

# Technical Appendix for “Bubbles and Total Factor Productivity”

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## Abstract

This note provides the technical details for the paper “Bubbles and Total Factor Productivity” published in AER Papers and Proceedings.

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# 1 The Model

Consider an infinite-horizon economy consisting of households and firms. There is no aggregate uncertainty, but firms face idiosyncratic productivity shocks. Time is discrete and denoted by  $t = 0, 1, 2, \dots$

## 1.1 Households

There is a continuum of identical households of unit mass. Each household is risk neutral and derives utility from a consumption stream  $\{C_t\}$  according to the utility function,  $\sum_{t=0}^{\infty} \beta^t C_t$ , where  $\beta = 1/(1+r)$  is the subjective discount factor. Households supply labor inelastically. The labor supply is normalized to one. Households trade firm stocks and risk-free bonds. The net supply of bonds is zero and the net supply of any stock is one. Because there is no aggregate uncertainty,  $r$  is equal to the risk-free rate (or interest rate) and also equal to the rate of the return of each stock.

## 1.2 Firms

There is a continuum of firms of unit mass. Firms are indexed by  $j \in [0, 1]$ . Each firm  $j$  combines labor  $N_t^j$  and capital  $K_t^j$  to produce output according to the following Cobb-Douglas production function:

$$Y_t^j = (A_t^j K_t^j)^\alpha (N_t^j)^{1-\alpha}, \quad \alpha \in (0, 1),$$

where  $A_t^j$  represents idiosyncratic productivity shocks. These shocks follow a Markov process with the state space  $\{A_1, A_2\}$  and with the transition probabilities given by:

$$\Pr(A_{t+1}^j = A_1 | A_t^j = A_1) = 1 - \lambda\rho, \quad (1)$$

$$\Pr(A_{t+1}^j = A_2 | A_t^j = A_2) = 1 - \rho, \quad (2)$$

where  $\rho, \lambda > 0$ . Assume that  $A_t^j$  is independent across firms and thus idiosyncratic risks wash out in the aggregate. Let  $A_0^j$  be drawn from the stationary distribution  $(1/(1+\lambda), \lambda/(1+\lambda))$ . Assume that  $A_1 > A_2$  and  $\rho < 1 - \rho\lambda$ , meaning that the chance of being productive is higher if the firm is relatively more productive in the previous period.

After solving the static labor choice problem, we obtain the operating profits:

$$A_t^j R_t K_t^j = \max_{N_t^j} (A_t^j K_t^j)^\alpha (N_t^j)^{1-\alpha} - w_t N_t^j, \quad (3)$$

where  $w_t$  is the wage rate and

$$R_t = \alpha \left( \frac{w_t}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}}. \quad (4)$$

After observing  $A_t^j$ , firm  $j$  may make investment  $I_t^j$  so that the law of motion for capital is given by:

$$K_{t+1}^j = (1-\delta) K_t^j + I_t^j, \quad (5)$$

where  $\delta > 0$  is the depreciation rate of capital. Assume that investment is subject to the following constraint:

$$-\mu K_t^j \leq I_t^j \leq A_t^j R_t K_t^j + L_t^j, \quad (6)$$

where  $\mu \in (0, 1-\delta)$  and  $L_t^j > 0$ . The first inequality captures the assumption that investment is partially irreversible. The second inequality says that firms can finance investment by internal funds and external borrowing. Assume that external equity is so costly that no firms will raise new equity to finance investment. For simplicity, we consider intratemporal loans as in Charles T. Carlstrom and Timothy Fuerst (1997), Urban J. Jermann and Vincenzo Quadrini (2010), and Miao and Wang (2011a). These loans are taken at the beginning of the period and repaid at the end of the period. They do not have interests.<sup>1</sup>

To capture financial frictions, we introduce credit constraints as in Miao and Wang (2011). In doing so, let  $V_t(K_t^j, A_t^j)$  denote the date- $t$  cum-dividends stock market value of firm  $j$  with assets  $K_t^j$  and the realized productivity shock  $A_t^j$ . Then we write the credit constraint as:

$$L_t^j \leq \beta E_t V_{t+1}(\xi K_t^j, A_{t+1}^j), \quad (7)$$

where  $E_t$  is the conditional expectation operator with respect to the shock  $A_{t+1}^j$ . The motivation of this constraint is similar to that in Nobuhiro Kiyotaki and John Moore (1997): Firm  $j$  pledges a fraction  $\xi \in (0, 1]$  of its assets (capital stock)  $K_t^j$  at the beginning of period  $t$  as the collateral. The parameter  $\xi$  may represent the degree of pledgeability or the extent of financial market

