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Three types of robust Ramsey problems in a linear-quadratic framework[☆]

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ABSTRACT

This paper studies robust Ramsey policy problems in a general discrete-time linear-quadratic framework when the Ramsey planner faces three types of ambiguity. This framework includes both exogenous and endogenous state variables. In addition, the equilibrium system from the private sector contains both backward-looking and forward-looking dynamics. We provide recursive characterizations and algorithms to solve for robust policy. We apply our method to a basic New Keynesian model of optimal monetary policy with persistent cost-push shocks. We find that (i) all three types of ambiguity make optimal monetary policy more history-dependent but with different reasons for each type; and (ii) they deliver qualitatively different initial responses of inflation and the output gap following a cost-push shock.

1. Introduction

The standard framework of Ramsey policy problems typically adopts the rational expectations hypothesis. Under this hypothesis, a Ramsey planner (or the Stackelberg leader) and a private sector (or the follower) share common beliefs about the state of the world and these beliefs coincide with the objective probability model of the state of the world. The rational expectations hypothesis has benefited economists and policymakers by not only providing sharp predictions but also achieving simplicity through imposing internal coherence of models. Considering the possibility of model misspecification, however, rational expectations become a particularly strong assumption since it is extremely difficult to achieve a coincidence of beliefs among diverse agents with different levels of knowledge about the true models. Following Hansen and Sargent (2001; 2008) and Anderson et al. (2003), one can view economic models as an approximation to the real world. Economic agents do not know the true models and their models may be misspecified. They are averse to model ambiguity and want to seek decision making that is robust to model misspecifications.[1]

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[1] Maccheroni, Marinacci and Rustichini (2006a, 2006b) and Strzalecki (2011) provide axiomatic foundations for the Hansen and Sargent approach. See Gilboa and Schmeidler (1989) for an alternative approach based on the maxmin expected utility model.

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The goal of our paper is to study how the Ramsey planner designs a robust policy in the presence of model ambiguity. In a Ramsey problem, there are two types of agents, the Ramsey planner and the private sector. One has to consider who faces ambiguity and what the agents are ambiguous about. In this paper, we follow Hansen and Sargent (2012) and consider three types of ambiguity. For all these types, only the Ramsey planner faces ambiguity.[2] Type I ambiguity refers to the case where the Ramsey planner has a concern for robustness about both the exogenous shock processes and expectations of the private sector. The planner chooses a robust policy based on the same distorted beliefs used by itself and the private sector. For type II ambiguity, the Ramsey planner does not trust its approximating model of the exogenous shock processes, but the private agents trust this approximating model. Finally, for type III ambiguity originally suggested by Woodford (2010), the planner fully trusts its approximating model of the exogenous shock processes. But it does not have full confidence about the private agents' beliefs. An important implication of these three types of ambiguity is that types II and III generate endogenous belief heterogeneity, while type I does not.

To solve the three types of robust Ramsey problem corresponding to the three types of ambiguity, we adopt a recursive formulation using the recursive saddle point method of Marcet and Marimon (2011). The key idea is to incorporate the Lagrange multiplier associated with the forward-looking constraints as a state variable. For the robust Ramsey problem, we also incorporate the planner's belief distortion (formally, the Radon–Nikodym derivative) as a state variable. For type I robust Ramsey problem, we show that the value function has a homogeneity property so that we can reduce the dimension of the state space. By suitably transforming the problem, we characterize the robust policy using the standard linear-quadratic method. In particular, we prove that the robust policy is a linear function of the predetermined state variable and the Lagrange multiplier, like in the standard Ramsey problem with rational expectations.

For types II and III robust Ramsey problems, we can also reduce the dimension of the state space by defining a belief-adjusted Lagrange multiplier as a state variable. However, we cannot transform types II and III robust Ramsey problem into a standard linear-quadratic form. In particular, the robust policy does not have a linear solution. We have to use a nonlinear solution method. We apply the second-order approximation around the non-stochastic steady state to derive numerical solutions (Schmitt-Grohe and Uribe, 2014). We implement this method using the Dynare software.[3]

We apply our methods to a basic New Keynesian model of optimal monetary policy with persistent cost-push shocks. We solve for robustly optimal monetary policies under three types of ambiguity and compare these policies with the standard optimal policy under rational expectations. In order to make comparisons of the three types of robust policy, we have to calibrate the robustness parameter consistently in the three types of robust Ramsey problem. We use the same detection error probability for discriminating between the approximating model and the endogenous worst-case model associated with a particular robustness parameter in each of the three types of robust Ramsey problem to calibrate this parameter.

Following Hansen and Sargent (2008), we compute the detection error probability using likelihood ratio tests. Since type I robust Ramsey problem yields a linear solution, we can use the Kalman filter to obtain the likelihood function. For types II and III robust Ramsey problem, however, we cannot use the Kalman filter since solutions are in a nonlinear form. Instead, we use particle filtering to compute the likelihood.

We find that all three types of ambiguity make the robustly optimal monetary policy more history-dependent than in the case of rational expectations, in line with Hansen and Sargent (2012); Woodford (2010), and Kwon and Miao (2012). Woodford (2000) points out that optimal commitment monetary policy under rational expectations is history-dependent. That is, it not only depends on the current state of the economy but also responds to past states. The intuition is that a history-dependent policy can affect the expectation of private agents and thus improves the performance of monetary policy. Why does optimal monetary policy under ambiguity become more history-dependent? Economic intuitions behind this feature are not the same across the three types of robust Ramsey problem. In type I robust Ramsey problem, increased history-dependence comes from the fact that the central bank (henceforth, CB) is concerned about the distortion of the cost-push shock. In types II and III robust Ramsey problem, the CB's incentive to better manage the expectations of the private sector is a major source of more history-dependent monetary policy. Under the worst-case beliefs, the CB and the private sector have disparate expectations so that the CB becomes more cautious to affect appropriately the private sector's expectations. Reflecting the CB's caution, the robustly optimal monetary policy becomes more history-dependent.

Even though three types of ambiguity share the property that inflation dynamics become more history-dependent, their implications for prices are different. Under the rational expectations hypothesis, optimal monetary policy implies the CB undoes all the effect of a cost-push shock and thus prices go back to the original level. However, this no longer applies to types II and III robust Ramsey problem. We find that the CB adjusts inflation more than under rational expectations and hence prices go below the original level in the long run. In these two types of robust Ramsey problem, the fact that the CB and the private sector have heterogeneous beliefs makes the dynamics of the price level deviate from that under rational expectations. By contrast, in type I robust Ramsey problem, both the CB and the private section share the same distorted beliefs about the cost-push shock, causing prices to go back to the original level in the long run.

We also show that the initial responses of inflation and output gap to a positive cost-push shock are disparate for different types of ambiguity. In type I robust Ramsey problem, the CB increases the initial responses of both inflation and

[2] See Karantounias (2013) and Orlik and Presno (2012) for the modeling of the case where the private agents face ambiguity but the policymaker does not.

[3] See Adjemian et al. (2011).

output. Under the worst-case beliefs, the CB worries that the cost-push shock is distorted in mean so that the CB responds as if the shock were greater compared to a shock in the approximating model. In type II robust Ramsey problem, while the CB increases the initial response of inflation, it decreases the initial response of output. The CB worries about the unfavorable distortion in the cost-push shock, which leads to an increase in the inflation response. On the other hand, the CB exploits the fact that the private sector fully trusts its approximating model. The cost-push shock in this model is believed less persistent than in the worst-case distorted model. As a result, the CB faces a smaller tradeoff between inflation and output. In type III robust Ramsey problem, the CB's initial response of inflation to a cost-push shock is less sensitive but the output responds more sensitively. The concern for robustness of the expectations of the private sector makes the CB manage the inflation expectations more cautiously. Thus, the CB faces a larger tradeoff. This result is in line with Woodford (2010) and Kwon and Miao (2012), even though they study type III ambiguity in the timeless perspective instead of the Ramsey framework.[4]

Our paper is closely related to Hansen and Sargent (2012). Hansen and Sargent study three types of ambiguity in a continuous-time basic New Keynesian model of monetary policy. One of our contributions is to extend their idea of three types of ambiguity to a Ramsey problem in a general discrete-time linear-quadratic framework. This framework includes both exogenous and endogenous state variables. In addition, the equilibrium system from the private sector contains both backward-looking and forward-looking dynamics. Their methods cannot be readily applied to our general framework. Our main contribution is to provide recursive characterizations and algorithms to solve for the robustly optimal policy. A nice feature of our recursive characterizations is that one can obtain numerical solutions easily using the standard perturbation method up to third-order approximations implemented by Dynare. As Aruoba et al. (2006) argue, the perturbation method can deliver a quite accurate solution and is an attractive compromise between accuracy, speed, and programming burden.

As a second contribution, we make comparisons of the three types of robust Ramsey problem by calibrating the context-specific robustness parameter using the same detection error probability, unlike Hansen and Sargent (2012) who use the same value of the robustness parameter for all three types of problems. Importantly, our finding of the differences in impulse responses is new and absent from their study.

Our paper is also related to Woodford (2010) and Kwon and Miao (2012). Woodford (2010) studies type III ambiguity using a basic New Keynesian model in the timeless perspective. Kwon and Miao (2012) generalize the Woodford model to a general linear-quadratic framework. They all show that robustly optimal policy in the timeless perspective is linear. However, the present paper shows that the robust Ramsey policy is not linear. One has to use a nonlinear method to derive numerical solutions.

The measure of model discrepancy in type III problem follows Woodford's (2010) formulation, which is different from the discounted relative entropy in Hansen and Sargent (2008). Motivated by the discussion on Woodford (2010) by Hansen and Sargent (2012), we also study a modified type III problem by adopting the relative entropy penalty term of Hansen and Sargent (2008). We find that numerical results are similar in our monetary policy application.

Finally, our paper is related to Anderson et al. (2003), Walsh (2004), Giordani and Söderlind (2004), Hansen and Sargent (2008, Chapter 16), Leitemo and Söderström (2008), Dennis (2008), and Olalla and Gomez (2011). The robust Ramsey models studied in these papers are similar to our type I problem. These papers introduce perturbations of the mean of the exogenous shock processes into the backward- and forward-looking constraints and a quadratic penalty into the objective function. As Hansen and Sargent (2012) point out, these models admit a better interpretation when described as type I ambiguity.

2. A linear-quadratic framework

2.1. Uncertainty and beliefs

Uncertainty is generated by a stochastic process of shocks $\{\varepsilon_t\}_{t=1}^{\infty}$ where ε_t is an $n_\varepsilon \times 1$ vector of independently and identically distributed standard normal random variable. Let's define $\varepsilon^t = \{\varepsilon_1, \ldots, \varepsilon_t\}$. At date t, both the Ramsey planner and the private sector have common information generated by ε^t and some initial state x_0. They may not have rational expectations in that their subjective beliefs may not coincide with the objective probability distribution governing exogenous shocks $\{\varepsilon_t\}_{t=1}^{\infty}$. One reason is that economic agents view their model as an approximation and thus may be concerned about model misspecification.

Model misspecification is described by a perturbation to the distribution of shocks. We follow Hansen and Sargent (2008) to represent probability distortions. Let $p(\varepsilon)$ denote the standard normal density of ε_t. Let $\mathbf{\Pi}$ and $\mathbf{\Pi}_t$ denote the induced distribution over the full state space and the induced joint distribution of ε^t, respectively. Assume that a distorted distribution is absolutely continuous with respect to the reference distribution $\mathbf{\Pi}$. We can then representing the belief distortion by Radon–Nikodym derivatives.

[4] Unlike in the Ramsey framework, there is an initial commitment for the planner and the planner choice must be self-consistent in the timeless perspective (see Woodford, 2010 for a further discussion).

