Supplementary Appendix to "Banking Bubbles and Financial Crises"

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This appendix contains four sections. Section 1 studies robustness of our model by analyzing alternative dividend policies and borrowing constraints. Section 2 considers the approach of Gertler and Karadi (2011) and Gertler and Kiyotaki (2010). Section 3 analyzes a specific discount window policy that can make the economy achieve the unique bubbly steady state equilibrium. Section 4 studies another type of equilibrium in which the reorganized bank upon default loses its reputation by losing part of its bubble value.

1 Robustness

This section addresses two issues. First, we have assumed a reduced-form dividend constraint in the paper. This constraint is used to prevent bankers from saving their way out of borrowing constraints. In Section 1.1 we will remove this constraint and introduce more primitive assumptions. We will show that it can be micro-founded. The second issue is related to the borrowing constraint. In Section 2.2 we will introduce three other types of borrowing constraints often used in the literature and show that the existence of a bubble is fairly robust in various contexts.

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1.1 Microfoundation of the Dividend Policy

We consider two approaches to endogenizing the dividend policy.[1] For both approaches, we introduce heterogeneity in banks. Suppose that there is a continuum of bankers and a continuum of identical workers within the family. Bankers are ex ante identical, but subject to idiosyncratic shocks described below. Given the affine structure of our model, we can conduct aggregation tractably without tracking the distribution of banks.

1.1.1 Stochastic Death of Banks

Our first approach to endogenizing the dividend policy draws on Gertler and Karadi (2011) and Gertler and Kiyotaki (2010). Suppose that each banker manages a bank and dies with a Poisson arrival rate ρdt between dates t and $t + dt$. The death shock is independent across all bankers. When the banker dies, he distributes all the bank's net worth as dividends. Workers randomly become bankers, keeping the number in each occupation constant. Each new banker receives a start up transfer from the family as a small constant fraction ω of the total net worth.

Let C_t^j represent a typical bank j's dividend rate at date t if $C_t^j \geq 0$. Bank j may raise new equity to avoid borrowing constraints by allowing $C_t^j < 0$. Suppose that external equity financing is so costly that bank j can only raise a limited amount of equity. For tractability, assume that the limit on the new equity is proportional to the bank's net worth, i.e.,

$$C_t^j \geq -\eta N_t^j, \text{ for } \eta > 0, \tag{1}$$

where N_t^j denotes bank j's net worth before dividend payout. Assume that bank j faces the following borrowing constraint:

$$D_t^j \leq V\left(\xi N_t^j, Q_t, B_t\right), \tag{2}$$

where D_t^j denotes bank j's deposits and $V(\cdot, Q_t, B_t)$ denotes bank j's value function. The flow-of-funds constraint is

$$dN_t^j = r_{kt}\left(N_t^j + D_t^j\right)dt - C_t^j dt - rD_t^j dt. \tag{3}$$

[1] Some researchers endogenize dividend policy by simply introducing convex adjustment costs (e.g., Jermann and Quadrini (2012)).

Bank j's balance sheet satisfies $D_t^j + N_t^j = S_t^j \geq 0$, where S_t^j denotes the loan volume. In equilibrium, the aggregate loan volume is equal to the aggregate capital stock, $\int S_t^j dj = K_t$.

Bank j's decision problem is to solve the following Bellman equation:

$$\begin{aligned} rV\left(N_t^j, Q_t, B_t\right) &= \max_{C_t^j, D_t^j} \; C_t^j + V_N\left(N_t^j, Q_t, B_t\right)\dot{N}_t^j + \dot{Q}_t V_Q\left(N_t^j, Q_t, B_t\right) \\ &\quad + \dot{B}_t V_B\left(N_t^j, Q_t, B_t\right) + \rho\left[N_t^j - V\left(N_t^j, Q_t, B_t\right)\right], \end{aligned}$$

subject to (1), (2), and (3).[2]

The following proposition provides conditions such that bubbly and bubbleless equilibria can coexist.

Proposition 1 *Consider the model with stochastic death. Suppose that*

$$\rho(1-\omega) - \eta - r > (r+\rho)\frac{\rho-\eta}{r+2\rho-\eta} > 0,$$

and

$$0 < \xi < \bar{\xi} \equiv \frac{(r+2\rho-\eta)(\rho(1-\omega)-\eta-r)}{(r+\rho)(\rho-\eta)} - 1 \leq 1.$$

Then both the bubbly and bubbleless steady states coexist and

$$Q^* = \frac{\rho-\eta}{\omega\rho+r} > 1, \;\; r_k^* - r = \frac{\rho(1-\omega)-\eta-r}{1+\xi\frac{\rho-\eta}{\omega\rho+r}} > 0,$$

$$Q^b = \frac{r+2\rho-\eta}{\rho-\eta-\xi(r+\rho)} > 1, \;\; r_k^b - r = \frac{(r+\rho)\left[\rho-\eta-\xi(r+\rho)\right]}{r+2\rho-\eta} > 0.$$

Proof of Proposition 1: Conjecture that $V\left(N_t^j, Q_t, B_t\right) = Q_t N_t^j + B_t$. Substituting this conjecture into the Bellman equation yields

$$\begin{aligned} rQ_t N_t^j + rB_t &= \max_{C_t^j, D_t^j} \; C_t^j + Q_t[r_{kt}N_t^j + (r_{kt}-r)\,D_t^j - C_t^j] \\ &\quad + \dot{Q}_t N_t^j + \dot{B}_t + \rho(N_t - Q_t N_t^j - B_t), \end{aligned} \tag{4}$$

