^*$ and Theorem 3.5, one can show that there is an $a^*(\underline{s}) > \underline{a}$ given in (18) such that

$$\begin{aligned}
\lambda^*(\{\underline{a}\} \times \mathbb{S}) &= \int_{\mathbb{A}\times\mathbb{S}} \mathbf{1}_{\{\underline{a}\}}(g(a, s))Q(s, \mathbb{S})\lambda^*(da, ds) = \lambda^*((a, s) \in \mathbb{A} \times \mathbb{S} : g(a, s) = \underline{a}) \\
&= \lambda^*([\underline{a}, a^*(\underline{s})] \times [\underline{s}, s^*]) > 0.
\end{aligned}$$

where the first equality follows from invariance of $\lambda^*$ and the last inequality from part (i). ■

**Proof of Theorem 3.7:**

The proof follows the idea in [13, Theorems 2 and 4] where there is no restriction on shock process $(s_t)$ such as stationarity and where $u$ is bounded, increasing and strictly concave, but not necessarily differentiable. I omit the details. The key observation is that the process $\beta^t(1 + r)^t u'(c_t)$, $t \geq 0$, is a nonnegative bounded supermartingale even for unbounded utility $u$ by Theorem 3.3. Moreover, a lemma [13, Lemma 2] used in the proof of [13, Theorem 4] is still valid for unbounded utility by Theorem 3.4 (iv). ■

# C Appendix: Comparative Statics Analysis

**Proof of Theorem 3.8:**

For weak monotonicity, I adapt arguments from [23]. So I only sketch the key step. For strict monotonicity, I exploit first-order conditions for interior solutions. Rewrite problem

(13) as

$$Tv(a,s;\beta,\underline{a},Q) = \sup_{a'\in\Gamma(a,s;\underline{a})} u((1+r)a+ws-a') + \beta E\left[v(a',s';\beta,\underline{a},Q)\mid s\right],$$

where $v:\mathbb{A}\times\mathbb{S}\to\mathbb{R}$ and $\Gamma(a,s;\underline{a}) = [\underline{a},(1+r)a+ws]$. Let

$$H(a,s,a';\beta,\underline{a},Q) = u((1+r)a+ws-a') + \beta\int_{\mathbb{S}} v(a',s';\beta,\underline{a},Q)Q(s,ds).$$

(i) For weak monotonicity in $\beta$, it suffices to show that $H(a,s,a';\beta)$ is supermodular in $(a,a',\beta)$ if $v(a',s';\beta)$ is supermodular in $(a',\beta)$. In fact, if $v(a',s';\beta)$ is supermodular in $(a',\beta)$, then for $\beta_2\geq\beta_1$ and $a_1',a_2'\in\mathbb{A}$,

$$v(a_1'\vee a_2',s';\beta_2) - v(a_1',s';\beta_2) \geq v(a_2',s';\beta_1) - v(a_1'\wedge a_2',s';\beta_1)$$

Multiplying the LHS by $\beta_2$ and the RHS by $\beta_1$ preserves the preceding inequality. Thus, $\beta E\left[v(a',s';\beta,Q)\mid s\right]$ is supermodular in $(a',\beta)$ and so is $H(a,s,a';\beta,Q)$.

For strict monotonicity in $\beta$, suppose $g(a,s;\beta^1) = g(a,s;\beta^2)$ for some $(a,s)\in\mathbb{A}\times\mathbb{S}$ such that $g(a,s;\beta^1) > \underline{a}$. Then by the first-order conditions,

$$\begin{aligned} u'((1+r)a+ws-g(a,s;\beta^1)) &= \beta^1 E\left[u'((1+r)g(a,s;\beta^1)+ws'-g(g(a,s;\beta^1),s';\beta^1))\mid s\right] \\ u'((1+r)a+ws-g(a,s;\beta^2)) &= \beta^2 E\left[u'((1+r)g(a,s;\beta^2)+ws'-g(g(a,s;\beta^2),s';\beta^2))\mid s\right] \end{aligned}$$

one can derive that

$$\begin{aligned} & \beta^1 E\left[u'((1+r)g(a,s;\beta^1)+ws'-g(g(a,s;\beta^1),s';\beta^1))\mid s\right] \\ = \; & \beta^2 E\left[u'((1+r)g(a,s;\beta^2)+ws'-g(g(a,s;\beta^2),s';\beta^2))\mid s\right] \\ = \; & \beta^2 E\left[u'((1+r)g(a,s;\beta^1)+ws'-g(g(a,s;\beta^1),s';\beta^2))\mid s\right]. \end{aligned}$$

Since $g(g(a,s;\beta^1),s';\beta^1) \leq g(g(a,s;\beta^1),s';\beta^2)$ if $\beta^1 < \beta^2$ as shown before, and since $u'$ is strictly decreasing, the above equality is a contradiction to the fact that $\beta^1<\beta^2$.

