Robust Financial Contracting and Investment*

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Abstract

We study how investors' preferences for robustness influence corporate investment, financing, and compensation decisions and valuation in a financial contracting model with agency. We characterize the robust contract and show that early liquidation can be optimal when investors are sufficiently ambiguity averse. We implement the robust contract by debt, equity, cash, and a financial derivative asset. The derivative is used to hedge against the investors' concern that the entrepreneur may be overly optimistic. Our calibrated model generates sizable equity premium and credit spread, and implies that ambiguity aversion lowers Tobin's q, the average investment, and investment volatility. The entrepreneur values the project at an internal rate of return of 3.5% per annum higher than investors do.

Keywords: Robustness, Overconfidence, Dynamic agency, q theory, Financial constraints, Ambiguity aversion, Asset pricing

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1 Introduction

In standard corporate finance models, the entrepreneur and investors are endowed with the same belief about the venture's productivity. In reality and entrepreneurial finance cases analyzed in MBA classrooms, the entrepreneur and investors rarely agree on the venture's cash-flow projection, as "unknown unknowns" are common for early-stage entrepreneurial firms. Early-stage ventures may be in uncharted waters with very limited data that investors and the entrepreneur can use to form their beliefs. Being concerned about the entrepreneur's productivity model, investors are ambiguity averse and want their decision rules to be robust against potentially misspecified models.[1]

In this paper, we study how investors's preferences for robustness influence corporate (investment, financing, payout, compensation, and liquidation) decisions and valuation by adopting the approach of Hansen and Sargent (2001, 2008) in the dynamic contracting model of DeMarzo, Fishman, He, and Wang (2012), henceforth, DFHW (2012). Our formulation is highly tractable yet rich enough for us to analyze how investors' ambiguity aversion influences corporate decisions and valuation in a dynamic agency framework from both conceptual and quantitative perspectives.

In our model, a risk-neutral entrepreneur has an exclusive access to a productive venture with an initial capital stock but is financially constrained. The entrepreneur's talent (skill) is described by a stochastic productivity model. Following Hayashi (1982) and the q-theory literature, we model the firm's production technology via standard investment dynamics and capital adjustment costs.

The entrepreneur needs to raise start-up funds from outside risk-neutral investors by writing a long-term contract, which specifies state-contingent on-going financing, investment, compensation, payout, and liquidation policies. After the venture is launched, the entrepreneur continuously chooses his hidden effort level, which affects the venture's expected productivity. As in DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (henceforth, BMPR) (2007), DeMarzo and Fishman (2007a), DFHW (2012), and Miao and Rivera (2016), the entrepreneur must be incentivized to exert the appropriate effort levels over time. Alternatively, we can interpret the agency problem as one in which the agent can divert output for his private benefit.

Unlike DFHW (2012), the investors in our model do not fully trust the entrepreneur's productivity model. We build on the multiplier preferences proposed by Hansen and Sargent (2001) to model the investors' objective by scaling the entropy penalty cost by a size-dependent process. This

[1]Knight (1921) is perhaps the first to point out the difference between risk (known unknowns) and uncertainty (unknown unknowns). Model uncertainty or ambiguity is often referred to as Knightian uncertainty. The Ellsberg paradox (Ellsberg (1961)) and the related experimental evidence demonstrate that people are typically averse to ambiguity. Hansen and Sargent (2001, 2008) interpret decision under ambiguity as robust decision. We adopt both interpretations interchangeably in this paper.

generalization allows us to focus on economically interesting cases where the investors' concerns for the entrepreneur's productivity model are always economically relevant regardless of firm size. The investors design a robust dynamic contract that maximizes their payoffs taking both their ambiguity aversion and the agency problem into account.

While the entrepreneur is confident about his model, investors view the entrepreneur's productivity model as an approximation to the unknown true model and form their own beliefs endogenously based on the venture's performance and their own degree of ambiguity aversion (preferences for robustness). The robust contract generates endogenous differences in beliefs: investors are pessimistic relative to the entrepreneur. Moreover, the risk-neutral investors behave as if they were endogenously risk averse for both agency and robustness reasons.

In our dynamic setting, the belief wedge between the entrepreneur and investors varies over time in response to the venture's performance. This wedge is captured by the change in the drift of the Brownian motion process, which can also be interpreted as the market price of model uncertainty (Anderson, Hansen, and Sargent (2003)). It directly influences optimal pay-performance sensitivity (PPS) and generates interesting asset-pricing implications for equity premium and credit spread.

As in other dynamic contracting models, the entrepreneur's continuation value is forward looking and also history keeping. As in DFHW (2012), the continuation value scaled by the firm's capital stock is the effective state variable. We show that, following good performances, the entrepreneur's scaled continuation value increases, the belief wedge decreases, and investors' endogenous risk aversion also decreases.

We find that the PPS in our model is a function of the entrepreneur's scaled continuation value depending on both agency and investors' preferences for robustness– this is in sharp contrast to the standard rational-expectations agency models, where the PPS is solely determined by the binding incentive constraint. Similar to Miao and Rivera (2016), we find that there are two endogenously determined regions for the PPS depending on whether the entrepreneur has done sufficiently well in the past, i.e., whether his scaled continuation value exceeds an endogenous cutoff value.

When the entrepreneur's scaled continuation value exceeds the cutoff, investors sufficiently trust the entrepreneur's model. Both the PPS and corporate investment increase with the scaled continuation value. Additionally, the incentive constraint does not bind. The optimal PPS takes a form resembling Merton's (1971) optimal portfolio rule. In particular, it increases with the market price of model uncertainty and decreases with investors' endogenous risk aversion and productivity volatility. As endogenous risk aversion decreases at a rate faster than the market price of uncertainty, the optimal PPS increases with the scaled continuation value.

When the scaled continuation value is below the endogenous cutoff, both investors' endogenous risk aversion and the belief wedge are sufficiently large. Investors want to significantly reduce their risk exposure but still have to induce the entrepreneur to exert effort. In this case, the agency problem dominates investors' concerns for the entrepreneur's model, causing the incentive constraint to bind as in the standard contracting models under rational expectations.

A striking result of our paper is that early liquidation can be optimal and occurs when investors lose confidence in the entrepreneur and the belief wedge is sufficiently large. In this case, investors voluntarily terminate the long-term contract with the entrepreneur even doing so yields a higher payoff to the entrepreneur than his outside option value. In reality, investors protect themselves against the entrepreneur's persistent poor performances with various protective measures, e.g., liquidation preferences and anti-dilution protection in venture capital (VC) term sheets.[2] Our early liquidation result is in sharp contrast with rational-expectations contracting models, e.g., DeMarzo and Sannikov (2006) and DFHW (2012), where investors fully trust the entrepreneur's model and liquidation is inefficient. In Miao and Rivera (2016), early liquidation does not take place either.

Our model's dynamic predictions are broadly consistent with VC contracting evidence documented by Kaplan and Strömberg (2003), who find that entrepreneur has stronger control rights when the firm performs well, while the investors have greater control over the project following a sequence of weak performances. Our explanation based on model uncertainty complements that advocated by the incomplete contracting literature (e.g., Aghion and Bolton (1992)).

We then implement the optimal contract by using debt, equity, cash, and a financial derivative, whose payoff is determined by the venture's productivity shocks. The key and novel component of our optimal security design is the time-varying demand for the derivative asset. This seemingly speculative demand arises from investors' concern for the entrepreneur's productivity model and is optimal for the venture. As the venture's performance improves, the entrepreneur's stake in the venture increases, its cash (financial slack) increases, investors becomes less pessimistic, the agency problem weakens, and corporate investment increases. Since in equilibrium the entrepreneur is more optimistic than investors about the venture's productivity, investors also optimally increase the venture's exposure to its own risk by increasing its long position in the derivative asset and the cumulative gains/losses contribute partly to finance future compensation to the entrepreneur.

We calibrate our model and quantify the implications of investors' concern about the entrepreneur's model for investment, compensation, and valuation (e.g., Tobin's average q, equity

[2]See Kaplan and Strömberg (2003), and Metrick and Yasuda (2010) on VC contracting.

premium, and credit spreads). We find that quantitative effects are significant based on plausible parameter values in DFHW (2012). The only new parameter in our model is the entropy penalty parameter θ that measures ambiguity aversion or concerns for robustness. We calibrate θ using the detection error probability methodology advocated by Anderson, Hansen, and Sargent (2003), and Hansen and Sargent (2008). We find that the mean of Tobin's q is 1.10 for our calibrated robustness model, which is 12% lower than the value (1.25) in DFHW (2012), and that the payout boundary is 1.12, which is more than 2.5 times the value (0.43) in DFHW (2012). The average investment rate and investment volatility in our model are respectively 3.5 and 0.2 percentage points lower than those in DFHW (2012). Moreover, the annual equity premium is 3.8%, even though investors are risk neutral, compared to zero equity premium in DFHW (2012). In sum, although moments matching is not our main goal in this paper, we show that our parsimonious model presents a plausible economic mechanism that generates quantitative predictitions in the right direction.

Our model also predicts that the equity premium and credit spread decrease with the level of investment and Tobin's average q. In this sense our model generates value premium, provides a framework to generate empirically observed relation between the investment and asset returns in asset markets (see, e.g., Zhang (2005), and Hou, Xue, and Zhang (2015)).

Finally, we use our model to shed light on a key valuation issue widely discussed in the VC community and also in MBA Entrepreneurial Finance classes: Why the internal rates of returns (IRRs) that VCs use to value projects are so high relative to the standard cost of capital calculations that we use to make capital-budgeting decisions? Our model offers one explanation. Rather than discouraging and trying to convince the entrepreneur and expressing their concerns for the entrepreneur's productivity model, investors can simply use the entrepreneur's belief model when quoting the cost of capital for the project to the entrepreneur. Doing so, the VCs achieves the same goal without making the entrepreneur feel that they do not trust the entrepreneur's productivity model. For our baseline calibration, our model predicts that the entrepreneur values the project at an annualized IRR that is 3.5% higher than investors do. These predictions are consistent with the empirical evidence documented by Malmendier and Tate (2005), who find that overconfident managers overestimate the returns to their investment projects and curtail investment when external funds are used. But our interpretation is different from Malmendier and Tate (2005). In our model the entrepreneur behaves as if he were more optimistic than investors because of investors' preferences for robustness.

Related literature. Our paper builds on the dynamic moral hazard/contracting literature in finance, e.g., DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007a, 2007b), BMPR (2007), Sannikov (2008), Biais, Mariotti, Rochet, and Villeneuve (2010), Williams (2009, 2011), DFHW (2012), and Ai and Bhandari (2020), among others.[3] Recently, there has been a growing interest in introducing robustness and model uncertainty/ambiguity to this literature.

Two approaches are widely used to model decision making under ambiguity in a continuous-time setup. One approach advocated by Chen and Epstein (2002) builds on the static maxmin expected utility model of Gilboa and Schmeidler (1989). Using this approach, Szydlowski and Yoon (2020) study a continuous-time contracting problem where the principal is ambiguity averse about the effort cost and Dumav (2017) studies a problem where both the principal and the agent are ambiguity averse. Dicks and Fulghieri (2020b) study a continuous-time contracting model within organizations, where the principal designs contracts with multiple agents who are exposed to multiple sources of uncertainty.

The other approach is based on the robust control or multiplier preference model of Hansen and Sargent (2001, 2008). Adopting this approach, Miao and Rivera (2016) introduce the investors' concern for the entrepreneur's model uncertainty into a financial contracting problem based on DeMarzo and Sannikov (2006). Liu (2020) also adopts this approach in a contracting model with investment, where the principal is ambiguity averse, but the agent is risk averse.

Our paper differs from Miao and Rivera (2016) and other aforementioned papers in three ways. First, we incorporate real investment decisions, calibrate our model, and generate quantitative predictions on corporate investment and Tobin's average q. Second, we show that it can be optimal for the firm to terminate the long-term contract early when the investors are sufficiently ambiguity averse. Upon early liquidation the investors make a lump-sum transfer payment to the entrepreneur larger than the entrepreneur's outside reservation value. Miao and Rivera (2016) focus on the case where early liquidation does not take place.

Third, we provide a new and intuitive financial implementation. Under model uncertainty, optimal securities must be designed to align the conflicts of interests between the investors and the entrepreneur with different endogenous beliefs. We introduce a financial asset whose payoffs are contingent on the firm's productivity shock. Since outside investors are endogenously more pessimistic than the entrepreneur, it is optimal for the firm to take a speculative position in this derivative asset and use the proceeds to compensate the entrepreneur (in promise). The speculative

[3] Broadly speaking, our model is also related to the limited-commitment-based contracting models, e.g., Albuquerque and Hopenhayn (2004), Quadrini (2004), and Clementi and Hopenhayn (2006), and Ai and Li (2015).

profits are paid out to creditors. In contrast, Miao and Rivera (2016) introduce special dividends, which can be received only by outside investors. These special dividends are used to hedge against model uncertainty. One implication of this implementation is that inside and outside equities are not pro-rata based.

Our paper is also related to some other papers that apply ambiguity and robustness to corporate finance. For example, Garlappi, Giammarino, and Lazrak (2017) develop a dynamic corporate investment problem where decisions have to be made collectively by a group of agents holding heterogeneous beliefs and recognize the ambiguity-like nature of corporate decisions. Dicks and Fulghieri (2019) develop a theory of systemic risk based on ambiguity aversion. Lee and Rajan (2020) study optimal security design when both the entrepreneur and investors are ambiguity averse. Malenko and Tsoy (2020) study security design under asymmetric information when the investor is ambiguity averse and show that the equilibrium security depends on the degree of uncertainty and the nature of private information. Dicks and Fulghieri (2020a) develop a theory of innovation waves, investor sentiment, and merger activity based on Knightian uncertainty.

2 Model

We formulate our robust financial contracting problem by introducing the principal's concern for model misspecification developed by Hansen and Sargent (2001), and Anderson, Hansen, and Sargent (2003) into the continuous-time dynamic-agency model with corporate investment proposed by DFHW (2012).

2.1 Neoclassic q Theory of Investment and Managerial Agency

We first present the setup of DFHW (2012). A firm employs capital to produce output, whose price is normalized to one. Let K_t and I_t denote the level of capital stock and gross investment rate at time $t \geq 0$, respectively. The firm capital stock (K_t) evolves according to

$$dK_t = (I_t - \delta K_t)\, dt\,, \tag{1}$$

where K_0 is given and $\delta \geq 0$ is the rate of depreciation.

The cost of investment includes both the cost of purchasing investment goods and capital adjustment costs, as is standard in the q-theory literature. Let $C(I, K)$ denote the cost function. We assume that $C(I, K)$ is smooth, increasing, and convex in I. To preserve the homogeneity property, we assume that $C(I, K)$ is homogeneous of degree one in I and K. We use i to denote the investment-capital ratio, $i_t = I_t/K_t$, and the homogeneity property allows us to express $C(I_t, K_t) = c(i_t)K_t$, where $c(i)$ is the scaled investment cost function.

Let dA_t denote the firm's productivity over time interval $[t, t+dt]$ and (A_t) be the corresponding cumulative productivity process. Assuming that the firm's output over $[t, t+dt]$ is proportional to the firm's capital stock, $K_t dA_t$, we express the firm's incremental cash flow dY_t as follows:

$$dY_t = K_t dA_t - C(I_t, K_t)dt\,. \tag{2}$$

We now introduce managerial agency. The firm is owned by investors who hire an entrepreneur (agent) to operate the firm. In contrast to the neoclassical model in which the productivity process (A_t) is exogenously specified, the productivity process in our model is affected by the entrepreneur's unobservable action, i.e., effort. The action $a_t \in [0,1]$ determines the expected rate of output per unit of capital, so that (A_t) satisfies

$$dA_t = a_t \mu dt + \sigma dZ_t^a, \tag{3}$$

where $\mu, \sigma > 0$ and Z^a is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}\,,\ \mathbb{P}_a)$. Here $\mathbb{P}_a$ is the probability measure induced by the entrepreneur's action a and $(\mathcal{F}_t)$ is the filtration generated by (A_t).[4]

The entrepreneur can enjoy private benefits at the rate of $\lambda\mu(1-a_t)dt$ per unit of the capital stock from the action a_t, where $\lambda \in [0,1]$. Due to linearity, our model is also equivalent to the binary effort setup in which the entrepreneur can either shirk, $a_t = 0$, or work, $a_t = 1$. Alternatively, we can interpret $1 - a_t$ as the fraction of cash flows that the entrepreneur diverts for his private benefits, with λ being the entrepreneur's net consumption per dollar diverted. In either case, a larger value of λ corresponds to a more severe agency problem. The choice of the entrepreneur's action is unobservable to the firm's investors, thereby creating moral hazard issues. The firm's investors only observe past and current cash flows and the investors' information set is represented by the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by (Y_t) or equivalently (A_t).

Investors have unlimited wealth and are risk-neutral with discount rate $r > 0$. The entrepreneur is also risk-neutral but more impatient than investors, i.e., the entrepreneur's discount rate γ is higher than r, $\gamma > r$. As in DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007a, 2007b), and DFHW (2012), by assuming $\gamma > r$, we avoid the scenario in which investors indefinitely postpone payments to the entrepreneur. This impatience may arise because the entrepreneur may

[4] All processes in the paper are assumed to be progressively measurable with respect to $(\mathcal{F}_t)$. Inequalities in random variables or stochastic processes are understood to hold almost surely. The triple $(A, Z^a, \mathbb{P}_a)$ is a weak solution to the stochastic differential equation (3).

have other attractive outside investment opportunities. The entrepreneur has no initial wealth and has limited liability, in that investors cannot pay negative wages at any time to the entrepreneur.