[2] Gertler and Karadi (2011) and Gertler and Kiyotaki (2010) assume that banker j consumes N_t^j only when he dies and $C_t^j = 0$ before death. Here, we assume that C_t^j is chosen endogenously.

subject to (1) and $D_t^j \leq \xi Q_t N_t^j + B_t$. It follows that as long as $Q_t > 1$, the equity constraint (1) will bind, i.e., $C_t^j = -\eta N_t^j$. In addition, as long as $r_{kt} > r$, the borrowing constraint (2) will bind so that $D_t^j = \xi Q_t N_t^j + B_t$. Under these assumptions, we substitute the decision rules into the Bellman equation and match coefficients to derive

$$
\begin{aligned}
rQ_t &= Q_t[r_{kt} + (r_{kt} - r)\xi Q_t] + \dot{Q}_t + (\rho - \eta)(1 - Q_t), \\
(r + \rho)B_t &= \dot{B}_t + Q_t (r_{kt} - r) B_t.
\end{aligned}
$$

We now aggregate banks' net worth. Denote the total net worth of all alive banks by $N_t = \int N_t^j dj$. Aggregating the discretized equation (3) and using a law of large numbers, we obtain

$$
N_{t+dt} = (1 - \rho dt) \int \left[N_t^j + \left((r_{kt} + \eta) N_t^j + (r_{kt} - r)\left(\xi Q_t N_t^j + B_t^j\right)\right) dt \right] dj + \omega N_t \rho dt.
$$

Deleting terms of order higher than dt and taking the limit as $dt \to 0$, we obtain

$$
\dot{N}_t = (r_{kt} + \eta) N_t + (r_{kt} - r)(\xi Q_t N_t + B_t) - \rho(1 - \omega)N_t.
$$

The preceding three differential equations for Q_t, B_t, and N_t together with the initial condition for N_0 and the usual transversality conditions constitute the equilibrium system.

Using this system we can derive the steady-state solution. We can also solve for the steady-state bubble size

$$
\begin{aligned}
(r_k^b - r)\frac{B}{N^b} &= [\rho(1 - \omega) - \eta - r] - (1 + \xi Q)(r_k^b - r) \\
&= [\rho(1 - \omega) - \eta - r] - \frac{r + \rho}{r + 2\rho - \eta}[(\rho - \eta) - \xi(r + \rho)] - \xi(r + \rho).
\end{aligned}
$$

The conditions in the proposition ensure that $Q^* > 1$, $Q^b > 1$, $r_k^b > r$, $r_k^* > r$, and $B > 0$. These inequalities verify that our previous assumptions $Q_t > 1$ and $r_{kt} > r$ hold in the neighborhood of a steady state. Q.E.D.

The assumption in the proposition implies that η cannot be too large. For example, if η is larger than ρ, then $Q^* < 0$, violating our previous assumptions. The intuition is that when η is sufficiently large, a bank can overcome the borrowing constraint by raising new equity to increase its loans. The first-best equilibrium can be achieved when the bank can raise K^{FB} of new equity and lend this amount to the non-financial firms at the rate r.

1.1.2 Risky Loans

Our second approach to endogenizing dividend policy is related to Jensen (1986). Jensen (1986) argues that managers may invest in risky projects due to agency problems. Thus too much cash in the firm may result in unprofitable investments that hurt shareholders. One way to take unnecessary cash from the firm is to increase the level of dividend payout.

To capture this idea in a simplest possible way, we assume that lending is risky. During the time interval between t and $t+dt$, bank j fails to receive loan repayments with Poisson rate θdt. The Poisson shock is independent across all banks. After observing the Poisson shock, bank j decides how much dividends to pay out and how much deposits to issue. Assume that the bank cannot raise new equity and faces the borrowing constraint (2). We will show that the bank chooses not to distribute any dividends before the arrival of a Poisson shock. The reason is that the loan is profitable and the bank prefers to lend as much as possible. After the arrival of a Poisson shock, the bank will not make any loan because the loan will be nonperforming. Instead, the bank chooses to distribute its net worth as dividends. We will also show that the model here is observationally equivalent to the baseline model of Section 2 in the aggregate.

Proposition 2 *Under condition*

$$0 < \xi < \frac{\theta}{r} - \frac{r}{\theta} - 1, \tag{5}$$

the model with risky loans and the baseline model in the paper deliver the same aggregate equilibrium dynamics

$$rQ_t = Q_t \left[r_{kt} + (r_{kt} - r)\, \xi Q_t \right] + \theta(1 - Q_t) + \dot{Q}_t, \tag{6}$$

$$rB_t = Q_t \left(r_{kt} - r \right) B_t + \dot{B}_t, \tag{7}$$

$$\dot{N}_t = \left(r_{kt} - \theta + (r_{kt} - r)\, \xi Q_t \right) N_t + (r_{kt} - r)\, B_t, \tag{8}$$

for (Q_t, B_t, N_t) *, where* r_{kt} *satisfies*

$$r_{kt} = \alpha\left(\left(\xi Q_t + 1\right) N_t + B_t\right)^{\alpha - 1} - \delta. \tag{9}$$

Proof of Proposition 2: Bank j's flow-of-funds constraint prior to the shock is given by

$$dN_t^j = r_{kt}\left(N_t^j + D_t^j\right) dt - dC_t^j - rD_t^j dt, \tag{10}$$

where N_t^j denotes the net worth before dividend payout and C_t^j denotes the cumulative dividend process which is increasing and right continuous with left limits satisfying $C_{0-}^j = 0$.[3] The assumption on the dividend process captures the fact that dividends may jump following a Poisson shock. The bank's balance sheet is given by[4]

$$N_t^j + D_t^j - dC_t^j = S_t^j \geq 0, \tag{11}$$

where S_t^j represents bank j's loans. In equilibrium the aggregate loan volume is equal to the aggregate capital stock, $\int S_t^j dj = K_t$.