(ii) For weak monotonicity in $Q$, it suffices to show that $H(a,s,a';Q)$ is supermodular in $(-a,-a',Q)$ if $v(a',s';Q)$ is supermodular in $(-a',Q)$ and in $(-a',s')$, and it is increasing in $s'$.[24] This is because $Tv(a,s;Q)$ is supermodular in $(-a,-a';Q)$ by [23, Lemma 1], and it is supermodular in $(-a',s')$ and increasing in $s'$ by Proposition A.1. Then one can adapt the argument in the proof of [23, Proposition 2].

If $v(a',s';Q)$ is supermodular in $(-a',Q)$, then for $Q_2\succeq_c Q_1$ and $a_1',a_2'\in\mathbb{A}$,

$$v(a_1'\wedge a_2',s';Q_2) - v(a_1',s';Q_2) \geq v(a_2',s';Q_1) - v(a_1'\vee a_2',s';Q_1)$$

Integrating the LHS with respect to $Q_2(s,ds')$ and the RHS with respect to $Q_1(s,ds')$ preserves the preceding inequality because $Q_2(s,\cdot)\succeq_{FSD} Q_1(s,\cdot)$ and $v(a',s';\cdot)$ is supermodular

[24]Note that $V$ is supermodular in $(-a,s)$ since $V_1(a,s)$ is decreasing in $s$ by Theorem 3.2 (ii).

in $(-a', s')$, which implies that as functions of $s'$ both the LHS and RHS are increasing in $s'$. Thus, $\beta \int v(a', s'; Q)Q(s, ds')$ is supermodular in $(-a', Q)$ and so is $H(a, s, a'; Q)$.

For strict monotonicity, suppose $g(a, s; Q^1) = g(a, s; Q^2)$ for some $(a, s) \in \mathbb{A} \times \mathbb{S}$ such that $g(a, s; ; Q^2) > \underline{a}$. Then one can similarly derive that

$$
\begin{aligned}
& \frac{1}{1+r} \int_{\mathbb{S}} V_1(g(a, s; Q^1), s'; Q^1) Q^1(s, ds') \\
= & \int_{\mathbb{S}} u'((1+r)g(a, s; Q^1) + ws' - g(g(a, s; Q^1), s'; Q^1)) Q^1(s, ds') \\
= & \int_{\mathbb{S}} u'((1+r)g(a, s; Q^2) + ws' - g(g(a, s; Q^2), s'; Q^2)) Q^2(s, ds') \\
= & \int_{\mathbb{S}} u'((1+r)g(a, s; Q^1) + ws' - g(g(a, s; Q^1), s'; Q^2)) Q^2(s, ds') \\
\leq & \int_{\mathbb{S}} u'((1+r)g(a, s; Q^1) + ws' - g(g(a, s; Q^1), s'; Q^1)) Q^2(s, ds') \\
= & \frac{1}{1+r} \int_{\mathbb{S}} V_1(g(a, s; Q^1), s'; Q^1) Q^2(s, ds').
\end{aligned}
$$

Since $V_1(g(a, s; Q^1), s'; Q^1)$ is strictly decreasing in $s'$ by Theorem 3.2 (ii), the above inequality is a contradiction to that fact that $Q^2(s, \cdot)$ strictly first-order stochastically dominates $Q^1(s, \cdot)$.

(iii) For weak monotonicity in $\underline{a}$, observing $\Gamma(a, s; \underline{a})$ is ascending in $\underline{a}$, it suffices to show that $H(a, s, a'; \underline{a})$ is supermodular in $(a, a', \underline{a})$ if $v(a', s'; \underline{a})$ is supermodular in $(a', \underline{a})$. This is immediate because the dependence of $H$ on $\underline{a}$ is through only $v(a', s'; \underline{a})$. ■

**Proof of Theorem 3.9:**

For parts (i)-(ii), I prove only monotonicity of $\mathbb{K}$ in $\alpha$. Then the rest follows from Theorem 3.8. Let $\alpha^1 < \alpha^2 < 1$. Then by Theorem 3.1, $\overline{a}^1$ satisfies the equation:

$$(r\overline{a}^1 + w\overline{s})^{\alpha^1 - 1} = \beta(1+r)E\left[(r\overline{a}^1 + ws')^{\alpha^1 - 1} \mid s\right].$$

