Supplementary Materials for
"Asset Bubbles, Collateral, and Policy Analysis"

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Section A provides proofs of all propositions in the main text. Section B provides proofs of results in Section 5.3 of the main text. Section C analyzes the impact of foreign purchases of domestic bonds. Section D describes data sources for Figure 1 in the main text.

A Proofs of all propositions

Proof of Proposition 1: We substitute the conjectured value function into the Bellman equation and write it as

$$
\begin{aligned}
& v_t(\tau_{jt})K_{jt} + p_t(\tau_{jt})H_{jt} - \varphi_t(\tau_{jt})B_{jt} \\
= & \max_{H_{jt},I_{jt},B_{jt+1}} R_{kt}K_{jt} + P_tH_{jt} - B_{jt} - \tau_{jt}I_{jt} - P_tH_{jt+1} + \frac{B_{jt+1}}{R_{ft}} \\
& + \beta \int \frac{\Lambda_{t+1}}{\Lambda_t} v_{t+1}(\tau) K_{jt+1} f(\tau) d\tau + \beta \int \frac{\Lambda_{t+1}}{\Lambda_t} p_{t+1}(\tau) H_{jt+1} f(\tau) d\tau \\
& - \beta \int \frac{\Lambda_{t+1}}{\Lambda_t} \varphi_{t+1}(\tau) B_{jt+1} f(\tau) d\tau
\end{aligned}
$$

subject to

$$K_{jt+1} = (1-\delta) K_{jt} + I_{jt} \tag{A.1}$$

$$\frac{B_{jt+1}}{R_{ft}} \leq \theta P_t H_{jt+1}, \tag{A.2}$$

$$H_{jt+1} \geq \omega H_{jt}, \tag{A.3}$$

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and

$$0 \leq \tau_{jt} I_{jt} \leq R_{kt} K_{jt} + \frac{B_{jt+1}}{R_{ft}} - B_{jt} - P_t (H_{jt+1} - H_{jt}). \tag{A.4}$$

Defining $Q_t = \beta \int \frac{\Lambda_{t+1}}{\Lambda_t} v_{t+1}(\tau) f(\tau) d\tau$ and plugging (A.1) into the Bellman equation above, we obtain (10) in the main text. More specifically, if $P_t > \beta \int \frac{\Lambda_{t+1}}{\Lambda_t} p_{t+1}(\tau) f(\tau) d\tau$, then each entrepreneur j wants to sell land so that $H_{jt+1} = \omega H_{jt}$ all all j. Then the land market cannot clear. But if $P_t < \beta \int \frac{\Lambda_{t+1}}{\Lambda_t} p_{t+1}(\tau) f(\tau) d\tau$, then each entrepreneur j wants to buy as much land as possible. Thus we must have $P_t = \beta \int \frac{\Lambda_{t+1}}{\Lambda_t} p_{t+1}(\tau) f(\tau) d\tau$. Similarly we must have $1/R_{ft} = \beta \int \frac{\Lambda_{t+1}}{\Lambda_t} \varphi_{t+1}(\tau) f(\tau) d\tau$.

Now the Bellman equation becomes

$$\begin{aligned} & v_t(\tau_{jt}) K_{jt} + p_t(\tau_{jt}) H_{jt} - \varphi_t(\tau_{jt}) B_{jt} \\ = & \max_{H_{jt+1}, I_{jt}, B_{jt+1}} R_{kt} K_{jt} + P_t H_{jt} - B_{jt} - \tau_{jt} I_{jt} + Q_t I_{jt} + Q_t (1-\delta) K_{jt}. \end{aligned} \tag{A.5}$$

(i) By (A.5) and (A.4), when $\tau_{jt} \leq Q_t$, we must have

$$\tau_{jt} I_{jt} = R_{kt} K_{jt} + \frac{B_{jt+1}}{R_{ft}} - B_{jt} - P_t (H_{jt+1} - H_{jt}). \tag{A.6}$$

In addition, it follows from (A.2) and (A.3) that both the borrowing and resaleability constraints bind. When $\tau_{jt} > Q_t$, it follows that $I_{jt} = 0$. Because B_{jt+1} and H_{jt+1} are canceled out in the objective of (A.5), the entrepreneur is indifferent among the feasible choices of B_{jt+1} and H_{jt+1}.

(ii) Substituting the decision rules in part (i) into (A.5) and matching coefficients, we obtain

$$v_t(\tau_{jt}) = \begin{cases} \frac{Q_t}{\tau_{jt}} R_{kt} + (1-\delta) Q_t & \text{if } \tau_{jt} \leq Q_t, \\ R_{kt} + (1-\delta) Q_t & \text{if } \tau_{jt} > Q_t \end{cases}, \tag{A.7}$$

$$p_t(\tau_{jt}) = \begin{cases} P_t + (1 - \omega + \omega\theta) \left(\frac{Q_t}{\tau_{jt}} - 1 \right) P_t & \text{if } \tau_{jt} \leq Q_t, \\ P_t & \text{if } \tau_{jt} > Q_t \end{cases}, \tag{A.8}$$

$$\varphi_t(\tau_{jt}) = \begin{cases} \frac{Q_t}{\tau_{jt}} & \text{if } \tau_{jt} \leq Q_t, \\ 1 & \text{if } \tau_{jt} > Q_t \end{cases}. \tag{A.9}$$

Using equation (10) in the main text of the paper, and the definition of Q_t, we can derive equations (11)-(13) in the main text. The transversality conditions follow from the infinite-horizon dynamic optimization problem, e.g., Ekeland and Scheinkman (1986). Q.E.D.

Proof of Proposition 2: By part (i) of Proposition 1, we can derive aggregate investment

$$I_t = \int_{\tau_{jt} \leq Q_t} \frac{1}{\tau_{jt}} \left[R_{kt} K_{jt} + (1 - \omega + \theta\omega) P_t H_{jt} - B_{jt} \right] dj.$$

Since τ_{jt} is independently and identically distributed and since K_{jt}, B_{jt}, and H_{jt} are predetermined, τ_{jt} is independent of these variables. By a law of large numbers, we obtain

$$\begin{aligned} I_t &= \frac{\int_{\tau_{jt} \leq Q_t} \frac{1}{\tau_{jt}} \left[R_{kt} K_{jt} + (1 - \omega + \theta\omega) P_t H_{jt} - B_{jt} \right] dj}{\int_{\tau_{jt} \leq Q_t} dj} \int_{\tau_{jt} \leq Q_t} dj \\ &= E\left[\frac{1}{\tau_{jt}} \left[R_{kt} K_{jt} + (1 - \omega + \theta\omega) P_t H_{jt} - B_{jt} \right] | \tau_{jt} \leq Q_t \right] \int_{\tau_{jt} \leq Q_t} dj \\ &= \int_{\tau_{jt} \leq Q_t} \frac{1}{\tau_{jt}} dj \left[R_{kt} \int K_{jt} dj + (1 - \omega + \theta\omega) P_t \int H_{jt} dj - \int B_{jt} dj \right] \\ &= \left[R_{kt} K_t + (1 - \omega + \theta\omega) P_t \right] \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau)\, d\tau, \end{aligned}$$

where we have used the market-clearing conditions to derive the last equality.

By equation (1) in the main text and the labor market-clearing condition,

$$1 = N_t = \int N_{jt} dj = \left(\frac{1-\alpha}{W_t} \right)^{\frac{1}{\alpha}} \int K_{jt} dj = \left(\frac{1-\alpha}{W_t} \right)^{\frac{1}{\alpha}} K_t.$$

From this equation, we can derive other equations in the proposition. Q.E.D.

Proof of Proposition 3: The right hand side of equation (20) in the main text is strictly decreasing in Q_f. When Q_f approaches the lower support of the distribution for τ_{jt}, the right-hand side approaches infinite. When Q_f approaches the upper support of the distribution, the right-hand side approaches $\beta\delta < 1 - \beta(1-\delta)$. Thus, by the Intermediate Value Theorem, there is a unique solution $Q_f \in (\tau_{\min}, \tau_{\max})$ to equation (20) in the main text. Condition (21) in the main text ensures that $C_f > 0$. Q.E.D.