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<sup>1</sup>Miao and Wang (2011a) study intertemporal bonds with interest payments and allow firms to save. This extension does not change our key insights.

imperfections. It is the key parameter for our analysis below. At the end of period  $t$ , the stock market value of the collateral is equal to  $\beta E_t V_{t+1}(\xi K_t^j, A_{t+1}^j)$ . The lender never allows the loan repayment  $L_t^j$  to exceed this value. If this condition is violated, then firm  $j$  may take loans  $L_t^j$  and walk away, leaving the collateralized assets  $\xi K_t^j$  behind. In this case, the lender runs the firm with the collateralized assets  $\xi K_t^j$  at the beginning of period  $t+1$  and obtains the smaller firm value  $\beta E_t V_{t+1}(\xi K_t^j, A_{t+1}^j)$  at the end of period  $t$ . Unlike Kiyotaki and Moore (1997), we have implicitly assumed that firm assets are not specific to a particular owner. Any owner can operate the assets using the same technology. Thus, the lender does not have to liquidate the collateralized assets in the event of default.

Following Miao and Wang (2011a), we may interpret the collateral constraint in (7) as an incentive constraint in an optimal contract between firm  $j$  and the lender with limited commitment.<sup>2</sup> Given a history of information at date  $t$  and after observing the idiosyncratic shock  $A_t^j$ , the contract specifies investments  $I_t^j$  and loans  $L_t^j$  at the beginning of period  $t$ , and repayments  $L_t^j$  at the end of period  $t$ . Firm  $j$  may default on debt at the end of period  $t$ . If it happens, then the firm and the lender renegotiate the loan repayment. In addition, the lender reorganizes the firm. Because of default costs, the lender can only seize a fraction  $\xi$  of capital  $K_t^j$ . The lender can run the firm with these assets at the beginning of period  $t+1$  after observing the productivity shock  $A_{t+1}^j$ . The date- $t$  stock market value of the firm is given by  $\beta E_t V_{t+1}(\xi K_t^j, A_{t+1}^j)$ . This value is the threat value (or the collateral value) to the lender at the end of period  $t$ . Following Jermann and Quadrini (2010), we assume that the firm has all the bargaining power in the renegotiation and the lender gets only the threat value. The key difference between our modeling and that of Jermann and Quadrini (2010) is that the threat value to the lender is the going concern value in our model, while Jermann and Quadrini (2010) assume that the lender liquidates the firm's assets and obtains the liquidation value in the event of default.

Enforcement requires that, the continuation value to the firm of not defaulting is not smaller

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<sup>2</sup>See Rui Albuquerque and Hugo A. Hopenhayn (2004) and Fernando Alvarez and Jermann (2000) for related contracting problems.

than the continuation value of defaulting, that is,

$$\begin{aligned} \beta E_t V_{t+1}(K_{t+1}^j, A_{t+1}^j) - L_t^j &\geq \\ \beta E_t V_{t+1}(K_{t+1}^j, A_{t+1}^j) - \beta E_t V_{t+1}(\xi K_t^j, A_{t+1}^j), \end{aligned}$$

where  $E_t$  is the conditional expectation operator with respect to  $A_{t+1}^j$ . This incentive constraint is equivalent to the collateral constraint in (7). This constraint ensures that there is no default in an optimal contract.

Firm value  $V_t(K_t^j, A_t^j)$  satisfies the following Bellman equation:

$$\begin{aligned} V_t(K_t^j, A_t^j) &= \max_{I_t^j} A_t^j R_t K_t^j - I_t^j \\ &+ \beta E_t V_{t+1}(K_{t+1}^j, A_{t+1}^j), \end{aligned} \tag{8}$$

subject to (5), (6) and (7).