In the discrete-time approximation, the flow-of-funds constraint prior to a Poisson shock is

$$N_{t+dt}^j = \left(N_t^j + D_t^j - dC_t^j\right) e^{r_{kt}dt} - e^{rdt} D_t^j. \tag{12}$$

But immediately after a Poisson shock, the loan is nonperforming so that the net worth becomes $N_{t+dt}^j = -e^{rdt} D_t^j$ or $N_t^j = -D_t^j$ in the continuous time limit. Clearly, to ensure nonnegative net worth, the bank will not demand any deposits, i.e., $D_t^j = 0$, because it has no earnings to repay deposits. In this case the bank is still solvent and its net worth is zero. It will continue to accept deposits and make loans in the future.

[3] For simplicity, we do not consider new equity issuance. This can be motivated by the fact that equity finance is more costly than debt finance.

[4] Note that when C_t^j is absolutely continuous in t, we can write $dC_t^j = c_t^j dt$ and (11) reduces to $N_t^j + D_t^j = S_t^j$ in the continuous time limit as $dt \to 0$.

Let $V(\cdot, Q_t, B_t)$ denote bank j's value function, which satisfies the Bellman equation

$$\begin{aligned}
& rV\left(N_t^j, Q_t, B_t\right) dt \\
= & \max_{dC_t^j, D_t^j} dC_t^j + V_N\left(N_t^j, Q_t, B_t\right) dN_t^j + V_Q\left(N_t^j, Q_t, B_t\right) dQ_t + V_B\left(N_t^j, Q_t, B_t\right) dB_t \\
& +\theta \max_{dC_t^j, D_t^j} \left[dC_t^j + V\left(-D_t^j, Q_t, B_t\right) - V\left(N_t^j, Q_t, B_t\right)\right] dt,
\end{aligned}$$

where the first max operator is subject to (2), (10), and

$$N_t^j + D_t^j - dC_t^j \geq 0, \tag{13}$$

and the second max operator is subject to (2) and (13).

Conjecture that

$$V\left(N_t^j, Q_t, B_t\right) = Q_t N_t^j + B_t. \tag{14}$$

Substituting this conjecture and (10) into the Bellman equation, we can deduce that if $Q_t > 1$, then $dC_t^j = 0$, and if $r_{kt} > r$, then the borrowing constraint (2) binds so that $D_t^j = \xi N_t^j + B_t$ before a Poisson shock. These conditions are satisfied in the neighborhood of the steady state given assumption (5). Immediately after a Poisson shock, the second optimization part in the preceding Bellman equation and (13) imply that $D_t^j = 0$ and $dC_t^j = N_t^j$. The bank pays out net worth as dividends when it is hit by a Poisson shock. If dividends were retained in the bank and lent to the non-financial firms, they would be lost because these loans are nonperforming. Thus the bank controlled by its shareholders would rather distribute all its net worth as dividends.

Substituting the preceding optimal decision rule and (10) into the Bellman equation and matching coefficients, we obtain equations (6) and (7). Define $N_t = \int N_t^j dj$. Aggregating (12) and using a law of large numbers, we obtain

$$\begin{aligned}
N_{t+dt} &= (1-\theta dt) \int \left[\left(N_t^j + D_t^j - dC_t^j\right) e^{r_{kt} dt} - e^{r dt} D_t^j\right] dj \\
&= (1-\theta dt) \left[N_t e^{r_{kt} dt} + \left(e^{r_{kt} dt} - e^{r dt}\right) \left(\xi N_t Q_t + B_t\right)\right].
\end{aligned}$$

Deleting terms of order higher than dt and taking the limit as $dt \to 0$, we obtain (8). In sum, the setup presented in this subsection delivers an observationally equivalent equilibrium system in the aggregate. Thus we can reinterpret our baseline model as the one in this subsection. Q.E.D.

1.2 Alternative Borrowing Constraints

In this subsection we study three different types of borrowing constraints and the conditions under which these borrowing constraints can generate a bubble. All these borrowing constraints come from optimal contracts between banks and households with limited commitment. But the details of contract forms are different. In particular, they differ in the off-equilibrium behavior after default. But they share the common feature that the market value of the bank enters the borrowing constraints. This feature is prone to the formation of a speculative bubble. The intuition is that if people believe that there is a bubble in the bank value, then the bubble relaxes the borrowing constraints. This allows the bank to attract more deposits, lend more, and make more profits, thereby raising the bank value. This supports the initial optimistic beliefs about a high bank value. Hence, bubbles are self-fulfilling.