Let $\hat{p}(\varepsilon|\varepsilon^t, x_0)$ denote an alternative one-step-ahead density for ε_{t+1} conditional on date t information. Form the likelihood ratio or the Radon–Nikodym derivative for one-step-ahead distributions:

$$m_{t+1} = \frac{\hat{p}\left(\varepsilon|\varepsilon^t, x_0\right)}{p(\varepsilon)}.$$

It satisfies the property

$$E_t[m_{t+1}] = 1, \tag{1}$$

where E_t denotes the conditional expectation operator with respect to the reference distribution $\mathbf{\Pi}$ given date t information. Recursively define a martingale $\{M_t\}$:

$$M_{t+1} = m_{t+1} M_t, \quad M_0 = 1. \tag{2}$$

M_t is a likelihood ratio of the joint densities of ε^t conditional on the initial information x_0 or the Radon–Nikodym derivative for joint distributions.

Following Hansen and Sargent (2008), we use relative entropy to measure the discrepancy between the distorted distribution and the reference distribution. Define the relative entropy (conditional on date zero information) of the distorted distribution associated with M_t over date t information as $E_0[M_t \ln M_t]$. Define the discounted entropy over an infinite horizon as

$$(1-\beta)E_0 \sum_{t=0}^{\infty} \beta^t M_t \ln M_t = \beta E_0 \sum_{t=0}^{\infty} \beta^t M_t E_t(m_{t+1} \ln m_{t+1}), \tag{3}$$

where $\beta \in (0, 1)$ is a discount factor and we have used (2) to derive the equality. Model ambiguity is described by a set of joint densities $\{M_t\}_{t=0}^{\infty}$ satisfying the following constraint:

$$\beta E_0 \sum_{t=0}^{\infty} \beta^t M_t E_t(m_{t+1} \ln m_{t+1}) \le \eta, \tag{4}$$

for some $\eta > 0$.

Woodford (2010) introduces a different measure of intertemporal entropy. First, he defines the conditional relative entropy of a one-step-ahead distribution given date t information as $E_t[m_{t+1} \ln m_{t+1}]$. He then defines the expected discounted entropy conditional on date zero information as

$$E_0 \sum_{t=0}^{\infty} \beta^t [E_t(m_{t+1} \ln m_{t+1})] = E_0 \sum_{j=0}^{\infty} \beta^t m_{t+1} \ln m_{t+1}. \tag{5}$$

In this case model ambiguity is described by a set of one-step-ahead densities $\{m_t\}_{t=1}^{\infty}$ satisfying the constraint

$$E_0 \sum_{t=0}^{\infty} \beta^t m_{t+1} \ln m_{t+1} \le \eta_0 \tag{6}$$

for some $\eta_0 > 0$.

2.2. Three types of robust Ramsey problem

Suppose that the equilibrium system from the private sector can be summarized by the following form:

$$\begin{bmatrix} I & 0 \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x_{t+1} \\ \hat{E}_t y_{t+1} \end{bmatrix} = \hat{A} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \hat{B} u_t + \begin{bmatrix} \widehat{C}_x \\ 0 \end{bmatrix} \varepsilon_{t+1}, \tag{7}$$

where $x_0 = \bar{x}_0$ is exogenously given and $\hat{E}_t$ denotes the conditional expectation operator given date t information based on the common beliefs of the private sector. The private sector's beliefs may not coincide with the "objective" probability distribution for $\{\varepsilon_t\}$, the reference distribution $\mathbf{\Pi}$. Here, x_t is an $n_x \times 1$ vector of predetermined variables in the sense defined in Klein (2000), y_t is an $n_y \times 1$ vector of non-predetermined or forward-looking variables, and u_t is an $n_u \times 1$ vector of instrument or control variables chosen by the Ramsey planner. We typically use x_t to represent the state of the economy, which may include productivity shocks, preference shocks, or capital stock. Note that x_t may include a component of unity in order to handle constants. The vector y_t represents endogenous variables such as consumption, inflation rate, and output. Examples of instruments u_t include interest rates and money growth rates. The equation for x_t is backward looking and represents the law of motion of state variables. The equation for y_t is forward looking and typically represents the first-order conditions from intertemporal optimization such as Euler equations.

All matrices in (7) are conformable. For simplicity, we suppose that the matrix on the left side of Eq. (7) is invertible,[5] so that we can multiply both sides of this equation by its inverse to obtain the system

$$\begin{bmatrix} x_{t+1} \\ \hat{E}_t y_{t+1} \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} B_x \\ B_y \end{bmatrix} u_t + \begin{bmatrix} C_x \\ 0 \end{bmatrix} \varepsilon_{t+1}, \tag{8}$$

where we have partitioned matrices conformably.

The Ramsey planner has the period loss function

$$L(x_t, y_t, u_t) = \frac{1}{2}\left[x_t', y_t'\right] \begin{bmatrix} Q_{xx}, & Q_{xy} \\ Q_{xy}' & Q_{yy} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \frac{1}{2} u_t' R u_t + \left[x_t', y_t'\right] \begin{bmatrix} S_x \\ S_y \end{bmatrix} u_t,$$

where the matrices R and

$$Q \equiv \begin{bmatrix} Q_{xx}, & Q_{xy} \\ Q_{xy}' & Q_{yy} \end{bmatrix}$$

are symmetric. In addition, suppose that Q is positive semidefinite.

If both the private sector and the Ramsey planner have rational expectations, then they have common beliefs which coincide with $\mathbf{\Pi}$, which is also the true distribution governing exogenous shocks $\{\varepsilon_t\}$. In this case, the Ramsey problem is given by

$$\max_{\{x_t, y_t, u_t\}} \; -E_0 \sum_{t=0}^{\infty} \beta^t L(x_t, y_t, u_t), \tag{9}$$

subject to (8) in which the conditional expectation operator $\hat{E}_t$ is equal to E_t, the conditional expectation operator with respect to $\mathbf{\Pi}$.

There is ample experimental and empirical evidence that documents the violation of the rational expectations hypothesis. We consider three approaches to the modeling of the departure from rational expectations in the policy analysis. These three approaches give rise to three types of robust Ramsey policy problem, corresponding to the three types of ambiguity analyzed by Hansen and Sargent (2012) in a continuous-time framework. In these problems, the Ramsey planner believes that the private sector experiences no model ambiguity. But the planner experiences ambiguity. They differ in what the planner is ambiguous about and how she evaluates the private sector's beliefs about the exogenous shocks.

Type I robust Ramsey problem. In this type of problem, the Ramsey planner has a set of models (probability distributions) centered on the reference model $\mathbf{\Pi}$, or the so called "approximating model" by Hansen and Sargent (2008). The Ramsey planner is uncertain about both the evolution of the exogenous processes and how the private sector views these processes. The planner thinks that the private sector knows a model that is distorted relative to the planner's approximating model. To cope with its ambiguity, the Ramsey planner chooses a worst-case model among a set containing the reference approximating model, while evaluating the private sector's forward-looking equations using that model. Formally, the Ramsey planner chooses $\{m_t\}$ to minimize and $\{x_t, y_t, u_t\}$ to maximize a multiplier criterion in the following program:[6]

$$\max_{\{x_t, y_t, u_t\}} \min_{\{m_{t+1}\}} \; -E_0 \sum_{t=0}^{\infty} \beta^t M_t L(x_t, y_t, u_t) + \beta\theta E_0 \sum_{t=0}^{\infty} \beta^t M_t m_{t+1} \ln m_{t+1}, \tag{10}$$

subject to (1), (2) and

$$\begin{bmatrix} x_{t+1} \\ E_t[m_{t+1} y_{t+1}] \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} B_x \\ B_y \end{bmatrix} u_t + \begin{bmatrix} C_x \\ 0 \end{bmatrix} \varepsilon_{t+1}. \tag{11}$$

The parameter $\theta > 0$ penalizes martingales $\{M_t\}$ with large relative entropies defined in (3). It may be regarded as the Lagrange multiplier for the constraint (4). Following Hansen and Sargent (2001),Hansen and Sargent (2008), instead of solving for the constraint problem subject to (4), we treat θ as a parameter, which measures the planner's degree of concern for possible departures from rational expectations, with a small value of θ implying a great degree of concern for robustness, while a large value of θ implies that only modest departures from rational expectations are considered plausible. When $\theta \to \infty$, the rational expectations analysis is obtained as a limiting case.

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5 The singular case can be handled by the QZ decomposition method, e.g., Klein (2000) and Sims (2002).

6 See Hansen and Sargent (2001),Hansen and Sargent (2008), Maccheroni, Marinacci and Rustichini (2006a), Maccheroni, Marinacci and Rustichini (2006b), and Strzalecki (2011) for interpretations and axiomatic foundations.

Type II robust Ramsey problem. In this type of problem, in the spirit of Hansen and Sargent (2008, Chapter 16), the Ramsey planner has a set of models surrounding an approximating model that the private sector completely trusts. The private sector's beliefs are represented by the Ramsey planner's approximating probability model $\mathbf{\Pi}$. The Ramsey planner chooses a worst-case probability model from its set of models, while evaluating the forward-looking equations for the private sector using the approximating model. Formally, type II Ramsey problem is described by

$$\max_{\{x_t, y_t, u_t\}} \min_{\{m_{t+1}\}} \; -E_0 \sum_{t=0}^{\infty} \beta^t M_t L(x_t, y_t, u_t) + \theta \beta E_0 \sum_{t=0}^{\infty} \beta^t M_t m_{t+1} \ln m_{t+1}, \tag{12}$$

subject to (1), (2) and

$$\begin{bmatrix} x_{t+1} \\ E_t \; y_{t+1} \end{bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} B_x \\ B_y \end{bmatrix} u_t + \begin{bmatrix} C_x \\ 0 \end{bmatrix} \varepsilon_{t+1}. \tag{13}$$

The interpretation of the parameter $\theta > 0$ is the same as in type I robust Ramsey problem. The objective functions in types I and II problems are the same. But the constraints in (11) and (13) are different.

Type III robust Ramsey problem. This type of problem is based on Woodford (2010). The Ramsey planner is assumed to have a single model of the exogenous processes $\{\varepsilon_t\}$ and thus no ambiguity along this dimension. Nevertheless, the planner faces ambiguity because it knows only that the private sector's model is within a set of probability models surrounding its own model. The Ramsey planner evaluates the private sector's forward-looking equation using a worst-case model and solves the following problem:

$$\max_{\{x_t, y_t, u_t\}} \min_{\{m_{t+1}\}} \; -E_0 \sum_{t=0}^{\infty} \beta^t L(x_t, y_t, u_t) + \theta E_0 \sum_{t=0}^{\infty} \beta^t m_{t+1} \ln m_{t+1}, \tag{14}$$

subject to (1) and (11). Unlike in type I robust Ramsey problem, here the parameter $\theta > 0$ penalizes one-step-ahead densities $\{m_t\}$ with the discounted entropy defined in (5). It may be regarded as the Lagrange multiplier for the constraint (6). Following Hansen and Sargent (2001),Hansen and Sargent (2008), instead of solving for the constraint problem subject to (6), we treat θ as a parameter, which measures the planner's degree of concern for possible departures from rational expectations, with a small value of θ implying a great degree of concern for robustness, while a large value of θ implies that only modest departures from rational expectations are considered plausible. When $\theta \to \infty$, the rational expectations analysis is obtained as a limiting case.

Type III$^+$ Robust Ramsey Problem. Hansen and Sargent (2012) argue that Woodford's measure of model discrepancy in (5) when formulated in continuous time is not the usual relative entropy used in the literature. We now use the entropy constraint in (4) and replace the penalty term in (14) with that in (10) or (12). We then obtain the following type III$^+$ robust Ramsey problem:

$$\max_{\{x_t, y_t, u_t\}} \min_{\{m_{t+1}\}} \; -E_0 \sum_{t=0}^{\infty} \beta^t L(x_t, y_t, u_t) + \beta \theta E_0 \sum_{t=0}^{\infty} \beta^t M_t m_{t+1} \ln m_{t+1}, \tag{15}$$

subject to (1), (2) and (11).