Proof of Propositions 4 and 5: Equation (11) in the main text implies that the bubbly steady-state Tobin's Q, denoted by Q_b, satisfies the equation,

$$\frac{\beta^{-1} - 1}{1 - \omega(1-\theta)} = \int_{\tau \leq Q_b} \frac{Q_b - \tau}{\tau} f(\tau) d\tau. \tag{A.10}$$

By the Intermediate Value Theorem, if

$$\frac{\beta^{-1} - 1}{1 - \omega(1-\theta)} < \tau_{\max} \int \frac{1}{\tau} f(\tau) d\tau - 1, \tag{A.11}$$

then (A.10) has a unique solution $Q_b \in (\tau_{\min}, \tau_{\max})$. We can then derive the steady-state rental rate of capital R_{kb} using equation (12) in the main text,

$$R_{kb} = \frac{1-\beta(1-\delta)}{\beta \int \max\left(\frac{1}{\tau}, \frac{1}{Q_b}\right) f(\tau) d\tau}. \tag{A.12}$$

We then use (18) in the main text to determine the steady-state capital stock K_b. The bubbly steady-state investment, output, and consumption are given by $I_b = \delta K_b$, $Y_b = K_b^{\alpha}$, and $C_b = Y_b - I_b$, respectively. We use equation (15) in the main text to determine the steady-state land price P,

$$\frac{P}{Y_b} = \frac{1}{1-\omega(1-\theta)} \left[\frac{\delta\alpha\beta}{1-\beta(1-\delta)} \frac{\int \max\left(\frac{1}{\tau}, \frac{1}{Q_b}\right) f(\tau) d\tau}{\int_{\tau \le Q_b} \frac{1}{\tau} f(\tau) d\tau} - \alpha \right]. \tag{A.13}$$

We need $P > 0$ and $C_b > 0$ for the existence of a bubbly steady state.

The proof consists of two parts.

Part I. Suppose that the bubbly and bubbly steady states coexist. We then prove Proposition 5 and the necessity of condition (24) in the main text.

Step 1. We prove $Q_b < Q_f$. By equation (20) in the main text,

$$1-\beta(1-\delta) = \beta\delta \frac{\int \max\left(\frac{1}{\tau}, \frac{1}{Q_f}\right) f(\tau) d\tau}{\int_{\tau \le Q_f} \frac{1}{\tau} f(\tau) d\tau}. \tag{A.14}$$

Equation (A.13) and $P > 0$ imply that

$$R_{kb} < \frac{\delta}{\int_{\tau \le Q_b} \frac{1}{\tau} f(\tau) d\tau}.$$

Combining equation (A.12) and the preceding inequality, we can derive that

$$\begin{aligned} 1-\beta(1-\delta) &= \beta R_{kb} \int \max\left(\frac{1}{\tau}, \frac{1}{Q_b}\right) f(\tau) d\tau \\ &< \beta\delta \left[1 + \frac{1-F(Q_b)}{Q_b \int_{\tau \le Q_b} \frac{1}{\tau} f(\tau) d\tau} \right]. \end{aligned}$$

Combining the preceding inequality with (A.14) yields

$$\beta\delta \frac{\int \max\left(\frac{1}{\tau}, \frac{1}{Q_f}\right) f(\tau) d\tau}{\int_{\tau \le Q_f} \frac{1}{\tau} f(\tau) d\tau} < \beta\delta \left[1 + \frac{1-F(Q_b)}{Q_b \int_{\tau \le Q_b} \frac{1}{\tau} f(\tau) d\tau} \right].$$

This inequality is equivalent to the following inequality:

$$\frac{1-F(Q_b)}{Q_b \int_{\tau \le Q_b} \frac{1}{\tau} f(\tau) d\tau} > \frac{1-F(Q_f)}{Q_f \int_{\tau \le Q_f} \frac{1}{\tau} f(\tau) d\tau}.$$

Thus $Q_b < Q_f$.

Step 2. We prove $R_{kf} > R_{kb}$. The steady-state version of equation (12) in the main text is given by

$$1 = \beta \left[(1-\delta) + R_k \int \max\left(\frac{1}{\tau}, \frac{1}{Q}\right) f(\tau) d\tau \right]. \tag{A.15}$$

The equation above implies that $R_{kf} > R_{kb}$ since $Q_b < Q_f$.

Step 3. Because $R_{kb} = \alpha K_b^{\alpha-1} < R_{kf} = \alpha K_f^{\alpha-1}$, we have $K_b > K_f$. Hence, $Y_b = K_b^\alpha > Y_f = K_b^\alpha$, $I_b = \delta K_b > I_f = \delta K_f$. In addition, equation (13) in the main text implies that $1/R_{fb} = \beta \int \max(\frac{Q_b}{\tau}, 1) f(\tau) d\tau < \beta \int \max(\frac{Q_f}{\tau}, 1) f(\tau) d\tau = 1/R_{ff}$, i.e., $R_{fb} > R_{ff}$.

Step 4. In the bubbly steady state, equation (11) in the main text implies that

$$1 = \beta \left[1 + (1 - \omega + \omega\theta) \int_{\tau \le Q_b} \frac{Q_b - \tau}{\tau} f(\tau) d\tau \right]. \tag{A.16}$$

Since $Q_b < Q_f$, condition (24) in the main text must hold. This proves the necessity of (24) in the main text as well as Proposition 5.

Part II. Now,we suppose that conditions (22), (23), and (24) in the main text hold. We then prove that the bubbly and bubbleless steady states coexist.

Step 1. The right-hand side of (A.10) is strictly increasing in Q_b. It is equal to 0 when $Q_b = \tau_{\min}$ and equal to $\tau_{\max} \int \frac{1}{\tau} f(\tau) d\tau - 1$ when $Q_b = \tau_{\max}$. If condition (A.11) holds, then (A.10) has a unique solution $Q_b \in (\tau_{\min}, \tau_{\max})$ by the Intermediate Value Theorem.

Step 2. By (A.16) and condition (24) in the main text, $Q_b < Q_f$, where Q_f is given by equation (20) in the main text. By Step 2 of Part I, $R_{kb} < R_{kf}$. Condition (23) in the main text implies that $R_{kf} > R_{kb} > \alpha\delta$. By Proposition 3, a bubbleless steady state exists.

Step 3. To show the existence of a bubbly steady state, we must show $C_b > 0$ and $P > 0$. Since

$$C_b = Y_b - I_b = Y_b - \delta K_b = K_b \left(\alpha K_b^{\alpha-1}/\alpha - \delta \right) = K_b \left(R_{kb}/\alpha - \delta \right),$$

condition (23) in the main text ensures that $C_b > 0$ holds. We now check $P > 0$. By (A.13),

$$\begin{aligned}
\frac{P}{Y_b} &= \frac{1}{1-\omega(1-\theta)} \left[\frac{\delta\alpha\beta}{1-\beta(1-\delta)} \frac{\int \max\left(\frac{1}{\tau}, \frac{1}{Q_b}\right) f(\tau) d\tau}{\int_{\tau \le Q_b} \frac{1}{\tau} f(\tau) d\tau} - \alpha \right] \\
&> \frac{1}{1-\omega(1-\theta)} \left[\frac{\delta\alpha\beta}{1-\beta(1-\delta)} \frac{\int \max\left(\frac{1}{\tau}, \frac{1}{Q_f}\right) f(\tau) d\tau}{\int_{\tau \le Q_f} \frac{1}{\tau} f(\tau) d\tau} - \alpha \right] \\
&= \frac{1}{1-\omega(1-\theta)} \left[\frac{\delta\alpha\beta}{1-\beta(1-\delta)} \frac{1-\beta(1-\delta)}{\beta\delta} - \alpha \right] \\
&= 0,
\end{aligned} \tag{A.17}$$

where the first inequality follows from $Q_f > Q_b$ by Step 2 of Part II and the second equality follows from equation (20) in the main text. Q.E.D.