Assume that the firm's capital stock K_t, investment I_t, and (cumulative) cash flow Y_t are observable and contractible. To maximize their value, investors offer a contract that specifies the firm's investment process (I_t), the cumulative compensation process (U_t), and termination time τ, all of which depend on the history of (Y_t). The entrepreneur's limited liability requires the compensation process (U_t) to be nondecreasing and right continuous with left limits.

Upon the contract's termination, investors recover L_t and the entrepreneur receives his outside reservation value, which we normalize to zero as in DFHW (2012) and Miao and Rivera (2016). We will relax this assumption in Section 3.3 and consider the possibility of early termination.

To preserve the homogeneity property, we assume that $L_t = \ell K_t$, where $\ell > 0$ represents the fixed fraction of the firm's capital stock that investors recover upon liquidation. We assume that ℓ is sufficiently small so that the investors do not want to liquidate the firm immediately as liquidation is inefficient.[5]

Let $\Phi = (I, U, \tau)$ denote a contract. For a given contract Φ, the entrepreneur chooses an action process $\{\hat{a}_t \in [0,1] : t \in [0, \tau]\}$ to maximize the following objective function:

$$W_t(\Phi, a) = \sup_{\hat{a}} \ \mathbb{E}_t^{\mathbb{P}_{\hat{a}}} \left[\int_t^{\tau} e^{-\gamma(s-t)} \left(dU_s + \lambda\mu(1-\hat{a}_s) K_s ds \right) \right], \tag{4}$$

where $\mathbb{E}_t^{\mathbb{P}_{\hat{a}}}$ denotes the conditional expectation operator with respect to the measure $\mathbb{P}_{\hat{a}}$ given the information set $\mathcal{F}_t$. The entrepreneur's objective function includes the present discounted value of compensation (the first term in (4)) and the potential private benefits from taking action $\hat{a} < 1$ (the second term in (4)). The contract Φ is incentive-compatible with respect to the entrepreneur's chosen action process (a_t) if (a_t) is a solution to the problem stated in (4).

At the time the contract is initiated, the firm has capital stock K_0 and the entrepreneur obtains the expected promised payoff W_0. The entrepreneur's W_0 will be determined by the relative bargaining power of the entrepreneur and investors when the contract is initiated. By varying W_0, we can obtain the entire feasible contract curve.

2.2 Robust Contracting Problem

In DFHW (2012) the investors and the entrepreneur share the same belief and both use the probability measure $\mathbb{P}_a$ to evaluate their expected payoffs. In our model, decision makers may be concerned about model misspecification. To capture the bias that the manager tends to be more confident

[5] Since the firm could always liquidate by disinvesting, it is natural to assume $\ell \geq c'(-\infty)$.

or optimistic about his ability and talent, we assume that the entrepreneur's belief is given by the probability measure $\mathbb{P}_a$. However, the investors do not fully trust the entrepreneur's probability model $\mathbb{P}_a$. Instead investors treat $\mathbb{P}_a$ as an approximation to the true unknown model. As a result, investors design the contract so that it is robust to their concerns that the entrepreneur's model can be misspecified.

Suppose that the investors consider alternative models as possible and all distorted beliefs are described by mutually absolutely continuous measures with respect to $\mathbb{P}_a$ over any finite time intervals. This assumption allows us to use Girsanov's Theorem for changing measures so that model ambiguity is about the drift of the diffusion process.[6] Define a *density generator* associated with an action process (a_t) as a real-valued process (h_t). Denote the set of density generators by $\mathcal{H}^a$. Given a process (h_t) in $\mathcal{H}^a$, we define another process $\left(\xi_t^h\right)$ as

$$\xi_t^h = \exp\left(-\int_0^t h_s dZ_s^a - \frac{1}{2}\int_0^t h_s^2 ds\right). \tag{5}$$

Under the assumption that $\int_0^t h_s^2 ds < \infty$ for all $t > 0$, ξ_t^h defined in (5) is a $(\mathbb{P}_a, \mathcal{F}_t)$-martingale, which plays an important role in our model.[7]

By Girsanov's Theorem, there is a measure $\mathbb{Q}_h$ corresponding to h defined on $(\Omega, \mathcal{F})$ such that ξ_t^h is the Radon-Nikodym derivative of $\mathbb{Q}_h$ with respect to $\mathbb{P}_a$ when restricted to $\mathcal{F}_t$, $d\mathbb{Q}_h/d\mathbb{P}_a|_{\mathcal{F}_t} = \xi_t^h$, and the process $\left(Z_t^h\right)$, defined by

$$Z_t^h = Z_t^a + \int_0^t h_s ds, \tag{6}$$

is a standard Brownian motion under measure $\mathbb{Q}_h$. We can identify any measure $\mathbb{Q}_h$ by either its density generator (h_t) or density process $\left(\xi_t^h\right)$. Denote the set of all such measures by $\mathcal{P}_a$.

Under measure $\mathbb{Q}_h$, the cumulative productivity (A_t) defined in (3) can be written as:

$$dA_t = a_t \mu dt + \sigma(dZ_t^h - h_t dt). \tag{7}$$

Accordingly, the cash flow process (Y_t) under the measure $\mathbb{Q}_h$ follows

$$dY_t = K_t(a_t\mu - \sigma h_t)dt - C(I_t, K_t)dt + K_t\sigma dZ_t^h. \tag{8}$$

Importantly, the firm's incremental output dY_t under measure $\mathbb{Q}_h$ now depends on both the entrepreneur's effort a_t and investors' concerns for the entrepreneur's model misspecification, which is described by h_t.

[6]See Epstein and Ji (2013) for a continuous-time model of ambiguity about both the drift and volatility without the mutually absolute continuity assumption.

[7]See Hansen et al (2006) for the technical details.

To incorporate the investors' concerns for the model misspecification, building on the robust control and variational utility models of Anderson, Hansen, and Sargent (2003), and Maccheroni, Marinacci, and Rustichini (2006a, 2006b), we represent the investors's preferences at time t as follows:

$$F_t(\Phi, a) = \inf_{\mathbb{Q}_h \in \mathcal{P}_a} \mathbb{E}_t^{\mathbb{Q}_h} \left[\int_t^\tau e^{-r(s-t)} dY_s + e^{-r(\tau-t)} L_\tau - \int_t^\tau e^{-r(s-t)} dU_s \right] + \mathcal{K}_t(\mathbb{Q}_h), \tag{9}$$

where $\mathcal{K}_t(\mathbb{Q}_h)$ represents an ambiguity index.[8] Intuitively speaking, investors want their decisions to be robust to the "worst-case" model $\mathbb{Q}_h$, which is chosen from the set $\mathcal{P}_a$ to minimize the sum of the expected present value of payoffs under $\mathbb{Q}_h$, which is the first term in (9), and the entropy penalty term $\mathcal{K}_t(\mathbb{Q}_h)$, which is determined by the degree of concerns for robustness.

Generalizing the relative entropy cost of Anderson, Hansen, and Sargent (2003), Hansen et al. (2006), and Hansen and Sargent (2008), we follow Maenhout, Vedolin, and Xing (2020) and specify the penalty term as

$$\mathcal{K}_t(\mathbb{Q}_h) = \mathbb{E}_t^{\mathbb{P}_a} \left[\int_t^\tau e^{-r(s-t)} \Theta_s d\phi \left(\frac{\xi_s^h}{\xi_t^h} \right) \right], \tag{10}$$

where $\phi(x) \equiv x \ln x$ and $\Theta_t > 0$ measures the size of the entropy cost. Maenhout, Vedolin, and Xing (2020) show that this specification implies

$$\mathcal{K}_t(\mathbb{Q}_h) = \frac{1}{2} \mathbb{E}_t^{\mathbb{Q}_h} \left[\int_t^\tau e^{-r(s-t)} \Theta_s h_s^2 ds \right], \tag{11}$$

and the utility process in (9) is time consistent. When Θ_t is constant for all t, (10) is reduced to the relative entropy penalty of Anderson, Hansen, and Sargent (2003).

To capture the intuitive idea that a larger firm should have a larger entropy penalty (measured in firm value or other dollar-based value), we assume that Θ_t increases with capital stock K_t. By making this assumption, we obtain the economically desirable outcome that the firm will never grow too large relative to the entropy cost or the ambiguity index. For tractability, we assume that Θ_t is proportional to K_t:

$$\Theta_t = \Theta(K_t) = \theta K_t, \tag{12}$$

where the parameter $\theta > 0$ measures the degree of investors' concerns for robustness. Following Maccheroni, Marinacci, and Rustichini (2006a, 2006b), we may also interpret $1/\theta$ as the degree of ambiguity aversion. We will use these two interpretations interchangeably in the paper.

[8]In a discrete time setting, Maccheroni, Marinacci, and Rustichini (2006a, 2006b), show that, by appropriately specifying $\mathcal{K}$, this model includes the multiple-priors model of Gilboa and Schmeidler (1989) and the multiplier model of Anderson, Hansen, and Sargent (2003), Hansen et al. (2006), and Hansen and Sargent (2008) as special cases.

The homogeneity assumption about Θ_t given in (12) allows us to simplify our analysis by reducing a two-dimensional robust contacting problem to a uni-dimensional one. Our modeling of Θ implies that the entropy cost of model uncertainty grows with firm size so that the investors' aversion to model uncertainty matters for the value of capital, e.g., when measured by Tobin's average q. The larger the value of θ, the larger the penalty cost given in (10), and hence the more the investors trust the entrepreneur's probability model. In the limit as $\theta \to \infty$, the investors have the same belief as the agent and our model becomes DFHW (2012).

In summary, we study the following contracting problem when the investors are concerned about the entrepreneur's model misspecification:

Problem 1 *(robust contract)*

$$P(K_0, W_0) = \sup_{(\Phi,a)} \; F_0\left(\Phi, a\right), \tag{13}$$

subject to (1), (2), (7), and the constraints that Φ *is incentive-compatible with respect to the action process* a *and* $W_0(\Phi, a) = W_0 \geq 0$, *where* $W_t(\Phi, a)$ *and* $F_t\left(\Phi, a\right)$ *for* $t \geq 0$ *are defined in (4) and (9), respectively,*

3 Solution

In this section, we first summarize the first-best (FB) solution under rational expectations without agency frictions and then derive the solution for the robust contract with agency.

Let q^{FB} denote Tobin's average q under the first best. As in Hayashi (1982), the homogeneity property implies that the marginal q is equal to the average q and

$$q^{FB} = c'\left(i^{FB}\right) = \max_i \; \frac{\mu - c\left(i\right)}{r + \delta - i}, \tag{14}$$

where $i = I/K$ and i^{FB} is the solution to (14). We require that $\mu < c(r+\delta)$ so that the firm's value under the first-best is well defined and given by (14). Both i^{FB} and q^{FB} are then constant. Let w_t denote the entrepreneur's scaled value: $w_t = W_t/K_t$. It is immediate to see that the investors' scaled value function is given by $p^{FB}\left(w\right) = q^{FB} - w$.

3.1 Robust Contract with Agency

As for our FB case, we focus on the contract in which the high effort level $a_t^* = 1$ is always optimal and hence should be implemented.[9] We then derive the robust contract heuristically and leave technical details including certain regulation conditions to Appendix A.

[9]See the sufficient condition in Proposition 3 in Appendix A.

By using the martingale representation theorem, we can express the entrepreneur's promised utility process (W_t), defined in (4), via the following controlled diffusion process under the high-level effort $a_t^* = 1$:

$$dW_t + dU_t = \gamma W_t dt + \beta_t K_t \left(dA_t - \mu dt\right) = \gamma W_t dt + \beta_t K_t \sigma dZ_t^a \,, \tag{15}$$

where β_t is endogenously determined. The left side of (15) is equal to the entrepreneur's incremental compensation at t, which is equal to the sum of the cash payment dU_t and the change in the entrepreneur's promised utility, dW_t. The right side of (15) states that the change of the entrepreneur's total compensation over time increment dt is equal to the sum of $\gamma W_t dt$ and a mean-zero diffusion term under the entrepreneur's belief $\mathbb{P}_a$.

To understand the determinant of β_t, we need to consider the possibility that the entrepreneur deviates from the FB effort level, $a_t^* = 1$. Intuitively, shirking (by choosing $a_t < 1$) gives the entrepreneur a private benefit $\lambda \mu K_t(1 - a_t)$ but lowers his promised utility in expectation by $\beta_t \mu K_t(1 - a_t)$ per unit of time due to the fall of the firm's expected output. To ensure that the high effort level is incentive compatible, the cost of shirking must be larger than the benefit for all t so that

$$\beta_t \geq \lambda \,. \tag{16}$$

This result follows DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007a), and DFHW (2012). However, because of the investors' concern for the entrepreneur's model misspecification, the solution for β is much different.

As investors do not trust the entrepreneur's model $\mathbb{P}_{a^*}$ induced by $a^* = 1$, their belief is represented by $\mathbb{Q}_h$ as discussed in Section 2.2. Under the belief $\mathbb{Q}_h$ and $a_t^* = 1$, we use Girsanov's Theorem to rewrite (15) as

$$dW_t = \left(\gamma W_t - \beta_t K_t \sigma h_t\right) dt - dU_t + \beta_t K_t \sigma dZ_t^h. \tag{17}$$

Importantly, the drift (expected change) of the entrepreneur's value over dt, $dW_t + dU_t$, viewed from the investors' perspective, (i.e., under $\mathbb{Q}_h$), is equal to the difference between $\gamma W_t dt$ and $\beta_t K_t \sigma h_t dt$. The wedge between the drift in (15) with that in (17), $\beta_t K_t \sigma h_t$, captures the effect of the investors' concerns for the entrepreneur's model misspecification.

Using (8), (9), and (11), we can rewrite the objective function under measure $\mathbb{Q}_h$ as

$$\mathbb{E}^{\mathbb{Q}_h}\left[\int_0^\tau e^{-rt}\left(K_t(\mu - \sigma h_t)dt - C(I_t, K_t)dt - dU_t\right) + e^{-r\tau} L_\tau + \frac{1}{2}\int_0^\tau e^{-rt}\Theta(K_t) h_t^2 dt\right]. \tag{18}$$

Then the value function $P(K_t, W_t)$ satisfies the following HJB equation in the interior region:

$$\begin{aligned} rP(K_t, W_t) &= \sup_{I_t, \beta_t} \inf_{h_t} \quad K_t(\mu - \sigma h_t) - C(I_t, K_t) + \frac{1}{2}\Theta(K_t) h_t^2 + P_W(K_t, W_t)(\gamma W_t - \beta_t K_t \sigma h_t) \\ &\qquad + P_K(K_t, W_t)(I_t - \delta K_t) + \frac{1}{2} P_{WW}(K_t, W_t)(\beta_t K_t \sigma)^2 \ , \end{aligned} \tag{19}$$

subject to the incentive constraint given in (16).

Compared with the HJB equation in DFHW (2012), there are four key differences due to model uncertainty. First, as investors have preferences for the contract to be robust, we must have an inf problem to determine h, which equivalently pins down the measure $\mathbb{Q}_h$. Second, there is a quadratic entropy penalty term $\Theta(K_t)h_t^2/2$, which generates a tradeoff for the choice of h_t. Third, the drift of productivity (A_t) under $\mathbb{Q}_h$ is lowered by σh_t from μ due to the belief distortion. Finally, the drift of the entrepreneur's promised utility (W_t) under $\mathbb{Q}_h$ is lowered by $\beta_t K_t \sigma h_t$ from γW_t.

To solve the dynamic programming problem in (19), we first consider how to determine h_t^*. On one hand, increasing h_t (which means that investors' belief $\mathbb{Q}_h$ is more distant from the entrepreneur's model $\mathbb{P}_{a^*}$) makes investors better off as doing so protects them against the entrepreneur's model misspecification. On the other hand, increasing h_t raises the entropy cost for investors. The first-order condition (FOC) for h_t equates the marginal entropy cost $\Theta(K_t)h_t$ with the marginal benefit $K_t\sigma[1 + \beta_t P_W(K_t, W_t)]$ of guarding against model misspecification.[10] Simplifying this FOC, we obtain

$$h_t^* = \frac{K_t \sigma [1 + \beta_t^* P_W(K_t, W_t)]}{\Theta(K_t)} \ . \tag{20}$$

Second, consider the optimal PPS β_t^*. Substituting (20) into the FOC for β_t yields the following expression for a candidate interior solution if $\beta_t > \lambda$:

$$\beta_t = \frac{P_W(K_t, W_t)}{\Theta(K_t) P_{WW}(K_t, W_t) - P_W(K_t, W_t)^2} \ . \tag{21}$$

As the incentive constraint (16) has to hold, the following equation fully characterizes the optimal sensitivity:

$$\beta_t^* = \max\left\{ \frac{P_W(K_t, W_t)}{\Theta(K_t) P_{WW}(K_t, W_t) - P_W(K_t, W_t)^2}, \lambda \right\} . \tag{22}$$

Third, the FOC for investment equates the marginal benefit of investing, $P_K(K_t, W_t)$ with the marginal cost of investing $C_I(I_t, K_t)$. Because $C_{II}(I, K) > 0$, this FOC implies the following implicit function for the optimal investment I_t^*:

$$C_I(I_t^*, K_t) = P_K(K_t, W_t) \ . \tag{23}$$

[10] Since $\Theta(K_t) > 0$, this condition is also sufficient for h_t^* to be the minimizer.