3. Type I robust Ramsey problem

3.1. Recursive formulation

Following Marcet and Marimon (2011) and Hansen and Sargent (2012), we characterize type I robust Ramsey problem in a recursive form. First, define the Lagrangian expression for (10) as

$$E_0 \sum_{t=0}^{\infty} \beta^t \{ M_t [-L(x_t, y_t, u_t) + \beta \theta m_{t+1} \ln m_{t+1}] \}$$
$$-E_0 \sum_{t=0}^{\infty} \beta^t M_t \mu'_{yt} (E_t [m_{t+1} y_{t+1}] - A_{yx} x_t - A_{yy} y_t - B_y u_t),$$

where $\beta^t M_t \mu_{yt}$ is the Lagrange multiplier associated with the forward-looking equation in (11) and the law of motion of the state variable is given by the upper block of Eq. (11), i.e.,

$$x_{t+1} = A_{xx} x_t + A_{xy} y_t + B_x u_t + C_x \varepsilon_{t+1}. \tag{16}$$

Then, introduce a new variable

$$\lambda_{yt+1} = \mu_{yt}, \tag{17}$$

and rewrite the above Lagrangian expression as

$$E_0 \sum_{t=0}^{\infty} \beta^t \{ M_t(-L(x_t, y_t, u_t) + \beta\theta m_{t+1} \ln m_{t+1}) + M_t \mu'_{yt}(A_{yx}x_t + A_{yy}y_t + B_y u_t) - \beta^{-1} M_t \lambda'_{yt} y_t \},$$

where we have used $M_{t+1} = M_t m_{t+1}$. Note that at time zero, we set $\lambda_{y0} = \mu_{y,-1} = 0$.

Now we are ready to write type I robust Ramsey problem in a recursive form

$$W(x_t, \lambda_{yt}, M_t) = \max_{y_t, u_t} \min_{m_{t+1}, \mu_{yt}} r(x_t, y_t, u_t, \lambda_{yt}, \mu_{yt}, M_t) + \beta E_t\left[W\left(x_{t+1}, \lambda_{yt+1}, M_{t+1}\right) + \theta M_t m_{t+1} \ln m_{t+1}\right],$$

subject to (1), (2), (16), and (17), where

$$r(x_t, y_t, u_t, \lambda_{yt}, \mu_{yt}, M_t) = -M_t L(x_t, y_t, u_t) + M_t \mu'_{yt}(A_{yx}x_t + A_{yy}y_t + B_y u_t) - \beta^{-1} M_t \lambda'_{yt} y_t.$$

Note that W satisfied the following linear homogeneity property:

$$W(x_t, \lambda_{yt}, M_t) = M_t V(x_t, \lambda_{yt}),$$

for some function V. We then derive

$$V(x_t, \lambda_{yt}) = \max_{y_t, u_t} \min_{m_{t+1}, \mu_{yt}} r(x_t, y_t, u_t, \lambda_{yt}, \mu_{yt}) + \beta E_t\left[m_{t+1} V\left(x_{t+1}, \lambda_{yt+1}\right) + \theta m_{t+1} \ln m_{t+1}\right], \tag{18}$$

subject to (1), (16), and (17), where

$$r(x_t, y_t, u_t, \lambda_{yt}, \mu_{yt}) = -L(x_t, y_t, u_t) + \mu'_{yt}(A_{yx}x_t + A_{yy}y_t + B_y u_t) - \beta^{-1} \lambda'_{yt} y_t.$$

This problem is a Robust control problem with backward-looking constraints. The state variables are (x_t, λ_{yt}) and the control variables are $(y_t, u_t, \mu_{yt}, m_{t+1})$.

As is well known from robust control theory (Hansen and Sargent, 2008), there is a connection to risk-sensitive control. We now derive this connection. From the first-order condition with respect to m_{t+1}, we can show that

$$m_{t+1} = \frac{\exp\left(\frac{-V\left(x_{t+1}, \lambda_{yt+1}\right)}{\theta}\right)}{E_t\left[\exp\left(\frac{-V\left(x_{t+1}, \lambda_{yt+1}\right)}{\theta}\right)\right]}, \tag{19}$$

where E_t denotes the conditional expectation operator given the state (x_t, λ_{yt}). This equation gives the worst-case density. Substituting it back to the preceding Bellman equation yields

$$V(x_t, \lambda_{yt}) = \max_{y_t, u_t} \min_{\mu_{yt}} r(x_t, y_t, u_t, \lambda_{yt}, \mu_{yt}) + \beta \mathcal{R}_t(V)\left(x_{t+1}, \lambda_{yt+1}\right), \tag{20}$$

subject to (16) and (17), where

$$\mathcal{R}_t(V)\left(x_{t+1}, \lambda_{yt+1}\right) = -\theta \ln\left[E_t \exp\left(\frac{-V\left(x_{t+1}, \lambda_{yt+1}\right)}{\theta}\right)\right].$$

The right-hand side of (20) is the objective function in a risk-sensitive control problem. It is not a standard risk-sensitive control problem because it involves both maximization and minimization. However, since first-order conditions are identical for both maximization and minimization, this problem can be solved using the method described in Hansen and Sargent (2008). In particular, the decision rule is linear and the value function is quadratic in terms of state variables.

3.2. Solution method

Define the new state vector $x_t^{*\prime} = (x'_t, \lambda'_{yt})$ and the new control vector $u_t^{*\prime} = (y'_t, u'_t, \mu'_{yt})$. We can then write the state transition equation as

$$x^*_{t+1} = A^* x^*_t + B^* u^*_t + C^* \varepsilon_{t+1}, \tag{21}$$

where

$$A^* = \begin{bmatrix} A_{xx} & 0 \\ 0 & 0 \end{bmatrix}, \; B^* = \begin{bmatrix} A_{xy} & B_x & 0 \\ 0 & 0 & I \end{bmatrix}, \; C^* = \begin{bmatrix} C_x \\ 0 \end{bmatrix}.$$

Conjecture that the value function takes the following form:

$$V(x_t, \lambda_{yt}) = -\frac{1}{2} x_t^{*\prime} P x_t^* - \frac{1}{2} d, \tag{22}$$

where P and d are to be determined. By (19) and (22), the worst-case likelihood ratio m_{t+1}^* satisfies

$$m_{t+1}^* \propto \exp\left[\frac{1}{2\theta}\varepsilon_{t+1}' C^{*\prime} P C^* \varepsilon_{t+1} + \frac{1}{\theta}\varepsilon_{t+1}' C^{*\prime} P (A^* x_t^* + B^* u_t^*)\right],$$

where $\propto$ means "proportional". Thus, the worst-case density satisfies

$$p_{t+1}^* = p_{t+1} m_{t+1}^* \propto \exp\left[-\frac{1}{2}\varepsilon_{t+1}'\left(I - \frac{1}{\theta}C^{*\prime}PC^*\right)\varepsilon_{t+1} \right.$$
$$\left. + \varepsilon_{t+1}'\left(I - \frac{1}{\theta}C^{*\prime}PC^*\right)\left(\theta I - C^{*\prime}PC^*\right)^{-1} C^{*\prime} P (A^* x_t^* + B_t^* u_t^*)\right].$$

This implies that p_{t+1}^* is also a normal density with mean $(\theta I - C^{*\prime}PC^*)^{-1} C^{*\prime} P (A^* x_t^* + B^* u_t^*)$ and covariance matrix $(I - \theta^{-1} C^{*\prime}PC^*)^{-1}$. In this computation we must assume that the matrix $(I - \theta^{-1}C^{*\prime}PC^*)$ is nonsingular.

Given (22), we can compute that

$$\mathcal{R}_t(V)\left(x_{t+1}^*\right) = -\theta \ln\left[E_t \exp\left(\frac{1}{2\theta} x_{t+1}^{*\prime} P x_{t+1}^* + \frac{1}{2\theta} d\right)\right]$$
$$= -\frac{1}{2}(A^* x_t^* + B^* u_t^*)'\left[P + PC^*\left(\theta I - C^{*\prime}PC^*\right)^{-1} C^{*\prime} P\right](A^* x_t^* + B^* u_t^*)$$
$$-\frac{\theta}{2}\ln\det\left(I - \frac{1}{\theta}C^{*\prime}PC^*\right)^{-1} - \frac{d}{2}.$$

Substituting this equation into (20), we can see that the objective function is quadratic. Given the linear constraint (21), the optimized value will be quadratic and the decision rule will be linear. This verifies the conjecture in (22). Matching coefficients in the Bellman equation determines the solution for P, d and decision rules.

Instead of using this method, we solve another robust control problem. To this end, define the return function as

$$r^*(x_t^*, u_t^*) = -\frac{1}{2} x_t^{*\prime} Q^* x_t^* - \frac{1}{2} u_t^{*\prime} R^* u_t^* - x_t^{*\prime} S^* u_t^*,$$

where

$$Q^* = \begin{bmatrix} Q_{xx} & 0 \\ 0 & 0 \end{bmatrix}, \; R^* = \begin{bmatrix} Q_{yy} & S_y & -A_{yy}' \\ S_y' & R & -B_y' \\ -A_{yy} & -B_y & 0 \end{bmatrix}, \; S^* = \begin{bmatrix} Q_{xy} & S_x & -A_{yx}' \\ \beta^{-1} I & 0 & 0 \end{bmatrix}.$$

The new control problem is given by

$$\max_{\{y_t, u_t\}} \min_{\{w_{t+1}, \mu_{yt}\}} E\sum_{t=0}^{\infty}\beta^t r^*(x_t^*, u_t^*) + \frac{\theta}{2} E\sum_{t=0}^{\infty}\beta^t w_{t+1}' w_{t+1}, \tag{23}$$

subject to

$$x_{t+1}^* = A^* x_t^* + B^* u_t^* + C^*(\varepsilon_{t+1} + w_{t+1}).$$

Adapting the arguments in Hansen and Sargent (2008), we can derive the following result. We omit its proof.

Proposition 1. *The decision rule and the value function for problem (23) are of the following form:*[7]

$$u_t^* = -F_u x_t^*, \;\; V^*(x_t^*) = -\frac{1}{2} x_t^{*\prime} P x_t^* - \frac{1}{2} d^*,$$

for some matrices F_u and P and constant d^*. The decision rule and the matrix P are the same as those derived from problem (18) or (20). The solution w_{t+1}^* to the problem (23) gives the worst-case mean distortion derived from problem (18),

$$w_{t+1}^* = \left(\theta I - C^{*\prime}PC^*\right)^{-1} C^{*\prime} P (A^* x_t^* + B_t^* u_t^*),$$

where $u_t^* = -F_u x_t^*$.

[7] Note that d^* is not equal to d.

We use the method proposed by Giordani and Söderlind (2004) to solve problem (23). Specifically, we rewrite this problem as a linear quadratic control problem

$$\max_{\{y_t,u_t\}}\min_{\{w_{t+1},\mu_{yt}\}} \; -\frac{1}{2}E\sum_{t=0}^{\infty}\beta\left(x_t^{*\prime}Q^*x_t^* + \frac{1}{2}\tilde{u}_t^{*\prime}\tilde{R}^*u_t^* + x_t^{*\prime}\tilde{S}^*u_t^*\right),$$

subject to

$$x_{t+1}^* = A^*x_t^* + \tilde{B}^*\tilde{u}_t^* + C^*\varepsilon_{t+1},$$

where

$$\tilde{R}^* = \begin{bmatrix} R^* & 0 \\ 0 & -\theta I \end{bmatrix},\; u_t^* = \begin{bmatrix} u_t^* \\ w_{t+1} \end{bmatrix},$$
$$\tilde{S}^* = \begin{bmatrix} S^* & 0 \end{bmatrix},\; \tilde{B}^* = \begin{bmatrix} B^* & C^* \end{bmatrix}.$$

Because the first-order conditions are the same for maximization and for minimization, we can use a standard method for solving a standard linear-quadratic control problem. The solution is of the following form:

$$x_{t+1}^* = \begin{bmatrix} x_{t+1} \\ \lambda_{yt+1} \end{bmatrix} = H\begin{bmatrix} x_t \\ \lambda_{yt} \end{bmatrix} + \begin{bmatrix} C_x \\ 0 \end{bmatrix}\varepsilon_{t+1},$$

$$\tilde{u}_t^* = \begin{bmatrix} y_t \\ u_t \\ \mu_{yt} \\ w_{t+1} \end{bmatrix} = -F\begin{bmatrix} x_t \\ \lambda_{yt} \end{bmatrix},$$

for some matrices H and F.