Proof of Proposition 6: Denote by F the cumulative distribution function of τ and define $J(Q_t) = \int_{\tau_{\min}}^{Q_t} \frac{1}{\tau} f(\tau) d\tau$. We can use Proposition 2 to show that the equilibrium system can be described by the following four difference equations:

$$C_t + K_{t+1} - (1-\delta)K_t = K_t^{\alpha}, \tag{A.18}$$

$$\frac{Q_t}{C_t} = \beta \frac{1}{C_{t+1}} \left\{ (1-\delta)Q_{t+1} + \alpha K_{t+1}^{\alpha-1} [Q_{t+1}J(Q_{t+1}) + 1 - F(Q_{t+1})] \right\}, \tag{A.19}$$

$$\frac{P_t}{C_t} = \beta \frac{P_{t+1}}{C_{t+1}} \left\{ 1 + (1-\omega+\omega\theta)[Q_{t+1}J(Q_{t+1}) - F(Q_{t+1})] \right\}, \tag{A.20}$$

$$K_{t+1} - (1-\delta)K_t = [\alpha K_t^{\alpha} + (1-\omega+\omega\theta)P_t] J(Q_t), \tag{A.21}$$

for four unknowns $\{K_t, C_t, Q_t, P_t\}$. Only K_t is predetermined. The other three variables are nonpredetermined.

Linearizing P_t around zero and log-linearizing Q_t, K_t and C_t around their bubbleless steady state values, we obtain

$$\begin{aligned} \frac{C_f}{K_f^{\alpha}}\hat{C}_t + \frac{K_f}{K_f^{\alpha}}\hat{K}_{t+1} - \frac{(1-\delta)K_f}{K_f^{\alpha}}\hat{K}_t &= \alpha \hat{K}_t, \\ \hat{Q}_t - \hat{C}_t &= -\hat{C}_{t+1} - (1-\alpha)[1-\beta(1-\delta)]\hat{K}_{t+1} + \beta \hat{Q}_{t+1}, \\ P_t &= \beta \left\{ 1 + (1-\omega+\omega\theta)[Q_f J(Q_f) - F(Q_f)] \right\} P_{t+1}, \\ \hat{K}_{t+1} - (1-\delta)\hat{K}_t &= \delta \frac{f(Q_f)}{J(Q_f)}\hat{Q}_t + \alpha\delta \hat{K}_t + (1-\omega+\omega\theta)\frac{J(Q_f)}{K_f}P_t, \end{aligned} \tag{A.22}$$

where $\hat{C}_t$, $\hat{K}_t$, and $\hat{Q}_t$ denote log-deviation from the steady state. We rewrite the system in the following matrix form:

$$B \begin{bmatrix} \hat{C}_{t+1} \\ \hat{K}_{t+1} \\ \hat{Q}_{t+1} \\ P_{t+1} \end{bmatrix} = G \begin{bmatrix} \hat{C}_t \\ \hat{K}_t \\ \hat{Q}_t \\ P_t \end{bmatrix},$$

where

$$B = \begin{bmatrix} 0 & -\frac{K_f}{K_f^{\alpha}} & 0 & 0 \\ -1 & -(1-\alpha)(1-\beta+\beta\delta) & \beta & 0 \\ 0 & 0 & 0 & B_{34} \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} \frac{C_f}{K_f^\alpha} & -\frac{(1-\delta)K_f}{K_f^\alpha} - \alpha & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1-\delta+\alpha\delta & \delta\frac{f(Q_f)}{J(Q_f)} & (1-\omega+\omega\theta)\frac{J(Q_f)}{K_f} \end{bmatrix},$$

with

$$B_{34} \equiv \beta\{1 + (1-\omega+\omega\theta)[Q_f J(Q_f) - F(Q_f)]\}.$$

It is straightforward to check that G is invertible.

To study the local dynamics around the bubbleless steady state, we study the eigenvalues of the matrix $M \equiv G^{-1}B$.

First, we check that 0 must be an eigenvalue of matrix M. Note that matrix B is singular because its columns 1 and 3 are linearly dependent. Thus $\det(M) = \det(M - 0 \cdot I) = 0$, implying that 0 is an eigenvalue.

Second, note that

$$\begin{aligned} \det(M - B_{34} \cdot I) &= \det[G^{-1}B - B_{34} \cdot G^{-1}G] \\ &= \det(G^{-1}) \cdot \det(B - B_{34}G) \\ &= 0. \end{aligned}$$

Thus $B_{34} = \beta\{1+(1-\omega+\omega\theta)[Q_f J(Q_f) - F(Q_f)]\}$ is an eigenvalue of matrix M. Let $\lambda_1 \equiv B_{34}$ denote this eigenvalue.

Third, we can show that the other two eigenvalues are positive real numbers, with one greater than 1 and the other smaller than 1. Let λ_2 and λ_3 denote these two eigenvalues. We can then write

$$\det(M - \lambda I) = \lambda(\lambda_1 - \lambda)(-\lambda^2 + b\lambda + c),$$

where

$$\begin{aligned} b \equiv\ & \frac{1}{d}\left\{[1+\beta(1-\delta+\alpha\delta)]\frac{C_f}{K_f^\alpha} + (1-\alpha)(1-\beta+\beta\delta)\delta\frac{f(Q_f)}{J(Q_f)}\frac{C_f}{K_f^\alpha}\right. \\ & \left. +(2-\delta)\alpha f(Q_f) + \alpha\delta\frac{f(Q_f)}{J(Q_f)}\right\}, \end{aligned}$$

$$c \equiv -\frac{1}{d}\left[\beta\frac{C_f}{K_f^\alpha} + \delta K_f^{1-\alpha}\frac{f(Q_f)}{J(Q_f)}\right],$$

$$d \equiv (1-\delta+\alpha\delta)\frac{C_f}{K_f^\alpha} + \alpha(1-\delta)f(Q_f) + \alpha\delta\frac{f(Q_f)}{J(Q_f)} > 0.$$

Since $c < 0$, it follows that $\lambda_2\lambda_3 > 0$. We can also show that

$$-1+b+c = \frac{\delta(1-\alpha)}{d}\frac{C_f}{Y_f}\left[(1-\beta)+(1-\beta+\beta\delta)\frac{f(Q_f)}{J(Q_f)}\right] > 0.$$

Thus the quadratic equation always has two real solutions, with one larger than 1 and the other smaller than 1.

Without loss of generality, we suppose that $\lambda_2 < 1 < \lambda_3$. We then have two eigenvalues (0 and λ_2) inside the unit circle and one (λ_3) outside the unit circle. Whether the local dynamic around the bubbleless steady state is determinate depends on whether $\lambda_1 = \beta\left\{1+(1-\omega+\omega\theta)\left[Q_f J(Q_f) - F(Q_f)\right]\right\}$ is smaller than 1. By Proposition 4, when both the bubbly and bubbleless steady states exist, condition (24) in the main text must hold, i.e.,

$$\lambda_1 = \beta\left\{1+(1-\omega+\omega\theta)\left[Q_f J(Q_f) - F(Q_f)\right]\right\} > 1,$$

implying that the matrix M has two eigenvalues outside the unit circle and two eigenvalues inside the unit circle. Since there are three nonpredetermined variables, this means that the bubbleless steady state is a saddle with indeterminacy of degree 1.

When only the bubbleless steady state exists, we must have

$$\lambda_1 = \beta\left\{1+(1-\omega+\omega\theta)\left[Q_f J(Q_f) - F(Q_f)\right]\right\} \leq 1.$$

If $\lambda_1 < 1$, then the matrix M has three eigenvalues inside the unit circles and one eigenvalue outside the unit circle, implying that the local dynamic is determinate. If $\lambda_1 = 1$, then (A.22) implies that $P_t = P_{t+1}$. Since $\lim_{t\to+\infty} P_t = 0$, it follows that $P_t = 0$ for all t. In both cases, there is a unique equilibrium, which is bubbleless. Q.E.D.