Thus,

$$1 = \beta(1+r)E\left[\frac{(r\overline{a}^1 + ws')^{\alpha^1 - 1}}{(r\overline{a}^1 + w\overline{s})^{\alpha^1 - 1}} \mid s\right] > \beta(1+r)E\left[\frac{(r\overline{a}^1 + ws')^{\alpha^2 - 1}}{(r\overline{a}^1 + w\overline{s})^{\alpha^2 - 1}} \mid s\right].$$

If $r > 0$, then the expression on the RHS of the inequality is strictly decreasing in $\overline{a}^1$. Thus the solution for $y$ to the equation:

$$(ry + w\overline{s})^{\alpha^2 - 1} = \beta(1+r)E\left[(ry + ws')^{\alpha^2 - 1} \mid s\right],$$

is strictly smaller than $\overline{a}^1$. Then since $\overline{a}^2$ is determined by the smallest solution to the above equation as shown in Theorem 3.1, $\overline{a}^2 < \overline{a}^1$.

For (iii), because $g(a, s; \beta, \underline{a})$ is increasing in $(\beta, \underline{a})$ by Theorem 3.8, the associated Markov operator $M^*_\Lambda$ can be ordered in the sense of first-order stochastic dominance (see (20)-(21)).

Moreover, the operator $M_\Lambda^*$ is monotone by monotonicity of $g$ and $Q$. Then, since $\lambda^*$ is the limit of $M_\Lambda^{*n}(\lambda_0)$ for any $\lambda_0 \in \mathcal{P}(\mathbb{A} \times \mathbb{S})$ by Theorem 3.6, monotonicity of $\lambda^*$ in $(\beta, \underline{a})$ follows from [23, Corollary 3]. Finally, for part (iv), continuity of $E_{\lambda^*}[a]$ follows from a straightforward application of [43, Theorem 12.13]. Monotonicity follows from parts (i)-(iii). ■

# D Appendix: Existence and Properties of Equilibria

**Proof of Lemma 4.1:**

Let $v : I \times \mathbb{A} \times \mathbb{S} \to \mathbb{R}$ be a Caratheodory function that is $\mathcal{B}(I)$-measurable for fixed $(a, s) \in \mathbb{A} \times \mathbb{S}$ and continuous for fixed $i \in I$. Consider consumer $i$'s decision problem:

$$Tv^i(a,s) = \max_{a' \in \Gamma(a,s)} u^i((1+r)a + ws - a') + \beta^i \int_{\mathbb{A}\times\mathbb{S}} v^i(a', s') Q^i(s, ds')$$

where $\Gamma(a,s) = [\underline{a}, (1+r)a + ws]$. Under Assumption 4, $Tv^i(a,s)$ is $\mathcal{B}(I)$-measurable for fixed $(a,s) \in \mathbb{A} \times \mathbb{S}$ by the Measurable Maximum Theorem (see [3, Theorem 17.18]). It is also continuous for fixed $i \in I$ by the usual Maximum Theorem. Thus it is a Caratheodory function. Because the fixed point of $T$, $V^i(a,s)$, satisfies $V^i(a,s) = \lim_{n\to\infty} T^n 0$, $V^i(a,s)$ is also $\mathcal{B}(I)$-measurable for fixed $(a,s) \in \mathbb{A} \times \mathbb{S}$. Note that Theorem 3.1 has demonstrated that $V^i$ is a continuous function for fixed $i \in I$ under Assumptions 1 (a)-(b), 2, and 6-7. Thus $V^i(a,s)$ is a Caratheodory function. Finally, apply the Measurable Maximum Theorem and the Maximum Theorem again to conclude that the optimal policy function $(i,a,s) \mapsto g(i,a,s)$ corresponding to the problem $TV^i(a,s) = V^i(a,s)$ is a Caratheodory function.