4. Type II robust Ramsey problem

4.1. Recursive formulation

We follow a similar strategy to derive a recursive formulation of type II robust Ramsey problem. We first construct the Lagrangian expression for (12)

$$E_0\sum_{t=0}^{\infty}\beta^t\{M_t[-L(x_t,y_t,u_t) + \beta\theta m_{t+1}\ln m_{t+1}]$$
$$-M_t\mu_{yt}'(E_t[y_{t+1}] - A_{yx}x_t - A_{yy}y_t - B_yu_t)\},$$

where $M_t\mu_{yt}$ is the Lagrange multiplier associated with the forward-looking equation in (11). We then define λ_{yt} as in (17) and rewrite the above Lagrangian as

$$E_0\sum_{t=0}^{\infty}\beta^t\{M_t[-L(x_t,y_t,u_t) + \beta\theta m_{t+1}\ln m_{t+1}]$$
$$+M_t\mu_{yt}'(A_{yx}x_t + A_{yy}y_t + B_yu_t) - \beta^{-1}M_{t-1}\lambda_{yt}'y_t\}.$$

At time zero, we set $\lambda_{y0} = \mu_{y,-1} = 0$.

Now, we can derive a recursive formulation of type II robust Ramsey problem:

$$W(x_t,\lambda_{yt},M_t) = \max_{y_t,u_t}\min_{m_{t+1},\mu_{yt}} \; r(x_t,y_t,u_t,\lambda_{yt},\mu_{yt},M_t)$$
$$+\beta E_t\left[W\left(x_{t+1},\lambda_{yt+1},M_{t+1}\right) + \theta M_t m_{t+1}\ln m_{t+1}\right],$$

subject to (1), (2), (16), and (17), where

$$r(x_t,y_t,u_t,\lambda_{yt},\mu_{yt},M_t)$$
$$= -M_tL(x_t,y_t,u_t) + M_t\mu_{yt}'(A_{yx}x_t + A_{yy}y_t + B_yu_t) - \beta^{-1}M_{t-1}\lambda_{yt}'y_t.$$

Let

$$W(x_t,\lambda_{yt},M_t) = M_tV(x_t,\lambda_{yt},m_t).$$

We the have

$$V(x_t, \lambda_{yt}, m_t) = \max_{y_t, u_t} \min_{m_{t+1}, \mu_{yt}} r(x_t, y_t, u_t, \lambda_{yt}, \mu_{yt}, m_t) + \beta E_t\left[m_{t+1} V\left(x_{t+1}, \lambda_{yt+1}, m_{t+1}\right) + \theta m_{t+1} \ln m_{t+1}\right],$$

subject to (1) and (16), where

$$r(x_t, y_t, u_t, \lambda_{yt}, \mu_{yt}, m_t) = -L(x_t, y_t, u_t) + \mu_{yt}'(A_{yx}x_t + A_{yy}y_t + B_y u_t) - \beta^{-1} m_t^{-1} \lambda_{yt}' y_t.$$

We can reduce the dimension of the state space by defining

$$\xi_{yt} = m_t^{-1} \lambda_{yt} = m_t^{-1} \mu_{yt-1}. \tag{24}$$

Now, the Bellman equation becomes

$$V(x_t, \xi_{yt}) = \max_{y_t, u_t} \min_{m_{t+1}, \mu_{yt}} r(x_t, y_t, u_t, \xi_{yt}, \mu_{yt}) + \beta E_t\left[m_{t+1} V\left(x_{t+1}, m_{t+1}^{-1} \mu_{yt}\right) + \theta m_{t+1} \ln m_{t+1}\right], \tag{25}$$

subject to (1) and (16), where

$$r(x_t, y_t, u_t, \xi_{yt}, \mu_{yt}) = -L(x_t, y_t, u_t) + \mu_{yt}'(A_{yx}x_t + A_{yy}y_t + B_y u_t) - \beta^{-1} \xi_{yt}' y_t. \tag{26}$$

Unlike the solution to type I robust Ramsey problem, here the decision rule is not linear and the value function is not quadratic. One way to solve type II robust Ramsey problem is to use a nonlinear method to solve the above dynamic programming problem. Another method is to use perturbation around $\theta = \infty$ or $\gamma = 1/\theta = 0$ (Hansen and Sargent, 2012; Anderson et al., 2012).

4.2. Solution method

Set up the Lagrangian expression for (25)

$$r(x_t, y_t, u_t, \xi_{yt}, \mu_{yt}) + \beta E_t\left[m_{t+1} V\left(x_{t+1}, m_{t+1}^{-1} \mu_{yt}\right) + \theta m_{t+1} \ln m_{t+1}\right] - \phi_t (E_t m_{t+1} - 1) - E_t \xi_{xt+1}'(x_{t+1} - A_{xx}x_t - A_{xy}y_t - B_x u_t),$$

where ϕ_t and ξ_{xt+1} are the Lagrange multipliers associated with (1) and (16). First-order conditions are given by[8]

$$m_{t+1}: \quad 0 = \beta\left[V\left(x_{t+1}, m_{t+1}^{-1}\mu_{yt}\right) - m_{t+1}^{-1}\mu_{yt}' V_2\left(x_{t+1}, m_{t+1}^{-1}\mu_{yt}\right) + \theta(1 + \ln m_{t+1})\right] - \phi_t, \tag{27}$$

$$y_t : 0 = -L_2(x_t, y_t, u_t) + A_{yy}'\mu_{yt} - \beta^{-1}\xi_{yt} + A_{xy}' E_t \xi_{xt+1}, \tag{28}$$

$$u_t : 0 = -L_3(x_t, y_t, u_t) + B_y'\mu_{yt} + B_x' E_t \xi_{xt+1}, \tag{29}$$

$$\mu_{yt} : 0 = A_{yx}x_t + A_{yy}y_t + B_y u_t + \beta E_t V_2\left(x_{t+1}, m_{t+1}^{-1}\mu_{yt}\right), \tag{30}$$

$$x_{t+1} : 0 = \beta m_{t+1} V_1\left(x_{t+1}, m_{t+1}^{-1}\mu_{yt}\right) - \xi_{xt+1}. \tag{31}$$

Envelope conditions are given by

$$V_1(x_t, \xi_{yt}) = -L_1(x_t, y_t, u_t) + A_{yx}'\mu_{yt} + E_t A_{xx}' \xi_{xt+1}, \tag{32}$$

$$V_2(x_t, \xi_{yt}) = -\beta^{-1} y_t. \tag{33}$$

Leading (33) by one period and substituting it into (27) and (30), we obtain

$$m_{t+1}: \quad 0 = \beta\left[V_{t+1} + \beta^{-1} m_{t+1}^{-1} \mu_{yt}' y_{t+1} + \theta(1 + \ln m_{t+1})\right] - \phi_t, \tag{34}$$

$$\mu_{yt} : 0 = A_{yx}x_t + A_{yy}y_t + B_y u_t - E_t y_{t+1}. \tag{35}$$

[8] By definition of L, we can compute

$$L_1(x_t, y_t, u_t) = Q_{xx}x_t + Q_{xy}y_t + S_x u_t,$$
$$L_2(x_t, y_t, u_t) = Q_{yy}y_t + Q_{xy}'x_t + S_y u_t,$$
$$L_3(x_t, y_t, u_t) = Ru_t + S_x'x_t + S_y'y_t.$$

Taking one period lag in Eq. (31) and using (32), we obtain

$$x_{t+1} : 0 = \beta m_t \left(-L_1(x_t, y_t, u_t) + A'_{yx}\mu_{yt} + E_t A'_{xx}\xi_{xt+1}\right) - \xi_{xt}. \tag{36}$$

Using (34) and (1), can derive

$$m_{t+1} = \frac{\exp\left(-\frac{1}{\theta}\left[V(x_{t+1}, \xi_{t+1}) + \beta^{-1}\xi'_{yt+1}y_{t+1}\right]\right)}{E_t \exp\left(-\frac{1}{\theta}\left[V(x_{t+1}, \xi_{t+1}) + \beta^{-1}\xi'_{yt+1}y_{t+1}\right]\right)}. \tag{37}$$

We then obtain a system of 8 Eqs, (28), (29), (35), (36), (24), (16), (37), and

$$V_t = r(x_t, y_t, u_t, \xi_{yt}, \mu_{yt}) + \beta E_t[m_{t+1}V_{t+1} + \theta m_{t+1}\ln m_{t+1}],$$

for 8 variables x_t, y_t, u_t, μ_{yt}, ξ_{xt}, ξ_{yt}, m_{t+1}, and V_t. The predetermined variables are x_t and ξ_{yt}. The other variables are nonpredetermined. We can use Dynare to solve for a second-order approximate solution.

4.3. An equivalence result

The following proposition shows that Type II robust Ramsey problem is equivalent to a standard Ramsey problem with recursive utility or risk-sensitive utility.

Proposition 2. *Type II robust Ramsey problem is equivalent to the following problem:*

$$\max_{\{x_t, y_t, u_t\}} V_0$$

subject to (13), where V_t satisfies

$$V_t = -L(x_t, y_t, u_t) + \beta\mathcal{R}_t(V_{t+1}), \text{ for all } t \geq 0.$$

We omit its proof. The basic idea is to define V_t as

$$\begin{aligned} V_t &= -L(x_t, y_t, u_t) + \min_{\{m_{t+1}: E_t m_{t+1} = 1\}} \beta(E_t m_{t+1}V_{t+1} + \theta m_{t+1}\ln m_{t+1}) \\ &= -L(x_t, y_t, u_t) + \beta\mathcal{R}_t(V_{t+1}), \end{aligned}$$

where the second line follows from a straightforward computation. By a dynamic programming argument (e.g., Hansen and Sargent, 2008 and Maccheroni, Marinacci and Rustichini, 2006a, b), V_0 satisfies

$$V_0 = \min_{\{m_{t+1}\}} \; -E_0\sum_{t=0}^{\infty}\beta^t M_t L(x_t, y_t, u_t) + \theta\beta E_0 M_t \sum_{t=0}^{\infty}\beta^t m_{t+1}\ln m_{t+1}.$$

Therefore, we obtain Proposition 2.

5. Type III robust Ramsey problem

5.1. Recursive formulation

Form the Lagrangian expression for problem (14):

$$E_0\sum_{t=0}^{\infty}\beta^t\left\{-L(x_t, y_t, u_t) + \theta m_{t+1}\ln m_{t+1} - \mu'_{yt}(E_t[m_{t+1}y_{t+1}] - A_{yx}x_t - A_{yy}y_t - B_y u_t)\right\},$$

where $\beta^t\mu_{yt}$ is the Lagrange multiplier associated with the forward-looking equation in (11). Define

$$\xi_{yt} = m_t\mu_{yt-1}, \tag{38}$$

and rewrite the above Lagrangian as

$$E_0\sum_{t=0}^{\infty}\beta^t\{-L(x_t, y_t, u_t) + \theta m_{t+1}\ln m_{t+1} + \mu'_{yt}(A_{yx}x_t + A_{yy}y_t + B_y u_t) - \beta^{-1}\xi'_{yt}y_t\}.$$

At time zero, we set $\xi_{y0} = m_0\mu_{y,-1} = 0$. Then the Bellman equation becomes

$$\begin{aligned} V(x_t, \xi_{yt}) = \max_{y_t, u_t}\min_{m_{t+1}, \mu_{yt}} \; & r(x_t, y_t, u_t, \xi_{yt}, \mu_{yt}) \\ & + E_t[\beta V(x_{t+1}, m_{t+1}\mu_{yt}) + \theta m_{t+1}\ln m_{t+1}], \end{aligned} \tag{39}$$

subject to (1) and (16), where

$$r(x_t, y_t, u_t, \xi_{yt}, \mu_{yt}) = -L(x_t, y_t, u_t) + \mu'_{yt}(A_{yx}x_t + A_{yy}y_t + B_y u_t) - \beta^{-1}\xi'_{yt}y_t.$$

Comparing this Bellman equation with that for type II robust Ramsey problem, we find that the period return function is identical. But the continuation values are different because the belief-adjustment is different as revealed by (24) and (38).