This equation shows that the marginal cost of investing is equal to marginal q, $P_K(K_t, W_t)$. Although robustness does not directly impact the investment FOC, it impacts investment indirectly via its effect on the endogenously determined marginal q.

Given the homogeneity property of our model, we use the lower-case letter to denote the corresponding upper-case letter scaled by the firm's contemporaneous capital stock. For example, we write $w_t = W_t/K_t$, $p(w_t) = P(K_t, W_t)/K_t$, $i_t = I_t/K_t$, and $du_t = dU_t/K_t$. We first focus on the case where optimal liquidation occurs when w_t reaches 0 for the first time. In Section 3.3, we show that early liquidation is also possible. The following proposition characterizes the optimal robust contract.

Proposition 1 *Let $a_t^* = 1$ be implemented. Suppose that $p(w)$ satisfies the following ordinary differential equation (ODE) on $[0, \overline{w}]$:*

$$rp(w) = \max_{\beta \geq \lambda, i} \ \mu - c(i) + (i - \delta)p(w) + (\gamma + \delta - i)wp'(w) + \frac{\sigma^2}{2}p''(w)\beta^2 - \frac{\sigma^2}{2\theta}\left[1 + p'(w)\beta\right]^2 \quad (24)$$

subject to the conditions: $p'(w) > -1$ on $[0, \overline{w})$, $p(0) = \ell$, and

$$p'(\overline{w}) = -1, \quad p''(\overline{w}) = 0. \quad (25)$$

Suppose that ℓ is sufficiently small and

$$\theta p''(w) - \left[p'(w)\right]^2 < 0 \quad on \quad [0, \overline{w}]. \quad (26)$$

Then: (i) For $w \in [0, \overline{w}]$, the investors' scaled value function is $p(w)$. The optimal sensitivity, $\beta^*(w)$, is given by

$$\beta^*(w) = \max\left\{\frac{p'(w)}{\theta p''(w) - [p'(w)]^2}, \lambda\right\} \quad (27)$$

and the worst-case density generator is given by

$$h^*(w) = \frac{\sigma}{\theta}[1 + \beta^*(w)p'(w)]. \quad (28)$$

The optimal investment rate $i^*(w)$ satisfies

$$c'(i^*(w)) = p(w) - wp'(w). \quad (29)$$

The entrepreneur's scaled continuation value w_t satisfies the following diffusion process:

$$dw_t = [(\gamma + \delta - i^*(w_t))w_t - \beta^*(w_t)\sigma h^*(w_t)]\,dt - du_t^* + \beta^*(w_t)\sigma dZ_t^{h^*}, \quad (30)$$

for $w_t \in [0, \overline{w}]$. The firm is liquidated whenever $w_t = 0$ for the first time. The optimal compensation $dU_t^* = K_t du_t^*$ keeps (w_t) reflecting at $\overline{w}$ and satisfies

$$du_t^* = \max\{w_t - \overline{w}, 0\}. \tag{31}$$

(ii) For $w > \overline{w}$, $p(w) = p(\overline{w}) - (w - \overline{w})$. The investors pay $w - \overline{w}$ immediately to the entrepreneur and the contract continues with the entrepreneur's new initial value $\overline{w}$.

In Appendix C we present the solution for the robust contract without agency and show that the corresponding scaled value function $p(w)$ is concave. By contrast, for the robust contract with agency, $p(w)$ may not be concave as shown in Miao and Rivera (2016). Inequality (26) is a sufficient condition for the optimality of β^*, which does not imply that $p(w)$ is concave.[11]

As in DFHW (2012), the key variable that determines the firm's performance and its financial constraint is the entrepreneur's scaled continuation value w. Unlike in DFHW (2012), there is an additional term $-\beta^*(w_t)\sigma h^*(w_t)$ in the drift of equation (30) under the investors' endogenous belief $\mathbb{Q}_{h^*}$. Under the entrepreneur's belief $\mathbb{P}_{a^*}$, equation (30) becomes

$$dw_t = (\gamma + \delta - i^*(w_t)) w_t dt - du_t^* + \beta^*(w_t)\sigma dZ_t^{a^*}. \tag{32}$$

By comparing the dynamics for w in (30) and (32), we see that the expected value of dw_t from investors' perspective is lower than the expected value of dw_t, as the wedge between the drifts of w_t is equal to $\beta^*(w_t)\sigma h^*(w_t)$, which we show is positive.

3.2 Properties of the Contract

We use Figure 1 to discuss the properties and intuition of the optimal robust contract.

Investors' scaled value function $p(w_t)$ and optimal investment-capital ratio $i^*(w_t)$. Panels A and B of Figure 1 plot the investors' scaled value function $p(w)$ and the investment-capital ratio $i^*(w)$, respectively. We find that $p(w)$ is globally concave. We are unable to prove this result generally, but show that it holds for the robust contract without agency for $\lambda = 0$ (see Appendix C). The intuition is as follows. In the absence of managerial agency, the first-best outcome is for investors to sell the firm to the entrepreneur if the entrepreneur is wealthy enough to purchase

[11] If $p(w)$ is not globally concave, the Bellman-Isaacs condition does not hold so that we cannot exchange the order of optimization (Hansen et al. (2006)). As a result we cannot offer an ex post Bayesian interpretation for the robust contract. We refer to Miao and Rivera (2016) for more discussions on this issue. However, in all our numerical examples below, we find that $p(w)$ is globally concave.

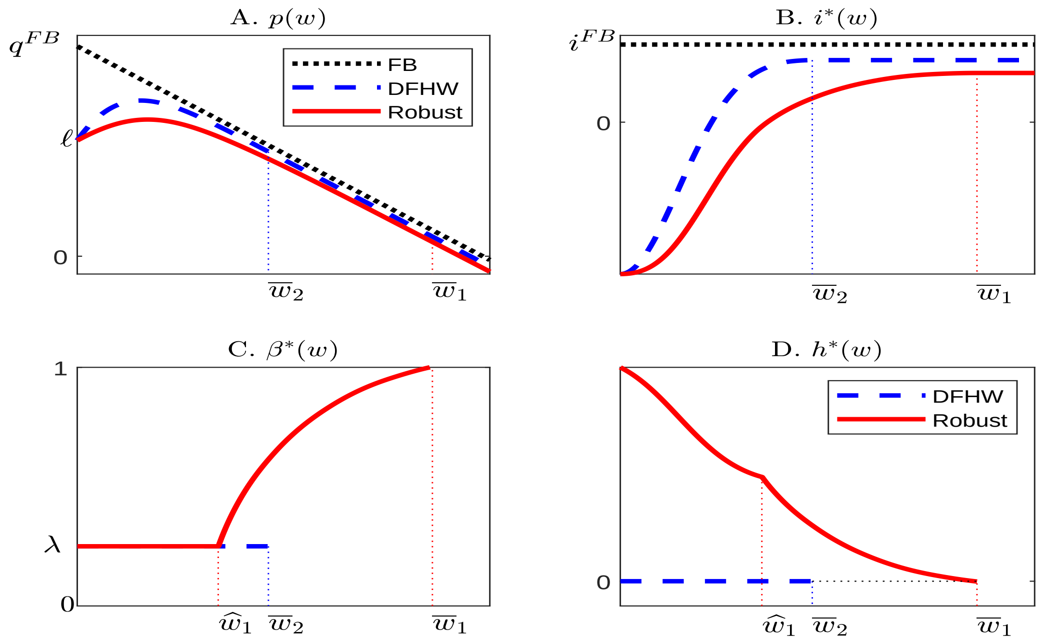

Figure 1: **Optimal robust contracting.** The optimal payout boundary for the robust contract is $\overline{w}_1$, larger than for the rational-expectations DFHW (2012) contract $\overline{w}_2$. For the robust contract, $\widehat{w}_1$ denotes the value of w above which $\beta^*(w) > \lambda$, i.e., the incentive constraint does not bind.

the firm. However, because the entrepreneur is financially constrained and also protected by limited liability, the entrepreneur cannot provide full insurance against investors' ambiguity about his model $\mathbb{P}_{a^*}$. Therefore, in equilibrium investors are endogenously averse to output fluctuation. Similar intuition also applies to the robust contract with agency with one difference. For a low w, the marginal benefit $p'(w)$ to the investors may increase with w when the firm is away from the liquidation boundary as discussed in Miao and Rivera (2016).

Panels A and B of Figure 1 show that the investors' concern for model misspecification lowers their scaled value $p(w)$ (see Proposition 2 in Appendix A) and reduce investment $i(w)$. Technically speaking, the only difference between the ODE for $p(w)$ in our model and that in DFHW (2012) is the last term on the right side of (24), which is negative and proportional to σ^2/θ. Intuitively, the greater the investors' concern for the entrepreneur's model misspecification (i.e., the lower the value of θ), the more pessimistic the investors are about their value (i.e., the lower their value $p(w)$). This also causes them to be more pessimistic about the marginal benefit of investment, leading to a lower investment rate $i^*(w)$.

PPS $\beta^*(w_t)$ and endogenous belief wedge $h^*(w_t)$. Panel C plots the firm's optimal $\beta^*(w)$. As in DFHW (2012), for incentive compatibility, the investors optimally expose the entrepreneur's value to sufficiently large volatility, i.e., $\beta_t \geq \lambda$. Unlike in DFHW (2012) where the incentive constraint always binds, i.e., $\beta_t = \lambda$, it can be optimal for the entrepreneur to be exposed more to the cash-flow uncertainty when the investors are ambiguity averse. We show that with sufficiently large values of w, we have $\beta_t > \lambda$.

Panel D plots the endogenous belief wedge $h^*(w)$, which is always positive. The intuition is that ambiguity aversion generates pessimistic beliefs so that the drift of productivity (A_t) perceived by the ambiguity averse investors is always lower than the drift under the entrepreneur's belief as shown in (32) and (30). Moreover, this belief wedge $h^*(w)$ decreases towards zero as w increases towards the payout boundary $\overline{w}$. The intuition is that as the entrepreneur's performance improves and hence w increases, the investors increasingly trust the entrepreneur's model more.

When w is sufficiently low, the firm's liquidation risk is high. The dominant consideration in this low-w region is to preserve the firm's going concern value by setting $\beta(w) = \lambda$ to minimize the risk of inefficient liquidation as in DFHW (2012).

In contrast, when w is sufficiently high, liquidation risk is much less of a concern. The optimal policy from the investors' perspective is to let the entrepreneur bear more cash-flow risk as the entrepreneur is risk neutral, but the investors are ambiguity averse. It is optimal for the robust contract to transfer uncertainty from the investors to the entrepreneur.

Therefore, there exists an endogenous cutoff value $\widehat{w}$, such that the incentive constraint binds, $\beta^*(w) = \lambda$, only when $w \in [0, \widehat{w}]$. In the region where $w \in (\widehat{w}, \overline{w})$, the incentive constraint does not bind, $\beta^*(w) > \lambda$, as the investors' concern for the entrepreneur's productivity model is the primary driving determinant for β^*.

In this region $(\widehat{w}, \overline{w})$, by substituting h^* given by (28) into (27), we may write $\beta^*(w_t)$ as follows:

$$\beta^*(w_t) = \frac{p'(w_t)}{\theta p''(w_t) - [p'(w_t)]^2} = \frac{h^*(w_t)}{\sigma \pi(w_t)}, \quad \text{for} \quad w_t \in (\widehat{w}, \overline{w}), \tag{33}$$

where $\pi(w)$ can be interpreted as a measure of investors' endogenous (absolute) risk aversion:

$$\pi(w) = \frac{p''(w)}{p'(w)}. \tag{34}$$

The investors' endogenous risk aversion $\pi(w)$ depends on the agency problem (λ), liquidation risk (σ), and the investors' concern for the entrepreneur's model misspecification.[12]

[12]Note that we do not have a minus sign in $p''(w_t)/p'(w_t)$ for the definition of absolute risk aversion. This is because here the entrepreneur's w is the investors' liability rather than asset. In particular, both $p''(w) < 0$ and $p'(w) < 0$ for $w \in (\widehat{w}, \overline{w})$.

Equation (33) resembles the optimal asset allocation rule in the dynamic portfolio choice literature (e.g., Merton (1971)). The analogy is as follows: (i) the belief wedge $h_t^* = h^*(w_t)$, which can be interpreted as the market price of model uncertainty as will be shown in Section 4, is analogous to the market price of risk in Merton (1971); (ii) $\pi(w_t) = p''(w_t)/p'(w_t)$ is analogous to the coefficient of absolute risk aversion in Merton (1971). The larger the wedge h_t^*, the more optimistic the entrepreneur is relative to the investors, the greater the investors benefit by letting the entrepreneur bear the firm's risk, i.e., the higher the value of β^*. Unlike Merton (1971), $h_t^* = h^*(w_t)$ and $\pi(w_t)$ are both endogenous and indeed perfectly correlated in our model.

As the entrepreneur's w_t increases (e.g., following good performances), the wedge h^* between the entrepreneur's model $\mathbb{P}_{a^*}$ and investors' belief $\mathbb{Q}_{h^*}$ decreases but importantly the investors' endogenous risk aversion $\pi(w_t) = p''(w_t)/p'(w_t)$ decreases even more. As a result, β_t^* *increases* with w_t as one can see from Figure 1. That is, once the entrepreneur has earned sufficiently high stake in the firm, i.e., $w_t \in (\widehat{w}, \overline{w})$, it is optimal to increase the entrepreneur's PPS $\beta^*(w_t)$ with w_t. In this way, the ambiguity-averse investors can transfer more uncertainty to the risk/ambiguity neutral entrepreneur. This PPS has critical implications for the firm's payout policy and its choice between hedging and speculation.

Payout boundary $\overline{w}$. Substituting the smooth-pasting condition, $p'(\overline{w}) = -1$, and the super-contact condition, $p''(\overline{w}) = 0$, given in (25) into (24), we obtain the following condition at $\overline{w}$:

$$rp\left(\overline{w}\right) = \max_i \ \mu - c\left(i\right) + (i - \delta)p\left(\overline{w}\right) - (\gamma + \delta - i)\overline{w}\,.$$

DFHW (2012) show that this condition can be equivalently expressed as the "steady-state" valuation condition:

$$p\left(\overline{w}\right) + \overline{w} = \max_i \ \frac{\mu - c\left(i\right) - \left(\gamma - r\right)\overline{w}}{r + \delta - i}. \tag{35}$$

Although the boundary conditions for $\overline{w}$ are the same as in DFHW (2012), the economics in our model is quite different from that in DFHW (2012).

In our model, at the payout boundary $\overline{w}$, it follows from (25), (27), and (28) that

$$h^*(\overline{w}) = 0 \quad \text{and} \quad \beta^*(\overline{w}) = 1\,, \tag{36}$$

The intuition for (36) is as follows. At the payout boundary $\overline{w}$, investors fully trust the entrepreneur's model and the belief wedge disappears: $h^*(\overline{w}) = 0$. Otherwise, it is always optimal for the investors to pay the entrepreneur in promise by increasing w rather than in cash. Since the entropy penalty cost is quadratic in h_t, the marginal entropy cost is linear in h_t and hence equals zero at $\overline{w}$ as $h^*(\overline{w}) = 0$. The marginal benefit is proportional to $1 + \beta_t P_W = 1 - \beta_t$ at $\overline{w} = 0$. Therefore, it is optimal to set $\beta_t = 1$ at the payout boundary $\overline{w}$ as doing so equates the marginal benefit with the zero marginal cost. The result $\beta^*(\overline{w}) = 1$ is fundamentally different from the result that the incentive constraint always binds, $\beta^*(w) = \lambda$ for all $w \in [0, \overline{w}]$, in standard contracting models under rational-expectations.

As a result, our model allows us to separately identify volatility σ and the agency parameter λ, which is infeasible in DFHW (2012), as $\lambda\sigma$ always appears multiplicatively in DFHW (2012). In contrast, in our model, it is optimal for the entrepreneur to be the entire residual claimant when the firm is at its payout boundary $\overline{w}$, regardless of the severity of agency, measured by λ.

Because $\beta^*(\overline{w}) = 1 > \lambda$, in order to deliver the same level of promised w, the investors need to further back-load compensation and delay cash compensation. That is, the endogenous payout boundary under robustness, $\overline{w}_1$, is larger than the boundary $\overline{w}_2$ in DFHW (2012), as we see in Figure 1.

So far, we have focused on the case where the investors always wait until the last moment to liquidate the firm. Next, we consider the possibility that the investors may choose to liquidate the firm before the entrepreneur's participation constraint $W_t \geq 0$ binds.

3.3 The Case with Early Liquidation

In DeMarzo and Sannikov (2006) and DFHW (2012), it is never optimal for the investors to liquidate the firm before the entrepreneur's participation constraint binds. The intuition is that with rational expectations the investors fully trust the entrepreneur's talent and is only worried about the agent's incentive problem. Therefore, the investors optimally delay using the stick (to inefficiently terminate the project) as much as possible.

However, in our model as the investors are ambiguity-averse, early liquidation can be optimal when the investors are sufficiently concerned about the entrepreneur's productivity. We show that it can be optimal for the investors to compensate the entrepreneur when the investors no longer consider the entrepreneur to be sufficiently productive. In this case, it is optimal for the investors to pay a lump-sum amount before the agent's participation constraint binds. This lump-sum payment can be viewed as a severance compensation or a golden parachute.

