Responses to "Toward Bubble Clarity: A Comment on Miao and Wang"

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Abstract

Tomohiro Hirano and Alexis Akira Toda (2025) argue that our model (Miao and Wang 2018) does not feature a rational bubble based on the traditional definition. Their comment is more about semantics rather than about substance. They do not find any error in Miao and Wang (2018). Their argument does not apply to the Miao-Wang model, which goes beyond the traditional definition. The bubble component in the Miao-Wang model incorporates a liquidity premium and cannot be ruled out by the transversality condition.

1 Introduction

Tomohiro Hirano and Alexis Akira Toda (2025) (henceforth HT) provide a comment paper on our article on bubbles (Miao and Wang 2018) (henceforth MW). In their paper, the authors prove the "nonexistence" of "rational bubbles" in the MW model based on the traditional definition of "rational bubble". They also aim to clarify the precise mathematical definition and the economic meaning of "rational bubble" in an accessible way to the general audience.

HT do not find any mathematical errors or logical problems with the economic mechanisms in Miao and Wang (2018). Their comment focuses exclusively on the use of the terminology "rational bubbles." In our view, their comment paper does not provide any new insights for two main reasons. First, their definition of rational bubbles is taken from the textbook by Miao (2014), who is one of the authors of Miao and Wang (2018). This definition is based on the seminal work of Manuel S. Santos and Michael Woodford (1997) and the earlier work of Jean Tirole (1985). We do not think researchers including MW have misunderstandings of this definition. Second, the Bubble Characterization Lemma is taken from Luigi Montrucchio (2004)

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and is not the original contribution by HT. This lemma essentially provides a simple condition to check the transversality condition. HT use this lemma to explain the "fundamental difficulty in attaching rational bubbles to dividend-paying assets" (7–8). This difficulty has already been explained by Miao and Wang (2018, Section I) in a different and more intuitive way. MW already show that the transversality condition is satisfied in their model and there is no need to apply the Bubble Characterization Lemma. Importantly, MW show that the transversality condition cannot rule out "bubbles" studied in MW (2018).

We think the main differences of opinions between us and HT lie in two key issues: (i) the definition of rational bubbles and (ii) the interpretation of the term "bubble " in MW. We now discuss these two issues and other related issues.

2 Rational Bubble in MW

HT do not need to use the Bubble Characterization Lemma or Proposition 1 to show the "nonexistence" of "rational bubbles" (defined in the traditional way) in MW (2018) because MW already point out that the usual transversality condition holds in their model. See equations (6) and (22) of MW (2018).

Let us use the simple discrete time setting in Miao and Wang (2018, Section 1) to explain the intuition. The transversality condition is

$$\lim_{T\to\infty} e^{-rT} V_{t+T} = 0, \tag{1}$$

where $r > 0$ is the discount rate and V_t denotes the cum-dividends stock price, which satisfies the pricing equation

$$V_t = D_t + e^{-r} V_{t+1}, \tag{2}$$

where $D_t > 0$ denotes dividends. Equation (1) corresponds to (5) in the HT paper. By the traditional definition (which is also stated in their comment paper), there is no "rational bubble" at time t if and only if (1) holds.

To understand the last claim, we can decompose V_t into

$$V_t = V_t^* + B_t,$$

where V_t^* is the "fundamental component" satisfying

$$V_t^* = \sum_{s=0}^{\infty} e^{-rs} D_{t+s}, \tag{3}$$

and $B_t \equiv \lim_{T\to\infty} e^{-rT} V_{t+T}$ is the bubble component satisfying

$$B_t = e^{-r} B_{t+1}. \tag{4}$$

Thus the transversality condition (1) implies $B_t = 0$ and rules out bubbles.

Notice that the traditional theory summarized above (or the exposition given in the HT paper) is derived from the asset pricing equation (2) alone without any additional restrictions. In particular, the pricing equation (4) for the bubble B_t naturally follows from (2).