### 1.3 Competitive Equilibrium

Let  $N_t = \int_0^1 N_t^j dj$ , and  $Y_t = \int_0^1 Y_t^j dj$  denote the aggregate labor demand, and aggregate output. Let  $K_{it} = \int_{A_t^j=A_i} K_t^j dj$  and  $I_{it} = \int_{A_t^j=A_i} I_t^j dj$  denote the aggregate capital stock and the aggregate investment for firms with productivity  $A_i$ ,  $i = 1, 2$ . Then a competitive equilibrium is defined as sequences of  $\{Y_t\}$ ,  $\{C_t\}$ ,  $\{K_{it}\}$ ,  $\{I_{it}\}$ ,  $\{N_t\}$ ,  $\{w_t\}$ ,  $\{R_t\}$ ,  $\{V_t(K_t^j, A_t^j)\}$ ,  $\{I_t^j\}$ ,  $\{K_t^j\}$ ,  $\{N_t^j\}$  and  $\{L_t^j\}$  such that households and firms optimize and markets clear in that  $N_t = 1$ ,

$$C_t + I_{1t} + I_{2t} = Y_t, \tag{9}$$

$$\begin{aligned} K_{1t+1} &= [(1 - \delta) K_{1t} + I_{1t}] (1 - \rho\lambda) \\ &+ [(1 - \delta) K_{2t} + I_{2t}] \rho, \end{aligned} \tag{10}$$

$$\begin{aligned} K_{2t+1} &= [(1 - \delta) K_{2t} + I_{2t}] (1 - \rho) \\ &+ [(1 - \delta) K_{1t} + I_{1t}] \lambda\rho. \end{aligned} \tag{11}$$

Before we study equilibria below, we first observe that problem (3) implies:

$$(1 - \alpha) \left( A_t^j K_t^j \right)^\alpha \left( N_t^j \right)^{-\alpha} = w_t,$$

$$R_t = \alpha \left( A_t^j K_t^j \right)^{\alpha-1} \left( N_t^j \right)^{-\alpha}.$$

Using these two equations and  $N_t = 1$ , we deduce the following equilibrium equations:

$$Y_t = (A_1 K_{1t} + A_2 K_{2t})^\alpha N_t^{1-\alpha} = (A_1 K_{1t} + A_2 K_{2t})^\alpha, \quad (12)$$

and

$$R_t = \alpha (A_1 K_{1t} + A_2 K_{2t})^{\alpha-1}. \quad (13)$$

## 2 Bubbleless Equilibrium

In a bubbleless equilibrium, we conjecture that firm value takes the following form:

$$V_t(K, A_i) = v_{it}K, \quad i = 1, 2, \quad (14)$$

where  $v_{it}$  is to be determined. Define

$$Q_{1t} = \beta [v_{1t+1}(1 - \rho\lambda) + v_{2t+1}\rho\lambda], \quad (15)$$

$$Q_{2t} = \beta [v_{1t+1}\rho + v_{2t+1}(1 - \rho)]. \quad (16)$$

Let the optimal investment rate for a firm with productivity  $A_l$  be  $i_{lt} = I_{lt}/K_{lt}$ ,  $l = 1, 2$ . Then we substitute (14), (15) and (16) into (8) to derive:

$$v_{1t}K_t^j = A_1 R_t K_t^j + (Q_{1t} - 1)i_{1t}K_t^j + (1 - \delta)Q_{1t}K_t^j,$$

$$v_{2t}K_t^j = A_2 R_t K_t^j + (Q_{2t} - 1)i_{2t}K_t^j + (1 - \delta)Q_{2t}K_t^j.$$

Matching coefficients yields:

$$v_{1t} = A_1 R_t + (Q_{1t} - 1)i_{1t} + (1 - \delta)Q_{1t},$$

$$v_{2t} = A_2 R_t + (Q_{2t} - 1)i_{2t} + (1 - \delta)Q_{2t}.$$

Plugging these two equations into (15) and (16) yields:

$$\begin{aligned} Q_{1t} &= \beta(1 - \rho\lambda)[A_1 R_{t+1} + (1 - \delta)Q_{1t+1} + (Q_{1t+1} - 1)i_{1t}] \\ &\quad + \beta\rho\lambda[A_2 R_{t+1} + (1 - \delta)Q_{2t+1} + (Q_{2t+1} - 1)i_{2t}], \end{aligned} \quad (17)$$

$$\begin{aligned}
Q_{2t} &= \beta\rho[A_1R_{t+1} + (1 - \delta)Q_{1t+1} + (Q_{1t+1} - 1)i_{1t}] \\
&\quad + \beta(1 - \rho)[A_2R_{t+1} + (1 - \delta)Q_{2t+1} + (Q_{2t+1} - 1)i_{2t}].
\end{aligned} \tag{18}$$

We next show how to solve for  $i_{1t}$  and  $i_{2t}$  below.