Proof of Proposition 7: See the proofs of Proposition 8 and Lemma 2. Q.E.D.

Proof of Proposition 8: Consider the equilibrium without bubble first. Let $\delta = 1$ and $\omega(1-\theta) = 0$. Equation (12) in the main text becomes

$$\frac{Q_t}{(1-s_t)Y_t} = \frac{\beta}{(1-s_{t+1})Y_{t+1}}\left[\frac{\alpha Y_{t+1}}{s_t Y_t}\int \max\left(\frac{Q_{t+1}}{\tau},1\right)f(\tau)d\tau\right].$$

We can further reduce the above equation to

$$\frac{Q_t}{1-s_t} = \frac{\beta}{1-s_{t+1}}\frac{\alpha}{s_t}\int \max\left(\frac{Q_{t+1}}{\tau},1\right)f(\tau)d\tau. \tag{A.23}$$

Equation (15) in the main text implies that

$$K_{t+1} = I_t = \alpha Y_t \int_{\tau\leq Q_t}\frac{1}{\tau}f(\tau)d\tau = s_t Y_t,$$

or

$$s_t = \alpha \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau. \tag{A.24}$$

Hence the system of two difference equations (A.23) and (A.24) determine the bubbleless equilibrium trajectories for Q_t and s_t. Clearly a constant steady state is a solution to the system. The following lemma shows that this is the unique local solution.

Lemma 1 *There is a unique local solution to the system of two equations (A.23) and (A.24), which is the bubbleless steady state.*

Proof: We use F to denote the cumulative distribution function of τ and define $H(Q_t) \equiv \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau$. In the steady state, equations (A.23) and (A.24) imply that

$$Q_f = \frac{\beta \alpha}{s_f} \left[Q_f H(Q_f) + 1 - F(Q_f) \right],$$

$$s_f = \alpha H(Q_f).$$

Then log-linearizing the system around this steady state, we obtain

$$\hat{Q}_t + \frac{s_f}{1 - s_f} \hat{s}_t = \frac{s_f}{1 - s_f} \hat{s}_{t+1} - \hat{s}_t + \beta \hat{Q}_{t+1},$$

$$\hat{s}_t = \frac{f(Q_f)}{H(Q_f)} \hat{Q}_t.$$

We rewrite the above two equations as

$$B_f \begin{bmatrix} \hat{s}_{t+1} \\ \hat{Q}_{t+1} \end{bmatrix} = G_f \begin{bmatrix} \hat{s}_t \\ \hat{Q}_t \end{bmatrix},$$

where

$$B_f = \begin{bmatrix} \frac{s_f}{1-s_f} & \beta \\ 0 & 0 \end{bmatrix},$$

$$G_f = \begin{bmatrix} \frac{1}{1-s_f} & 1 \\ 1 & -\frac{f(Q_f)}{H(Q_f)} \end{bmatrix}.$$

In order to understand the local dynamics around this steady state, we need to study the two eigenvalues of the matrix $G_f^{-1} B_f$. They are 0 and λ_f where

$$\lambda_f = \frac{\frac{s_f}{1-s_f} \frac{f(Q_f)}{H(Q_f)} + \beta}{\frac{1}{1-s_f} \frac{f(Q_f)}{H(Q_f)} + 1}.$$

It is straightforward that $0<\lambda_f<1$ since $0<\frac{f(Q_f)}{H(Q_f)}\frac{s_f}{1-s_f}<\frac{1}{1-s_f}\frac{f(Q_f)}{H(Q_f)}$ and $0<\beta<1$. This means the two eigenvalues are both inside the unit circle. Therefore there is a unique local solution to the system since both s_t and Q_t are nonpredetermined. ■

The lemma above shows that the steady state is the unique solution to the system of two equations (A.23) and (A.24) for s_t and Q_t in the neighborhood of the bubbleless steady state. We then use these two equations to determine Q_f by

$$\frac{1}{\beta}-1=\frac{1-F(Q_f)}{Q_f\int_{\tau\leq Q_f}\frac{1}{\tau}f(\tau)d\tau}. \tag{A.25}$$

Since $\lim_{Q_f\to\tau_{\min}}\frac{1-F(Q_f)}{Q_f\int_{\tau\leq Q_f}\frac{1}{\tau}f(\tau)d\tau}=+\infty$ and $\lim_{Q_f\to\tau_{\max}}\frac{1-F(Q_f)}{Q_f\int_{\tau\leq Q_f}\frac{1}{\tau}f(\tau)d\tau}=0$, by the Intermediate Value Theorem, there is a unique solution in $(\tau_{\min},\tau_{\max})$. Once Q_f is determined, then the saving rate is given by

$$s_f=\alpha\beta\int\max\left(\frac{1}{\tau},\frac{1}{Q_f}\right)f(\tau)d\tau. \tag{A.26}$$

from equation (A.23).

We need $s_f\in(0,1)$ for a bubbleless equilibrium to exist. This condition is equivalent to (21) in the main text. This is because that equation implies that $\frac{1}{\int_{\tau\leq Q_f}\frac{1}{\tau}f(\tau)d\tau}>\alpha$. By equation (A.24), $s_f=\alpha\int_{\tau\leq Q_f}\frac{1}{\tau}f(\tau)\,d\tau<1$.

We compute the life-time utility as

$$U_f(K_0)=\sum_{t=0}^{\infty}\beta^t[\ln(1-s_f)+\ln(Y_t)]=\frac{\ln(1-s_f)}{1-\beta}+\alpha\sum_{t=0}^{\infty}\beta^t\ln(K_t),$$

where s_f is given by equation (A.26) and

$$\ln(K_{t+1})=\ln(s_fY_t)=\ln(s_f)+\alpha\ln(K_t).$$

Hence, the welfare for any given K_0 is given by

$$U_f(K_0)=\frac{\ln(1-s_f)}{1-\beta}+\frac{\alpha}{1-\alpha\beta}\left[\frac{\beta}{1-\beta}\ln(s_f)+\ln(K_0)\right]. \tag{A.27}$$

Next, consider the equilibrium path to the bubbly steady state. Equation (A.23) still holds. Equation (15) in the main text becomes

$$K_{t+1}=I_t=(\alpha Y_t+P_t)\int_{\tau\leq Q_t}\frac{1}{\tau}f(\tau)d\tau=s_tY_t.$$

Dividing by Y_t on the two sides of this equation yields

$$s_t = (\alpha + p_t) \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau, \tag{A.28}$$

where $p_t = P_t / Y_t$. Using $C_t = (1 - s_t) Y_t$, we rewrite equation (11) in the main text as

$$\frac{p_t}{1 - s_t} = \beta \frac{p_{t+1}}{1 - s_{t+1}} \int \max\left(\frac{Q_{t+1}}{\tau}, 1\right) f(\tau) d\tau. \tag{A.29}$$

Therefore, the system of three difference equations (A.23), (A.28) and (A.29) determine three sequences for s_t, p_t, and Q_t. Clearly, the steady state is a solution to this system. The following lemma shows that it is a unique local solution.