5.2. Solution method

Set up the Lagrangian expression for (39)

$$r(x_t, y_t, u_t, \xi_{yt}, \mu_{yt}) + E_t[\beta V(x_{t+1}, m_{t+1}\mu_{yt}) + \theta m_{t+1} \ln m_{t+1}] - \phi_t(E_t m_{t+1} - 1)$$
$$-E_t \xi'_{xt+1}(x_{t+1} - A_{xx}x_t - A_{xy}y_t - B_x u_t),$$

where ϕ_t and ξ_{xt+1} are the Lagrange multipliers associated with (1) and (16). First-order conditions are given by

$$m_{t+1} : 0 = \beta \mu'_{yt} V_2(x_{t+1}, m_{t+1}\mu_{yt}) + \theta(1 + \ln m_{t+1}) - \phi_t, \tag{40}$$

$$y_t : 0 = -L_2(x_t, y_t, u_t) + A'_{yy}\mu_{yt} - \beta^{-1}\xi_{yt} + E_t A'_{xy}\xi_{xt+1}, \tag{41}$$

$$u_t : 0 = -L_3(x_t, y_t, u_t) + B'_y \mu_{yt} + B'_x E_t \xi_{xt+1}, \tag{42}$$

$$\mu_{yt} : 0 = A_{yx}x_t + A_{yy}y_t + B_y u_t + \beta E_t m_{t+1} V_2(x_{t+1}, m_{t+1}\mu_{yt}), \tag{43}$$

$$x_{t+1} : 0 = \beta V_1(x_{t+1}, m_{t+1}\mu_{yt}) - \xi_{xt+1}. \tag{44}$$

Envelope conditions are given by

$$V_1(x_t, \xi_{yt}) = -L_1(x_t, y_t, u_t) + A'_{yx}\mu_{yt} + A'_{xx}E_t\xi_{xt+1}, \tag{45}$$

$$V_2(x_t, \xi_{yt}) = -\beta^{-1} y_t. \tag{46}$$

Leading (46) by one period and substituting it into (40) and (43) yields

$$m_{t+1} : 0 = \theta(1 + \ln m_{t+1}) - \mu'_{yt} y_{t+1} - \phi_t, \tag{47}$$

$$\mu_{yt} : 0 = A_{yx}x_t + A_{yy}y_t + B_y u_t - E_t m_{t+1} y_{t+1} \tag{48}$$

Taking one period lag in (44) and using (45) yields

$$x_{t+1} : \xi_{xt} = \beta\left(-L_1(x_t, y_t, u_t) + A'_{yx}\mu_{yt+1} + A'_{xx}E_t\xi_{xt+1}\right). \tag{49}$$

Using (1) and (47), we can derive

$$m_{t+1} = \frac{\exp\left(\theta^{-1}\mu'_{yt}y_{t+1}\right)}{E_t\left[\exp\left(\theta^{-1}\mu'_{yt}y_{t+1}\right)\right]}. \tag{50}$$

We then obtain a system of 7 Eqs. (41) (42), (48), (49), (16), (38), and (50) for 7 variables x_t, y_t, u_t, μ_{yt}, ξ_{xt}, ξ_{yt}, and m_{t+1}. The predetermined state variables are x_t and ξ_{yt}. The other variables are nonpredetermined. We can use perturbation methods up to third-order approximations implemented in Dynare to solve the system numerically.

6. Type III$^+$ robust Ramsey problem

6.1. Recursive formulation

Form the Lagrangian expression for problem (15):

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{-L(x_t, y_t, u_t) + \beta\theta M_t m_{t+1} \ln m_{t+1} - \mu'_{yt}(E_t[m_{t+1}y_{t+1}] - A_{yx}x_t - A_{yy}y_t - B_y u_t)\right\},$$

where $\beta^t \mu_{yt}$ is the Lagrange multiplier associated with the forward-looking equation in (11). We then define

$$\xi_{yt+1} = \mu_{yt} m_{t+1}$$

and rewrite the above Lagrangian as

$$E_0 \sum_{t=0}^{\infty} \beta^t \{-L(x_t, y_t, u_t) + \beta\theta M_t m_{t+1} \ln m_{t+1} + \mu'_{yt}(A_{yx}x_t + A_{yy}y_t + B_y u_t) - \beta^{-1}\xi'_{yt}y_t\}.$$

At time zero, we set $\xi_{y0} = m_0 \mu_{y,-1} = 0$.

Now we can derive a recursive formulation of type III$^+$ robust Ramsey problem

$$V(x_t, \xi_{yt}, M_t) = \max_{y_t, u_t} \min_{m_{t+1}, \mu_{yt}} r(x_t, y_t, u_t, \xi_{yt}, \mu_{yt})$$
$$+\beta E_t\left[V\left(x_{t+1}, \xi_{yt+1}, M_{t+1}\right) + \theta M_t m_{t+1} \ln m_{t+1}\right], \tag{51}$$

subject to (1) and (16) where

$$r(x_t, y_t, u_t, \xi_{yt}, \mu_{yt}) = -L(x_t, y_t, u_t) + \mu'_{yt}(A_{yx}x_t + A_{yy}y_t + B_y u_t) - \beta^{-1}\xi'_{yt}y_t.$$

6.2. Solution method

Set up the Lagrangian expression for (51)

$$r(x_t, y_t, u_t, \xi_{yt}, \mu_{yt}) + E_t[\beta V(x_{t+1}, m_{t+1}\mu_{yt}, M_t m_{t+1}) + \beta\theta M_t m_{t+1}\ln m_{t+1}]$$
$$-\phi_t(E_t m_{t+1} - 1) - E_t\xi'_{xt+1}(x_{t+1} - A_{xx}x_t - A_{xy}y_t - B_x u_t),$$

where ϕ_t and ξ_{xt+1} are the Lagrange multipliers associated with (1) and (16). First-order conditions are given by

$$m_{t+1} : 0 = \beta\mu'_{yt}V_2(x_{t+1}, m_{t+1}\mu_{yt}, M_t m_{t+1}) + \beta M_t V_3(x_{t+1}, m_{t+1}\mu_{yt}, M_t m_{t+1}) + \beta\theta M_t(1 + \ln m_{t+1}) - \phi_t, \tag{52}$$

$$y_t : 0 = -L_2(x_t, y_t, u_t) + A'_{yy}\mu_{yt} - \beta^{-1}\xi_{yt} + A'_{xy}E_t\xi_{xt+1}, \tag{53}$$

$$u_t : 0 = -L_3(x_t, y_t, u_t) + B'_y\mu_{yt} + B'_x E_t\xi_{xt+1}, \tag{54}$$

$$\mu_{yt} : 0 = A_{yx}x_t + A_{yy}y_t + B_y u_t + \beta E_t m_{t+1}V_2(x_{t+1}, m_{t+1}\mu_{yt}, M_t m_{t+1}), \tag{55}$$

$$x_{t+1} : 0 = \beta V_1(x_{t+1}, m_{t+1}\mu_{yt}, M_t m_{t+1}) - \xi_{xt+1}. \tag{56}$$

Envelope conditions are given by

$$V_1(x_t, \xi_{yt}, M_t) = -L_1(x_t, y_t, u_t) + A'_{yx}\mu_{yt} + A'_{xx}E_t\xi_{xt+1}, \tag{57}$$

$$V_2(x_t, \xi_{yt}, M_t) = -\beta^{-1}y_t, \tag{58}$$

$$V_3(x_t, \xi_{yt}, M_t) = \beta E_t m_{t+1}V_3\left(x_{t+1}, \xi_{yt+1}, M_{t+1}\right) + \beta\theta E_t m_{t+1}\ln m_{t+1} \tag{59}$$

Leading (58) by one period and substituting it into (52) and (55) yields

$$m_{t+1} : 0 = \beta M_t V_3(x_{t+1}, m_{t+1}\mu_{yt}, M_t m_{t+1}) + \beta\theta M_t(1 + \ln m_{t+1}) - \mu'_{yt}y_{t+1} - \phi_t, \tag{60}$$

$$\mu_{yt} : 0 = A_{yx}x_t + A_{yy}y_t + B_y u_t - E_t m_{t+1}y_{t+1} \tag{61}$$

Taking one period lag in (56) and using (57) yields

$$x_{t+1} : \xi_{xt} = \beta\left(-L_1(x_t, y_t, u_t) + A'_{yx}\mu_{yt+1} + A'_{xx}E_t\xi_{xt+1}\right). \tag{62}$$

Using (1) and (60), we can derive

$$m_{t+1} = \frac{\exp\left(\beta^{-1}\theta^{-1}M_t^{-1}\mu'_{yt}y_{t+1} - \theta^{-1}V_{3,t+1}\right)}{E_t\left[\exp\left(\beta^{-1}\theta^{-1}M_t^{-1}\mu'_{yt}y_{t+1} - \theta^{-1}V_{3,t+1}\right)\right]}, \tag{63}$$

where $V_{3,t+1} \equiv V_3\left(x_{t+1}, \xi_{yt+1}, M_{t+1}\right)$ and M_t is defined in a recursive way

$$M_t = m_t M_{t-1}. \tag{64}$$

We then obtain a system of 9 Eqs. (16), (38), (53), (54), (59), and (61)–(64) for 9 variables x_t, y_t, u_t, μ_{yt}, ξ_{xt}, ξ_{yt}, $V_{3,t}$, m_t, and M_t. The predetermined state variables are x_t, ξ_{yt}, and M_t. The other variables are nonpredetermined. We can use perturbation methods up to third-order approximations implemented in Dynare to solve the system numerically. Compared to Woodford's type III problem, type III$^+$ problem is more complicated to solve since it involves two more variables V_{3t} and M_t. In particular, M_t is non-stationary.

7. Applications to monetary policy

In this section we apply our general theory to the study of robustly optimal monetary policy in a basic New Keynesian model. The objective function of the central bank (or CB) under rational expectations is given by

$$\max_{\{x_t, \pi_t\}} -\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\left[\pi_t^2 + \lambda(x_t - x^*)^2\right], \tag{65}$$

where π_t and x_t denote inflation and the output gap, respectively, β is the discount factor of the private sector and λ is a weight the CB places on the stabilization of the output gap variability. Here $x^* \geq 0$ denotes the distortion in the objective of the CB towards a positive output gap.

Under rational expectations, the CB faces the following New Keynesian Phillips Curve (hereafter, NKPC):

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} + z_t. \tag{66}$$

where β is a discount factor shared with the policymaker. Here, z_t denotes the cost-push shock which is assumed to follow an AR(1) process

$$z_t = \rho_z z_{t-1} + \sigma_z \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, 1), \tag{67}$$

where $\rho_z \in [0, 1)$ denotes the AR(1) coefficient and $\sigma_z > 0$ represents the standard deviation of a new innovation in cost-push shocks. Here, ε_t is an independently and identically distributed standard normal random variable.