## 2.1 The First Best

We start with the first best equilibrium in which the credit constraint never binds. In this case,  $Q_{1t} = 1$  and  $Q_{2t} < 1$ . Because  $Q_{1t} = 1$ , the optimal investment level for any firm with productivity  $A_1$  is indeterminate. Only aggregate investment  $I_{1t}$  for these firms is determined in equilibrium. We shall focus on the symmetric case in which  $i_{1t} = I_{1t}/K_{1t}$  for any high productivity firm. Because  $Q_{2t} < 1$ ,  $i_{2t} = -\mu$ . Equation (17) becomes

$$\begin{aligned}
1 &= \beta(1 - \rho\lambda)[A_1R_{t+1} + (1 - \delta)] \\
&\quad + \beta\rho\lambda[A_2R_{t+1} + (1 - \delta)Q_{2t+1} - \mu(Q_{2t+1} - 1)],
\end{aligned} \tag{19}$$

The first best equilibrium is characterized by four equations (10), (11), (18), and (19) for four variables  $Q_{2t}$ ,  $I_{1t} = i_{1t}K_{1t}$ ,  $K_{1t}$ , and  $K_{2t}$ , where  $R_t$  is given by (13). The usual transversality condition must also be satisfied. Once these variables are determined, we can then solve for consumption  $C_t$  and output  $Y_t$  using (9) and (12).

## 2.2 Inefficient Bubbleless Equilibrium

In an inefficient bubbleless equilibrium,  $Q_{1t} > 1$  and  $Q_{2t} \leq 1$  in the neighborhood of a steady state. In this case, any firm with productivity  $A_1$  chooses the maximal investment level and the credit constraint binds. Its optimal investment satisfies:

$$I_t^j = A_1R_tK_t^j + \xi Q_{1t}K_t^j. \tag{20}$$

Aggregate investment for these firms is given by:

$$I_{1t} = A_1R_tK_{1t} + \xi Q_{1t}K_{1t} = i_{1t}K_{1t}.$$

If  $Q_{2t} < 1$ , then the investment rate of any firm with productivity  $A_2$  reaches the lower bound  $-\mu$ . If  $Q_{2t} = 1$ , then its investment rate is indeterminate. Only the aggregate investment level

$I_{2t}$  of these firms is determined in equilibrium. We shall focus on the symmetric case in which  $i_{2t} = I_{2t}/K_{2t}$  for any low productivity firm. The bubbleless equilibrium is characterized by four equations, (10), (11), (17) and (18) for four variables  $Q_{1t}, I_{2t}, K_{1t}$  and  $K_{2t}$  if  $Q_{2t} = 1$ , and for  $Q_{1t}, Q_{2t}, K_{1t}$  and  $K_{2t}$  if  $Q_{2t} < 1$ . Moreover, the usual transversality condition must be satisfied.

### 3 Bubbly Equilibrium

In a bubbly equilibrium, we conjecture that firm value takes the following form:

$$V_t(K, A_i) = v_{it}K + b_{it}, \quad (21)$$

where  $v_{it}$  and  $b_{it}$  are to be determined. Due to limited liability, stock prices cannot be negative. Thus, we require  $b_{it} > 0$  and interpret it as a bubble. Define  $Q_{1t}$  and  $Q_{2t}$  as in (15) and (16) and define

$$B_{1t} = \beta [b_{1t+1}(1 - \rho\lambda) + b_{2t+1}\rho\lambda], \quad (22)$$

$$B_{2t} = \beta [b_{2t+1}(1 - \rho) + b_{1t+1}\rho]. \quad (23)$$

We will construct a bubbly equilibrium in which  $Q_{1t} > 1$ , and  $Q_{2t} < 1$  around a steady state. Substituting the conjecture (21) into the Bellman equation (8), we find that optimal investment satisfies:

$$I_t^j = \begin{cases} A_1 R_t K_t^j + \xi Q_{1t} K_t^j + B_{1t} & A_t^j = A_1 \\ -\mu K_t^j & A_t^j = A_2 \end{cases}. \quad (24)$$