Lemma 2 *There exists a unique solution $s_t = s_b$, $p_t = p_b$ and $Q_t = Q_b$ for all t to the system of three equations (A.23), (A.28) and (A.29) in the neighborhood of the bubbly steady state.*

Proof: Substituting (A.28) into (A.23) and (A.29), we obtain

$$\frac{Q_t}{1 - (\alpha + p_t) \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau} = \frac{\beta \int \max\left(\frac{Q_{t+1}}{\tau}, 1\right) f(\tau) d\tau}{1 - (\alpha + p_{t+1}) \int_{\tau \leq Q_{t+1}} \frac{1}{\tau} f(\tau) d\tau} \frac{\alpha}{(\alpha + p_t) \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau},$$

$$\frac{p_t}{1 - (\alpha + p_t) \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau} = \frac{\beta p_{t+1} \int \max\left(\frac{Q_{t+1}}{\tau}, 1\right) f(\tau) d\tau}{1 - (\alpha + p_{t+1}) \int_{\tau \leq Q_{t+1}} \frac{1}{\tau} f(\tau) d\tau}.$$

As before, denote $J(Q_t) = \int_{\tau \leq Q_t} \frac{1}{\tau} f(\tau) d\tau$. Then $\int \max\left(\frac{Q_{t+1}}{\tau}, 1\right) f(\tau) d\tau = Q_{t+1} J(Q_{t+1}) + 1 - F(Q_{t+1})$. At the bubbly steady state, the two equations above imply that

$$1 = \beta \left[Q_b J(Q_b) + 1 - F(Q_b)\right],$$

$$\alpha = Q_b (\alpha + p_b) J(Q_b).$$

We log-linearize these two difference equations around the bubbly steady state and obtain

$$\begin{aligned} & \left[1 + \frac{s_b}{1 - s_b} \frac{p_b}{\alpha + p_b}\right] \hat{p}_t + \frac{s_b}{1 - s_b} \frac{f(Q_b)}{J(Q_b)} \hat{Q}_t \\ = & \left[1 + \frac{s_b}{1 - s_b} \frac{p_b}{\alpha + p_b}\right] \hat{p}_{t+1} + \left[\beta \frac{\alpha}{\alpha + p_b} + \frac{s_b}{1 - s_b} \frac{f(Q_b)}{J(Q_b)}\right] \hat{Q}_{t+1}, \end{aligned}$$

$$\begin{aligned} & \frac{1}{1 - s_b} \frac{p_b}{\alpha + p_b} \hat{p}_t + \left[1 + \frac{1}{1 - s_b} \frac{f(Q_b)}{J(Q_b)}\right] \hat{Q}_t \\ = & \frac{s_b}{1 - s_b} \frac{p_b}{\alpha + p_b} \hat{p}_{t+1} + \left[\beta \frac{\alpha}{\alpha + p_b} + \frac{s_b}{1 - s_b} \frac{f(Q_b)}{J(Q_b)}\right] \hat{Q}_{t+1}. \end{aligned}$$

We rewrite the two equations above in the following form

$$B_b \begin{bmatrix} \hat{p}_{t+1} \\ \hat{Q}_{t+1} \end{bmatrix} = G_b \begin{bmatrix} \hat{p}_t \\ \hat{Q}_t \end{bmatrix},$$

where

$$B_b = \begin{bmatrix} 1 + \frac{s_b}{1-s_b}\frac{p_b}{\alpha+p_b} & \beta\frac{\alpha}{\alpha+p_b} + \frac{s_b}{1-s_b}\frac{f(Q_b)}{J(Q_b)} \\ \frac{s_b}{1-s_b}\frac{p_b}{\alpha+p_b} & \beta\frac{\alpha}{\alpha+p_b} + \frac{s_b}{1-s_b}\frac{f(Q_b)}{J(Q_b)} \end{bmatrix}$$

and

$$G_b = \begin{bmatrix} 1 + \frac{s_b}{1-s_b}\frac{p_b}{\alpha+p_b} & \frac{s_b}{1-s_b}\frac{f(Q_b)}{J(Q_b)} \\ \frac{1}{1-s_b}\frac{p_b}{\alpha+p_b} & 1 + \frac{1}{1-s_b}\frac{f(Q_b)}{J(Q_b)} \end{bmatrix}.$$

As before, we need to check the eigenvalues of the matrix $G_b^{-1}B_b$. The characteristic function of the matrix $G_b^{-1}B_b$ is $\lambda^2 + b\lambda + c$ where

$$b \equiv -\frac{1}{d}\left[\beta\left(\frac{\alpha}{\alpha+p_b}\right)^2 + \frac{s_b}{1-s_b}\frac{p_b}{\alpha+p_b}\frac{f(Q_b)}{J(Q_b)} + 1 + \frac{1}{1-s_b}\frac{f(Q_b)}{J(Q_b)} + \frac{s_b}{1-s_b}\frac{p_b}{\alpha+p_b}\right] < 0,$$

$$c \equiv \frac{1}{d}\left[\beta\frac{\alpha}{\alpha+p_b} + \frac{s_b}{1-s_b}\frac{f(Q_b)}{J(Q_b)}\right] > 0,$$

$$d \equiv 1 + \frac{1}{1-s_b}\frac{f(Q_b)}{J(Q_b)} + \frac{s_b}{1-s_b}\frac{p_b}{\alpha+p_b} > 0.$$

We then prove the following two facts: (1) $0 < c < 1$; (2) $1 + b + c > 0$.

(1) Claim $0 < c < 1$.

Since

$$\begin{aligned} 0 \le \frac{s_b}{1-s_b}\frac{f(Q_b)}{J(Q_b)} &\le \frac{1}{1-s_b}\frac{f(Q_b)}{J(Q_b)}, \\ 0 < \beta\frac{\alpha}{\alpha+p_b} &< 1, \end{aligned}$$

then

$$\begin{aligned} 0 &< \frac{s_b}{1-s_b}\frac{f(Q_b)}{J(Q_b)} + \beta\frac{\alpha}{\alpha+p_b} \\ &< 1 + \frac{1}{1-s_b}\frac{f(Q_b)}{J(Q_b)} \\ &< 1 + \frac{1}{1-s_b}\frac{f(Q_b)}{J(Q_b)} + \frac{s_b}{1-s_b}\frac{p_b}{\alpha+p_b}, \end{aligned}$$

which implies $0 < c < 1$.

(2) Claim $1 + b + c > 0$.

We use the definition of b and c to compute

$$1+b+c=\frac{\beta}{d}\frac{\alpha}{\alpha+p_b}\frac{p_b}{\alpha+p_b}>0.$$

Given these four facts, if the two roots (denoted by λ_1 and λ_2) are real numbers, they must both be positive because $0<\lambda_1\lambda_2=c<1$ and $\lambda_1+\lambda_2=-b>0$. Since $1+b+c>0$, the two roots must be smaller than 1, otherwise $\lambda_1\lambda_2>1$. If the two eigenvalues are complex numbers, it follows from $0<c=\lambda_1\lambda_2<1$ that they must be inside the unit circle. We conclude that, in both cases, there is a unique local solution for p_t and Q_t since both are nonpredetermined variables. The solution is the bubbly steady state. We then use (A.28) to determine the solution for s_t, which is also the steady state value. ■

Now we compute the bubbly equilibrium welfare for any given initial non-steady-state capital stock K_0 :

$$U_b(K_0)=\frac{\ln(1-s_b)}{1-\beta}+\frac{\alpha}{1-\alpha\beta}\left[\frac{\beta}{1-\beta}\ln(s_b)+\ln(K_0)\right]. \tag{A.30}$$

We then compare $U_f(K_0)$ and $U_b(K_0)$. Note that the life-time utility in equation (25) of the main text as a function of the saving rate s is concave and has a maximum at $s=\alpha\beta$. Using (19) of the main text and $\delta=1$, $s_f=\alpha/R_{kf}=\alpha\int_{\tau\leq Q_f}\frac{1}{\tau}f(\tau)\,d\tau$. By assumption, $s_f>\alpha$. By Proposition 5, $R_{kb}<R_{kf}$. Thus $s_b=\alpha/R_{kb}>\alpha/R_{kf}=s_f>\alpha>\alpha\beta$. Hence, $U_f(K_0)>U_b(K_0)$.

Another sufficient condition for $U_f(K_0)>U_b(K_0)$ is $\tau\in[0,1]$. Under this condition,

$$\int\max\left(\frac{1}{\tau},\frac{1}{Q_f}\right)f(\tau)\,d\tau>1,$$

so that $s_b>s_f=\alpha\beta\int\max\left(\frac{1}{\tau},\frac{1}{Q_f}\right)f(\tau)\,d\tau>\alpha\beta$, where the equation for s_f follows from (A.26).