By contrast, in the model of Miao and Wang (2018), dividends D_t are endogenously generated in a production economy with endogenous credit constraints so that B_t follows a pricing equation different from (4). MW show that equation (2) takes a form of the Bellman equation subject to the endogenous credit constraints and several additional constraints. They show that the stock price (or the stock market value of the firm) can be decomposed into

$$V_t = Q_t K_t + B_t, \tag{5}$$

where $Q_t K_t$ is the fundamental component as is well known in the standard Tobin's Q theory and B_t satisfies

$$B_t = e^{-r} B_{t+1} \left(1 + \ell_{t+1}\right), \tag{6}$$

where $\ell_{t+1} > 0$ represents a liquidity premium. MW show that a pure bubble also follows the same pricing equation (6), which resembles the money bubble equation in the monetary theory literature surveyed by Lagos, Rocheteau, and Wright (2017). Thus, MW interpret B_t as a bubble component, which will be discussed further below. They also show that the transversality condition (1) cannot rule out bubbles in (6) in the sense that the model features multiple equilibria. One type of equilibria is the fundamental equilibrium with $B_t = 0$ for all t. The other type is the bubbly equilibrium with $B_t > 0$ for all t. There are also other types, e.g., the stochastic bubble equilibrium. There are two steady states with $B = 0$ and $B > 0$. The bubbleless steady state with $B = 0$ has indeterminacy of degree 1 and the bubbly steady state with $B > 0$ is a saddle point. All these features resemble those in the bubble literature, (e.g. Tirole 1985 and Weil 1987).

3 Definition of Rational Bubble

It is intuitive and generally agreed that an asset bubble is defined as the difference between the asset market price and its fundamental value. The main difficulty is how to define fundamental value. As Tirole (1985, 1091) points out, "in some cases, the usual notion of market fundamental and bubble is not fully satisfactory." He then discusses two reasons why it may be so. The first is related to an illusion in bubble accounting. Tirole (1985) gives an example in which the fundamental value itself is a bubble. The second is related to the distinction between the financial market fundamental and the real market fundamental. Both reasons are relevant for the MW model.

First, dividends are endogenous in the MW model. In footnote 1 of MW (2018), they point out:

> A stock price bubble is defined as the difference between a stock's market value and its fundamental value, e.g., the discounted value of exogenously given dividends in exchange economies (Santos and Woodford 1997). It is subtle to apply this definition to our model because dividends are endogenously generated through investment and production and because bubbles help generate dividends. (MW, 2590 n1)

In the MW model, a firm's borrowing capacity is limited by its own market value. If all agents speculate that firm value contains a bubble component B_t, then the firm can borrow more funds, which can finance more investment, increase dividend payment, and raise firm value. This higher firm value then supports the initial belief that firm value contains a bubble. Because of this positive feedback loop mechanism, B_t satisfies the pricing equation (6). Unlike the traditional pricing equation (4), the bubble component commands a liquidity premium ℓ_{t+1}.

Second, the component $Q_t K_t$ in firm value in equation (5) can be intuitively interpreted as the real fundamental value because this is the traditional fundamental value measured in the Tobin's Q theory of the firm.

We should add a third reason for the difficulty of the traditional definition of the fundamental value. The traditional definition relies on the no-arbitrage pricing equation (2), or more generally, equation (1) in HT:

$$P_t = \frac{1}{\pi_t} E_t \left[\pi_{t+1} \left(P_{t+1} + D_{t+1} \right) \right], \tag{7}$$

where π_t is a state price deflator.[1] Then we have the decomposition,

$$P_t = \lim_{T\to\infty} \frac{1}{\pi_t} E_t \left[\sum_{s=t+1}^{T} \pi_s D_s \right] + \underbrace{\lim_{T\to\infty} \frac{E_t \left[\pi_T P_T\right]}{\pi_t}}_{B_t}, \tag{8}$$

$$B_t = E_t \left[\frac{\pi_{t+1}}{\pi_t} B_{t+1} \right], \tag{9}$$

The first component is the fundamental value and the second component B_t represents the traditional bubble.