Thus,  $i_{2t} = I_{2t}/K_{2t} = -\mu$  and

$$I_{1t} = i_{1t}K_{1t} = A_1 R_t K_{1t} + \xi Q_{1t} K_{1t} + B_{1t}/(1 + \lambda). \quad (25)$$

Manipulating the Bellman equation (8) and matching coefficients, we obtain equations (17), (18), and

$$B_{1t} = \beta [B_{1t+1}Q_{1t+1}(1 - \rho\lambda) + \rho\lambda B_{2t+1}], \quad (26)$$

$$B_{2t} = \beta [B_{2t+1}(1 - \rho) + \rho B_{1t+1}Q_{1t+1}]. \quad (27)$$

The bubbly equilibrium is characterized by six equations (10), (11), (17), (18), (26), and (27) for six variables,  $Q_{1t}, Q_{2t}, K_{1t}, K_{2t}, B_{1t}$ , and  $B_{2t}$ . In addition, the usual transversality condition must be satisfied.



## 4 Steady States

To analyze the existence of equilibrium discussed in the previous section, we shall focus on the steady state. To facilitate analysis, we shall set  $A_1 = 1$  and  $A_2 = 0$ . We use a variable without a subscript  $t$  to denote its steady state value. The crucial parameter for our analysis is  $\xi$ , the degree of pledgeability or the fraction of assets recovered by the lender in the event of default. We shall show below that there are three cutoff values  $\xi_1, \xi_2$ , and  $\xi_3$  such that four cases can happen:

1. If  $\xi > \xi_1$ , then the economy achieves the first best equilibrium in which  $Q_1 = 1$  and  $Q_2 < 1$ .
2. If  $\xi_2 < \xi < \xi_1$ , then there is a unique bubbleless equilibrium in which  $Q_1 > 1$  and  $Q_2 < 1$ .
3. If  $\xi_3 < \xi < \xi_2$ , then there is a bubbly equilibrium in which  $Q_1 > 1$  and  $Q_2 < 1$ . There is also a bubbleless equilibrium in which  $Q_1 > 1$  and  $Q_2 < 1$ .
4. If  $\xi < \xi_3$ , then there is a bubbleless equilibrium in which  $Q_1 > 1$  and  $Q_2 = 1$ . There is also a bubbly equilibrium in which  $Q_1 > 1$  and  $Q_2 < 1$ . Note that TFP rises in a bubbly equilibrium only in this case.

### 4.1 First Best

Let  $Q_1 = 1$  and  $Q_2 < 1$ . Then  $i_2 = -\mu$ . Using (19) and (18), we obtain

$$1 = \beta(1 - \rho\lambda) [R + (1 - \delta)] + \beta\rho\lambda[\mu(1 - Q_2) + (1 - \delta)Q_2],$$

and

$$Q_2 = \beta\rho[R + (1 - \delta)] + \beta(1 - \rho)[\mu(1 - Q_2) + (1 - \delta)Q_2].$$

Solving yields:

$$\begin{aligned} Q_2 &= \frac{\rho/\beta + \mu(1 - \rho - \lambda\rho)}{(1 - \lambda\rho)/\beta - (1 - \rho - \lambda\rho)(1 - \delta - \mu)}, \\ R &= \frac{1 - \beta\rho\lambda[\mu + (1 - \delta - \mu)Q_2]}{\beta(1 - \rho\lambda)} - 1 + \delta. \end{aligned} \tag{28}$$

It is easy to show that  $Q_2 < 1$ .

Using (10) and (11), we obtain

$$K_1 = (1 - \delta + i_1)K_1(1 - \rho\lambda) + \rho(1 - \delta - \mu)K_2,$$

$$K_2 = (1 - \delta - \mu)K_2(1 - \rho) + \lambda\rho(1 - \delta + i_1)K_1.$$

Solving these two equations yields

$$\frac{K_1}{K_2} = \frac{\mu + \delta + \rho - \rho(1 + \lambda)(\delta + \mu)}{\lambda\rho}, \quad (29)$$

$$i_1 = \frac{1}{1 - \rho\lambda} \left[ 1 - \frac{\lambda\rho^2(1 - \delta - \mu)}{\mu + \delta + \rho - \rho(1 + \lambda)(\delta + \mu)} \right] - 1 + \delta. \quad (30)$$

For the credit constraint to not bind, we need

$$R + \xi > i_1.$$

This inequality gives the cutoff value  $\xi_1$ :

$$\xi_1 = i_1 - R, \quad (31)$$

where  $R$  and  $i_1$  are given by equations (28) and (30), respectively. When  $\xi > \xi_1$ , the economy achieves the first best steady state. Finally, equation (13) implies that

$$R = \alpha(K_1)^{\alpha-1}. \quad (32)$$

Using this equation and equation (28) delivers  $K_1$ . Using (29) gives  $K_2$ .