7.1. Robust Ramsey policy

Now consider robust Ramsey policy. We use the methods introduced in Sections 3–5 to derive this policy for the three types of robust Ramsey monetary policy problem. For type I ambiguity, we can derive the first-order conditions

$$\pi_t = \beta^{-1}(\mu_{\pi t} - \mu_{\pi,t-1}), \tag{68}$$

$$\mu_{\pi t} = -\frac{\lambda\beta}{\kappa}(x_t - x^*), \tag{69}$$

where $\mu_{\pi t}$ is the Lagrange multiplier associated with distorted NKPC

$$\pi_t = \kappa x_t + \beta E_t[m_{t+1}\pi_{t+1}] + z_t. \tag{70}$$

The above first-order conditions are the same as in the case of rational expectations. The difference is that the distribution of the shock process is distorted. Substituting (68) and (69) into (70), we obtain a difference equation for $\mu_{\pi t}$. Unlike in the case of rational expectations, type I ambiguity causes the expectation in this difference equation to be distorted. Under the worst-case distribution, the density is given by

$$m_{t+1} = \frac{\exp\left(\frac{-1}{\theta}V(z_{t+1}, \mu_{\pi t})\right)}{E_t\left[\exp\left(\frac{-1}{\theta}V(z_{t+1}, \mu_{\pi t})\right)\right]},$$

where V is the value function for type I ambiguity. This means that the CB attaches more weight to states when continuation values are low. As we show in Section 3.2, V is quadratic, m_{t+1} is a normal density, and type I robust policy is linear in the state variables z_t and $\mu_{\pi t}$. The state variable $\mu_{\pi t}$ encodes the history and generates history dependency of the optimal monetary policy.

For type II ambiguity, the first-order conditions are given by (69),

$$\pi_t = \beta^{-1}(\mu_{\pi t} - \xi_t), \tag{71}$$

$$\phi_t = \beta V(z_{t+1}, \xi_{t+1}) - \xi_{t+1}\pi_{t+1} + \beta\theta(1 + \ln m_{t+1}), \tag{72}$$

where $\mu_{\pi t}$ and ϕ_t are the Lagrange multipliers associated with (66) and (1), respectively, and ξ_t is the belief-adjusted Lagrange multiplier defined as

$$\xi_t = m_t^{-1}\mu_{\pi t-1}, \ t \geq 1, \quad \xi_0 = \mu_{\pi,-1} = 0.$$

Here, V is the value function for type II ambiguity. Using (1) to eliminate ϕ_t yields

$$m_{t+1} = \frac{\exp\left(-\frac{1}{\theta}\left[V(z_{t+1}, \xi_{t+1}) - \beta^{-1}\xi_{t+1}\pi_{t+1}\right]\right)}{E_t \exp\left(-\frac{1}{\theta}[V(z_{t+1}, \xi_{t+1}) - \beta^{-1}\xi_{t+1}\pi_{t+1}]\right)}.$$

This equation implies that the CB puts more weight on the states with low continuation values. Type II robust policy is a nonlinear function of the state variables, the shock z_t and the belief-adjusted Lagrange multiplier ξ_t.

Substituting (71) into (66), we obtain a difference equation for ξ_t. To solve this equation, we need to know the belief distortion represented by the density m_t, which in turn must be solved jointly with the value function V.

For type III ambiguity, the first-order conditions are still given by (69) and (71), where $\mu_{\pi t}$ is the Lagrange multiplier associated with the distorted NKPC (70) and

$$\xi_t = m_t\mu_{\pi t-1}, \ t \geq 1, \ \xi_0 = \mu_{\pi,-1} = 0.$$

We can solve for the distorted belief as

$$m_{t+1} = \frac{\exp\left(\theta^{-1}\mu_{\pi t}\pi_{t+1}\right)}{E_t\left[\exp\left(\theta^{-1}\mu_{\pi t}\pi_{t+1}\right)\right]}. \tag{73}$$

Eqs. (69)–(71), and (73), together with the definition of ξ_t determine π_t, x_t, m_t, ξ_t, and $\mu_{\pi t}$. We can eliminate $\mu_{\pi t}$ and represent the type III robust policy as a nonlinear function of the predetermined state variables z_t and ξ_t.

Note that Eq. (73) implies that when the Lagrange multiplier $\mu_{\pi t}$ is positive, the CB's concern for robustness causes it to assign higher probabilities to more inflationary states. Similarly, when $\mu_{\pi t} < 0$, the CB worries that less inflationary or more deflationary states are more likely than under its approximating model. Clearly, the above equations imply that type III robust policy is nonlinear. This is different from the linear robust policy in the timeless perspective studied by Woodford (2010) and Kwon and Miao (2012).

Finally, for type III$^+$ ambiguity, the first-order conditions also include (69) and (71) as for type III, and the definition of ξ_t is also the same. What is different is the belief distortion:

$$m_{t+1} = \frac{\exp\left(\beta^{-1}\theta^{-1}M_t^{-1}\mu_{\pi t}\pi_{t+1} - \theta^{-1}V_3(z_{t+1}, \xi_{t+1}, M_{t+1})\right)}{E_t\left[\exp\left(\beta^{-1}\theta^{-1}M_t^{-1}\mu_{\pi t}\pi_{t+1} - \theta^{-1}V_3(z_{t+1}, \xi_{t+1}, M_{t+1})\right)\right]}, \tag{74}$$

where $V_3(z_t, \xi_t, M_t)$ comes from the envelope condition with respect to M_t and is determined recursively by

$$V_3(z_t, \xi_t, M_t) = \beta E_t m_{t+1} V_3(z_{t+1}, \xi_{t+1}, M_{t+1}) + \beta\theta E_t m_{t+1} \log m_{t+1}. \tag{75}$$

From (74) we can see that the CB not only cares about the future but also takes the history of belief distortions into account when formulating the worst-case scenario. The first term in the exponent shows that the smaller M_t is, the more the CB worries about the future state with higher commitment value and higher inflation. The second term says that the CB assigns higher probability to a state with lower value of $V_3(z_{t+1}, \xi_{t+1}, M_{t+1})$, which can be interpreted as the marginal value of M_{t+1}.

7.2. Calibration

To illustrate the quantitative impact of a concern for robustness, we take the same parameter values as in Woodford (2010) and Kwon and Miao (2012): $\beta = 0.99$, $\kappa = 0.2$, $\lambda = 0.08$, $x^* = 0.05$.[9] Also we assume that $\rho_z = 0.5$ and $\sigma_z = 0.02$.

Now the only remaining parameter to be calibrated is θ, which measures the degree of concerns for robustness. We apply the detection error probability method proposed by Hansen and Sargent (2008). The detection error probability gives the probability that an econometrician cannot correctly figure out the true data generating process (DGP) after observing a series of data, especially when she has two competing candidates of the DGP. If two models (or DGPs) are almost identical, the detection error probability is close to 50%, which implies that there is roughly a 50–50 chance to make an error about which model generates an observed series of data. In other words, it is almost impossible to differentiate the two models. The detection error probability becomes close to zero when two competing models are very different so that the econometrician can almost always detect the true DGP. Specifically, the detection error probability can be computed using log-likelihood ratios

$$\frac{1}{2}\Pr\left(\log\frac{L_A}{L_W} > 0 \middle| W\right) + \frac{1}{2}\Pr\left(\log\frac{L_W}{L_A} > 0 \middle| A\right),$$

where $L_A(L_W)$ denotes the likelihood of model $A(W)$. One can consider A represents an approximating model and W means the worst-case one. $\Pr(\cdot|A)$ denotes the probability conditional on the hypothesis that model A is a true one.

For type I robust Ramsey problem, the detection error probability can be easily computed using the Kalman filter since the solution is linear and the shock process is Gaussian.[10] Since we solve type II and type II robust Ramsey problems using the second-order approximation method, solutions to these types are not linear any more. Not surprisingly, it is generally very difficult to find the exact likelihood even if the shock process is Gaussian. Therefore, we approximate likelihoods using particle filtering. The particle filter enables us to evaluate the likelihood numerically via the sequential Monte Carlo algorithm when a model is non-linear and/or non-Gaussian. Fernández-Villaverde and Rubio-Ramírez (2007) show that particle filtering can be used to calculate likelihoods of DSGE models when solved with non-linear methods, particularly in their case, a second-order approximation. More recently, Bidder and Smith (2012) apply this method to compute the detection error probability. Appendix A details our computation procedure.

We calibrate θ such that the detection error probability is approximately 10% for each type of robust Ramsey problem. We use simulated time series of 150 periods and simulate 1,200 times to calculate the detection error probability.[11] Also we use 100,000 particles to evaluate the likelihood. Note that since type I ambiguity yields a linear solution, we use the Kalman filter to obtain the likelihood function in a state space framework. Table 1 shows calibrated values of θ.

[9] Hansen and Sargent (2012) also used similar values.

[10] See Giordani and Söderlind (2004) for detailed information.

[11] Note that there is a tradeoff between accuracy and speed as the number of particles increases. Exploiting Matlab parallel computing with 32 cores, it took about 40 minutes for 1200 simulations using 100,000 particles to compute the detection error probability corresponding to a single value of θ. Bidder and Smith (2012) use as many as 160,000 particles. We also checked the detection error probability corresponding to our calibrated value of θ using 160,000 particles but the difference between the two results was negligible.

Table 1
Calibrated values of θ.

Type I	Type II	Type III	Type III$^+$
0.0069	0.7611	0.0151	0.0156

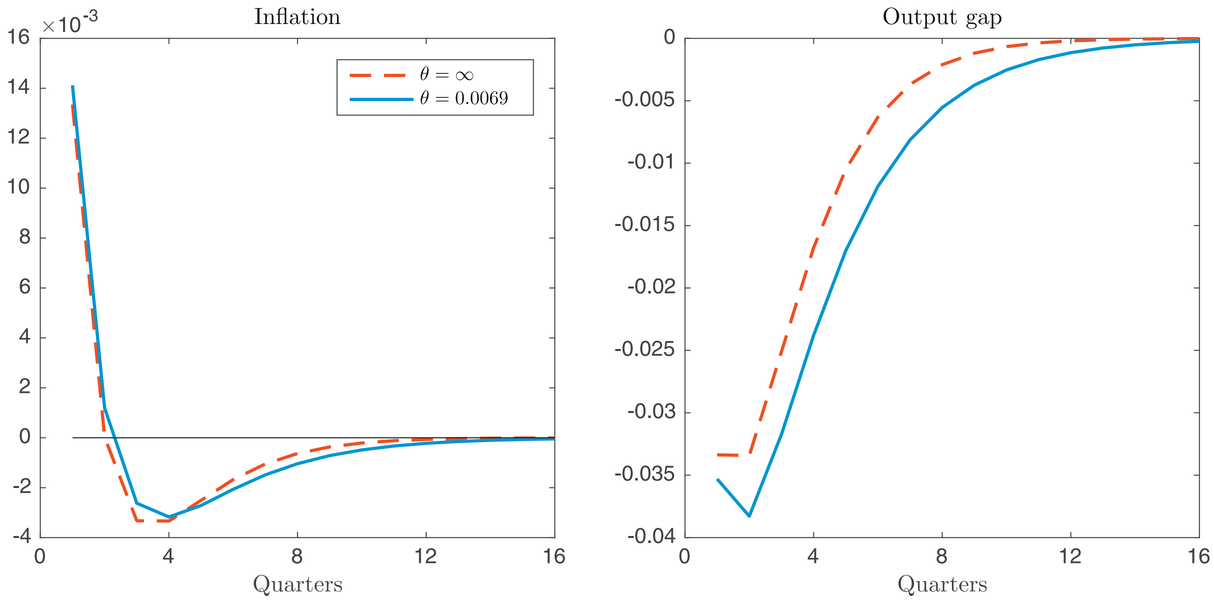

Fig. 1. Impulse responses of inflation and output under type I ambiguity.
Note: Dashed lines represent impulse responses under rational expectations. Solid lines plot impulse responses under type I ambiguity. The time period is quarterly.