1.2.1 Self-Enforcing Constraints

We start with the self-enforcing constraints (Kehoe and Levine (1993), Alvarez and Jermann (2000), and Hellwig and Lorenzoni (2009)). For simplicity, we introduce these constraints in the baseline model of Section 2 of the paper. Suppose that at any time t the banker can steal the whole net worth N_t and a fraction $\xi \in (0,1]$ of deposits and run away.[5] There is no explicit punishment on the banker. But the implicit punishment is that the banker will lose his reputation and cannot attract any new deposits or borrow in the future. Suppose that the banker can still lend to non-financial firms using his net worth. We now study the optimal contract such that the bank will never default.

We first derive the value to the banker if he chooses to default and run away. Let $V_t^a(N_t)$ denote this value when the bank has net worth N_t at date t. It satisfies the discretized Bellman equation

$$V_t^a(N_t) = \max_{C_t^a} \ C_t^a dt + e^{-rdt} V_{t+dt}^a(N_{t+dt}),$$

[5] We allow $\xi = 1$ or $\xi \in (0,1)$. It seems reasonable to assume that stealing deposits is harder than stealing net worth since deposits are more closely monitored by depositors.

subject to $C_t^a \geq \theta N_t$, and $N_{t+dt} = e^{r_k dt} N_t - C_t^a dt$. Conjecture that $V_t^a(N_t) = Q_t N_t$. Substituting this conjecture into the Bellman equation and taking the limit as $dt \to 0$, we find that when $Q_t > 1$, $C_t^a = \theta N_t$ and Q_t satisfies the differential equation

$$rQ_t = Q_t r_{kt} + \theta(1 - Q_t) + \dot{Q}_t. \tag{15}$$

Note that the transversality condition $\lim_{T\to\infty} e^{-rT} V_T^a(N_T) = 0$ rules out a bubble in the value $V_t^a(N_t)$.

Next we derive the incentive constraint. If the banker steals bank assets $(N_t + \xi D_t)$ and runs away, his outside value is given by $V_t^a(N_t + \xi D_t)$. The banker will not default if the value of staying in the bank, $V_t(N_t)$, is not smaller than the outside value, i.e.,

$$V_t(N_t) \geq V_t^a(N_t + \xi D_t). \tag{16}$$

This delivers a self-enforcing borrowing constraint.

We conjecture and verify that $V_t(N_t) = Q_t N_t + B_t$. In this case it follows from (16) that

$$Q_t N_t + B_t \geq Q_t(N_t + \xi D_t) \Longrightarrow D_t \leq \frac{B_t}{\xi Q_t}. \tag{17}$$

This constraint implies that the bank demanded deposits are limited by the size of the bubble (normalized by ξQ_t). Martin and Ventura (2012) study a similar type of borrowing constraints for firms instead of banks.

Proposition 3 *Consider the model with self-enforcing borrowing constraints. Suppose that $(1+\xi)r < \theta$. Then the bubbly and bubbleless steady states coexist and*

$$r < r_k^b = (1+\xi)r < r_k^* = \theta,\ Q^* = \frac{\theta}{r} > Q^b = \frac{\theta}{\theta - \xi r} > 1.$$

Moreover, in the neighborhood of either steady state, the equilibrium system for (N_t, Q_t, B_t) is given *by (15) and*

$$rB_t = (r_{kt} - r)B_t/\xi + \dot{B}_t,$$
$$\dot{N}_t = r_{kt}N_t - \theta N_t + (r_{kt} - r) B_t / (\xi Q_t),$$

where $r_{kt} = \alpha (B_t / (\xi Q_t) + N_t)^{\alpha - 1} - \delta.$

Proof of Proposition 3: Given the conjectured value function, the bank's Bellman equation becomes

$$r (Q_t N_t + B_t) = \max_{C_t^b} \{C_t^b + Q_t[r_{kt}N_t + (r_{kt} - r) D_t - C_t^b] + \dot{Q}_t N_t + \dot{B}_t\} \tag{18}$$

subject to (17) and

$$C_t^b \geq \theta N_t. \tag{19}$$

We guess that in a neighborhood of a steady state, $Q_t > 1$ and $r_{kt} > r$. We then verify this claim by deriving the equilibrium system and its steady states. Given the guess, both constraints in (19) and (17) bind. This implies that $C_t^b = \theta N_t$ and $D_t = B_t / (\xi Q_t)$. Substituting this decision rule back into the Bellman equation and matching coefficients, we obtain the equilibrium system in the proposition.

Using the equilibrium system, we can easily derive that there is a bubbleless steady state in which $B = 0$ and r_k^* and Q^* are given in the proposition. In addition, there is a bubbly steady state in which r_k^b and Q^b are given in the proposition, and

$$\frac{B}{N^b} = \frac{\theta - r_k^b}{r_k^b - r} Q^b = \frac{\theta - (1+\xi)r}{\xi r} \frac{\theta}{\theta - \xi r},$$

which is positive by assumption. From these solutions, we can easily check that if the assumption in the proposition holds, $r_k^* > r_k^b > r$ and $Q^* > Q^b > 1$. These relations also hold in the neighborhood of a steady state, verifying our previous assumptions. Q.E.D.

As (17) shows, a dollar of extra bubble leads to $1/(Q_t \xi)$ dollars of extra deposits. This generates $(r_{kt} - r)/(Q_t \xi)$ dollars of extra profits and hence net worth. Multiplying by the shadow price Q_t of the net worth yields the extra benefit $(r_{kt} - r)/\xi$ of an additional dollar of the bubble. This explains the differential equation for B_t in the proposition as an asset pricing equation. The extra benefit generated by the bubble supports the initial optimistic belief about the market value of the bank. If no one believes in a bubble, then $B_t = 0$ is also a solution to the asset pricing equation for the bubble. The differential equations for Q_t and N_t then constitute the system for the bubbleless equilibrium.