In the analysis below, we assume  $\xi < \xi_1$ . Under this assumption, the credit constraint binds whenever a more productive firm chooses to invest. Thus,  $Q_{1t} > 1$  around the steady state.

## 4.2 Bubbleless Steady State

There are two cases. First,  $Q_1 > 1$  and  $Q_2 < 1$ . In this case,

$$I_{1t} = R_t K_{1t} + \xi Q_{1t} K_{1t}, \quad I_{2t} = -\mu K_{2t}.$$

Thus,

$$i_{1t} = R_t + \xi Q_{1t}, \quad i_{2t} = -\mu.$$

It follows from equations (10) and (11) that

$$\begin{aligned} K_1 &= [(1 - \delta)K_1 + (RK_1 + \xi Q_1 K_1)](1 - \rho\lambda) \\ &\quad + (1 - \delta - \mu)K_2\rho, \end{aligned} \quad (33)$$

$$\begin{aligned} K_2 &= (1 - \delta - \mu)K_2(1 - \rho) \\ &\quad + \lambda\rho[(1 - \delta)K_1 + (RK_1 + \xi Q_1 K_1)]. \end{aligned} \quad (34)$$

Using these two equations, we obtain  $K_1/K_2$  given in (29). By equations (17) and (18),

$$\begin{aligned} Q_1 &= \beta(1 - \rho\lambda)[Q_1 R + (1 - \delta)Q_1 + (Q_1 - 1)\xi Q_1] \\ &\quad + \beta\rho\lambda[\mu(1 - Q_2) + (1 - \delta)Q_2] \end{aligned} \quad (35)$$

$$\begin{aligned} Q_2 &= \beta\rho[Q_1 R + (1 - \delta)Q_1 + (Q_1 - 1)\xi Q_1] \\ &\quad + \beta(1 - \rho)[(1 - \delta)Q_2 + \mu(1 - Q_2)] \end{aligned} \quad (36)$$

By (33),

$$1 = (1 - \delta + R + \xi Q_1)(1 - \rho\lambda) + (1 - \delta - \mu)\frac{K_2}{K_1}\rho \quad (37)$$

The above three equations can be used to solve for three variables  $Q_1$ ,  $Q_2$  and  $R$  as functions of  $\xi$ . In particular, we can use equations (35) and (36) to derive

$$\frac{Q_2}{\beta\rho} - \frac{Q_1}{\beta(1 - \rho\lambda)} = \left[ \frac{1 - \rho}{\rho} - \frac{\rho\lambda}{1 - \rho\lambda} \right] [\mu(1 - Q_2) + (1 - \delta)Q_2]. \quad (38)$$

For  $Q_1 > 1$ , we need  $\xi < \xi_1$ , where  $\xi_1$  is the cutoff value such that  $Q_1 = 1$ . It is given by (31). For  $Q_2 < 1$ , we need  $\xi > \xi_3$ , where  $\xi_3$  is the cutoff value such that  $Q_2 = 1$ . We shall solve this cutoff value explicitly below.

Turn to the second case in which  $Q_1 > 1$  and  $Q_2 = 1$ . Setting  $Q_2 = 1$  in (38) yields:

$$Q_1 = \frac{1 - \rho\lambda}{\rho} + \frac{\beta(\rho + \lambda\rho - 1)(1 - \delta)}{\rho}. \quad (39)$$

Setting  $Q_2 = 1$  in (35), we can solve for  $R$ :

$$R = \frac{Q_1 - \beta\rho\lambda(1 - \delta)}{\beta(1 - \rho\lambda)Q_1} - (1 - \delta) - (Q_1 - 1)\xi. \quad (40)$$

We then use (32) to determine  $K_1$ .