The intuition is that the investors are so worried about the entrepreneur's productivity (model) that it is better off for the investors to "buy out" the entrepreneur so that the firm can be liquidated sooner than later. This early liquidation is triggered in equilibrium by the investors' belief that the entrepreneur's productivity is really low not because of the agency problem.

Formally, the investors offer $W_\tau > 0$ to the entrepreneur at the termination time τ. For a given contract Φ, the entrepreneur chooses an action process $\{\hat{a}_t \in [0,1] : t \in [0,\tau]\}$ to maximize the following objective function:

$$W_t(\Phi, a) = \sup_{\hat{a}} \ \mathbb{E}_t^{\mathbb{P}_{\hat{a}}} \left[\int_t^\tau e^{-\gamma(s-t)} \left(dU_s + \lambda\mu(1-\hat{a}_s) K_s ds \right) + e^{-\gamma(\tau-t)} W_\tau \right]. \tag{37}$$

The investors' payoff at time t is as follows:

$$F_t\left(\Phi, a\right) = \inf_{\mathbb{Q}_h \in \mathcal{P}_a} \ \mathbb{E}_t^{\mathbb{Q}_h} \left[\int_t^\tau e^{-r(s-t)} (dY_s - dU_s) + e^{-r(\tau-t)} (L_\tau - W_\tau) \right] + \mathcal{K}_t(\mathbb{Q}_h). \tag{38}$$

Then the robust contract with early liquidation is defined in the same away as in Problem 1 where $W_t(\Phi, a)$ and $F_t\left(\Phi, a\right)$ for $t \geq 0$ are given in (37) and (38), respectively. As in Proposition 1, we consider the case in which the high effort ($a^* = 1$) is implemented.

Let $\underline{w}$ denote the entrepreneur's scaled promised utility $w_\tau = W_\tau / K_\tau$ when the investors terminate the contract at time τ. Then $\underline{w}$ satisfies

$$\underline{w} = \inf\left\{ w \geq 0 : p\left(w\right) = \ell - w \right\}.$$

Intuitively, at the liquidation time τ, the investors offer $\underline{w} \geq 0$ to the entrepreneur who is willing to accept it as the payoff is higher than his reservation value of zero. The investors obtain $p\left(\underline{w}\right) = \ell - \underline{w}$ upon liquidation. For $w > \underline{w}$, Proposition 1 applies and will not be stated here again.

Figure 2 shows that as we decrease θ from 1.0 to 0.2, the investors become more concerned about the entrepreneur's (productivity) model misspecification, liquidate the project sooner, and pay out to the entrepreneur later (larger $\overline{w}$).

Importantly, rather than waiting to terminate the entrepreneur's employment when w reaches his reservation value of zero for $\theta = 1.0$, the investors choose to terminate the contract earlier when w reaches $\underline{w} = 0.07$ for $\theta = 0.2$. This is because it takes fewer negative shocks for more ambiguity-averse investors to lose confidence in the entrepreneur's productivity and hence it is optimal for them to liquidate the project sooner despite paying a liquidating payoff $\underline{w} = 0.07$, which is higher than the entrepreneur's outside option value of zero.

Additionally, the investors' value $p(w)$ decreases, investment $i(w)$ falls, and the equilibrium belief wedge $h^*(w)$ increases for all levels of w (see Panels A, B, and D) as investors become more concerned about the entrepreneur's productivity.

What is not obvious is that as θ decreases from 1.0 to 0.2, the PPS β^* decreases for a given level of w. Intuitively, liquidation occurs sooner at $\underline{w}_2 = 0.07$ for $\theta = 0.2$, and in order to reduce liquidation risk, the optimal contract with $\theta = 0.2$ calls for a lower uncertainty exposure $\beta^*(w)$

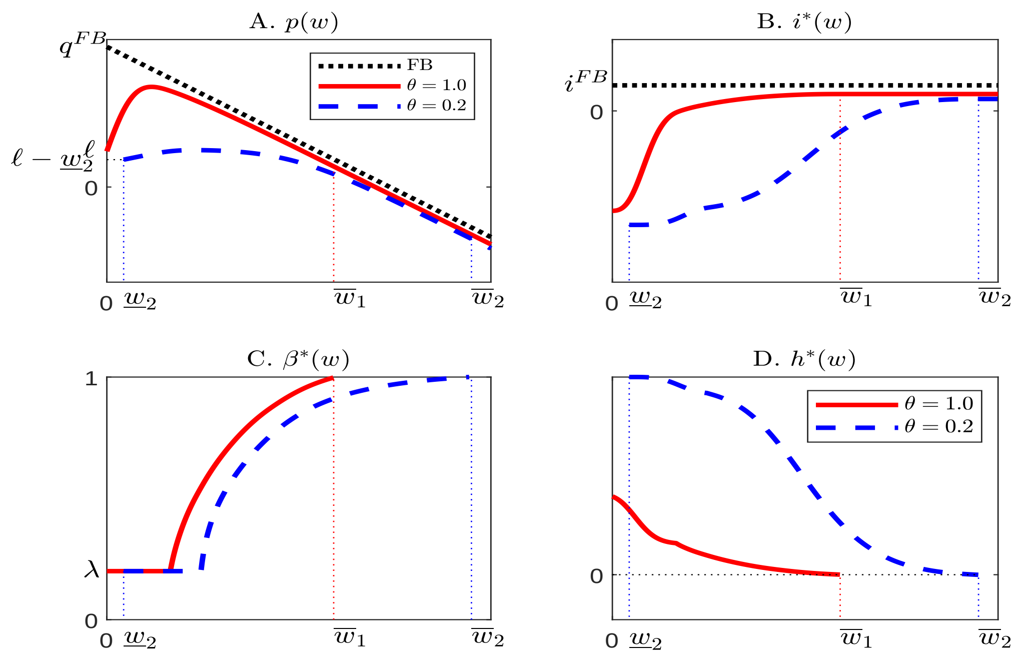

Figure 2: **Equilibrium early liquidation ($\underline{w} > 0$).** As we decrease θ from 1 to 0.2, the liquidation boundary increases from zero to $\underline{w}_2 = 0.07$.

for any $w \in [\underline{w}_2, \overline{w}_1]$ than the contract with $\theta = 1.0$ does. Moreover, the investors with $\theta = 0.2$ delay payout to the entrepreneur as paying the entrepreneur in promise rather than in cash is more efficient for the firm both because the entrepreneur is more optimistic about the firm's prospect and also because it is optimal for the firm to keep more slack in the long-term contracting relationship when the investors are more concerned about the entrepreneur's productivity.

4 Security Design and Financial Implementation

In this section we show that our robust contracting solution can be implemented by dynamically managing the firm's financial slack (e.g., cash reserves), choosing a capital structure with a mix of debt, inside equity, and outside equity, and dynamically trading a derivative contingent on the firm's output.[13] Our implementation builds on DeMarzo and Sannikov (2006), BMPR (2007), DeMarzo and Fishman (2007a, 2007b), DFHW (2012), Miao and Rivera (2016), among others. What is new in our implementation is the entrepreneur's speculative hedging demand that arises endogenously from the equilibrium belief wedge between the entrepreneur and investors. For expositional simplicity,

[13] It is well known in the security design literature that implementation is not unique. We provide an economically intuitive implementation that uses simple and standard contracts.

we study financial implementation for the case without early liquidation, as the case where early liquidation is optimal is similar.

4.1 Security Design and Pricing

Let B_t, S_t, and M_t denote the market value of the firm's debt, equity, and its cash balance at time t. Our goal is to determine these values and design a firm's dynamic financing policies based on the optimal contract.

Corporate Liquidity and Risk Management. We denote the scaled cash reserves by $m_t = M_t/K_t$ in $[0, \overline{m}]$, where $\overline{m}_t = \overline{w}_t/\lambda$ is the endogenous payout boundary. In the interior region where the firm's cash balance M_t lies inside $(0, \overline{m}_t K_t]$, the firm dynamically engages in liquidity and risk management. The firm earns interest on its cash balance M_t at the risk-free rate r.

The firm also has access to a liquid financial asset that is perfectly correlated with its productivity shock $dZ_t^{h^*}$. We refer to this asset as the derivative asset and define its payoff structure as follows: An investor who has a unit of long position in this derivative asset at time t receives a stochastic payoff $dZ_t^{h^*}$ at $t + dt$ under $\mathbb{Q}_{h^*}$. As investors have no wealth constraints and their equilibrium belief is given by the measure $\mathbb{Q}_{h^*}$, all contingent claims can thus be priced in a perfectly competitive market under $\mathbb{Q}_{h^*}$. Therefore, it is costless for investors to take either a long or a short position in this derivative asset as doing so is a mean-zero preserving spread for risk-neutral investors under $\mathbb{Q}_{h^*}$. However, because the entrepreneur is relatively more optimistic than investors, the entrepreneur is willing to pay for this derivative asset as this asset provides a positive expected payoff $h_t^* dt$ to him.

Let $x_t K_t$ denote the firm's demand for this hedging asset at t. Over $[t, t + dt]$, the firm's cash-flow exposure to this hedging asset dH_t is then given by the product of $x_t K_t$ and $dZ_t^{h^*}$, in that

$$dH_t = x_t K_t dZ_t^{h^*}, \tag{39}$$

where x_t is to be determined.

Equity. Because the ambiguity-averse investors are well diversified and financially unconstrained, they are the marginal investors and therefore the firm's equity value is given by

$$S_t = \mathbb{E}_t^{\mathbb{Q}_{h^*}} \left[\int_t^\tau e^{-r(v-t)} dU_v^S \right], \tag{40}$$

where $\{U_v^S : t \leq v \leq \tau\}$ is the undiscounted cumulative dividend. Its increment is given by

$$dU_t^S = dU_t^* / \lambda, \tag{41}$$

where dU_t^* is the equilibrium payout to the entrepreneur characterized in Proposition 1. By defining dividends in this way, we naturally conclude that the entrepreneur receives λ fraction of the firm's total equity, as $dU_t^* = \lambda dU_t^S$ under all circumstances. This dividend payment interpretation specified in (41) is widely used in the literature, e.g., DeMarzo and Sannikov (2006), BMPR (2007), DeMarzo and Fishman (2007a, 2007b), and Miao and Rivera (2016).

Debt. Similarly, we value the firm's debt as follows:

$$B_t = \mathbb{E}_t^{\mathbb{Q}_{h^*}} \left[\int_t^\tau e^{-r(s-t)} U_s^B ds + e^{-r(\tau-t)} \ell K_\tau \right], \tag{42}$$

where τ is the endogenous stochastic default time. The firm's debt service over $[t, t+dt]$ is given by $U_t^B dt$, where

$$U_t^B = (\mu - c(i_t^*)) K_t + x_t h_t^* K_t - (\gamma - r) M_t. \tag{43}$$

The debt service U_t^B per unit of time is locally deterministic. The first term in (43) is the expected operating cash flow. Because debt is senior to equity, it is intuitive that debt holders receive this *expected* payoff.[14]

The second term $x_t h_t^* K_t$ is from the firm's speculative position via the derivative asset. Because the entrepreneur beliefs that the derivative asset has a positive drift of $h_t^* dt$ while investors think the drift of this asset is zero, it is efficient for the firm to take the speculative position $x_t h_t^* K_t$ and distribute this amount (in expectation) to debt holders. We will provide more discussions on this point latter.

The last term in (43) is the adjustment due to the firm's performance measured by its cash balance M_t. Because the entrepreneur's required rate of return is γ and the discount-rate wedge between the entrepreneur and investors is $\gamma - r > 0$, the firm keeps $(\gamma - r)M_t$ each period for future dividend payments to shareholders.

Whenever the firm misses its debt payment schedule, creditors have the right to liquidate the firm. In equilibrium, the firm only defaults on its debt when it exhausts its cash balance, meaning that the firm declares default at time τ where $\tau = \inf\{s \geq 0 : M_s = 0\}$. Creditors collect all liquidation proceeds ℓK_τ and equity investors are completely wiped out at the moment of liquidation τ, as implied by the absolute priority rule (APR).

There are two key differences in debt value given in (42) and equity value given in (40). First, debt is senior to equity as bond holders collect the firm's entire liquidation proceeds ℓK_τ and

[14] Of course, we could also attribute this as part of payout to outside equity investors. Since we are treating inside and outside equity investors symmetrically, we allocate the expected free cash flows to debt holders.

shareholders are completely wiped out upon liquidation (e.g., the APR holds in our model.) Second, debt payment is locally deterministic with coupon payment $U_t^B dt$ while cash-flow to equity is stochastic in general.

Cash Balance. The firm's cash balance is the nexus connecting the firm's various decisions. Specifically, in the interior region $(0, \overline{m}_t K_t]$, the firm's cash holding M_t evolves as follows:

$$dM_t = \underbrace{K_t dA_t}_{\text{sales}} + \underbrace{rM_t dt}_{\text{interest}} - \underbrace{C(I_t, K_t)dt}_{\text{Cap Ex costs}} - \underbrace{U_t^B dt}_{\text{coupon}} + \underbrace{dH_t}_{\text{speculation}} - \underbrace{dU_t^S}_{\text{dividends}} . \tag{44}$$

Equation (44) connects the firm's dynamically changing cash holding M, an item on the balance sheet, to its three Cash-Flow Statements as follows:

1. cash flows from operating activities, $K_t dA_t$;
2. cash flows from investing activities, given by interest income rM_t, minus investment costs $C(I_t, K_t)$, and minus debt coupon payment at the rate of U_t^B;
3. cash flows from financing activities, which include dividend payments dU_t^S to equity investors and speculation gains or losses dH_t.

The firm pays out dividends, meaning $dU_t^S > 0$, if and only if the firm's cash holding M_t exceeds $\overline{m} K_t$, and the dividend policy is to distribute the firm's "excess" cash so that m_t reflects away from $\overline{m}$ into the interior region. As we discussed earlier, the dividend payment specification dU_t^S allocates the entrepreneur a (non-tradable) λ fraction of the firm's total payout to equity, so that inside equity implements the optimal compensation contract for the entrepreneur.

For the above capital structure to implement the robust contract characterized in Proposition 1, we use equations (30), (44), $W_t = w_t K_t$, and $M_t = W_t/\lambda$ to solve for the firm's hedge position:

$$x_t^* = \left(\frac{\beta_t^* - \lambda}{\lambda}\right)\sigma . \tag{45}$$

This equations shows that, when the incentive constraint binds ($\beta_t^* = \lambda$ by Proposition 1), the firm holds no hedging asset.

When the incentive constraint does not bind ($\beta_t^* > \lambda$), we have $x_t^* > 0$, which means that the firm takes a long position in its own cash-flow risk via the hedging asset. This is in contrast to the conventional wisdom that a financially constrained firm should hedge against its risk by taking a short position in the hedging asset. The intuition for our results is as follows. Because in equilibrium the entrepreneur is more optimistic than investors about the productivity, it is optimal for the firm to expose itself more to its own risk, as the firm compensates the entrepreneur via λ fraction of equity, which is cost effective.

From investors' point of view, the expected profit from speculation is zero because

$$\mathbb{E}_t^{\mathbb{Q}_{h^*}}[dH_t] = \mathbb{E}_t^{\mathbb{Q}_{h^*}}\left[x_t^* K_t dZ_t^{h^*}\right] = 0\,. \tag{46}$$

However, the speculation is profitable from the entrepreneur's point of view. Since investors do not trust the entrepreneur's model, their beliefs $\mathbb{Q}_{h^*}$ are different from the entrepreneur's $\mathbb{P}_{a^*}$. As we discussed earlier, the entrepreneur is endogenously more optimistic than the investors.

Formally, we use (6) to compute

$$\mathbb{E}_t^{\mathbb{P}_{a^*}}[dH_t] = \mathbb{E}_t^{\mathbb{P}_{a^*}}[x_t^* K_t dZ_t^{h^*}] = \mathbb{E}_t^{\mathbb{P}_{a^*}}\left[x_t^* K_t \left(dZ_t^{a^*} + h_t^* dt\right)\right] = x_t^* K_t h_t^* dt > 0, \tag{47}$$

where $h_t^* > 0$ can be interpreted as the market price of uncertainty (Anderson, Hansen and Sargent (2003)). The speculative profits represent free cash flows which are extracted by the firm via a higher coupon rate U_t^B paid to creditors.Doing so alleviates agency issues. If the profits were kept in the firm, the entrepreneur could divert them for his private benefits.

Under rational expectations, BMPR (2007) show that their optimal contract can be implemented by debt, equity, and cash reserves without speculative demand, i.e., $x_t^* = 0$. DeMarzo and Sannikov (2006), and DFHW (2012) offer different implementations. Under model uncertainty, Miao and Rivera (2016) introduce special dividends to the implementation, which play a different role of speculative demand in this paper. It is well known in the literature that financial implementation is generally not unique.

One robust feature of any financial implementation is that there exists a key state variable to capture the firm's financial slack, which is tied to the entrepreneur's promised utility W_t. Here we choose cash reserves $M_t = W_t/\lambda$. We could have chosen an alternative, e.g., credit line, as in DeMarzo and Sannikov (2006), and DeMarzo and Fishman (2007a, 2007b).

Next, we discuss the implications of our financial implementation. In doing so, we equivalently express the contracting solution in Proposition 1 as various functions of the new state variable, the scaled cash balance $m = M/K = w/\lambda \in (0, \overline{w}/\lambda)$. For example, we write $\beta^*(\lambda m)$, $h^*(\lambda m)$, and $x^*(m)$ by (45). Let $s(m_t)$ and $b(m_t)$ denote the scaled equity and debt values: $s(m_t) = S_t/K_t$ and $b(m_t) = B_t/K_t$.