7.3. Numerical results

Figs. 1–3 plot the impulse responses of inflation and the output gap following a positive unexpected one standard deviation cost-push shock for the three types of robust Ramsey problem and the Ramsey problem under rational expectations with $\theta = \infty$. We do not report the results for type III$^+$ ambiguity here, which are available upon request. The reason is that types III and III$^+$ robust Ramsey problems give very similar results under our calibrated values of *theta*, though type III$^+$ ambiguity implies a slightly more history dependence. The results are substantially different only when θ is sufficiently small.

An important finding is that optimal monetary policy becomes more history-dependent when the policymaker faces model ambiguity irrespective of its type. More history dependent monetary policy implies that it takes a longer time for the policymaker to return inflation and the output gap to their steady state levels. Although monetary policy becomes more history dependent for all three types of ambiguity, there are subtle differences in reasons behind this result. For type I ambiguity, the CB faces ambiguity about both the shock process and private agents' expectations. As we show in Section 3.1, the private sector's expectations in the difference equation for $\mu_{\pi t}$ or the commitment value are distorted under the worst-case beliefs. This makes the robustly optimal monetary policy more history dependent than that under rational expectations.

For type II ambiguity, the CB does not suffer from ambiguity about private agents' beliefs but it does not trust its approximating model of the exogenous shock process. The CB believes that the cost-push shock is more persistent than under rational expectations, causing the robust monetary policy more history dependent. Note that even though the Ramsey policymaker fully trusts the private sector's beliefs, its concern for robustness leads to a change in the policy rule for inflation as shown in (71). Type II ambiguity generates endogenous belief heterogeneity. The state variable that encodes the history is the belief-adjusted Lagrange multiplier associated with the NKPC. The evolution of this state variable explains the more history dependence of Type II robust policy.

For type III ambiguity, the CB fully trusts its approximating model of the shock process, but is uncertain about agents' beliefs. More history-dependence comes from the CB's concern for robustness of agents' beliefs. This can better manage agents' expectations as Woodford (2010) argues. The state variable that encodes history is the belief-adjusted Lagrange multiplier associated with the distorted NKPC. Like Type II ambiguity, type III ambiguity also generates endogenous belief heterogeneity. But the belief adjustment and the evolution of the belief-adjusted Lagrange multiplier are different.

Even though all three types of ambiguity share the common feature of making monetary policy more history dependent, the implications for the price level are not the same across the three types. The left panel of Fig. 4 plots the impulse

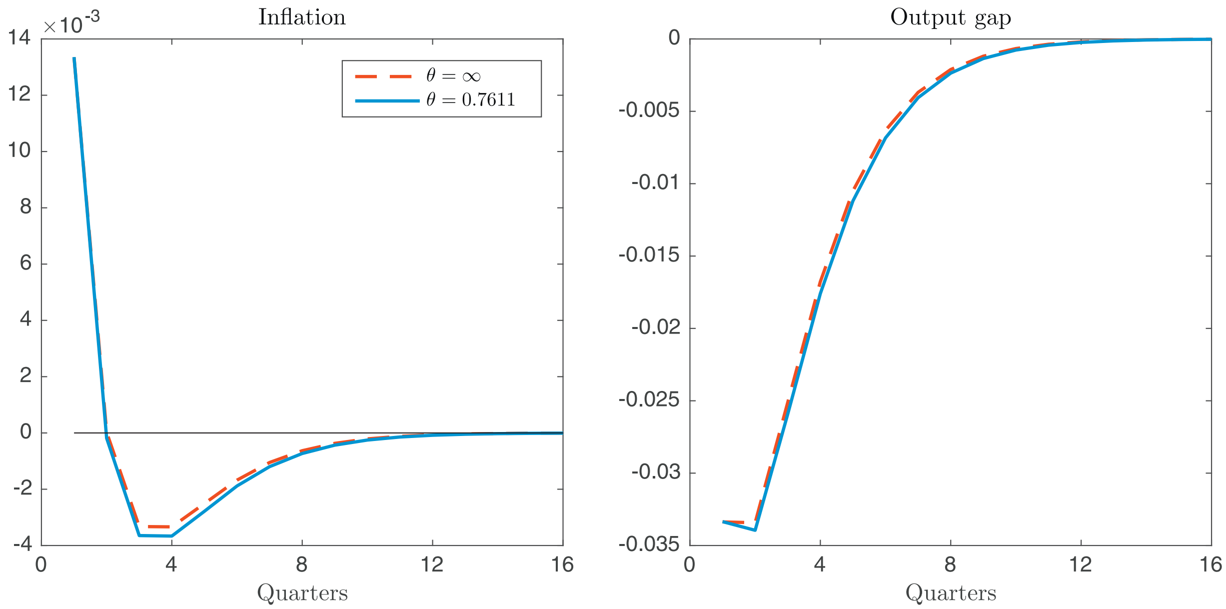

Fig. 2. Impulse responses of inflation and output under type II ambiguity. Note: Dashed lines represent impulse responses under rational expectations. Solid lines plot impulse responses under type II ambiguity. The time period is quarterly.

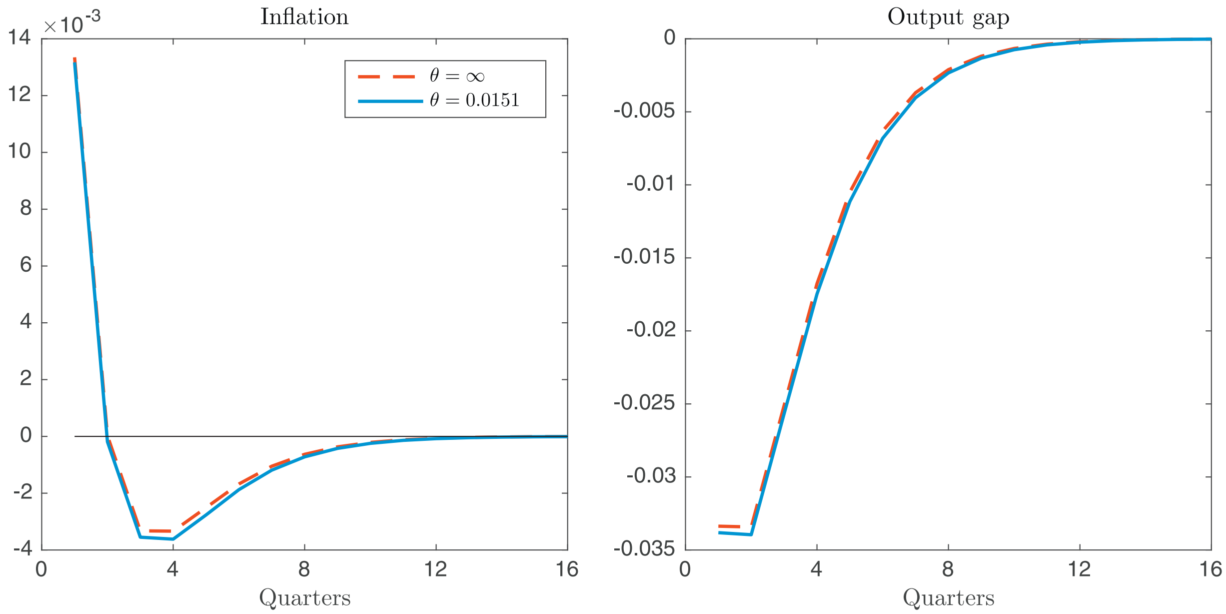

Fig. 3. Impulse responses of inflation and output under type III ambiguity. Note: Dashed lines represent impulse responses under rational expectations. Solid lines plot impulse responses under type III ambiguity. The time period is quarterly.

responses of the price level under type I ambiguity (solid line) and under rational expectations (dashed line). The robust policy for inflation implies that once a cost-push shock hits the economy, it is optimal to undo all the changes in the price level so that it returns to its steady state level. This implies that type I robustly optimal monetary policy is price level targeting, just like in the case of rational expectations. The middle and right panels of Fig. 4 plot the impulse responses for types II and III ambiguity. The robust policy indicates that in response to a positive cost-push shock the CB adjusts the inflation rate to a degree so that the price level goes below its original level. This implies that under types II and III ambiguity, price level targeting is not an optimal policy any more. Note that our finding is in line with Woodford (2010) and Kwon and Miao (2012). The difference is that they find this feature of monetary policy under type III ambiguity in the timeless perspective instead of the Ramsey framework as in this paper.

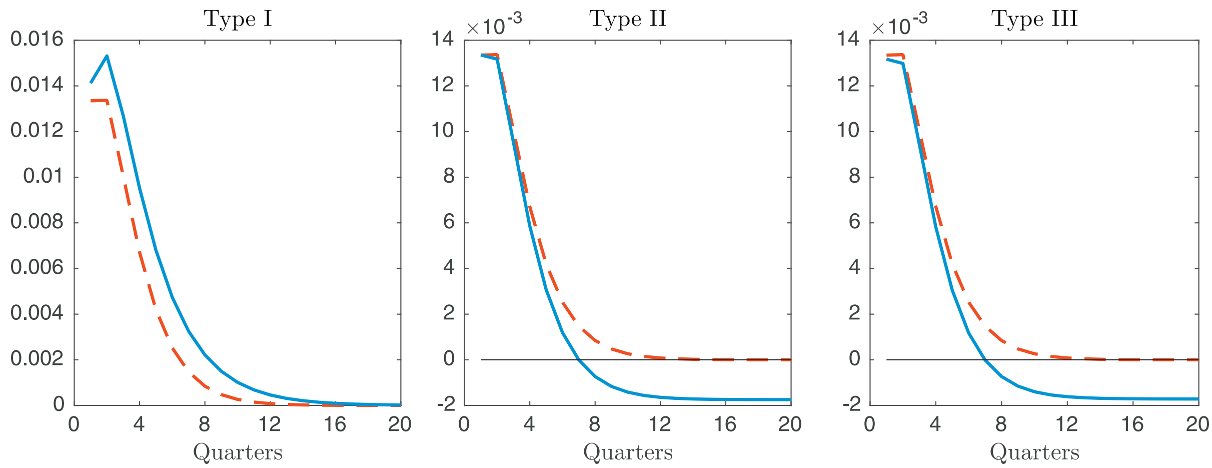

Fig. 4. Impulse responses of the price level. Note: Dashed lines represent impulse responses under rational expectations. Solid lines plot impulse responses under three types of ambiguity. The time period is quarterly.

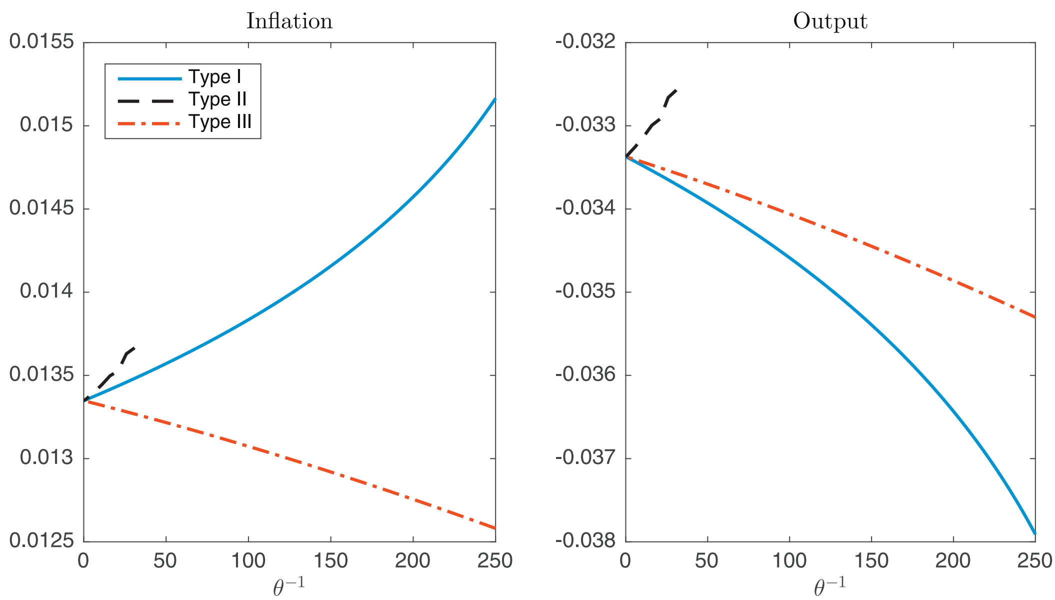

Fig. 5. Initial responses of inflation and output. Note: The horizontal axis represents the degree of concerns for robustness θ^{-1}. The case of $\theta^{-1} = 0$ corresponds to rational expectations.