We now solve for  $K_2$  and  $i_2$  using equations (10) and (11). Substituting  $i_1 = RK_1 + \xi Q_1 K_1$  into the steady-state version of these two equations yields:

$$\begin{aligned} K_1 &= [(1 - \delta)K_1 + RK_1 + \xi Q_1 K_1] (1 - \rho\lambda) \\ &\quad + (1 - \delta + i_2)K_2\rho, \end{aligned} \quad (41)$$

$$\begin{aligned} K_2 &= (1 - \delta + i_2)K_2(1 - \rho) \\ &\quad + \lambda\rho[(1 - \delta)K_1 + RK_1 + \xi Q_1 K_1]. \end{aligned} \quad (42)$$

Eliminating  $i_2$  yields:

$$\frac{K_1}{\rho} - \frac{K_2}{1 - \rho} = \frac{(1 - \lambda\rho - \rho)(1 - \delta + R + \xi Q_1)}{\rho(1 - \rho)} K_1. \quad (43)$$

Using (39) and (40), we can compute

$$1 - \delta + R + \xi Q_1 = \frac{1}{\beta(1 - \rho\lambda)} - \frac{\rho\lambda(1 - \delta)}{(1 - \rho\lambda)Q_1} + \xi.$$

Plugging this equation into (43) yields:

$$K_2 = \left[ (1 - \rho) - (1 - \lambda\rho - \rho) \left( \frac{1}{\beta(1 - \rho\lambda)} - \frac{\rho\lambda(1 - \delta)}{(1 - \rho\lambda)Q_1} + \xi \right) \right] \frac{K_1}{\rho}. \quad (44)$$

We then use (41) to solve for  $i_2$ :

$$i_2 = \frac{1 - (1 - \delta + R + \xi Q_1)(1 - \rho\lambda)}{\frac{K_2}{K_1}\rho} - 1 + \delta. \quad (45)$$

To be consistent with equilibrium, we require  $i_2 \geq -\mu$ . Note that  $K_2/K_1$  decreases in  $\xi$  and  $R + \xi Q_1$  increases in  $\xi$ . But we can show that  $i_2$  decreases in  $\xi$ . Define the cutoff value  $\xi_3$  such that  $i_2 = -\mu$ . We then deduce that if  $\xi < \xi_3$ ,  $i_2 \geq \mu$ .

To solve for  $\xi_3$ , we set  $i_2 = -\mu$  in (45) to derive:

$$(1 - \mu - \delta) \frac{K_2}{K_1} \rho = 1 - (1 - \delta + R + \xi_3 Q_1)(1 - \rho\lambda). \quad (46)$$

When  $i_2 = -\mu$ ,  $K_1/K_2$  is given by (29). Using this equation and the expressions for  $Q_1$  and  $R$  in (39) and (40), we deduce that  $\xi_3$  satisfies:

$$\frac{\lambda\rho^2(1 - \mu - \delta)}{\mu + \delta + \rho - \rho(1 + \lambda)(\delta + \mu)} = 1 - \left( \frac{Q_1 - \beta\rho\lambda(1 - \delta)}{\beta(1 - \rho\lambda)Q_1} + \xi_3 \right) (1 - \rho\lambda).$$

### 4.3 Bubbly Steady State

We now solve for a bubbly equilibrium in which  $Q_1 > 1$  and  $Q_2 < 1$ . In this case,

$$I_1 = RK_1 + \xi Q_1 K_1 + B_1 / (1 + \lambda), \quad i_2 = -\mu. \quad (47)$$

In addition,  $Q_1$  and  $Q_2$  satisfy (35) and (36) and hence (38). Use (26) and (27) to derive

$$B_1 = \beta[Q_1 B_1 (1 - \rho\lambda) + \rho\lambda B_2], \quad (48)$$

$$B_2 = \beta[B_2 (1 - \rho) + \rho B_1 Q_1]. \quad (49)$$

These two equations can be used to solve for:

$$Q_1 = \frac{\beta^{-1}}{1 - \rho\lambda + \rho\lambda \frac{\rho}{\beta^{-1} - 1 + \rho}} > 1, \quad (50)$$

$$B_2 = \frac{\rho}{(1 - \rho\lambda)(1 - \beta) + \beta\rho} B_1.$$

We can easily check that  $Q_1 > 1$ . By assumption  $\rho < 1 - \rho\lambda$ , we deduce that  $B_2 < B_1$ . We now use (38) to solve for  $Q_2$ :

$$Q_2 = \frac{\rho Q_1 + \beta\mu(1 - \rho - \lambda\rho)}{(1 - \lambda\rho) - (1 - \delta - \mu)\beta(1 - \rho - \lambda\rho)}. \quad (51)$$

Notice that both  $Q_1$  and  $Q_2$  are independent of  $\xi$ . We need to impose assumptions on parameters  $\beta, \rho, \lambda, \mu$  such that  $Q_2 < 1$ .