Graphic Illustration. Panels A and B of Figure 3 plot the scaled speculative demand $x^*(m)$ and the scaled speculation profits $x^*(m)\,h^*(\lambda m)$, respectively, for two values of θ. Panels C and D present the corresponding sensitivity $\beta^*(\lambda m)$ and belief wedge $h^*(\lambda m)$.

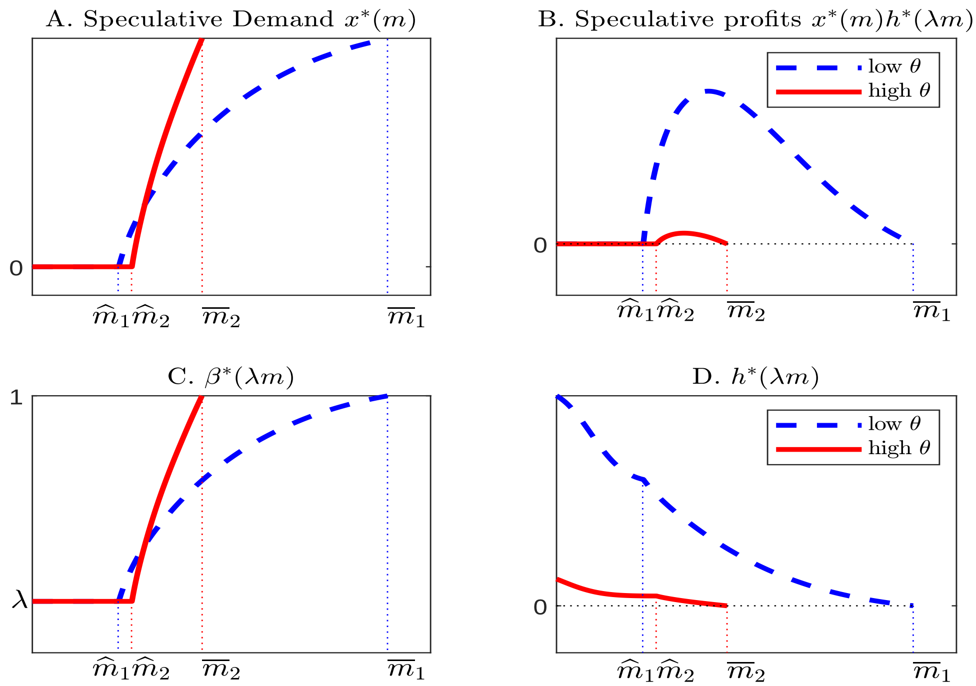

Figure 3: **Speculative demand, speculative profits, sensitivity, and belief wedge for high and low values of θ.**

Note that the entrepreneur speculates only when cash holding m is sufficiently high. When m is low, investors are more concerned about agency problems and the risk of liquidation so that the entrepreneur's incentive constraint binds, i.e. $\beta^*(\lambda m) = \lambda$ and hence $x^*(m) = 0$, as in rational expectations models.

However, when $m \geq \widehat{m} = \widehat{w}/\lambda$, the entrepreneur's incentive constraint no longer binds. Because the firm has sufficient financial slack, it has more room to exploit the endogenous belief dispersion between the entrepreneur and investors via speculation, as the firm can buffer losses better via its savings. For this reason, the speculative demand $x^*(m)$ increases with m. As one may expect, the speculation profit first increases with m but then decreases with m after reaching a sufficiently high level of m. This is due to the countervailing force that the investors increasingly trust the entrepreneur's model. As a result, the market price of model uncertainty $h^*(\lambda m)$ decreases with the firm's financial slack m. In the limit, when $m = \overline{m} = \overline{w}/\lambda$, $h^*(\lambda\overline{m}) = 0$, i.e., the two beliefs coincide and hence there is no speculation profit. In sum, the speculation profit is hump shaped and equal to zero when m is either sufficiently low or high.

Figure 3 also shows that when the investors are more ambiguity averse (a smaller θ), the speculative profit is higher (Panel B). This is because the belief wedge is larger which implies that the market price of uncertainty $h^*(\lambda m)$ is also larger (Panel D.)

The sensitivity β^* is not monotonic in θ. A firm with a higher value of θ pays out sooner to shareholders (a lower value of $\overline{w}$), and optimally increases exposes the entrepreneur's exposure to liquidation risk when its cash balance m is sufficiently high. This is because as $w \to \overline{w}$, the investors' endogenous risk aversion approaches zero and β^* approaches unity. In contrast, when w is sufficiently close to the liquidation boundary, the PPS β^* is always (weakly) lower when investors are less concerned about the entrepreneur's model. This is because the going concern value for a firm with a higher value of θ is larger and as a result investors become more prudent by lowering $\beta^*(w)$ when w is not high. This explains why we have a crossing for β^* for the high and low θ cases.

Next, we summarize our implementation by linking a firm's balance sheet to its cash flow statements.

4.2 Enterprise Value, Tobin's Average q, and Balance Sheet

The firm's enterprise value is the present value of the firm's cash flows under the investor's endogenously determined belief $\mathbb{Q}_{h^*}$. Note that here we are netting out the compensation to the entrepreneur as it is a transfer from investors to the entrepreneur. That is,

$$V_t = \mathbb{E}_t^{\mathbb{Q}_{h^*}} \left[\int_t^{\tau} e^{-r(s-t)} dY_s + e^{-r(\tau-t)} \ell K_\tau \right]. \tag{48}$$

Thus the firm's enterprise value V_t is equal to the sum of the present value of the cash flows generated by the firm's project and the liquidation value. In the first best case, liquidation is never optimal and the firm's enterprise value is simply equal to $q^{FB} K_t$ where q^{FB} is the firm's Tobin's average q given in (14).

We show that the firm's enterprise value is equal to the sum of its debt and equity values minus cash. Creditors and (inside and outside) equity investors together own the firm including its going-concern value and cash balances. Using outside investors endogenously determined belief $\mathbb{Q}_{h^*}$, we may calculate the firm's enterprise value, V_t, as follows:

$$V_t = B_t + S_t - M_t \,. \tag{49}$$

Note that the firm's enterprise value V_t is the sum of the firm's net debt, $B_t - M_t$ and the market value of its equity held by both outside equity investors and the entrepreneur. We give the proof of equation (49) in Appendix A.

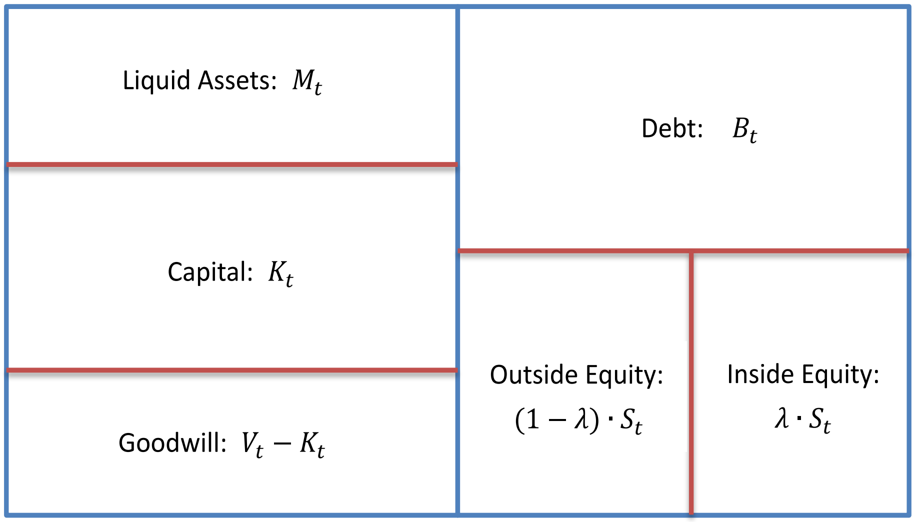

Figure 4: **Balance Sheet (Market Value).**

Figure 4 illustrates the firm's balance sheet. On the liability side, the firm's market value of debt is B_t and its equity value is S_t. The fraction λ of its equity is held by the entrepreneur and the remaining $1-\lambda$ fraction is held by outside equity investors. On the asset side, the firm's cash holding is M_t, the firm's capital stock is K_t, and the goodwill is $V_t - K_t$, where V_t is the firm's enterprise value defined in equation (49). At each time t, the two sides of the balance sheet are equal, i.e., $M_t + V_t = B_t + S_t$ as given in (49).

We further link the investors' value function $P(K_t, W_t)$ from the contracting model to the firm's enterprise value V_t in our financial implementation as follows:

$$V_t = P(K_t, W_t) + \mathbb{E}_t^{\mathbb{Q}_{h^*}}\left[\int_t^{\tau} e^{-r(s-t)} dU_s^*\right] - \mathcal{K}_t\left(\mathbb{Q}_h^*\right) = P(K_t, W_t) + \lambda S_t - \mathcal{K}_t\left(\mathbb{Q}_{h^*}\right). \tag{50}$$

Therefore, we obtain

$$V_t - \lambda S_t = P(K_t, W_t) - \mathcal{K}_t\left(\mathbb{Q}_{h^*}\right). \tag{51}$$

The right side of (51) is equal to the investors' value $P(K_t, W_t)$ minus the penalty due to investors' ambiguity aversion. The left side of (51) is equal to the firm's total enterprise value minus the market value of the entrepreneur's inside equity. In equilibrium, as the contracting and implementation formulations are equivalent and dual to each other, the two sides of (51), using the same belief and the same discount rate r to value the investors' total value, have to be equal.

We define average q as the ratio of firm value to the capital stock $q_a(t) = V_t/K_t$. Equation (49) implies that $q_{a,t} = s(m_t) + b(m_t) - m_t$. In Appendix B we provide an ODE characterization for

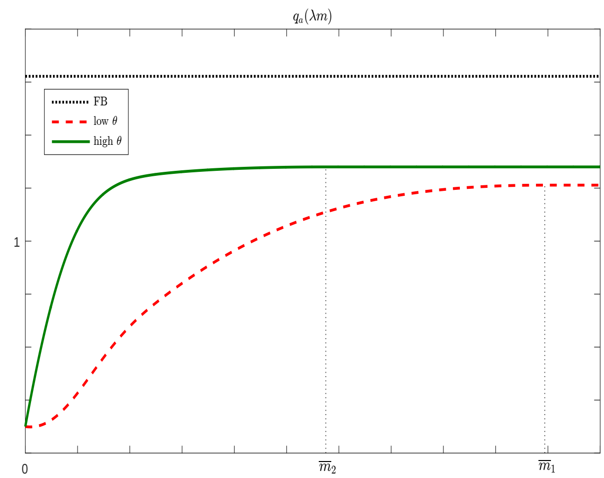

Figure 5: **Tobin's average q.** The agency problem and investors' concern for the entrepreneur's model lower Tobin's average q from the first-best value q^{FB}. Additionally, Tobin's average q increases with cash balance m for a given level of θ. The more the investors are concerned about the entrepreneur's productivity model the lower the firm's average q.

$q_a(w) = q_a(\lambda m)$. Figure 5 presents average q for the robust contract with agency for two values of θ and compare with the first-best q^{FB}. Tobin's average q increases with cash balances m and ambiguity aversion lowers average q. For all cases average q starts at $q_a(0) = \ell$ and increases to the payout boundary such that $q_a'(\overline{w}) = 0$.

Finally, we conclude our financial implementation by discussing the initial capital structure.

Initial Capital Structure. The firm with an initial capital stock K_0 sets up its capital structure at time 0 by raising debt with market value of B_0 and issuing equity to outside investors with market value of $(1-\lambda)S_0$. Because the firm needs to motivate the financially constrained entrepreneur to exert effort, it allocates λ fraction of its equity as the long-term compensation package to the entrepreneur. The firm's initial cash balance is $M_0 = B_0 + S_0 - V_0$.

5 Quantitative Analysis

In this section, we analyze our model's quantitative implications for asset pricing and corporate policies.

5.1 Equity Premium and Default Risk

The investors' concern about the entrepreneur's productivity model generates an endogenous market price of model uncertainty $h^*(\lambda m_t) > 0$. By construction, equity premium is zero in DFHW (2012). In contrast, the market price of uncertainty generates a positive equity premium in our model. We define the conditional expected equity premium under measure $\mathbb{P}_{a^*}$, $e(m_t)$, as follows:

$$e(m_t) \equiv \frac{1}{S_t dt}\mathbb{E}_t^{\mathbb{P}_{a^*}}\left(\frac{dU_t^*}{\lambda} + dS_t\right) - r\,,$$

where m_t is the scaled cash balance. Similar to Miao and Rivera (2016), we can show that

$$e(m_t) = h^*(\lambda m_t)\frac{s'(m_t)}{s(m_t)}\frac{\sigma\beta^*(\lambda m_t)}{\lambda}\,. \tag{52}$$

We follow BMPR (2007) and use the credit spread on a consol bond to measure default risk. The consol bond pays one dollar at each date until the firm defaults at τ and nothing afterwards. Let Δ_t denote the credit spread on this consol bond. Equilibrium pricing implies

$$\int_t^\infty e^{-(r+\Delta_t)(s-t)}ds = \mathbb{E}_t^{\mathbb{Q}_{h^*}}\left[\int_t^\tau e^{-r(s-t)}ds\right]. \tag{53}$$

It is helpful to introduce D_t, the market value of a default-contingent contract that pays the contract holder one dollar at default time τ. Using investors' endogenous belief $\mathbb{Q}_{h^*}$, we have

$$D_t = \mathbb{E}_t^{\mathbb{Q}_{h^*}}\left[e^{-r(\tau-t)}\right] \quad \text{for} \quad t\in[0,\tau]\,. \tag{54}$$

Using (54) to solve Δ_t defined in equation (53),[15] we obtain $\Delta_t = rD_t/(1-D_t)$, which implies that the credit spread Δ_t increases with the value of the default-contingent contact, D_t.

Figure 6 presents the equity premium $e(m)$ and the credit spread $\Delta(m)$ as functions of the level of cash reserves m for three different values of θ. We find that both the equity premium and credit spread increase with investors' concern for the entrepreneur's productivity $(1/\theta)$ and decreases with cash balance (m). Intuitively, investors' concern for the entrepreneur's productivity causes equity value under the endogenous belief $\mathbb{Q}_{h^*}$ to be lower than under the reference belief $\mathbb{P}_{a^*}$ as ambiguity-averse investors demands an expected return premium for holding risky equity.

Unlike in Miao and Rivera (2016), the equity premium and credit risk are also related to investment and average q. Agency and investors' concern for the entrepreneur's productivity together determine corporate investment, the equity premium, and credit spread. Our model predicts that firms that invest more tend to have a lower equity premium, lower credit spread, and higher cash balances (or more financial slack). These results are broadly in line with empirical studies by Zhang (2005), Hou, Xue, and Zhang (2015), and Chordia et al. (2017).

[15] By using $\mathbb{E}_t^{\mathbb{Q}_{h^*}}\left(\int_t^\tau e^{-r(s-t)}ds\right) = \int_t^\infty e^{-r(s-t)ds}\left[1-\mathbb{E}_t^{\mathbb{Q}_{h^*}}\left(e^{-r(\tau-t)}\right)\right] = (1-D_t)/r$ and the definition for the credit spread, $\int_t^\infty e^{-(r+\Delta_t)(s-t)}ds = 1/(r+\Delta_t)$, we obtain $\Delta_t = rD_t/(1-D_t)$.

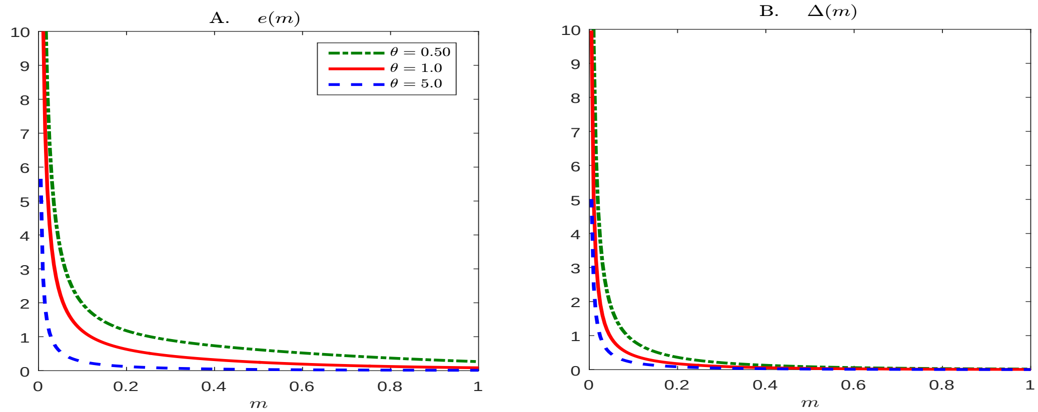

Figure 6: **Equity premium** $e(m)$ **and credit spread** $\Delta(m)$**.** The equity premium and credit spread are higher when investors are more concerned about the entrepreneur's productivity model (i.e., when θ is lower.) Also, both $e(m)$ and $\Delta(m)$ decrease with cash balance m.