1.2.2 The Gertler-Karadi-Kiyotaki Approach

Gertler and Karadi (2011) and Gertler and Kiyotaki (2010) assume that any banker j can divert a fraction of bank assets, $\xi\left(D_t^j + N_t^j\right)$, and transfer the diverted assets to the family. Moreover, the banker cannot use these assets to lend in the future. As a result, the outside option value for banker j is equal to $\xi\left(D_t^j + N_t^j\right)$. In this case the incentive constraint is given by

$$\xi\left(D_t^j + N_t^j\right) \leq V_t\left(N_t^j\right). \tag{20}$$

In Section 2 we will show that this borrowing constraint cannot generate a bubble in the model of Section 1.1.1 with stochastic death.[6] We will also show that it can generate a bubble in the baseline model of Section 2 in the paper. The bubble affects the steady-state prices only, but not the steady-state real allocation.

This conclusion also holds for the risky loan model of Section 1.1.2 when (20) is used as the borrowing constraint, since this model and the baseline model of Section 2 in the paper are observationally equivalent in the aggregate. To allow the bubble to affect real allocation, we need to model a bank's dividend policy in a richer way. We now modify the model in Section 1.1.2 such that a bubble will affect the bank's dividend policy and raise its dividend payout. Suppose that between time t and $t + dt$, when a Poisson shock arrives with the arrival rate θdt, a fraction $\mu \in (0, 1)$ of bank j's loans get repaid with probability $(1 - \pi)$, but none of the loans gets repaid with probability $\pi \in (0, 1)$. The setup in Section 1.1.2 is a special case with $\pi = 1$.

We then characterize equilibrium in the following proposition.

[6] It also cannot generate a bubble in the discrete-time models of Gertler and Karadi (2011) and Gertler and Kiyotaki (2010). We thank an anonymous referee for pointing this out to us.

Proposition 4 *Consider the model with risky loans and borrowing constraints (20). Assume that* $1-\frac{r}{\theta}>\mu>\frac{r}{\theta\pi}$ *and* $\pi\in(0,1)$*. Then the bubbly and bubbleless steady states coexist and*

$$Q^* = \frac{\theta\pi}{r} > \frac{1}{\mu} > 1,\ r_k^* - r = \frac{\theta(1-\mu(1-\pi))-r}{\theta\pi} r\xi > 0,$$

$$\frac{1}{\mu} > Q^b = \frac{\theta}{\theta - r} > 1, \quad 0 < r_k^b - r = \frac{\xi r(\theta - r)}{\theta} < r_k^* - r.$$

In the neighborhood of the bubbly steady state, the bubbly equilibrium system is given by

$$\begin{aligned}
\dot{N}_t &= -\theta N_t + N_t r_{kt} + \left[\left(Q_t N_t + B_t\right)/\xi - N_t\right](r_{kt} - r), \\
rQ_t &= \dot{Q}_t + Q_t(r_{kt} + (Q_t/\xi - 1)(r_{kt} - r)) + \theta(1 - Q_t), \\
rB_t &= \dot{B}_t + Q_t(r_{kt} - r)B_t/\xi,
\end{aligned}$$

where $r_{kt} = \alpha\left(\left(Q_t N_t + B_t\right)/\xi\right)^{\alpha-1} - \delta$*. In the neighborhood of the bubbleless steady state, the bubbleless equilibrium system is given by*

$$\begin{aligned}
rQ_t &= \dot{Q}_t + Q_t(r + Q_t(r_{kt} - r)/\xi) + \theta\pi(1 - Q_t) + \theta(1-\pi)Q_t(\mu - 1), \\
\dot{N}_t &= -\theta N_t + N_t r + Q_t N_t(r_{kt} - r)/\xi + \theta(1-\pi)\mu N_t,
\end{aligned}$$

where $r_{kt} = \alpha\left(Q_t N_t/\xi\right)^{\alpha-1} - \delta$*.*

Proof of Proposition 4: Bank j's value function $V\left(N_t^j, Q_t, B_t\right)$ satisfies the Bellman equation:

$$\begin{aligned}
& rV\left(N_t^j, Q_t, B_t\right)dt \qquad (21) \\
= & \max_{dC_t^j, D_t^j} \left\{dC_t^j + V_N\left(N_t^j, Q_t, B_t\right)dN_t^j\right\} + V_Q\left(N_t^j, Q_t, B_t\right)dQ_t + V_B\left(N_t^j, Q_t, B_t\right)dB_t \\
& + (1-\pi)\theta dt \max_{dC_t^j, D_t^j}\left\{dC_t^j + V\left(\left(N_t^j + D_t^j - dC_t^j\right)\mu - D_t^j, Q_t, B_t\right) - V\left(N_t^j, Q_t, B_t\right)\right\} \\
& + \pi\theta dt \max_{dC_t^j, D_t^j}\left\{dC_t^j + V\left(-D_t^j, Q_t, B_t\right) - V\left(N_t^j, Q_t, B_t\right)\right\},
\end{aligned}$$

where the first max operator is subject to (10), (13), and (20), and the second and third max operators are subject to (20) and (13). Clearly, for the third maximization part, we must have

$D_t^j = 0$ and $dC_t^j = N_t^j$ since net worth must be nonnegative. In the second maximization part, immediately after the Poisson shock and before the interest repayment, the bank receives a fraction μ of principals, but repays deposits D_t^j. Thus, its net worth is $\left(N_t^j + D_t^j - dC_t^j\right)\mu - D_t^j$.