I now use induction on $t$ to prove that $(i,\omega) \mapsto a_{t+1}(i,\omega)$ is $\mathcal{B}(I) \otimes \mathcal{F}_t$-measurable. First, as functions of $(i,\omega)$, $a_0^i$ and $s_0^i$ are $\mathcal{B}(I) \otimes \mathcal{F}_0$-measurable. It follows from [3, Lemma 4.48] that the function $(i,\omega) \mapsto (i, a_0(i), s_0(i))$ (from $I \times \Omega$ to $I \times \mathbb{A} \times \mathbb{S}$) is $(\mathcal{B}(I) \otimes \mathcal{F}_0, \mathcal{B}(I) \otimes \mathcal{B}(\mathbb{A}) \otimes \mathcal{B}(\mathbb{S}))$-measurable. Note that $g$ is $\mathcal{B}(I) \otimes \mathcal{B}(\mathbb{A}) \otimes \mathcal{B}(\mathbb{S})$-measurable because $g$ is a Caratheodory function (see [3, Lemma 4.50]). Thus, by [3, Lemma 4.51], $a_1^i = a_1(i,\omega) = g(i, a_0(i), s_0(i))$ is $\mathcal{B}(I) \otimes \mathcal{F}_0$-measurable.

Suppose now that $(i,\omega) \mapsto a_t(i,\omega)$ is $\mathcal{B}(I) \otimes \mathcal{F}_{t-1}$-measurable. Of course, it is also $\mathcal{B}(I) \otimes \mathcal{F}_t$-measurable. Then observing that $(i,\omega) \mapsto s_t^i(\omega)$ is $\mathcal{B}(I) \otimes \mathcal{F}_t$-measurable, the argument in the preceding paragraph can be carried out in the same way so that $(i,\omega) \mapsto a_{t+1}(i,\omega)$ is $\mathcal{B}(I) \otimes \mathcal{F}_t$-measurable. Finally, it follows immediately from the budget constraint (1) that $(i,\omega) \mapsto c_t(i,\omega)$ is $\mathcal{B}(I) \otimes \mathcal{F}_t$-measurable. ■

**Proof of Lemma 4.3:**

Noting that Lemma 4.2 implies that the aggregate transition function $\overline{\lambda}$ is given by

$$\overline{\lambda}(a, s, A \times B) = \int_I \mathbf{1}_A(g^i(a,s)) Q^i(s, B) \phi(di), \; A \times B \in \mathcal{B}(\mathbb{A}) \times \mathcal{B}(\mathbb{S}),$$

it suffices to show that

$$\begin{aligned}
\overline{\lambda}'(A\times B) &= \int_{\mathbb{A}\times\mathbb{S}}\int_I \mathbf{1}_A(g^i(a,s))Q^i(s,B)\phi(di)\overline{\lambda}(da,ds)\\
&= \int_I\int_{\mathbb{A}\times\mathbb{S}}\int_I \mathbf{1}_A(g^i(a,s))Q^i(s,B)\phi(di)\lambda^{*i}(da,ds)\phi(di)\\
&= \int_I\int_I\int_{\mathbb{A}\times\mathbb{S}} \mathbf{1}_A(g^i(a,s))Q^i(s,B)\lambda^{*i}(da,ds)\phi(di)\phi(di)\\
&= \int_I\int_I \lambda^{*i}(A\times B)\phi(di)\phi(di) = \int_I \lambda^{*i}(A\times B)\phi(di),
\end{aligned}$$

where the second equality follows from (12) by hypothesis, the third equality from the Fubini Theorem, and the fourth from the invariance of $\lambda^{*i}$. ■

**Proof of Theorem 4.1:**

I will construct a recursive stationary equilibrium. Then a stationary equilibrium can be obtained in the way described in Section 4.2. Note that the resource constraint (11) follows from aggregating the budget constraint (1) and using homogeneity of $F$.

Recall that the equilibrium wage rate $w = w(r) = F_2(F_1^{-1}(r+\delta),1) > 0$. Let $\mathcal{Y}$ be the set of all $r \in (-\delta, 1/\beta_{\min}-1)$ such that $r\underline{a} + w\underline{s} \geq \varepsilon > 0$. Clearly, $\mathcal{Y} \neq \varnothing$ (since $0 \in \mathcal{Y}$) and Assumptions 6-7 hold for all $r \in \mathcal{Y}$.