Finally, we compare $U_f(K_f)$ and $U_b(K_b)$. In any steady state, $K=I=sK^{\alpha}$. It follows that the steady-state capital stock satisfies $K=s^{1/(1-\alpha)}$. Using (25) in the main text, we can then compute the steady-state welfare as

$$U=\frac{1}{1-\beta}\left[\ln(1-s)+\frac{\alpha}{1-\alpha}\ln(s)\right].$$

It is a concave function of s and is maximized at $s=\alpha$. Since $s_b>s_f>\alpha$, it follows that $U_f(K_f)>U_b(K_b)$.   Q.E.D.

B Proof of results in Section 5.3

We first prove the following result.

Proposition 1 *The equilibrium system for the economy with transaction taxes is given by the following equations:*

$$\begin{aligned}(1+\phi)\,P_t &= \beta\frac{\Lambda_{t+1}}{\Lambda_t}P_{t+1}\Bigg\{1+\phi+\theta\int_{\frac{1-\phi-\theta}{1+\phi-\theta}Q_{t+1}<\tau\leq Q_{t+1}}\frac{Q_{t+1}-\tau}{\tau}f(\tau)d\tau \qquad \text{(B.1)}\\ &+\int_{\tau\leq\frac{1-\phi-\theta}{1+\phi-\theta}Q_{t+1}}\left[\left[\theta\omega+(1-\omega)(1-\phi)\right]\frac{Q_{t+1}-\tau}{\tau}-2\phi\,(1-\omega)\right]f(\tau)d\tau\Bigg\},\end{aligned}$$

$$\begin{aligned}I_t &= R_{kt}K_t\int_{\tau\leq Q_t}\frac{1}{\tau}f\,(\tau)\,d\tau+P_tH_t\theta\int_{\frac{1-\phi-\theta}{1+\phi+\theta}Q_t<\tau\leq Q_t}\frac{1}{\tau}f\,(\tau)\,d\tau\\ &+P_tH_t\left[\omega\theta+(1-\omega)\,(1-\phi)\right]\int_{\tau\leq\frac{1-\phi-\theta}{1+\phi-\theta}Q_t}\frac{1}{\tau}f\,(\tau)\,d\tau, \qquad \text{(B.2)}\end{aligned}$$

and equations (12), (13), (16), (17), and (18) in the main text for nine variables $\{C_t,\ I_t,\ Y_t,\ K_{t+1},\ W_t,\ R_{kt},\ R_{ft},\ Q_t,\ P_t\}$*. The usual transversality conditions hold.*

Proof: We conjecture that entrepreneur j's value function takes the form, $V_t(\tau_{jt},K_{jt},H_{jt},B_{jt})=v_t(\tau_{jt})K_{jt}+p_t(\tau_{jt})H_{jt}-\varphi_t(\tau_{jt})B_{jt}$, where v_t, p_t and φ_t are functions to be determined and satisfy

$$(1+\phi)P_t=\beta\frac{\Lambda_{t+1}}{\Lambda_t}\int p_{t+1}(\tau_{jt+1})dj, \qquad \text{(B.3)}$$

$$\frac{1}{R_{ft}}=\beta\frac{\Lambda_{t+1}}{\Lambda_t}\int \varphi_{t+1}(\tau_{jt+1})dj. \qquad \text{(B.4)}$$

Denote $Q_t=\beta\frac{\Lambda_{t+1}}{\Lambda_t}\int v_{t+1}(\tau_{jt+1})dj$ as Tobin's marginal Q. Given the preceding conjecture, we can rewrite the Bellman equation as

$$\begin{aligned}& v_t(\tau_{jt})K_{jt}+p_t(\tau_{jt})H_{jt}-\varphi_t(\tau_{jt})B_{jt} \qquad \text{(B.5)}\\ =& \max_{I_{jt},H_{jt+1}} R_{kt}K_{jt}-B_{jt}+(Q_t-\tau_{jt})\,I_{jt}+Q_t(1-\delta)K_{jt}\\ & -P_t(H_{jt+1}-H_{jt})-P_t\phi|H_{jt+1}-H_{jt}|+(1+\phi)P_tH_{jt+1},\end{aligned}$$

subject to (A.2), (A.3), $I_{jt}\geq 0$, and

$$R_{kt}K_{jt}-\tau_{jt}I_{jt}-P_t(H_{jt+1}-H_{jt})+\frac{B_{jt+1}}{R_{ft}}-B_{jt}-\phi P_t|H_{jt+1}-H_{jt}|\geq 0. \qquad \text{(B.6)}$$

Note that terms related to B_{jt+1} are canceled out in the Bellman equation.

We first consider a low-τ entrepreneur with $\tau_{jt} \leq Q_t$. This entrepreneur would like to invest as much as possible until (B.6) and (A.2) bind. Thus,

$$\tau_{jt} I_{jt} = R_{kt} K_{jt} + \frac{B_{jt+1}}{R_{ft}} - B_{jt} - P_t \left(H_{jt+1} - H_{jt}\right) - \phi P_t \left|H_{jt+1} - H_{jt}\right|,$$

and

$$\frac{B_{jt+1}}{R_{ft}} = \theta P_t H_{jt+1}.$$

Substituting this investment rule into the preceding Bellman equation yields

$$\begin{aligned} & v_t(\tau_{jt}) K_{jt} + p_t(\tau_{jt}) H_{jt} - \varphi_t(\tau_{jt}) B_{jt} \qquad \text{(B.7)} \\ = & \max_{H_{jt+1} \geq 0} \; R_{kt} K_{jt} - B_{jt} - P_t (H_{jt+1} - H_{jt}) - P_t \phi |H_{jt+1} - H_{jt}| + Q_t (1-\delta) K_{jt} \\ & + \left(\frac{Q_t}{\tau_{jt}} - 1\right) \left[R_{kt} K_{jt} - B_{jt} + \theta P_t H_{jt+1} - P_t (H_{jt+1} - H_{jt}) - P_t \phi |H_{jt+1} - H_{jt}|\right] \\ & + (1+\phi) P_t H_{jt+1}, \end{aligned}$$

subject to (A.3).

Now consider the choice of H_{jt+1}. We claim that entrepreneur j will never buy land (i.e. $H_{jt+1} > H_{jt}$) because this would imply that the marginal benefit of holding one more unit of land is negative, i.e., $-(1+\phi-\theta)\left(\frac{Q_t}{\tau_{jt}} - 1\right) P_t < 0$. It must be the case that $H_{jt+1} \leq H_{jt}$. We can then compute the marginal benefit of holding one more unit of land as

$$\left[2\phi - (1-\phi-\theta)\left(\frac{Q_t}{\tau_{jt}} - 1\right)\right] P_t.$$

This expression is positive when $\tau_{jt} > \frac{1-\phi-\theta}{1+\phi-\theta} Q_t$. In this case, entrepreneur j will keep buying until $H_{jt+1} = H_{jt}$. However, when $\tau_{jt} < \frac{1-\phi-\theta}{1+\phi-\theta} Q_t$, the marginal benefit of holding one more unit of land is negative so that entrepreneur j prefers to sell as much land as possible until $H_{jt+1} = \omega H_{jt}$. In sum, optimal land holdings are given by

$$H_{jt+1} = \begin{cases} H_{jt} & \text{when } \frac{1-\phi-\theta}{1+\phi-\theta} Q_t < \tau_{jt} \leq Q_t \\ \omega H_{jt} & \text{when } \tau_{jt} \leq \frac{1-\phi-\theta}{1+\phi-\theta} Q_t \end{cases}.$$

Substituting the decision rule for H_{jt+1} above into the Bellman equation in (B.7), we can simplify the Bellman equation. In particular, for $\tau_{jt} \leq \frac{1-\phi-\theta}{1+\phi-\theta} Q_t$, the value function satisfies

$$\begin{aligned} & v_t(\tau_{jt}) K_{jt} + p_t(\tau_{jt}) H_{jt} - \varphi_t(\tau_{jt}) B_{jt} \qquad \text{(B.8)} \\ = & \frac{Q_t}{\tau_{jt}} \left(R_{kt} K_{jt} - B_{jt}\right) + Q_t (1-\delta) K_{jt} \\ & + \left[(1-\phi)(1-\omega) + \left(\frac{Q_t}{\tau_{jt}} - 1\right)\left[\theta\omega + (1-\omega)(1-\phi)\right] + \omega(1+\phi)\right] P_t H_{jt}, \end{aligned}$$

where $\left[(1-\phi)(1-\omega)+\left(\frac{Q_t}{\tau_{jt}}-1\right)(1-\omega)(1-\phi)\right]P_tH_{jt}$ is the investment financed by selling a fraction $(1-\omega)$ of the current land holdings net of transaction tax, $\left(\frac{Q_t}{\tau_{jt}}-1\right)\theta\omega P_tH_{jt}$ is the investment financed by borrowing using a fraction ω of the current land holdings as collateral, and $\omega(1+\phi)P_tH_{jt}$ is the shadow value of the land.