Conjecture that $V\left(N_t^j, Q_t, B_t\right) = Q_t N_t^j + B_t$. Substituting this conjecture into the Bellman equation yields

$$\begin{aligned} & r\left(Q_t N_t^j + B_t\right) dt \\ = & \max_{dC_t^j, D_t^j} \left\{ dC_t^j + Q_t \left[r_{kt}\left(N_t^j + D_t^j\right) dt - dC_t^j - rD_t^j dt \right] \right\} + N_t^j dQ_t + dB_t \\ & + (1-\pi)\theta dt \max_{dC_t^j, D_t^j} \left\{ dC_t^j + Q_t \left[\left(N_t^j + D_t^j - dC_t^j\right)\mu - D_t^j \right] - Q_t N_t^j \right\} \\ & + \pi\theta dt \left(N_t^j - Q_t N_t^j\right). \end{aligned} \tag{22}$$

First, we derive the bubbly equilibrium system. Guess that $1 < Q_t < 1/\mu$ and $r_{kt} > r$ in the neighborhood of the bubbly steady state. It follows that $dC_t = 0$ and the borrowing constraint (20) binds so that $D_t^j = \left(Q_t N_t^j + B_t\right)/\xi - N_t^j$ for the first maximization part. But $dC_t^j = N_t^j$ and $D_t^j = 0$ for the second and third maximization parts. Substituting this decision rule back into the Bellman equation, we can derive that

$$\begin{aligned} r\left(Q_t N_t^j + B_t\right) dt = & Q_t \left[r_{kt} N_t^j dt + (r_{kt} - r)\left(\left(Q_t N_t^j + B_t\right)/\xi - N_t^j\right) dt \right] + N_t^j dQ_t + dB_t \\ & + (1-\pi)\theta dt \left(N_t^j - Q_t N_t^j\right) + \pi\theta dt \left(N_t^j - Q_t N_t^j\right). \end{aligned}$$

Matching coefficients of N_t^j and B_t on the two sides of the equation yields the differential equations for Q_t and B_t in the proposition.

To derive the law of motion for the aggregate net worth, we aggregate (12) to obtain

$$\begin{aligned} N_{t+dt} = & (1-\theta dt) \int \left[N_t^j e^{r_{kt} dt} + \left(e^{r_{kt} dt} - e^{rdt}\right)\left(\left(Q_t N_t^j + B_t\right)/\xi - N_t^j\right) \right] dj \\ & + (1-\pi)\theta dt \int \left(N_t^j e^{r_{kt} dt} - N_t^j\right) dj + \pi\theta dt \int \left(N_t^j e^{r_{kt} dt} - N_t^j\right) dj. \end{aligned}$$

Noting $e^{r_{kt} dt} = 1 + r_{kt} dt$ and $e^{rdt} = 1 + rdt$, eliminating terms of order higher than dt, and then taking the limit as $dt \to 0$, we obtain the differential equation for N_t as in the proposition.

Aggregating (11), we derive aggregate capital as $K_t = (Q_t N_t + B_t)/\xi$. Thus the capital return $r_{kt} = \alpha K_t^{\alpha-1} - \delta = \alpha\left((Q_t N_t + B_t)/\xi\right)^{\alpha-1} - \delta$.

Next, we derive the bubbleless equilibrium system. Setting $B_t = 0$ in (22), we obtain

$$\begin{aligned} rQ_t N_t^j dt &= \max_{dC_t^j, D_t^j} \left\{ dC_t^j + Q_t \left[r_{kt} \left(N_t^j + D_t^j \right) dt - dC_t^j - rD_t^j dt \right] \right\} + N_t^j dQ_t \\ &\quad + (1-\pi)\theta dt \max_{dC_t^j, D_t^j} \left\{ dC_t^j + Q_t \left[\left(N_t^j + D_t^j - dC_t^j \right) \mu - D_t^j \right] - Q_t N_t^j \right\} \\ &\quad + \pi \theta dt \left(N_t^j - Q_t N_t^j \right). \end{aligned}$$

Guess that $Q_t > 1/\mu > 1$ and $r_{kt} > r$ in the neighborhood of the bubbleless steady state. It follows that for the first maximization problem, $dC_t^j = 0$ and the borrowing constraint (25) binds so that $D_t^j = (Q_t/\xi - 1) N_t^j$. But, for the second maximization problem, $dC_t^j = 0$ and $D_t^j = 0$. Substituting this decision rule into the Bellman equation yields

$$\begin{aligned} rQ_t N_t^j dt &= Q_t \left[r_{kt} N_t^j + (r_{kt} - r)(Q_t/\xi - 1) N_t^j dt \right] + N_t^j dQ_t \\ &\quad + (1-\pi)\theta dt \left(Q_t N_t^j \mu - Q_t N_t^j \right) + \pi\theta dt \left(N_t^j - Q_t N_t^j \right). \end{aligned}$$