Fig. 5 shows the initial responses of inflation and output gap for various values of the robustness parameter θ. The inverse θ^{-1} describes the degree of concerns for robustness or ambiguity aversion. The initial responses under rational expectations correspond to $\theta^{-1} = 0$. Note that when θ is smaller than around 0.3, the solution reaches the breakdown point pointed out by Hansen and Sargent (2008) and hence we restrict solution for type II problem in a small interval.

An interesting and important finding is that different types of ambiguity deliver different initial responses to a positive cost-push shock. For type I ambiguity, the initial responses of inflation and the output gap are larger than those under rational expectations. Since type I ambiguity is related to the overall model misspecification, in the worst case scenario, the CB worries that the cost-push shock is distorted in mean so that the CB responds more aggressively as if the shock were greater compared to that in the approximating model. At the same time the CB concerns that the private agents' expectations are distorted to a higher level too.

For type II ambiguity, the initial response of inflation is also greater than under rational expectations. The difference between type I and type II lies in the initial response of the output gap. While type I ambiguity makes the initial response of the output gap greater than under rational expectations, type II ambiguity leads to a less responsive output gap. The intuition is the following: The CB worries about the unfavorable distortion in the cost-push shock, leading to an increase in the inflation response. On the other hand, the CB exploits the fact that the private sector fully trusts its model and in its model the cost-push shock is believed less persistent than under the CB's distorted beliefs. As a result, the CB faces a smaller tradeoff between inflation and output.

Finally, for type III ambiguity, the CB's initial response of inflation to a positive cost-push shock is less sensitive but the initial response of the output gap is more sensitive. The concern for robustness of the expectation of the private sector makes the CB manage the inflation expectation more cautiously. In other words, the CB puts more effort to stabilize inflation and the inflation expectation since it worries that the private agents' inflation expectations are biased upward. In this case, the CB faces a larger tradeoff between inflation and the output gap. This result is in line with Woodford (2010) and Kwon and Miao (2012). In these papers, they focus on the linear class of policy rules in the timeless perspective. Our finding shows that their finding is still valid in the Ramsey framework.

8. Conclusion

In this paper we study three types of robust Ramsey problems corresponding to three types of ambiguity analyzed by Hansen and Sargent (2012) in a general linear-quadratic framework. We provide recursive characterizations and algorithms to solve the robust Ramsey policy. We apply our methods to a basic New Keynesian model of optimal monetary policy with persistent cost-push shocks.

There are three main findings. First, robust Ramsey monetary policy for all three types of ambiguity generates more history dependence than under rational expectations. Second, in response to a positive cost-push shock, the price level eventually returns to the initial level for type I ambiguity and rational expectations. But the sign of the initial price level effect is eventually reversed for types II and III ambiguity. Third, the initial response of inflation is more aggressive for types I and II ambiguity than for rational expectations, but it is less aggressive for type III ambiguity. The initial response of the output gap is more aggressive for types I and III ambiguity than for rational expectations, but it is less responsive for type II ambiguity.

Our analysis highlights the importance of modeling who faces ambiguity and what the policymaker is ambiguous about because different types of ambiguity generate very different policy responses to shocks. For future research, it would be interesting to study micro-founded models of robust policy. Adam and Woodford (2012) solve an optimal monetary policy problem under type III ambiguity with a micro-founded model. They use linear approximation methods to solve the model. It would be interesting to study different types of ambiguity in a micro-founded model with non-linear solutions.

Appendix A. Computing detection error probabilities

We can write the solution to a robust Ramsey problem in a state space representation:

$$s_t = F(s_{t-1}, \varepsilon_t),$$
$$\tilde{y}_t = G(s_t),$$

where s_t and $\tilde{y}_t$ denote unobservable hidden states and endogenous variables at time t, respectively. $F(\cdot)$ and $G(\cdot)$ are system and observation functions, respectively, which are possibly non-linear. Here, ε_t denotes a system innovation. To apply the particle filtering algorithm, we assume that the endogenous variables are observable with measurement errors:

$$y_t = G(s_t) + \upsilon_t, \tag{A.1}$$

where υ_t is a measurement error.

Note that according to Bayes' theorem the likelihood of $y^t \equiv \{y_0, y_1, \ldots, y_t\}$ under the hypothesis of $\theta \in \Theta$ is given by

$$p\left(y^t\,;\,\theta\right) = p(y_0\,;\,\theta)\prod_{k=1}^{t} p\left(y_k \,\middle|\, y^{k-1}\,;\,\theta\right), \tag{A.2}$$

where $p(y_0\,;\theta)$ denotes a prior probability which is assumed to be known. Generally, one cannot compute the likelihood analytically. The key idea of the particle filtering algorithm is to approximate the likelihood through Monte Carlo simulation.

More concretely, note that one can rewrite the last term of (A.2) as

$$p\left(y_k \,\middle|\, y^{k-1}\,;\,\theta\right) = \int p\left(y_k \,\middle|\, s_k\,;\,\theta\right) p\left(s_k \,\middle|\, y^{k-1}\,;\,\theta\right) \mathrm{d}s_k,$$

which can be approximated using discrete samples:

$$p\left(y_k \,\middle|\, y^{k-1}\,;\,\theta\right) \approx \frac{1}{N}\sum_{i=1}^{N} p\left(y_k \,\middle|\, s_k^i\,;\,\theta\right) p\left(s_k^i \,\middle|\, y^{k-1}\,;\,\theta\right).$$

Now let $\{w_k^i, s_k^i\}_{i=1}^N$ be a swarm of particles such that s_t^i is randomly drawn from $p(s_k^i \mid s_{k-1}^i; \theta)$ and also the importance weight w_t^i is computed recursively[12] by

$$w_t^i = p\left(y_k \mid s_k^i; \theta\right) \frac{w_{k-1}^i}{\sum w_{k-1}^i}.$$

It is important to notice that $\{w_k^i, s_k^i\}_{i=1}^N$ is a discrete approximation to the distribution

$$p\left(y_k \mid s_k; \theta\right) p\left(s_k \mid y^{k-1}; \theta\right)$$

which is justified by the law of large numbers as the number of particles N increases. Thus, one can find that the likelihood function can be approximated by

$$p\left(y_k \mid y^{k-1}; \theta\right) \approx \frac{1}{N} \sum_{i=1}^N w_t^i.$$

Notice that if one resamples s_k^j from $\{w_k^i, s_k^i\}_{i=1}^N$ with replacement one can update a swarm of particles to $\{N^{-1}, s_k^j\}_{j=1}^N$. This procedure is called the sampling importance resampling.[13]

To sum up, we implement the particle filtering algorithm using the following procedure:

- FOR $t = 2 : T$, given a swarm of particles $\{w_t^i, s_t^i\}_{i=1}^N$, draw $s_{t+1}^i \sim p(s \mid s_t^i; \theta)$.
- Update $w_{t+1}^i = p(y_{t+1} \mid s_{t+1}^i; \theta) w_t^i (\sum w_t^i)^{-1}$.
- Compute the conditional likelihood $L_{t+1} = N^{-1} \sum w_{t+1}^i$.
- Resample $\{w_{t+1}^i, s_{t+1}^i\}_{i=1}^N$ from $\{w_{t+1}^j, s_{t+1}^j\}_{j=1}^N$ with replacement such that $s_{t+1}^i \sim i.i.d. \{w_{t+1}^i, s_{t+1}^i\}_{i=1}^N$ and $w_{t+1}^i = N^{-1}$.
- $t = t + 1$.
- END FOR.

While implementing particle filtering, an issue needing our caution is that in type II robust Ramsey problem, the worst-case shock process is different from the reference process. Note that for type III, the shock process is not distorted but only beliefs of a Ramsey planner about private agents' expectations are distorted. To derive the worst-case distribution, one can do Monte-Carlo simulations and compute the one-step-ahead likelihood ratio, $m_{t+1}(s_{t+1})$. Then one can draw the distorted shock according to the distorted probability. The problem is that it makes the computing time very long. Instead we take a simpler and more practical way to draw distorted shocks. By the second-order approximation we can express m_{t+1} in terms of state variables. Then we can compute the conditional mean of the distorted shock by

$$E_t[m_{t+1} e_{t+1}] = E_t[G_m(s_t, e_{t+1}) e_{t+1}] \tag{A.3}$$

where $G_m(\cdot)$ denotes decision rule for m_{t+1} given s_t. The distortion in mean at time $t + 1$ depends on state variables at time t and a innovation shock e_{t+1} so that it is now easy to compute. Bidder and Smith (2012) also take a similar approach.

After finding the conditional mean, we generate the worst-case shocks for each state by randomly drawing a number from the normal distribution $\mathcal{N}(E_t[m_{t+1} e_{t+1}], 1)$. In other words, we simply assume that the worst-case distribution distorts only the mean of the shock process. We confirm this assumption by Monte-Carlo simulations following Bidder and Smith (2012). We find that the variance of the worst-case shock is not different from that of the reference shock process and we cannot not find any significant difference between two distributions except for the mean. The gain from reducing computing time, however, is very significant.

Finally, the likelihood can be computed as

$$L(y_T; \theta) = \prod_{t=1}^T L_t.$$

In order to implement particle filtering, we first generate a set of simulated time series using the solution to the robust Ramsey problem. In our model, there are three hidden state variables $(z_t, \mu_{\pi t}, \xi_t)$ and two observable variables π_t and

[12] By using $q(s_k^i \mid y^{k-1})$ as the importance sampling distribution of $p(s_k^i \mid y^k)$ one can find that the importance weight w_t^i can be expressed recursively:

$$w_k^i \propto \frac{p\left(s_k^i \mid y^k\right)}{q\left(s_k^i \mid y^{k-1}\right)} \propto \frac{p\left(y_k \mid s_k^i\right) p\left(s_k^i \mid s_{k-1}^i\right)}{q\left(s_k^i \mid s_{k-1}^i, y^{t-1}\right)} \frac{p\left(s_{k-1}^i \mid y^{k-1}\right)}{q\left(s_{k-1}^i \mid y^{k-2}\right)} \propto w_{k-1}^i p\left(y_k \mid s_k^i\right).$$

Since $q(s_k^i \mid y^{k-1})$ can be chosen arbitrarily, one can easily sample s_k^i such that

$$s_k^i \ \underset{\sim}{i.i.d.}\ q\left(s_k \mid y^{k-1}\right) \Rightarrow \hat{q}\left(s \mid y^{k-1}\right) = \frac{1}{N} \sum \delta_{s_k^i}(s),$$

where $\delta(\cdot)$ is the Dirac delta function.

[13] There are various approaches to resampling but in this paper we follow an algorithm called systemic resampling suggested by Kitagawa (1996).

x_t. We assume that the measurement error of each observable variable follows a normal distribution with the standard deviation of 20% unconditional counterpart of each variable, i.e.,

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = G(s_t) + \begin{bmatrix} \sigma_x e_{x,t} \\ \sigma_\pi e_{\pi,t} \end{bmatrix}, \quad \begin{bmatrix} e_x \\ e_\pi \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

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