For $\frac{1-\phi-\theta}{1+\phi-\theta}Q_t<\tau_{jt}\leq Q_t$, the value function satisfies

$$
\begin{aligned}
& v_t(\tau_{jt})K_{jt}+p_t(\tau_{jt})H_{jt}-\varphi_t(\tau_{jt})B_{jt} \\
= \quad & \frac{Q_t}{\tau_{jt}}\left(R_{kt}K_{jt}-B_{jt}\right)+Q_t(1-\delta)K_{jt}+\left[\left(\frac{Q_t}{\tau_{jt}}-1\right)\theta+(1+\phi)\right]P_tH_{jt},
\end{aligned}
\tag{B.9}
$$

where $\left(\frac{Q_t}{\tau_{jt}}-1\right)\theta P_tH_{jt}$ is the investment financed by borrowing with the current land holdings as collateral, and $(1+\phi)P_tH_{jt}$ is the shadow value of the land.

Next, consider a high-τ entrepreneur with $\tau_{jt}>Q_t$. In this case, investing is unprofitable so that $I_{jt}=0$. The Bellman equation in (B.5) becomes

$$
\begin{aligned}
& v_t(\tau_{jt})K_{jt}+p_t(\tau_{jt})H_{jt}-\varphi_t(\tau_{jt})B_{jt} \\
= \quad & \max_{H_{jt+1}\geq 0}\ R_{kt}K_{jt}-B_{jt}-P_t(H_{jt+1}-H_{jt})-P_t\phi|H_{jt+1}-H_{jt}| \\
& +Q_t(1-\delta)K_{jt}+(1+\phi)P_tH_{jt+1},
\end{aligned}
\tag{B.10}
$$

subject to (A.3). If $H_{jt+1}\leq H_{jt}$, then the marginal benefit of holding one more unit of land is $2\phi P_t>0$. In this case, the entrepreneur will increase land holdings until $H_{jt+1}=H_{jt}$. If $H_{jt+1}\geq H_{jt}$, then the terms related to H_{jt+1} are canceled out in the preceding Bellman equation. This means that the entrepreneur is indifferent among any feasible choices of $H_{jt+1}\geq H_{jt}$. We can then rewrite (B.10) as

$$
v_t(\tau_{jt})K_{jt}+p_t(\tau_{jt})H_{jt}-\varphi_t(\tau_{jt})B_{jt}=R_{kt}K_{jt}+(1+\phi)P_tH_{jt}-B_{jt}+Q_t(1-\delta)K_{jt}. \tag{B.11}
$$

Matching coefficients of K_{jt}, H_{jt} and B_{jt} on the two sides of equations (B.8), (B.9), and (B.11), respectively, we can derive expressions for $v_t(\tau_{jt})$, $p_t(\tau_{jt})$, and $\varphi_t(\tau_{jt})$. Substituting these expressions into (B.3), (B.4) and using the definition of Q_t, we obtain equations (B.1) and (12) and (13) in the main text after some manipulation.

Using a law of large numbers, we compute aggregate investment as

$$\begin{aligned}
I_t &= \int_{\tau_{jt}\leq\frac{1-\phi-\theta}{1+\phi-\theta}Q_t}\frac{1}{\tau_{jt}}\Big[R_{kt}K_{jt}-B_{jt}+\left[\theta\omega+(1-\omega)(1-\phi)\right]P_tH_{jt}\Big]dj \\
&\quad+\int_{\frac{1-\phi-\theta}{1+\phi-\theta}Q_t<\tau_{jt}\leq Q_t}\frac{1}{\tau_{jt}}\Big[R_{kt}K_{jt}-B_{jt}+\theta P_tH_{jt}\Big]dj \\
&= \int_{\tau_{jt}\leq\frac{1-\phi-\theta}{1+\phi-\theta}Q_t}\frac{1}{\tau_{jt}}dj\left[R_{kt}\int K_{jt}dj+\left[\theta\omega+(1-\omega)(1-\phi)\right]P_t\int H_{jt}dj-\int B_{jt}dj\right] \\
&\quad+\int_{\frac{1-\phi-\theta}{1+\phi-\theta}Q_t<\tau_{jt}\leq Q_t}\frac{1}{\tau_{jt}}dj\left[R_{kt}\int K_{jt}dj+\theta P_t\int H_{jt}dj-\int B_{jt}dj\right] \\
&= R_{kt}K_t\int_{\tau\leq Q_t}\frac{1}{\tau}f(\tau)\,d\tau \\
&\quad+P_t\left[\left[\omega\theta+(1-\omega)(1-\phi)\right]\int_{\tau\leq\frac{1-\phi-\theta}{1+\phi-\theta}Q_t}\frac{1}{\tau}f(\tau)\,d\tau+\theta\int_{\frac{1-\phi-\theta}{1+\phi-\theta}Q_t<\tau\leq Q_t}\frac{1}{\tau}f(\tau)\,d\tau\right],
\end{aligned}$$

where we have used the market-clearing condition $H_t = 1$. We can also derive equations (16), (17), and (18) in the main text as before. It is known from the literature that the transversality conditions are part of the necessary and sufficient conditions for optimality in infinite-horizon problems. Q.E.D.

By adapting the proof of Proposition 4, we can show that there is a unique bubbly steady state in which $P_t = P > 0$ for all t if and only if

$$\begin{aligned}
1 &< \beta\Bigg\{1+\theta\int_{\frac{1-\phi-\theta}{1+\phi-\theta}Q_f<\tau\leq Q_f}\frac{Q_f-\tau}{\tau(1+\phi)}f(\tau)d\tau \\
&\quad+\int_{\tau\leq\frac{1-\phi-\theta}{1+\phi-\theta}Q_f}\left[\left[\theta\omega+(1-\omega)(1-\phi)\right]\frac{Q_f-\tau}{\tau(1+\phi)}-\frac{2\phi(1-\omega)}{1+\phi}\right]f(\tau)d\tau\Bigg\},
\end{aligned}$$

where Q_f is determined by (20) in the main text. Since the right-hand side of the above inequality is decreasing in ϕ, when ϕ is sufficiently high the inequality is violated. As a result, a land bubble cannot exist.

C Foreign purchases of bonds

We now use our model to study the impact of capital inflow through foreign purchases of domestic private bonds on asset bubbles. It is often argued that the increased capital flows as a result the global saving gluts were an important reason for the US housing bubbles (see e.g., Bernanke's celebrated speech on "the global saving glut and the U.S. current account deficit" on March 10, 2005, and Greenspan's testimony at the Financial Crisis Inquiry Commission in April 2010). We now extend our model to investigate this hypothesis by assuming that investors from the rest of worlds buy $\bar{B}_t$ domestic bonds. We will focus the bubbly equilibrium only. With capital inflow, the bond market-clearing condition is given by

$$\int_0^1 B_{jt}dj = \bar{B}_t, \tag{C.1}$$

By Proposition 1 in the main text and the bond market-clearing condition (C.1), we can derive aggregate investment

$$I_t = [R_{kt}K_t + (1-\omega+\omega\theta)P_t - \bar{B}_t] \int_{\tau \le Q_t} \frac{1}{\tau} f(\tau) d\tau. \tag{C.2}$$

The resource constraint becomes

$$C_t + I_t + \bar{B}_t = Y_t + \frac{\bar{B}_{t+1}}{R_{ft}}. \tag{C.3}$$

The other equilibrium conditions described in Proposition 2 in the main text remain unchanged.