Matching coefficients of N_t^j yields the differential equation for Q_t as in the proposition. Aggregating individual flow-of-funds constraint in (10) yields

$$\begin{aligned} N_{t+dt} &= (1-\theta dt) \int \left[N_t^j e^{r_{kt} dt} + \left(e^{r_{kt} dt} - e^{r dt} \right) \left(Q_t N_t^j + B_t \right) \right] dj \\ &\quad + (1-\pi)\theta dt \int \mu N_t^j e^{r_{kt} dt} dj + \pi\theta dt \int \left(N_t^j e^{r_{kt} dt} - N_t^j \right) dj. \end{aligned}$$

Taking the limit as $dt \to 0$, we obtain the differential equation for N_t in the proposition. Aggregating (11) yields aggregate capital $K_t = Q_t N_t/\xi$. Thus $r_{kt} = \alpha K_t^{\alpha-1} - \delta = \alpha (Q_t N_t/\xi)^{\alpha-1} - \delta$.

Finally, we use the bubbly and bubbleless equilibrium systems to derive the steady states. Given the assumption in the proposition, we can easily check that $Q^* > 1/\mu > 1$ and $r_k^* > r$ in the bubbleless steady state and $1 < Q_b < 1/\mu$ and $r_{kb} > r$ in the bubbly steady state. This verifies that our previous guess in the neighborhood of a steady state is valid. We can also use the law of motion for the aggregate net worth to compute the steady-state size of the bubble as

$$\frac{B}{N} = \frac{r_k^b - \theta}{r_k^b - r} + 1 - \frac{Q^b}{\xi} = \frac{\theta^2 - 2r\theta}{\xi r\left(\theta - r\right)},$$

which is positive by assumption. Q.E.D.

To interpret this proposition, consider the bubbleless equilibrium first. The key is to understand the bank's decision problem. Recall that Q_t represents the shadow value of the net worth. When $Q^* > 1/\mu > 1$, in the neighborhood of the bubbleless steady state, bank j prefers to keep its cash inside the bank instead of paying out as dividends. The bank pays out all its net worth as dividends only when the bank is hit by a Poisson shock such that none of its loans gets repaid with probability π. In this case the bank does not demand any deposit because it has no profits to repay deposit. But prior to a Poisson shock, the bank prefers to make loans as much as possible since $r_{kt} > r$ and demands deposits until the borrowing constraint (20) binds.

Next we consider the bubbly equilibrium. The existence of a banking bubble relaxes the borrowing constraint (20) and lowers the shadow value of the net worth since $Q^b < Q^*$. Since $1 < Q^b < 1/\mu$, in the neighborhood of the bubbly steady state, keeping cash inside the bank is less valuable than paying out as dividends when a Poisson shock arrives no matter whether the loan will be repaid. In this case the bank will not demand any deposit because it has no profits to repay deposit. By contrast, prior to a Poisson shock, the bank does not pay dividends, makes loans as much as possible since $r_{kt} > r$, and demands deposits until the borrowing constraint (20) binds.

In summary, the frequency of dividend payout is $\pi\theta$ in the bubbleless equilibrium, but it is θ in the bubbly equilibrium. Because of the difference in the payout policy between the bubbly and bubbleless equilibria, the differential equations for Q_t and N_t in the bubbleless equilibrium cannot be obtained by simply setting $B_t = 0$ in the bubbly equilibrium. In fact, unlike in the baseline model and other models studied previously, setting $B_t = 0$ in the bubbly equilibrium system does not deliver a bubbleless equilibrium.

Note that when $\pi = 1$, $r_k^* = r_k^b$; so the bubble does not affect the steady-state capital stock. In this case the bank's payout policy does not change fundamentally in the presence of a bubble. By contrast, the randomness of the jump size with probability $\pi \in (0, 1)$ makes the model here richer and is the key element that generates different payout policies in the bubbly and bubbleless equilibria. It also leads to the result that $r_k^b < r_k^*$, which implies that the steady-state capital stock in the bubbly equilibrium is higher than that in the bubbleless equilibrium.

1.2.3 Depositors with Full Bargaining Power

In the baseline model we have assumed that the banker can default and refuse to repay deposits. Upon default, the depositors reorganize the bank and bargain over deposit repayments. We have also assumed that the bank has full bargaining power and then obtained the incentive constraint in the paper

$$D_t \leq V_t\left(\xi N_t\right), \tag{23}$$

Now we suppose that the depositors have full bargaining power. In this case bank j does not get anything after default. The incentive constraint in the discrete-time approximation is

$$\begin{aligned} & e^{r_{kt}dt}N_t^j + \left(e^{r_{kt}dt} - e^{rdt}\right)D_t^j - N_{t+dt}^j + e^{-rdt}V_{t+dt}\left(N_{t+dt}^j\right) \\ \geq \; & e^{r_{kt}dt}N_t^j + e^{r_{kt}dt}D_t^j - N_{t+dt}^j, \end{aligned}$$

where we have removed bank index j for notational simplicity. Simplifying the preceding inequality yields $e^{rdt}D_t^j \leq e^{-rdt}V_{t+dt}\left(N_{t+dt}^j\right)$. In the continuous-time limit, the incentive constraint is

$$D_t^j \leq V_t\left(N_t^j\right). \tag{24}$$

Alternatively, adapting Gertler and Karadi (2011) and Gertler and Kiyotaki (2010), we can interpret the preceding incentive constraint simply as follows. Banker j can steal the deposits D_t^j and run away. The outside option value is D_t^j. The banker has no incentive to steal the deposits if the value of staying in the bank, $V_t\left(N_t^j\right)$, is not smaller than the outside option value.