Use (35) to solve for  $R$ :

$$R = \frac{Q_1 - \beta\rho\lambda[\mu(1 - Q_2) + (1 - \delta)Q_2]}{\beta(1 - \rho\lambda)Q_1} - (1 - \delta) - (Q_1 - 1)\xi. \quad (52)$$

Thus,  $R$  is decreasing in  $\xi$ . Using equations (10) and (11) to eliminate  $I_{1t}$ , we obtain:

$$\frac{K_1}{K_2} = \frac{\mu + \delta + \rho - \rho(1 + \lambda)(\delta + \mu)}{\lambda\rho}.$$

Plugging (47) into (10) yields:

$$K_1 = \left[ (1 - \delta)K_1 + \left( A_1 R K_1 + \xi Q_1 K_1 + \frac{B_1}{\lambda + 1} \right) \right] (1 - \rho\lambda) + (1 - \delta - \mu)K_2\rho. \quad (53)$$

Manipulating yields:

$$1 = \left(1 - \delta + R + \xi Q_1 + \frac{B_1/K_1}{\lambda + 1}\right) (1 - \rho\lambda) + (1 - \delta - \mu)\rho \frac{K_2}{K_1}. \quad (54)$$

Plugging (52) into the above equation yields:

$$\begin{aligned} \frac{B_1/K_1}{\lambda + 1} &= \frac{1 - (1 - \delta - \mu)\rho \frac{K_2}{K_1}}{1 - \rho\lambda} - 1 + \delta - R - \xi Q_1 \\ &= \frac{1 - (1 - \delta - \mu)\rho \frac{K_2}{K_1}}{1 - \rho\lambda} - \frac{Q_1 - \beta\rho\lambda[\mu(1 - Q_2) + (1 - \delta)Q_2]}{\beta(1 - \rho\lambda)Q_1} - \xi. \end{aligned} \quad (55)$$

Since  $K_2/K_1$ ,  $Q_1$ , and  $Q_2$  are independent of  $\xi$ ,  $B_1/K_1$  is a decreasing affine function of  $\xi$ . We define the cutoff value  $\xi_2$  such that  $B_1/K_1 = 0$ . Then if  $\xi < \xi_2$ , then  $B_1/K_1 > 0$  and we obtain a bubbly equilibrium.

Note that we must have  $\xi_2 < \xi_1$  because the credit constraint does not bind for  $\xi > \xi_1$ . Also note that when  $\xi = \xi_2$ ,  $B_1 = B_2 = 0$  and the bubbly equilibrium reduces to a bubbleless equilibrium with  $Q_1 > 1$  and  $Q_2 < 1$ . Since we have shown that  $Q_1 > 1$  and  $Q_2 = 1$  in a bubbleless equilibrium for any  $\xi < \xi_3$ , we must have  $\xi_3 < \xi_2$ .

## 5 TFP

We can compute the steady state TFP as

$$TFP = \left(\frac{A_1 K_1 + A_2 K_2}{K_1 + K_2}\right)^\alpha.$$

Thus, to show that TFP rises in a bubbly equilibrium, we only need to show that  $K_1/K_2$  rises too. Using equations (10) and (11), we can show that

$$\frac{K_1}{K_2} = \frac{1 - \lambda\rho - (1 - \delta + I_2/K_2)(1 - \rho - \rho\lambda)}{\lambda\rho}.$$

If  $Q_2 = 1$  in a bubbleless equilibrium, then it is possible that aggregate investment for the less productive firms does not reach the lower bound so that  $I_{2t} > -\mu K_{2t}$ . If  $Q_2 < 1$  and  $I_{2t} = -\mu K_{2t}$  in a bubbly equilibrium, then the above equation reveals that  $K_1/K_2$  is higher in a bubbly equilibrium than in a bubbleless equilibrium given the assumption  $1 - \rho - \rho\lambda > 0$ , and hence TFP rises in a bubbly equilibrium. This result can hold only in the last case described at the beginning of Section 4.