5.2 Parameter Values

Next, we calibrate our model. We choose the widely used quadratic adjustment cost function:

$$c(i) = i + \frac{\psi}{2}i^2 \,, \tag{55}$$

where ψ is a positive parameter (Hayashi (1982)). The first-best optimal investment i and average q are given by

$$i^{FB} = r + \delta - \sqrt{(r+\delta)^2 - \frac{2}{\psi}\left[\mu - (r+\delta)\right]}, \quad q^{FB} = 1 + \psi i^{FB},$$

where we assume that $(r+\delta)^2 - 2\left(\mu - (r+\delta)\right)/\psi > 0$. Clearly, assuming $\ell < q^{FB}$ is sufficient to ensure that liquidation is never optimal in the first-best contract.

Table 1 reports the parameter values used for our baseline calculation, which is based on the baseline case in DFHW (2012). Specifically, we set the annual risk-free rate at $r = 4.6\%$, the annual mean and volatility of the iid productivity to $\mu = 20\%$ and $\sigma = 26\%$. For capital stock, we set the liquidation recovery at $\ell = 0.97$ per unit of capital, the capital adjustment parameter at $\psi = 2$, and the annual depreciation rate of capital stock at $\delta = 12.5\%$. Finally, we set the entrepreneur's annual discount rate at $\gamma = 5\%$, and the agency parameter at $\lambda = 0.2$.

The new parameter in our model is the key robustness parameter θ. Following Anderson, Hansen, and Sargent (2003), and Hansen and Sargent (2008), we use the detection error probability method to calibrate this parameter to 2.8.

Specifically, recall that $\mathbb{P}_{a^*}$ is the entrepreneur's belief where the effort level $a_t^* = 1$ for all t, which is an approximating model to an unknown true model, and $\mathbb{Q}_{h^*}$ is the investors' equilibrium

Table 1: **Baseline parameter values.** All parameter values are continuously compounded and annualized whenever applicable.

Parameters	Symbol	Value
Risk-free rate	r	4.6%
Depreciation rate of capital stock	δ	12.5%
Mean of productivity shock	μ	20%
Volatility of productivity shock	σ	26%
Entrepreneur's discount rate	γ	5%
Adjustment cost parameter	ψ	2.0
Agency parameter	λ	0.2
Liquidation recovery value	ℓ	0.97
Robustness parameter	θ	2.8

belief, also the worst-case measure for the robust contract with agency. The likelihood ratio between these two measures is given by the following Radon-Nikodym derivative:

$$\left.\frac{d\mathbb{Q}_{h^*}}{d\mathbb{P}_{a^*}}\right|_{\mathcal{F}_t} = \xi_t^{h^*} = \exp\left[-\int_0^t h^*(w_s)dZ_s^{a^*} - \frac{1}{2}\int_0^t \left(h^*(w_s)\right)^2 ds\right], \tag{56}$$

where h^* is given in Proposition 1. The detection-error probability is defined as

$$\alpha_T(\theta) = \frac{1}{2}\mathbb{P}_{a^*}\left[\ln(\xi_T^{h^*}) \geq 0\}\right] + \frac{1}{2}\mathbb{Q}_{h^*}\left[\ln(\xi_T^{h^*}) \leq 0\right] \tag{57}$$

for some large T. This probability $\alpha_T(\theta)$ describes the likelihood of wrongly rejecting the true model, which could be either the entrepreneur's approximating model $\mathbb{P}_{a^*}$ or the investors' endogenous belief $\mathbb{Q}_{h^*}$, based on a finite sample of T years. We calculate $\alpha_T(\theta)$ using Monte Carlo simulations as there is no analytical expression.

We consider two cases with $T = 10$ and $T = 20$ years. Figure 7 plots the relation between $\alpha_T(\theta)$ and the robustness parameter θ. First, consider the special case where investors fully trust the entrepreneur's productivity level ($\theta = \infty$). In this case, there is only one model on the table ($\mathbb{P}_{a^*}$ is true and the same as $\mathbb{Q}_{h^*}$). If this model is rejected, it must be erroneous. That is, the detection error probability is one, which is the value we obtain from (57).

In contrast, when investors are concerned about the entrepreneurs' model (a finite value of θ), there is an economic and statistically meaningful difference between $\mathbb{P}_{a^*}$ and $\mathbb{Q}_{h^*}$. The lower the value of θ, the more significant the difference between the two models is. This suggests that the likelihood of erroneously rejecting the true model decreases as the investors become more concerned about the entrepreneur's productivity model (i.e, as θ decreases.) Indeed, Figure 7 confirms our intuition as $\alpha_T(\theta)$ increases with θ for a fixed T.

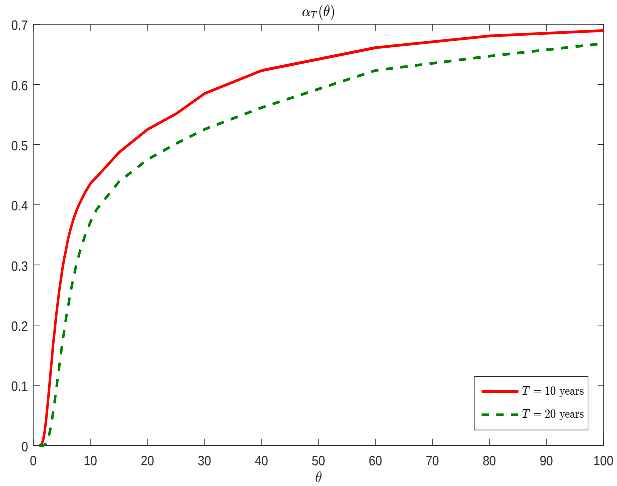

Figure 7: **Detection error probabilities $\alpha_T(\theta)$ as a function of the parameter θ.** We plot $\alpha_T(\theta)$ for two cases: $T = 10$ and $T = 20$ years. All other parameter values are reported in Table 1.

Anderson, Hansen, and Sargent (2003) suggest that modelers choose θ to target the implied $\alpha_T(\theta)$ to at least 10%. By targeting $\alpha_T(\theta)$ at 10%, we obtain $\theta = 2.8$ for $T = 10$ years and $\theta = 4.1$ for $T = 20$ years. Note that with a longer sample (20 versus 10 years), the true model is less likely to be rejected. Therefore, to attain the same level of detection error probability, e.g., 10%, we need investors to be less concerned about the entrepreneur's model for a longer sample. This explains why $\theta = 4.1$ for $T = 20$ and $\theta = 2.8$ for $T = 10$ as $\alpha_{20}(4.1) = \alpha_{10}(2.8)$. For our baseline calculation, we choose $\theta = 2.8$ and $T = 10$ (which seems sufficiently long for an entrepreneurial project.)

5.3 Quantitative Results

As in DFHW (2012), we simulate our model at the monthly frequency and generate sample paths that lasts 20 years unless the firm is liquidated, i.e., ending at $\min\{20, \tau\}$. Each simulation starts with $w_0 = \arg\max p(w)$. That is, we assume that the investors have all the bargaining power. We repeat the simulation 5,000 times and report various (sample) moments in Table 2.

Column (1) of Table 2 replicates the results in DFHW (2012) which corresponds to $\theta = \infty$ in our model. Column (6) presents the results for our baseline calibration with $\theta = 2.8$. The differences between columns (1) and (6) are entirely caused by the investors' concern for the entrepreneur's productivity model. Investors' concern for the entrepreneur's model delays payout to the entrepreneur, reduces the average level and volatility of the investment-capital ratio, lowers Tobin's average q, raises the average credit spread, and induces a large equity premium of 3.79%

Table 2: **Quantitative comparative static results with respect to θ and λ.** All other parameter values are reported in Table 1. Under the first best, $q^{FB} = 1.31$ and $i^{FB} = 0.16$.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Robustness parameter θ	∞	4.1	4.1	4.1	2.8	**2.8**	2.8
Agency parameter λ	0.2	0.1	0.2	0.4	0.1	**0.2**	0.4
Payout boundary $\overline{w}$	0.43	0.99	1.02	1.10	1.10	**1.12**	1.18
Investment reduction (%), $i^{FB} - i^*(w)$	6.14	8.31	8.70	9.48	9.46	**9.63**	9.89
Investment volatility (%), $\beta^*(w)\sigma i^{*\prime}(w)$	1.48	0.97	1.26	1.43	0.96	**1.24**	1.35
Firm value reduction, $q^{FB} - q_a(w)$	0.06	0.19	0.18	0.17	0.23	**0.21**	0.19
Equity premium (%), $e(m)$	0.00	1.91	2.38	4.38	2.93	**3.79**	7.42
credit spread (%), $\Delta(m)$	1.75	0.77	2.42	13.04	0.86	**2.84**	17.34

per annum, even though investors are risk neutral.

Table 2 also presents comparative static results with respect to θ and λ. As the agency problem becomes more severe when λ increases, the firm becomes more financially constrained and hence payout boundary $\overline{w}$ increases, underinvestment distortion $i^{FB} - i^*(w)$ becomes more severe, investment volatility $\beta^*(w)\sigma i^{*\prime}(w)$ increases, and both the equity premium and credit spread increase. The effect on firm value reduction $q^{FB} - q(w)$ is subtle. For both $\theta = 4.1$ and $\theta = 2.8$ cases, the firm value reduction decreases and hence firm's average q increases with λ.

The last monotonicity result is the opposite of that in DFHW (2012). The difference is entirely caused by investors' concern for the entrepreneur's model. The intuition is as follows. For incentive compatibility considerations, a higher value of λ causes the entrepreneur's w to be more exposed to cash flow risk. While this higher risk exposure raises the risk of liquidation, it also allows investors to let the more optimistic entrepreneur own a larger fraction of the firm at a cheaper valuation under their belief $\mathbb{Q}_{h^*}$. The net effect is that a firm's average q is higher if investors' concern for the entrepreneur's model is sufficiently large, for both $\theta = 2.8$ and $\theta = 4.1$ cases.

Now consider the effect of changing the investors' concern for the entrepreneur's productivity model (θ) by comparing columns (1), (3), and (6). As investors become more concerned about the entrepreneur's model (i.e., as θ decreases), the firm delays its payout ($\overline{w}$), under-investment ($i^{FB} - i^*(w)$) and firm value loss ($q^{FB} - q_a(w)$) become more significant, and both the equity premium and credit spread increase. These results are intuitive. The effect of changing θ on investment volatility is less obvious. Investors who are more concerned about the entrepreneur's model (a lower θ), are more averse to inefficient liquidation and hence an optimal contract calls for a less volatile investment, especially when the entrepreneur's w is sufficiently large (see Panel C of

Figure 3).

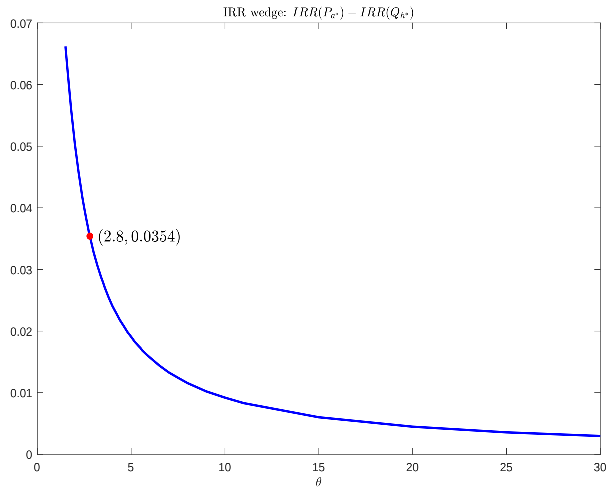

Figure 8: **Entrepreneur's equilibrium overconfidence and the IRR wedge due to the entrepreneur's and investors' belief wedge:** $IRR(\mathbb{P}_{a^*}) - IRR(\mathbb{Q}_{h^*})$**.** Note that $IRR(\mathbb{Q}_{h^*}) = r$. We use parameter values for the baseline case reported in Table 1 and set w_0 at $\arg\max_w p(w)$. For our baseline calibration with $\theta = 2.8$, the implied IRR wedge is 3.5% per annum.

The quantitative effects of investors' concern for the entrepreneur's model are significant for both quantities and prices. For example, with our calibrated value of $\theta = 2.8$, the mean of Tobin's average q is only 1.10, which is significantly lower than $q^{FB} = 1.31$ by 0.21, compared with the value of 1.25 for $\theta = \infty$; the average investment reduction is 9.63% and the investment volatility is 1.24%, compared with 6.14% and 1.48% for $\theta = \infty$; and the equilibrium equity premium is 3.8% per annum rather than zero for $\theta = \infty$. Our analysis shows that with plausible values for the robustness parameter θ, the quantitative predictions are drastically different from those in the standard model with $\theta = \infty$.

Our model also sheds light on different return expectations for the entrepreneur and investors in equilibrium. In the next subsection, we show that different returns can be quite sizable, especially for early stage projects and/or first-time (less experienced) entrepreneurs.

5.4 Internal Rate of Return (IRR)

A key valuation issue widely discussed in the VC industry and also in MBA Entrepreneurial Finance classes is why the IRRs that VCs use to value projects are so high relative to the standard cost of capital calculations (even with large βs) that we use to make investment decisions. There are various explanations, e.g., VCs' skill/value-add, network, and market power. One common explanation is that entrepreneurs tend to be overly optimistic and are often unrealistic about their projections. Rather than discouraging and trying to convince the entrepreneur and expressing their concerns for the entrepreneur's productivity model, investors can simply use the entrepreneur's belief model when quoting the cost of capital for the project to the entrepreneur. Doing so, the VCs achieves the same goal without making the entrepreneur feel that they do not trust the entrepreneur's productivity model $\mathbb{P}_{a^*}$.

We now calculate the project's IRR based on the entrepreneur's belief $\mathbb{P}_{a^*}$. That is, we compute $IRR(\mathbb{P}_{a^*})$ by solving the following equation:

$$V_0 = \mathbb{E}^{\mathbb{P}_{a^*}} \left(\int_0^\tau e^{-IRR(\mathbb{P}_{a^*})\cdot s} dY_s + e^{-IRR(\mathbb{P}_{a^*})\cdot \tau} \ell K_\tau \right) ,$$

where V_0 is the firm's enterprise value given by (48). Since investors are break-even in our model, the IRR for the risk-neutral deep pocketed investors is equal to the risk-free rate r under their equilibrium belief $\mathbb{Q}_{h^*}$, i.e., $IRR(\mathbb{Q}_{h^*}) = r$. In Appendix B, we verify the following result:

$$IRR(\mathbb{P}_{a^*}) - IRR(\mathbb{Q}_{h^*}) = \frac{\sigma h^*(w) \left[1 + \beta^*(w) v'(w)\right]}{v(w)}, \quad w \in [0, \overline{w}] .$$

Using the ODE for $v(w)$ given in Appendix B, we plot the IRR wedge $IRR(\mathbb{P}_{a^*}) - IRR(\mathbb{Q}_{h^*})$ as a function of θ in Figure 8.

This figure shows how the IRR wedge decreases as the investors become less concerned about the entrepreneur's productivity model. For each level of θ, we set $w_0 = \arg\max_w p(w)$ by assigning all the bargaining power to the investors in this calculation. At our calibrated value of $\theta = 2.8$, this IRR wedge is 3.54% per annum, which is quite large. In the limit as $\theta \to \infty$ (as in the rational expectations model), the belief wedge disappears, which in turn drives the IRR wedge to zero. As investors become more concerned about the entrepreneur's model (as θ decreases), the belief wedge $\mathbb{P}_{a^*} - \mathbb{Q}_{h^*}$ widens, which in turn causes the IRR wedge $IRR(\mathbb{P}_{a^*}) - IRR(\mathbb{Q}_{h^*})$ to increase.

6 Conclusion

We have developed a continuous-time contracting model with investment in which investors are concerned about the entrepreneur's productivity model. We characterize the robust contract and show that early liquidation can be optimal when investors are sufficiently ambiguity averse. We provide a novel implementation of the robust contract by debt, equity, cash, and a financial derivative asset. The derivable asset is used to hedge against the investors' concern that the entrepreneur's productivity model may be misspecified. Our calibrated model generates sizable equity premium and credit spread. We also find that (i) ambiguity aversion lowers Tobin's q, the average investment rate, and investment volatility; (ii) the equity premium and the credit spread decrease with Tobin's q, and (iii) the entrepreneur values the project at an internal rate of return 3.5% per annum higher than investors do.

Appendices

A Proofs

To ensure that the model is well posed, we impose the following conditions on K_t and the entrepreneur's compensation process U_t. Specifically, under the equivalent measure $\mathbb{Q}_h$ induced by $h \in \mathcal{H}^a$, where $\mathcal{H}^a$ is the set of density generators associated with effort $a \in [0,1]$, we require

$$\mathbb{E}^{\mathbb{Q}_h}\left[\int_0^T (e^{-rt}K_t)^2 dt\right] < \infty, \quad \forall T > 0, \quad \text{and} \quad \lim_{T\to\infty} \mathbb{E}^{\mathbb{Q}_h}\left(e^{-rT}K_T\right) = 0, \tag{A.1}$$

and

$$\mathbb{E}^{\mathbb{Q}_h}\left[\left(\int_0^\tau e^{-\gamma s} dU_s\right)^2\right] < \infty\,. \tag{A.2}$$

Proof of Proposition 1. Our proof of Proposition 1 essentially follows the proof of Proposition 1 in DFHW (2012) and the proof of Propositions 1 and 2 in Miao and Rivera (2016). We thus omit the proof here.