The problem is that in many models with market frictions, the no-arbitrage pricing equation (7) may not hold. This equation holds in models with debt (or solvency) constraints and with borrowing constraints (e.g., Miao (2014)). But it does not hold in models with short-sales constraints. Stephen F. LeRoy and Jan Werner (2001) give examples in which there exists arbitrage in models with short-sale constraints. Hua He and David M. Modest (1995) derive asset pricing equations different from (7) in the presence of transactions costs or other trading constraints. One can also give many other examples of models with endogenous borrowing constraints for which the asset pricing equation takes a different form than (7) (e.g., leverage/collateral constraints of Geanakopolos (2003) and Fostel and Geanakopolos (2008), and margin constraints of Garleanu and Pedersen (2011)). For these models, the fundamental value as the present value may not be well defined because the state price π_t or the stochastic discount factor is nontrivial to specify. In particular, the law of one price may fail in these models.

In summary, we think using (3) or (8) to define the fundamental value and the rational bubble has limitations. Adhering solely to this definition is too restrictive, stifling the advancement of innovative theoretical frameworks. We agree with HT that there are many notions of bubbles in the literature. In our view, for rational bubbles, rationality is important in the sense that all agents must have rational expectations. There is no behavioral bias or heterogeneous beliefs. A rational bubble is not backed by any fundamental payoffs and thus satisfies a pricing equation like (4) or (9). It may also satisfy equation like (6) due to the liquidity premium (e.g., the MW model and the Lagos-Wright model). These equations reveal that if $B_{t+1} = 0$, then $B_t = 0$, unlike the stock price equation (2) or (7), for which $P_{t+1} = 0$ does not imply $P_t = 0$ due to the presence of dividends $D_t > 0$ for all t. Moreover, speculation is important

[1] There may exist multiple state price deflators under incomplete markets. See Santos and Woodford (1997) for a discussion.

for the emergence of a rational bubble. If all agents believe or speculate that the asset contains a bubble ($B_{t+1} > 0$), then the bubble may be supported in equilibrium ($B_t > 0$) under certain conditions. The model of Miao and Wang (2018) has all these features.

4 Interpretation of B_t in (6).

There is no confusion in saying the value of a U.S. dollar is a pure rational bubble because a dollar's fundamental value is zero. Any positive value (in terms of Euros, say) of a U.S. dollar (or an intrinsically useless asset) represents a bubble. But even for the money bubble, its pricing equation may not satisfy (4), e.g., in the monetary theory of Lagos Rocheteau, and Wright (2017). In Section V. A of Miao and Wang (2018), they introduce a pure bubble asset like money. They show that the pricing equation satisfies an equation like (6) (see their Proposition 8, 2614). The transversality condition cannot rule out the pure bubble due to the presence of the liquidity premium.

A key contribution of Miao and Wang (2018) is to show that the component B_t in the stock price (5) follows the same pricing equation (6) for the money bubble denoted by P_{mt}, which is different from the traditional equation (4). Based on this fact, it is reasonable to call B_t a bubble component, just like the pure money bubble P_{mt}. Proposition 9 of MW (2018, 2615) also show that the equilibrium can determine the total size of the bubble $B_t + P_{mt}$, but not its composition. Thus, if one agrees that the money bubble P_{mt} is a rational bubble, then B_t should also be. Moreover, equation (6) has an intuitive interpretation based on speculation or in HT's own words "speculation backed by nothing." There is no fundamental payoffs to support (6). If all agents believe that there is no future value for the bubble component, $B_{t+1} = 0$, then it has no value today $B_t = 0$. Thus, a bubbleless equilibrium with $B_t = 0$ for all t can exist. But if all agents speculate that there is future value $B_{t+1} > 0$, then the value can be supported as $B_t > 0$. The bubble component cannot be ruled out by the transversality condition, because it has an additional benefit by providing liquidity to the firm. The liquidity premium ℓ_{t+1} is captured in (6) and is generated by speculation. If people believe $B_{t+1} = 0$, then the liquidity premium ℓ_{t+1} plays no role. In this sense, it is not a fundamental payoff or an exogenously given dividend.