Comparing the equilibrium system with that in benchmark model in the main text, only equations (C.2) and (C.3) are different. Therefore, in the bubbly steady state, $\{I_b, Y_b, K_b, W_b, R_{kb}, R_{fb}, Q_b\}$ are all the same as in the benchmark model in the main text. We can derive the bubbly steady-state land price P with capital inflow as

$$P = \frac{1}{1-\omega(1-\theta)} \left[\left(\frac{\delta\alpha\beta}{1-\beta(1-\delta)} \frac{\int \max\left(\frac{1}{\tau}, \frac{1}{Q_b}\right) f(\tau)d\tau}{\int_{\tau \le Q_b} \frac{1}{\tau} f(\tau)d\tau} - \alpha \right) Y_b + \bar{B} \right]. \tag{C.4}$$

We can see that the land price P increases with $\bar{B}$. In addition, a dollar capital inflow increases the land price by $\frac{1}{1-\omega(1-\theta)} > 1$ dollars. This means that there is an multiplier effect of capital inflow on land bubbles in the steady state.

Finally we use the resource constraint to derive

$$C_b = Y_b - I_b + \bar{B}\left(\frac{1}{R_{fb}} - 1\right) \ge Y_b - I_b \tag{C.5}$$

Since $R_{fb} \le 1$ and the inequality is strict for $\omega > 0$, the households receive a net income transfer in the bubbly steady state. This implies capital inflow will be welfare improving in the bubbly steady state.

In summary, we have the following proposition characterizing the bubbly equilibrium with capital inflow.

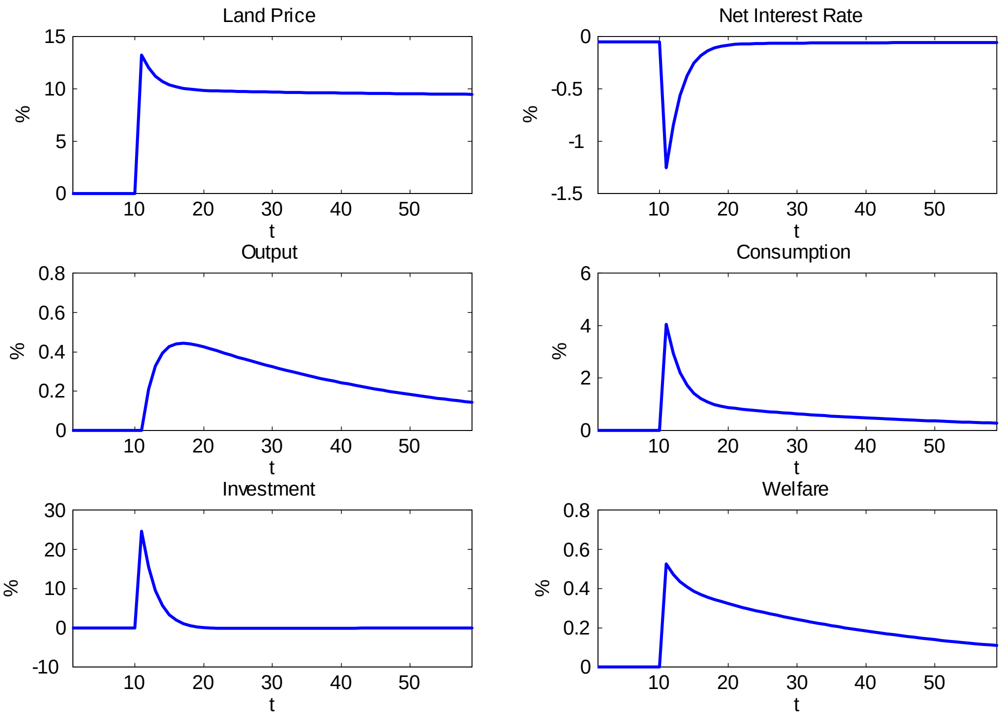

Figure 1: Transition paths of the bubbly equilibrium in response to a gradual increase in foreign purchases of bonds. Parameters values are given by $\alpha = 0.3$, $\beta = 0.99$, $\delta = 0.025$, $\eta = 5.7$, $\omega = 0.2$, $\theta = 0.75$, $\bar{B} = 1$, and $\rho = 0.6$.

Proposition 2 *Suppose that the assumptions in Proposition 4 in the baseline model without capital inflow hold. Then there always exists a bubbly equilibrium in which the values of* $\{I_b, Y_b, K_b, W_b, R_{kb}, R_{fb}, Q_b\}$ *are the same as those in Section 3.4 of the main text and* C_b *and* $P > 0$ *are given by equation (C.5) and (C.4).*

We now study the transition path. We assume that the economy is initially in the bubbly steady state without capital flows until period 10. In period 11, the capital account is opened. We set the same parameters as in Section 4.2 of the main text. The steady-state capital inflow is set to $\bar{B} = 1$, which corresponds about 1/3 of the initial level of bubbly steady-state output. Along the transition path, $\bar{B}_t = \left(1 - \rho^t\right) \bar{B}$ where $\rho \in (0, 1)$.

Figure 1 plots the net interest rate $R_{ft} - 1$ and the percentage deviations of the land price P_t, output Y_t, consumption C_t, investment I_t, and capital inflows $\bar{B}_t$ from their bubbly steady-state values without capital flows. The increase in the capital flow raises the demand for domestic bonds, thereby raising the bond price and lowering the net interest rate. The decline in the interest rate allows the household to substitute bonds for land, thereby raising the land price dramatically on impact. Alternatively, the decline in the interest rate lowers the discount rate for land and hence raises the land price. As the capital inflow gradually rises to its steady state level $\bar{B}$, the interest rate gradually rises back to its steady-state value. This induces the household to gradually shift investments in land to investments in bonds. Thus the land price gradually falls back to its steady-state value and the land price overshoots its long-run value on impact. The new steady-state land price is higher than its old steady state value due to the capital inflow. With capital inflow, the economy is able to finance higher consumption and investment simultaneously because the net interest rate is negative. The welfare as measured by discounted future utility increases accordingly. In the long run, output and investment return to its bubbly steady-state levels without capital inflow, but consumption reaches a permanently higher level. In sum, Figure 1 confirms the hypothesis of the global saving glut that foreign capital inflow will reduce interest rate and push up asset prices. But unlike the conventional wisdom, capital inflow is beneficial to the recipient economy.

D Data description for Figure 1 in the main text

We download the data from the Department of Economics of Queen's University via the link www.econ.queensu.ca/files/other/House_Price_indices%20(OECD).xls. All series are quarterly and seasonally adjusted. The data are defined as follows.

1. The nominal house price index of the US is the all-transaction index (estimated using sales price and appraisal data) from Federal Housing Finance Agency (FHFA).

2. The nominal house price index of Japan is the nationwide urban land price index from the Japan Real Estate Institute.

3. The nominal house price index of Spain is the average price per square meter of private housing (more than one year old) from the Bank of Spain.

4. The nominal house price index of Greece is the price per square meter of residential properties (all flats) in urban areas from the Bank of Greece.

5. The real house price index used in Figure 1 is the above nominal house price index deflated by the private consumption deflator. The average real index in 2000 is normalized to 100.

6. The price-income ratio used in Figure 1 is the ratio of the nominal house price index to the nominal per capita disposable income. The sample average is normalized to 100.

7. The price-rental ratio used in Figure 1 is the ratio of the nominal house price index to the rent component of the consumer price index. The sample average is normalized to 100.