Proof of Equation (49). Using the process (17) for the entrepreneur's continuous utility W_t, we obtain:

$$d(e^{-rt}W_t) = e^{-rt}dW_t - re^{-rt}W_t dt = e^{-rt}\left[\left((\gamma - r)W_t - \beta_t^* K_t \sigma h_t^*\right)dt - dU_t^* + \beta_t^* K_t \sigma dZ_t^{h^*}\right].$$

Integrating over the interval $[t, T\wedge\tau]$ for any $T > t$, we obtain

$$\begin{aligned}\int_t^{T\wedge\tau} d(e^{-rs}W_s) \quad &= e^{-r(T\wedge\tau)}W_{T\wedge\tau} - e^{-rt}W_t \\ &= \int_t^{T\wedge\tau} e^{-rs}\left[\left((\gamma - r)W_s - \beta_s^* K_s \sigma h_s^*\right)ds - dU_s^* + \beta_s^* K_s \sigma dZ_s^{h^*}\right].\end{aligned}$$

Rewriting the preceding equation yields

$$e^{-r(T\wedge\tau)}W_{T\wedge\tau} = e^{-rt}W_t + \int_t^{T\wedge\tau} e^{-rs}\left[\left((\gamma - r)W_s - \beta_s^* K_s \sigma h_s^*\right)ds - dU_s^* + \beta_s^* K_s \sigma dZ_s^{h^*}\right].$$

By using $\tau = \inf\{t \geq 0 : W_t = 0\}$, which implies $W_\tau = 0$, letting $T \to \infty$, using $M_t = W_t/\lambda$, and calculating the expectation under the measure $\mathbb{Q}_{h^*}$, we obtain:

$$M_t = \mathbb{E}_t^{\mathbb{Q}_{h^*}}\left[\int_t^\tau e^{-r(s-t)}\left(\frac{dU_s^*}{\lambda} - (\gamma - r)M_s ds + \frac{\beta_s^* K_s \sigma h_s^*}{\lambda} ds\right)\right]. \tag{A.3}$$

Summing up the firm's bond value B_t and equity value S_t, we obtain:

$$\begin{aligned}
B_t+S_t &= \mathbb{E}_t^{\mathbb{Q}_{h^*}}\left[\int_t^{\tau} e^{-r(s-t)}\left[K_s(\mu-c(i_s^*))-(\gamma-r)M_s\right]ds+\int_t^{\tau} e^{-r(s-t)}\left(\frac{\beta_s^*-\lambda}{\lambda}\right)K_s\sigma h_s^* ds\right] \\
&\quad +\mathbb{E}_t^{\mathbb{Q}_{h^*}}\left[e^{-r(\tau-t)}\ell K_\tau\right]+\mathbb{E}_t^{\mathbb{Q}_{h^*}}\left[\int_t^{\tau} e^{-r(s-t)}\frac{1}{\lambda}dU_s^*\right] \qquad \text{(A.4)} \\
&= \mathbb{E}_t^{\mathbb{Q}_{h^*}}\left[\int_t^{\tau} e^{-r(s-t)}\left(K_s(\mu-c(i_s^*))-K_s\sigma h_s^*\right)ds\right]+\mathbb{E}_t^{\mathbb{Q}_{h^*}}\left[e^{-r(\tau-t)}\ell K_\tau\right] \\
&\quad +\mathbb{E}_t^{\mathbb{Q}_{h^*}}\left[\int_t^{\tau} e^{-r(s-t)}\left(\frac{dU_s^*}{\lambda}-(\gamma-r)M_s ds+\frac{\beta_s^* K_s\sigma h_s^*}{\lambda}ds\right)\right] \qquad \text{(A.5)} \\
&= \mathbb{E}_t^{\mathbb{Q}_{h^*}}\left[\int_t^{\tau} e^{-r(s-t)}dY_s+e^{-r(\tau-t)}\ell K_\tau\right]+M_t \qquad \text{(A.6)} \\
&= V_t+M_t\,, \qquad \text{(A.7)}
\end{aligned}$$

where (A.4) uses the present-value formulas for S_t and B_t given in equations (40) and (42), (A.5) rearranges (A.4), and (A.6) uses the expression (A.3) for M_t and the expression (48) for the firm's enterprise value, V_t. We now have proved the identity $V_t = B_t + S_t - M_t$ stated in (49).

Proposition 2 *Let assumptions in Proposition 1 hold. Then the scaled value function $p(w)$ increases with θ for $w \in (0, \overline{w})$.*

Proof. Our proof builds on DeMarzo and Sannikov (2006), and Miao and Rivera (2016). Using the envelope theorem and differentiating (24) with respect to θ, we obtain

$$\begin{aligned}
r\frac{\partial p(w)}{\partial\theta} &= (\gamma+\delta-i^*(w))w\frac{\partial p'(w)}{\partial\theta}+(i^*(w)-\delta)\frac{\partial p(w)}{\partial\theta}+\frac{\sigma^2\left[\beta^*(w)\right]^2}{2}\frac{\partial p''(w)}{\partial\theta} \\
&\quad -\frac{\sigma^2\left[\beta^*(w)p'(w)+1\right]\beta^*(w)}{\theta}\frac{\partial p'(w)}{\partial\theta}+\frac{\sigma^2\left[1+\beta^*(w)\,p'(w)\right]^2}{2\theta^2}.
\end{aligned}$$

Using (28) for $h^*(w)$, we rewrite the preceding equation as

$$\begin{aligned}
(r+\delta-i^*(w))\frac{\partial p(w)}{\partial\theta} &= \left[(\gamma+\delta-i^*(w))w-\beta^*(w)\,\sigma h^*(w)\right]\frac{\partial p'(w)}{\partial\theta} \\
&\quad +\frac{\left[\beta^*(w)\sigma\right]^2}{2}\frac{\partial p''(w)}{\partial\theta}+\frac{\left[h^*(w)\right]^2}{2}.
\end{aligned}$$

We write the dynamics of w_t as

$$dw_t = \left[(\gamma+\delta-i^*(w_t))w_t-\beta^*(w_t)\,\sigma h^*(w_t)\right]dt - du_t^* + \beta^*(w_t)\,\sigma dZ_t^{h^*},$$

where Z^{h^*} is the standard Brownian motion under the measure $\mathbb{Q}_{h^*}$. Using the Feynman-Kac formula, we can show

$$\frac{\partial p(w)}{\partial\theta} = \mathbb{E}_t^{\mathbb{Q}_{h^*}}\left[\int_t^{\tau} e^{-\int_t^s g(w_u)du}\frac{\left[h^*(w)\right]^2}{2}ds\middle| w_t=w\right]>0,$$

where $g(w)=r+\delta-i^*(w)$. Thus, $p(w)$ is an increasing function of θ for any given $w\in(0,\overline{w})$.

Proposition 3 *The sufficient and necessary condition of implementing the highest effort $a^* = 1$ is*

$$\begin{aligned}\frac{(p(w)-wp'(w)-1)^2}{2\psi} &\leq (\delta+r)p(w)-p'(w)[(\gamma+\delta)w-\lambda\mu]+\frac{\sigma^2}{2\theta} \\ &\quad -\frac{\sigma^2}{2}\max_{\beta\leq\lambda}\left[\left(p''(w)-\frac{p'(w)^2}{\theta}\right)\beta^2-\frac{2p'(w)}{\theta}\beta\right]. \end{aligned} \tag{A.8}$$

Proof. Our proof builds on DeMarzo and Sannikov (2006), DFHW (2012), and Miao and Rivera (2016). The high effort level is optimal if and only if

$$\begin{aligned} rp(w) &\geq \sup_{i,\beta\leq\lambda} \left[-\lambda\mu p'(w)-c(i)\right]+(\gamma+\delta-i)wp'(w) \\ &\quad +(i-\delta)p(w)+\frac{\sigma^2\beta^2 p''(w)}{2}-\frac{\sigma^2}{\theta}\left(\frac{1}{2}+p'(w)\beta+\frac{1}{2}\left[p'(w)\beta\right]^2\right). \end{aligned} \tag{A.9}$$

Substituting the adjustment cost $c(i)=i+\psi i^2/2$ and the optimal rate $i^*(w)=(p(w)-wp'(w)-1)/\psi$ into the inequality given in (A.8), we obtain (A.9) stated in Proposition 3.

B Asset Pricing Formulas

ODE for the scaled equity value $s(m_t)$. Under the measure $\mathbb{Q}_{h^*}$, the scaled cash reserves m_t evolves as follows:

$$dm_t=(\gamma+\delta-i_t^*)m_t dt-du_t^*/\lambda+\frac{\beta_t^*}{\lambda}\sigma\left(dZ_t^{h^*}-h_t^*dt\right). \tag{B.1}$$

Using Ito's Lemma, we obtain the following ODE for $s(m_t)=S_t/K_t$ in the region $m_t\in[0,\overline{w}/\lambda]$:

$$rs(m)=(i^*(\lambda m)-\delta)s(m)+\frac{(\beta^*(\lambda m)\sigma)^2}{2\lambda^2}s''(m)+\left[(\gamma+\delta-i_t^*)m-\frac{\beta^*(\lambda m)\sigma}{\lambda}h^*(\lambda m)\right]s'(m),$$

subject to the boundary conditions $s(0)=0$ and $s'(\overline{w}/\lambda)=1$.

ODE for the scaled debt value $b(m_t)$. The ODE for $b(m_t)=B_t/K_t$ for $m_t\in[0,\overline{w}/\lambda]$ is

$$\begin{aligned} rb(m) &= \mu-c(i^*(\lambda m))-(\gamma-r)m+\frac{\beta^*(\lambda m)-\lambda}{\lambda}h^*(\lambda m)\sigma+(i^*(\lambda m)-\delta)b(m) \\ &\quad +\left[(\gamma+\delta-i_t^*)m-\frac{\beta^*(\lambda m)\sigma}{\lambda}h^*(\lambda m)\right]b'(m)+\frac{(\beta^*(\lambda m)\sigma)^2}{2\lambda^2}b''(m), \end{aligned}$$

subject to the boundary conditions subject to the boundary conditions $b(0)=\ell$ and $b'(\overline{w}/\lambda)=0$.

Price of $D_t=D(m_t)$ defined in (54). Using essentially the same argument as for $s(m_t)$ and $b(m_t)$, we obtain the following ODE for $D_t=D(m_t)$ for $m\in[0,\overline{w}/\lambda]$:

$$rD(m)=\left[(\gamma+\delta-i_t^*)m-\frac{\beta^*(\lambda m)\sigma}{\lambda}h^*(\lambda m)\right]D'(m)+\frac{[\beta^*(\lambda m)\sigma]^2}{2\lambda^2}D''(m),$$

subject to the boundary conditions $D(0)=1$ and $D'(\overline{w}/\lambda)=0$.

ODE for the scaled market value of firm $v(m_t)$. Let $v(w) = V_t/K_t$. Then $v(w)$ solves the following ODE:

$$\begin{aligned} rv(w) &= \mu - c(i^*(w)) - h^*(w)\sigma + [i^*(w) - \delta]v(w) \\ &\quad + [(\gamma + \delta - i_t^*)w - \beta^*(w)\sigma h^*(w)]v'(w) + \frac{[\beta^*(w)\sigma]^2}{2}v''(w), \end{aligned} \tag{B.2}$$

subject to the boundary conditions $v(0) = \ell$ and $v'(\overline{w}) = 0$. Since Tobin's average q is given by $q_t = V_t/K_t$, we may write q_t as a function of w_t, i.e., $q_a(w_t)$, where $q_a(w) = v(w)$.

ODE for the entrepreneur's $IRR(\mathbb{P}_{a^*})$. By recognizing that $\int_0^t e^{-IRR(\mathbb{P}_{a^*})\cdot s}dY_s + e^{-IRR(\mathbb{P}_{a^*})t}V_t$ is a martingale under $\mathbb{P}_{a^*}$ and using $V_t = v(w_t)K_t$ and the property that a martingale's drift is zero, we obtain the following differential equation:

$$IRR(\mathbb{P}_{a^*})v(w) = \mu - c(i^*(w)) + [i^*(w) - \delta]v(w) + [(\gamma + \delta - i_t^*)w]v'(w) + \frac{[\beta^*(w)\sigma]^2}{2}v''(w). \tag{B.3}$$

Since investors break even under their belief $\mathbb{Q}_{h^*}$, we have $IRR(\mathbb{Q}_{h^*}) = r$. Subtracting (B.2) from (B.3) on both sides and using $IRR(\mathbb{Q}_{h^*}) = r$, we obtain the following IRR wedge:

$$IRR(\mathbb{P}_{a^*}) - IRR(\mathbb{Q}_{h^*}) = \frac{\sigma h^*(w)\left[1 + \beta^*(w)v'(w)\right]}{v(w)}, \quad w \in [0, \overline{w}].$$

C Robust Contract without Agency

When $\lambda \to 0$, the incentive constraint (16) becomes irrelevant and the solution characterized in Proposition 1 reduces to the robust contract without agency. For this case, we show that the scaled value function $p(w)$ is globally concave, $h^*(w) > 0$, and $\beta^*(w) > 0$.

Under the belief $\mathbb{Q}_h$, we use (6) to rewrite (15) as

$$dW_t = (\gamma W_t - \beta_t K_t \sigma h_t)\,dt - dU_t - \lambda\mu(1 - a_t)K_t dt + \beta_t K_t \sigma dZ_t^h. \tag{C.1}$$

Using (2) and the Girsanov theorem, we can rewrite the objective function in (13) under $\mathbb{Q}_h$ as

$$\mathbb{E}^{\mathbb{Q}_h}\left[\int_0^\tau e^{-rt}\left(K_t(a_t\mu - \sigma h_t) - C(I_t, K_t)\right)dt - \int_0^\tau e^{-rt}dU_t + e^{-r\tau}\ell K_\tau + \frac{1}{2}\int_0^\tau e^{-rt}\Theta(K_t)h_t^2 dt\right].$$

Investors' value function $P(K, W)$ satisfies the HJB equation

$$\begin{aligned} rP(K_t, W_t)\,dt &= \sup_{I_t, dU_t, a_t, \beta_t}\ \inf_{h_t}\ K_t(a_t\mu - \sigma h_t)dt - C(I_t, K_t)dt - dU_t + \frac{1}{2}h_t^2\Theta(K_t)dt \\ &\quad + P_W(K_t, W_t)\left[(\gamma W_t - \beta_t K_t \sigma h_t)\,dt - dU_t - \lambda\mu(1 - a_t)K_t dt\right] \\ &\quad + P_K(K_t, W_t)(I_t - \delta K_t)dt + \frac{1}{2}P_{WW}(K_t, W_t)(\beta_t K_t \sigma)^2\,dt. \end{aligned} \tag{C.2}$$

Since $dU_t \geq 0$, the FOC for dU_t implies $P_W(K_t, W_t) \geq -1$. Additionally, if $P_W(K_t, W_t) > -1$, we have $dU_t = 0$.

Further simplifying the HJB equation (C.2), we obtain

$$\begin{aligned} rP(K_t, W_t) &= \sup_{I_t, a_t, \beta_t} \inf_{h_t} \; K_t(a_t\mu - \sigma h_t) - C(I_t, K_t) + \frac{1}{2}h_t^2\Theta(K_t) \\ &\quad + P_W(K_t, W_t)\left[(\gamma W_t - \beta_t K_t \sigma h_t) - \lambda\mu(1 - a_t)K_t\right] \\ &\quad + P_K(K_t, W_t)(I_t - \delta K_t) + \frac{1}{2}P_{WW}(K_t, W_t)(\beta_t K_t \sigma)^2 . \end{aligned} \tag{C.3}$$

The FOC for h_t is

$$h_t^* = \frac{K_t\sigma[1 + \beta_t P_W(K_t, W_t)]}{\Theta(K_t)} . \tag{C.4}$$

Since $\Theta(K_t) > 0$, this condition is also sufficient for h_t^* to be the minimizer. Substituting this expression into the HJB equation (C.3) yields

$$\begin{aligned} rP(K_t, W_t) &= \sup_{I_t, a_t, \beta_t} \; K_t[a_t\mu - \lambda\mu(1 - a)P_W(K_t, W_t)] - \frac{K_t^2\sigma^2}{2\Theta(K_t)} - C(I_t, K_t) \\ &\quad + (I_t - \delta K_t)P_K(K_t, W_t) + \gamma W_t P_W(K_t, W_t) - \frac{K_t^2\sigma^2 P_W(K_t, W_t)\beta_t}{\Theta(K_t)} \\ &\quad + \frac{K_t^2\sigma^2\left[\Theta(K_t)P_{WW}(K_t, W_t) - P_W(K_t, W_t)^2\right]\beta_t^2}{2\Theta(K_t)}. \end{aligned} \tag{C.5}$$

If $\Theta(K_t)P_{WW}(K_t, W_t) - P_W(K_t, W_t)^2 < 0$, then using the HJB equation (C.5), we obtain the following FOC for β^*:

$$\beta_t^* = \frac{P_W(K_t, W_t)}{\Theta(K_t)P_{WW}(K_t, W_t) - P_W(K_t, W_t)^2}.$$

Note that $a_t^* = 1$ is optimal for all t. Since $C(I, K)$ is convex in I, the first-order condition $P_K(K_t, W_t) = C_I(I_t, K_t)$ implies the optimal investment rule: $I_t^* = C_I^{-1}\left(P_K(K_t, W_t), K_t\right).$

Substituting the solutions β_t^*, I_t^* and a_t^* into (C.5), we obtain a partial differential equation for $P(K, W)$. Let $w = W/K$ and $p(w) = P(K, W)/K$. Using the homogeneity property, we characterize the robust contract without agency in the following proposition.