5 Bubble Characterization Lemma

HT use the Bubble Characterization Lemma to show the "nonexistence" of stock price bubble in Miao and Wang (2018). This lemma is basically an implication of the transversality condition for the stock price that satisfies pricing equation (7) in endowment economies. This lemma can rule out bubbles that satisfy equation (9) or (4). But it cannot rule out bubbles that satisfy equation (6) as in the MW model. In the MW model, bubbles allow a firm to borrow more funds so that it can pay more dividends.

Their claim on page 10 that "bubbles in dividend-paying assets can never occur so long as the price-dividend ratio (or the dividend yield) converges to a positive constant" has qualifications. The qualification is that the asset pricing equation takes the form like (7) and the bubbles they defined must satisfy a pricing equation like (9), because the Bubble Characterization Lemma is based on these two equations (see Montrucchio 2004). If one of the equations does not hold, this lemma may not apply.

6 Broader Economic Implications

On page 10, HT claim that

"It is more appropriate to interpret Miao and Wang (2018) and others as multiple equilibria in asset pricing models, where there are two steady states, one with high stock prices and the other with low stock prices. In both steady states, stock prices always reflect fundamentals, but self-fulfilling expectations determine which steady state is reached. In fact, Miao and Wang (2015, 772) state "one may also interpret it as a self-fulfilling component or a belief component if one wants to avoid using the term 'bubble.'"

In fact, the MW model has features closer to the rational bubble model of Tirole (1985). For example, there are two steady states: one is bubblebless with $B = 0$ and the other is bubbly with $B > 0$. The local equilibria around the bubbleless steady state have indeterminacy of degree 1. The bubbly steady state is a saddle point and the local equilibrium around this steady state is unique. The bubble can collapse as in Weil (1987) and there is a stochastic bubble equilibrium.

Of course the MW model is related to the multiple equilibria literature (e.g., Farmer 1999). This literature typically has a unique steady state. The multiplicity is generated by the failure

of the Blanchard-Khan condition in the sense that the number of predetermined variable is less than the number of stable eigenvalues for the linearized model around the steady state. One can broadly interpret that the bubble literature belongs to the multiple equilibria literature because there are typically multiple equilibria in the bubble literature, e.g., the Tirole (1985) model. Moreover, all pure bubble models (e.g., the monetary models of Samuelson (1958) and Lagos, Rocheteau, and Wright (2017)) feature multiple equilibria: one is the monetary equilibrium and the other is the non-monetary (also called fundamental or autarky) equilibrium.

We view that the multiplicity of equilibria is a key strength of rational bubble theory. After all, the literature of rational bubbles is motivated by the apparent excessive fluctuation in asset prices relative to the measured fundamental. In a quantitative study we authored with Zhiwei Xu (2015) show the framework developed by MW can successfully replicate some key moments of stock prices including their excessive volatility and the comovement with the real economy (Miao, Wang and Xu 2015). MW try to be open to different interpretations of a model without affecting its economic substance. The ultimate purpose of a theory or a model is to explain the real world economic phenomena. Simply judging a model's contribution by whether it has used the traditional definition of rational bubble strikes us as unreasonable.

7 Reply to the Open Letter to Miao and Wang

HT's article contains a section "Open Letter to Miao and Wang," with three questions enumerated. Our answer is yes to all questions given the qualification that the rational bubble is defined in the traditional sense. In particular, HT's Lemma 1 is taken from Montrucchio (2004) and HT's Lemma 2 is a continuous time version of Montrucchio (2004). Proposition 1 directly follows from Lemmas 1 and 2. All these results are based on the traditional definition of rational bubble. As we explained in previous sections, the bubble component in MW does not follow the traditional asset pricing equation like (9). It follows a very similar equation to that in the New Monetary Economics literature with a liquidity premium term. As a result, HT's Lemmas 1 and 2, and Proposition 1 do not imply there is no "rational bubble" as defined in MW. However, if one wants to stick to the traditional definition, then there is no "rational bubble" because MW already show that the transversality condition is satisfied. There is no need to use Lemmas 1 and 2 and Proposition 1 in HT to show this simple fact in the MW

model.