*Step 1. Show the properties of the capital supply function under Assumptions 6-7.* For all $r \in \mathcal{Y}$ and $r < 1/\beta^i - 1$ for $\phi$-a.e. $i$, equation (7) and Assumption 5 imply that $F_1(K,1) = r+\delta < 1/\beta^i - 1 + \delta \leq 1/\beta_{\min} - 1 + \delta$. The latter expression is equal to $F_1(\underline{K})$ by the definition in Remark 4. It follows from Assumption 5 and (8) that $K \geq \underline{K}$ and $w = F_2(K,1) \geq F_2(\underline{K},1) = \underline{w}$, where $\underline{w}$ is defined in Remark 2.4. Given (6), Remark A.1 after the proof Theorem 3.4 (iii) implies that the optimal asset accumulation policy is monotonic in the realization of shocks. Furthermore, all results in Section 3 still apply. In particular, there exists an invariant distribution $\lambda^{*i}$ for $\phi$-a.e. $i$. Lemma 4.1 implies that $\lambda^{*i}(r)$ is $\mathcal{B}(I)$-measurable so that one can apply Lemma 4.3 to obtain the invariant aggregate distribution $\overline{\lambda}(r) = \int_I \lambda^{*i}(da,ds;r)\phi(di)$. Let the aggregate capital supply be $K^s(r) = \int_{\mathbb{A}\times\mathbb{S}} a\overline{\lambda}(da,ds;r)$, $r \in \mathcal{Y}$. Note that the Feldman-Gilles construction ensures that the labor market clears, i.e., (10) holds, and that $\overline{\lambda}$ is a nonrandom distribution.

Consider a typical consumer and suppress the agent index. By Theorem 3.9, $E_{\lambda^*}[a]$ is a continuous function of $r$ for $r < 1/\beta - 1$ and $r \in \mathcal{Y}$. Furthermore, by Lemma 3.1 and Theorem 3.7, it goes to infinity as $r$ increases to $1/\beta - 1$, and goes to $\underline{a} \leq 0$ as $r$ decreases below to the value $r^0$ such that

$$u'(r^0\underline{a} + w\overline{s}) \geq \beta E\left[V_1(\underline{a},s') \mid \overline{s}\right] = \beta(1+r^0)E\left[u'(r^0\underline{a} + ws') \mid \overline{s}\right],$$

where the last equality follows from the envelope condition (14) and $g(\underline{a},s') = \underline{a}$ ($\underline{a} \leq g(\underline{a},s') \leq g(\underline{a},\overline{s}) = \underline{a}$). $r^0$ must be less than $1/\beta - 1$ because $u'(r^0\underline{a}+w\overline{s}) < E\left[u'(r^0\underline{a}+ws') \mid \overline{s}\right]$ from the concavity of $u$ and $w > 0$.

The above properties are inherited by the aggregate capital supply $K^s(r) = \int_I \int_{\mathbb{A}\times\mathbb{S}} a\lambda^{*i}(da, ds; r)\phi(di)$ as illustrated in Figure 1. Specifically, $K^s(r)$ approaches infinity as $r$ goes up to $1/\beta^i - 1$ for positive $\phi$-measure of $i$ and approaches $\underline{a}$ as $r$ goes below $r^{0i}$ for $\phi$-a.e. $i$.

*Step 2. Show the properties of the capital demand function.* The capital demand function is given by $K^d(r) = F_1^{-1}(r + \delta)$, which is continuous and decreasing in $r$ such that $\lim_{r\to\infty} K^d(r) = 0$ and $\lim_{r\to-\delta} K^d(r) = \infty$ as illustrated in Figure 1.

*Step 3. Show the existence of an equilibrium.* From steps 1-2 and the Intermediate Value Theorem, there is an interest rate $r^* \in \mathcal{Y}$ such that capital market clears, i.e., $K^d(r) = K^s(r)$ (see Figure 1). Moreover, $r^* < 1/\beta^i - 1$ for $\phi$-a.e. $i$. Let $w^* = F_2(F_1^{-1}(r^* + \delta), 1)$. Let $(V^i, g^i)_{i\in I}$ be the value and policy functions and $\overline{\lambda}$ be the invariant aggregate distribution, established in section 3, corresponding to $r^*$ and $w^*$. Then $((V^i, g^i)_{i\in I}, (r^*, w^*), \overline{\lambda})$ constitutes a stationary equilibrium in a recursive form.

To establish the second statement in the theorem, suppose that there exists a stationary equilibrium in which $r\underline{a} + w\underline{s} > 0$ and $\beta^i(1+r) \geq 1$ for positive $\phi$-measure of $i$. Then Theorem 3.7 implies that $a^i_{t+1} \overset{a.s.}{\to} \infty$ as $t \to \infty$ for these consumers. This implies that the aggregate capital stock satisfies $\int_I a^i_t \phi(di) \to \infty$ as $t \to \infty$ since each consumer's asset holdings are bounded below by $\underline{a}$. On the other hand, by Assumption 5 the maximal sustainable capital stock $K_{\max}$ is finite. It follows from the resource constraint (11) that $\int_I a^i_t \phi(di)$ must be finite for all $t$, which leads to a contradiction. ■

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