Proposition 4 *(robust contract without agency) Suppose that $p(w)$ satisfies the ODE on $[0, \overline{w}]$:*

$$\begin{aligned} rp(w) &= \max_{\beta, i} \; \mu - c(i) + (i - \delta)p(w) + (\gamma + \delta - i)wp'(w) \\ &\quad + \frac{\sigma^2}{2}\left(p''(w) - \frac{p'(w)^2}{\theta}\right)\beta^2 - \frac{\sigma^2 p'(w)}{\theta}\beta - \frac{\sigma^2}{2\theta}, \end{aligned} \tag{C.6}$$

subject to the conditions $p'(w) > -1$ on $[0, \overline{w})$ and

$$p(0) = \frac{\mu - c(i^*(0)) - \sigma^2/(2\theta)}{r + \delta - i^*(0)}, \quad p'(\overline{w}) = -1, \quad p''(\overline{w}) = 0, \tag{C.7}$$

where $i^*(w)$ satisfies (29). Suppose that $\ell < p(0)$ and $\theta p''(w) - [p'(w)]^2 < 0$ hold. Then (i) for $w \in [0, \overline{w}]$, the optimal sensitivity is given by

$$\beta^*(w) = \frac{p'(w)}{\theta p''(w) - [p'(w)]^2}, \tag{C.8}$$

and the worst-case density generator is given by

$$h^*(w) = \frac{\sigma[1 + \beta^*(w) p'(w)]}{\theta} = \frac{\sigma p''(w)}{\theta p''(w) - [p'(w)]^2}. \tag{C.9}$$

The firm is never liquidated and the principal's scaled value function is $p(w)$. The optimal investment rate is $i^*(w)$. The entrepreneur always exerts high effort level $a_t^* = 1$ and his scaled continuation value w_t satisfies

$$dw_t = [(\gamma + \delta - i^*(w_t))w_t - \beta^*(w_t)\sigma h^*(w_t)]\,dt - du_t^* + \beta^*(w_t)\sigma dZ_t^{h^*}, \tag{C.10}$$

for $w_t \in [0, \overline{w}]$. The optimal compensation $dU_t^* = K_t du_t^*$ keeps (w_t) reflecting at $\overline{w}$ and satisfies

$$du_t^* = \max\{w_t - \overline{w}, 0\}. \tag{C.11}$$

(ii) For $w > \overline{w}$, $p(w) = p(\overline{w}) - (w - \overline{w})$. The principal pays $w - \overline{w}$ immediately to the entrepreneur and the contract continues with the entrepreneur's new initial value $\overline{w}$.

The following result shows that $p(w)$ is concave for the robust contact without agency.

Proposition 5 *Let assumptions in Proposition 4 hold. The scaled value function $p(w)$ is concave, $h^*(w) > 0$, and $\beta^*(w) \in [0, 1]$ in the region where $(0, \overline{w})$.*

Proof: Substituting the optimal $i^*(w)$ and the optimal sensitivity $\beta^*(w)$ into (C.6), we obtain:

$$(r + \delta)p = \mu + \frac{(p - wp' - 1)^2}{2\psi} + (\gamma + \delta)wp' - \frac{\sigma^2}{2\theta}\frac{(p')^2}{\theta p'' - (p')^2} - \frac{\sigma^2}{2\theta}. \tag{C.12}$$

Differentiating (C.12), we obtain

$$\begin{aligned}(r + \delta)p' \;=\; & \frac{-(p - wp' - 1)wp''}{\psi} + (\gamma + \delta)\left(p' + wp''\right) \\ & -\frac{\sigma^2}{2\theta}\frac{2p'p''\left[\theta p'' - (p')^2\right] - (p')^2(\theta p''' - 2p'p'')}{\left[\theta p'' - (p')^2\right]^2}.\end{aligned} \tag{C.13}$$

Evaluating (C.13) at the payout boundary $\overline{w}$, and using $p'(\overline{w}) = -1$ and $p''(\overline{w}) = 0$, we obtain

$$\frac{\sigma^2}{2}p'''(\overline{w}) = \gamma - r > 0.$$

Therefore, $p''(\overline{w}-\epsilon)>0$ for small $\epsilon>0$. Let $q(w)=p(w)-wp'(w)$. We may rewrite (C.12) as

$$(r+\delta)\,q=\mu+\frac{(q-1)^2}{2\psi}+(\gamma-r)\,wp'-\frac{\sigma^2}{2\theta}\frac{(p')^2}{\theta p''-(p')^2}-\frac{\sigma^2}{2\theta}.$$

Suppose that there exists $\widetilde{w}<\overline{w}$ such that $p''(\widetilde{w})=0$. Choose the largest $\widetilde{w}$ such that $p''(\widetilde{w}+\epsilon)<0$ for a small $\epsilon>0$. Evaluating the above equation at $\widetilde{w}$, we obtain

$$(r+\delta)\,q(\widetilde{w})=\mu+\frac{(q(\widetilde{w})-1)^2}{2\psi}+(\gamma-r)\,\widetilde{w}p'(\widetilde{w}).$$

Since $p'(w)>-1$, $q(w)<p(w)+w<q^{FB}$. Recall that

$$(r+\delta)\,q^{FB}=\mu+\frac{\left(q^{FB}-1\right)^2}{2\psi}.$$

Therefore, $(\gamma-r)\,\widetilde{w}p'(\widetilde{w})<0$, which implies $p'(\widetilde{w})<0$. Evaluating (C.13) at $\widetilde{w}$, we obtain

$$(r+\delta)\,p'(\widetilde{w})=(\gamma+\delta)\,p'(\widetilde{w})+\frac{\sigma^2}{2}\frac{p'''(\widetilde{w})}{[p'(\widetilde{w})]^2}.$$

As $\gamma>r$, the preceding equation implies $p'''(\widetilde{w})>0$. This leads to a contradiction as by assumption $p''(\widetilde{w})=0$ and $p''(\widetilde{w}+\epsilon)<0$. Therefore, $p(w)$ is concave in $w\in(0,\overline{w})$.

Since $p''(w)<0$, it follows from (C.9) that $h^*(w)>0$. Since $p(w)$ takes the largest value at $w=0$ and since $p''(w)<0$, we deduce that $p'(w)<0$ for $w>0$. It follows from (C.8) that $\beta^*(w)\in[0,1]$.

Panels A and B of Figure 9 present the principal's scaled value function $p(w)$ and investment rule $i^*(w)$ for two different values of θ in the robust contract without agency. Panels C and D of Figure 9 present the optimal sensitivity $\beta^*(w)$ and the corresponding worst-case density generator $h^*(w)$. The straight lines in Figure 9 represent the solutions in the first-best case. We find that the robust scaled value function is concave. Moreover, when the principal is more ambiguity averse, i.e., when θ is smaller, his scaled value function is smaller, the payout boundary is larger, and the investment-capital ratio is lower. This means that ambiguity aversion is costly to the principal (investors), delays payout to the entrepreneur, and causes underinvestment. The $i(w)$ increases with the entrepreneur's scaled continuation value w for $w\in[0,\overline{w}]$.

Panels C and D of Figure 9 show that, when the principal is more ambiguity averse, his belief is distorted more in the sense that the worst-case density generator $h^*(w)$ is shifted upward more. The density generator is positive so that the principal puts more weight on worse outcomes. This is because the local mean of the Brownian motion is shifted downward under the principal's worst-case belief by (6). The density generator decreases to zero at the payout boundary. At $w=\overline{w}$, the boundary conditions $p'(\overline{w})=-1$ and $p''(\overline{w})=0$ imply that $h^*(\overline{w})=0$ and $\beta^*(\overline{w})=1$.

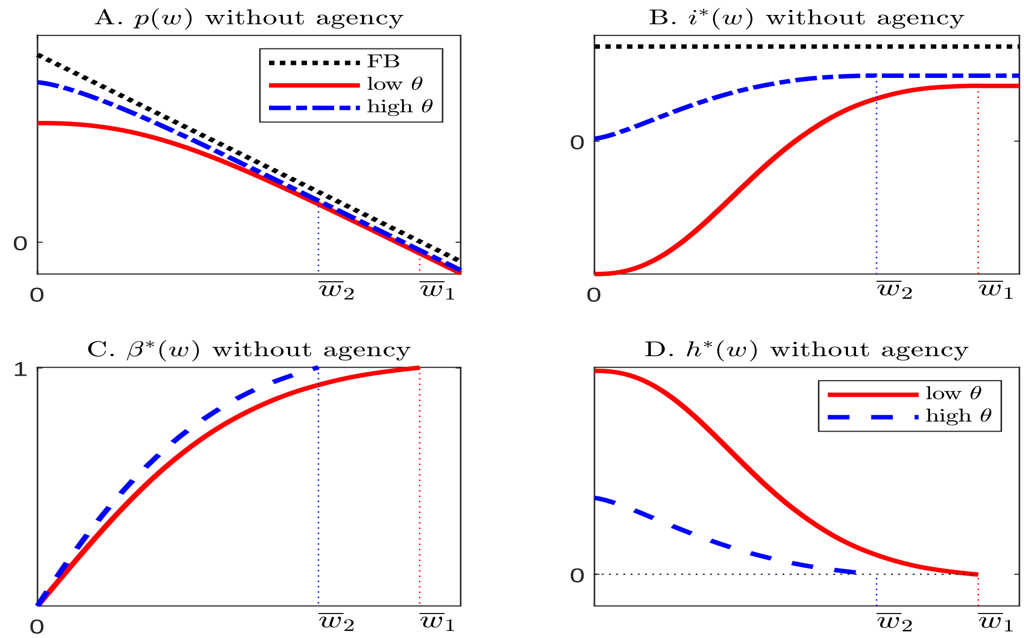

Figure 9: **Optimal robust contract without agency.** Investors' concerns for the entrepreneur's model lower $p(w)$ and $i^*(w)$ from the first-best levels. For a given level of θ, the sensitivity $\beta^*(w)$ and $i^*(w)$ increase with w, while the belief wedge $h^*(w)$ decreases with w.

The optimal sensitivity $\beta^*(w)$ starts at zero when $w = 0$ and increases to one at the payout boundary. This means that, in the robust contract without agency, the entrepreneur is partially exposed to the productivity uncertainty and cannot fully insure the ambiguity averse principal for low values of w. When w is larger, the entrepreneur can absorb more uncertainty. At the payout boundary, the entrepreneur fully absorbs model uncertainty.

References

Aghion, Philippe, and Patrick Bolton, 1992, An Incomplete Contracts Approach to Financial Contracting, *Review of Economic Studies* 77, 338-401.

Ai, Hengjie, and Anmol Bhandari, 2020, Asset pricing with endogenously uninsurable tail risk, *Econometrica*, forthcoming.

Ai, Hengjie, and Rui Li, 2015, Investment and CEO Compensation under Limited Commitment, *Journal of Financial Economics* 116, 452-472.

Albuquerque, Rui, and Hugo A. Hopenhayn, 2004, Optimal lending contracts and firm dynamics, *Review of Economic Studies* 71, 285-315.

Anderson, Evan W., Lars Peter Hansen, and Thomas J. Sargent, 2003, A quartet of semigroups for model specification, robustness, prices of risk, and model detection, *Journal of the European Economic Association* 1, 68-123.

Biais, Bruno, Thomas Mariotti, Guillaume Plantin, and Jean-Charles Rochet, 2007, Dynamic security design: Convergence to continuous time and asset pricing implications, *Review of Economic Studies* 74, 345-390.

Biais, Bruno, Thomas Mariotti, Jean-Charles Rochet, and Stéphane Villeneuve, 2010, Large risks, limited liability, and dynamic moral hazard, *Econometrica* 78, 73-118.

Chen, Zengjing, and Larry Epstein, 2002, Ambiguity, risk, and asset returns in continuous time, *Econometrica* 70, 1403-1443.

Chordia, Tarun, Amit Goyal, Yoshio Nozawa, Avanidhar Subrahmanyam, and Qing Tong, 2017, Are capital market anomalies common to equity and corporate bond markets? an empirical investigation, *Journal of Financial and Quantitative Analysis* 52, 1301-1342.

Clementi, Gian Luca, and Hugo A. Hopenhayn, 2006, A theory of financing constraints and firm dynamics, *Quarterly Journal of Economics* 121, 229-265.

DeMarzo, Peter M., and Michael J. Fishman, 2007a, Agency and optimal investment dynamics, *Review of Financial Studies* 20, 151-188.

DeMarzo, Peter M., and Michael J. Fishman, 2007b, Optimal long-term financial contracting, *Review of Financial Studies* 20, 2079-2128.

DeMarzo, Peter M., Michael J. Fishman, Zhiguo He, and Neng Wang, 2012, Dynamic agency and the q theory of investment, *Journal of Finance* 67, 2295-2340.

DeMarzo, Peter M., and Yuliy Sannikov, 2006, Optimal security design and dynamic capital structure in a continuous-time agency model, *Journal of Finance* 61, 2681-2724.

Dicks, David L., and Paolo Fulghieri, 2019, Uncertainty aversion and systemic risk, *Journal of Political Economy* 127, 1118-1155.

Dicks, David L., and Paolo Fulghieri, 2020a, Uncertainty, Investor Sentiment, and Innovation, *Review of Financial Studies*, forthcoming.

Dicks, David L., and Paolo Fulghieri, 2020b, Uncertainty and Contracting in Organizations, Working paper, Baylor University and University of North Carolina.

Dumav, Martin, 2017, Continuous-Time Contracting with Ambiguous Perceptions, Working paper, Universidad Carlos III de Madrid.

Ellsberg, Daniel, 1961, Risk, ambiguity and the savage axiom, *Quarterly Journal of Economics* 75, 643-669.

Epstein, Larry G., and Shaolin Ji, 2013, Ambiguous volatility and asset pricing in continuous time, *Review of Financial Studies* 26, 1740-1786.

Garlappi, Lorenzo, Ron Giammarino, and Ali Lazrak, 2017, Ambiguity and the corporation: Group disagreement and underinvestment, *Journal of Financial Economics*, 125, 417-433.

Gilboa, Itzhak, and David Schmeidler, 1989, Maxmin expected utility with non-unique prior, *Journal of Mathematical Economics* 18, 141-153.

Hansen, Lars Peter, and Thomas J. Sargent, 2001, Robust control and model uncertainty, *American Economic Review* 91, 60-66.

Hansen, Lars Peter, and Thomas J. Sargent, 2008, *Robustness.* Princeton University Press.

Hansen, Lars Peter, Thomas J. Sargent, Gauhar Turmuhambetova, and Noah Williams, 2006, Robust control and model misspecification, *Journal of Economic Theory* 128, 45-90.

Hayashi, Fumio, 1982, Tobin's marginal q and average q: A neoclassical interpretation, *Econometrica* 50, 213-224.

Hou, Kewei, Chen Xue, Lu Zhang, 2015, Digesting anomalies: An investment approach, *Review of Financial Studies* 28, 650-705.

Kaplan, Steven N., and Per Strömberg, 2003, Financial contracting theory meets the real world: An empirical analysis of venture capital contracts, *Review of Economic Studies* 70, 281-315.

Knight, Frank H., 1921, *Risk, uncertainty and profit.* Vol. 1, Houghton Mifflin, New York.

Lee, Seokwoo, and Uday Rajan, 2020, Robust Security Design, Working paper, University of Michigan

Liu, Yufeng, 2020, Optimal contracts with ambiguity and risk taking, Working paper, Kellogg School of Management

Maccheroni, Fabio, Massimo Marinacci, and Aldo Rustichini, 2006a, Ambiguity aversion, robustness and the variational representation of preferences, *Econometrica* 74, 1447-1498.

Maccheroni, Fabio, Massimo Marinacci, and Aldo Rustichini, 2006b, Dynamic variational preferences, *Journal of Economic Theory* 128, 4-44.

Maenhout, Pascal J., Andrea Vedolin, and Hao Xing, 2020, Generalized robustness and dynamic pessimism, NBER Working Paper No. 26970.

Malenko, Andrey, and Anton Tsoy, 2020, Asymmetric Information and Security Design under Knightian Uncertainty, Working paper, University of Michigan.

Malmendier, Ulrike, and Geoffrey Tate, 2005, CEO overconfidence and corporate investment, *Journal of Finance* 60, 2661-2700.

Merton, Robert C., 1971, Optimal consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory* 3, 373-413.

Metrick, Andrew, and Ayako Yasuda, 2010, The economics of private equity funds, *The Review of Financial Studies* 23, 2303-2341.

Miao, Jianjun, and Alejandro Rivera, 2016, Robust contracts in continuous time, *Econometrica* 84, 1405-1440.

Quadrini, Vincenzo, 2004, Investment and liquidation in renegotiation-proof contracts with moral hazard, *Journal of Monetary Economics* 51, 713-751.

Sannikov, Yuliy, 2008, A continuous-time version of the principal-agent problem, *Review of Economic Studies* 75, 957-984.

Szydlowski, Martin, and Ji Hee Yoon, 2020, Ambiguity in Dynamic Contracts, Working paper, University of Minnesota.

Williams, Noah, 2009, On dynamic principal-agent problems in continuous time, Working paper, University of Wisconsin at Madison.

Williams, Noah, 2011, Persistent private information, *Econometrica* 79, 1233-1274.

Zhang, Lu, 2005, The value premium, *Journal of Finance* 60, 67-103.