8 Open Letter to Hirano and Toda

To support a constructive scientific debate, let us raise some queries to Hirano and Toda.

1. MW (2018) already point out that a rational bubble cannot exist according to the traditional definition by the transversality conditions (see their Section I). The transversality conditions are listed on the top of page 2594, equation (6) on page 2597, equation (22) on page 2602. Are these statements correct? Do you acknowledge that we are already aware of the traditional definition of rational bubble?

2. As mentioned above, MW already show that the transversality conditions rule out rational bubble according to the traditional definition. There is no need to apply the Bubble Characterization Lemma, Lemmas 1 and 2, and Proposition 1 in HT to prove this fact. Is this claim correct? Can the Bubble Characterization Lemma rule out the asset bubble defined in MW (2018) that incorporates liquidity premium?

3. Do you find any mathematical or logical errors in MW (2018)?

4. The key contribution of MW (2018) is to provide a novel pricing equation for the bubble attached to an intrinsically useless asset (e.g., fiat money) or an asset that pays positive dividends (e.g., stocks). It is different from the traditional pricing equation. Is this pricing equation mathematically correct? Do you agree that the pure (money) bubble in MW (2018) is a bubble even though it does not follow the traditional definition of rational bubble? If yes, then why isn't the stock price bubble in MW a bubble given that MW follows the same pricing equation?

5. Your main critique seems to be that MW (2018) should not use the adjective "rational" in front of "bubble" because the bubble in MW (2018) does not follow the traditional pricing equation. Is that correct? As we already explained, our interpretation of "rational" is different. We think it means that individuals are rational or the rational expectations hypothesis is satisfied. It does not mean the bubble should satisfy the traditional pricing equation like (9). To avoid the confusion with the traditional definition, one may call the

rational bubble in MW (2018) the MW-type rational bubble or liquidity-based rational bubble. Would you object to our using "rational bubble " irrespective of any modifiers we include?

9 Conclusion

HT argue that the MW model does not feature a rational bubble according to the traditional definition. They do not find any error in MW and do not provide any new results or new insights into our understandings of the bubble literature. In this reply, we have argued that the traditional definition of rational bubble has limitations for several reasons. Some of the reasons are discussed by Tirole (1985). The traditional definition applies to some models, but not to others with financial or market frictions, e.g., the MW model in which stock price bubbles help generate dividends. The Bubble Characterization Lemma does not apply to the MW model based on the MW definition of bubble. In the MW model, the bubble component satisfies a pricing equation different than the traditional one, and thus the usual transversality condition fails to rule out the bubble component in the MW model. This equation is the same as that for the pure bubble asset like money, which incorporates liquidity premium, so that the interpretation of the bubble component can be justified. The critiques by HT based on the traditional definition of rational bubble do not apply to the MW model, which goes beyond the traditional definition.

Finally, to echo the first paragraph in HT, the development of new theory is often accompanied with proposals of new definitions. From the traditional Arrow-Debreu general equilibrium to the competitive equilibrium with financial frictions, we are all in the general equilibrium framework. Researchers often introduce new assumptions or new elements to the traditional model. As a result, some definitions must be refined and new concepts are introduced. Similarly, for the asset bubble literature, we view the traditional definition of rational bubble to be too restrictive. The meaning of rational bubble does not require it strictly satisfy the traditional asset pricing equation. This is true for the money bubble in the New Monetary Economics literature on liquidity premium (e.g., Lagos, Rocheteau, and Wright 2017) and is also true for the stock price bubble developed by MW (2018). To avoid the confusion with the traditional definition, one may call the rational bubble in MW (2018) the MW-type rational bubble or

liquidity-based rational bubble.

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