Note that (24) is a special case of (23) with $\xi = 1$. By (5) in Proposition 4, if $1 < \theta/r - r/\theta - 1$, then the bubbly and bubbleless steady states coexist in the baseline model. However, there is an undesirable feature of this case because Propositions 3 and 4 imply that $r_k^b = r_k^*$ when $\xi = 1$. This means that the capital stock and hence real allocation are identical in both the bubbly and bubbleless steady states. We can use the setup in the previous subsection to overcome this issue, except that (20) is replaced with (24). We characterize equilibrium in the following proposition.

Proposition 5 *Consider the model with risky loans and borrowing constraints (24). Assume that* $1 > \frac{\theta - r}{r+\theta} > \mu > \frac{r}{\pi\theta}$. *Then the bubbly and bubbleless steady states coexist and*

$$\begin{aligned} Q^* &= \frac{\theta\pi}{r} > \frac{1}{\mu},\ 0 < r_k^* - r = r\frac{\theta(1-\mu(1-\pi)) - r}{r + \theta\pi}, \\ \frac{1}{\mu} &> Q^b = \frac{r+\theta}{\theta - r} > 1,\ 0 < r_k^b - r = \frac{r(\theta - r)}{r+\theta} \leq r_k^* - r. \end{aligned}$$

In the neighborhood of the bubbly steady state, the bubbly equilibrium system is given by

$$rQ_t = \dot{Q}_t + Q_t\left(r_{kt} + Q_t\left(r_{kt} - r\right)\right) + \theta\ \left(1 - Q_t\right),$$

$$rB_t = \dot{B}_t + Q_t\left(r_{kt} - r\right)B_t,$$

$$\dot{N}_t = -\theta N_t + N_t r_{kt} + \left(Q_t N_t + B_t\right)\left(r_{kt} - r\right),$$

where $r_{kt} = \alpha\left(Q_t N_t + B_t + N_t\right)^{\alpha-1} - \delta$. *In the neighborhood of the bubbleless steady state, the bubbleless equilibrium system is given by*

$$rQ_t = \dot{Q}_t + Q_t\left(r_{kt} + Q_t\left(r_{kt} - r\right)\right) + (1-\pi)\,\theta\left(\mu - 1\right)Q_t + \pi\theta\ \left(1 - Q_t\right),$$

$$\dot{N}_t = -\theta N_t + N_t r_{kt} + Q_t N_t\left(r_{kt} - r\right) + \theta\left(1 - \pi\right)\mu N_t,$$

where $r_{kt} = \alpha\left(Q_t N_t + N_t\right)^{\alpha-1} - \delta$.

Proof of Proposition 5: The proof is similar to that of Proposition 9. We conjecture and verify that $V\left(N_t^j, Q_t, B_t\right) = Q_t N_t^j + B_t$. The Bellman equation is still given by (22), but the incentive constraint (20) is replaced with the following:

$$D_t \leq V\left(N_t^j, Q_t, B_t\right). \tag{25}$$

First, we derive the bubbly equilibrium system. Guess that $1 < Q_t < 1/\mu$ and $r_{kt} > r$ in a neighborhood of the bubbly steady state. It follows that $dC_t = 0$ and the borrowing constraint (25) binds so that $D_t^j = Q_t N_t^j + B_t$ for the first maximization part. But, $dC_t^j = N_t^j$ and $D_t^j = 0$ for the second maximization part. Substituting this decision rule back into the Bellman equation, we can derive that

$$\begin{aligned} & r\left(Q_t N_t^j + B_t\right) dt \\ = \ & Q_t\left[r_{kt} N_t^j dt + (r_{kt} - r)\left(Q_t N_t^j + B_t\right) dt\right] + N_t^j dQ_t + dB_t \\ & + (1-\pi)\,\theta dt\ \left(N_t^j - Q_t N_t^j\right) + \pi\theta dt\ \left(N_t^j - Q_t N_t^j\right). \end{aligned}$$

Matching coefficients of N_t^j on the two sides of the equation yields the differential equations for Q_t and B_t in the proposition.

To derive the law of motion for the aggregate net worth, we aggregate (12) and use a law of large numbers to obtain

$$\begin{aligned} N_{t+dt} \ = \ & (1-\theta dt)\int\left[N_t^j e^{r_{kt}dt} + \left(e^{r_{kt}dt} - e^{rdt}\right)\left(Q_t N_t^j + B_t\right)\right] dj \\ & + (1-\pi)\,\theta dt \int\left(N_t^j e^{r_{kt}dt} - N_t^j\right) dj + \pi\theta dt \int\left(N_t^j e^{r_{kt}dt} - N_t^j\right) dj. \end{aligned}$$

Noting $e^{r_{kt}dt} \approx 1 + r_{kt}dt$ and $e^{rdt} \approx 1 + rdt$, eliminating terms of order higher than dt, and then taking the limit as $dt \to 0$, we obtain the differential equation for N_t as in the proposition. Aggregating (11), we derive aggregate capital as

$$K_t = (1-\theta dt)\int\left(N_t^j + D_t^j\right) dj.$$

When $dt \to 0$, $K_t = N_t + Q_t N_t + B_t$. Thus the capital return $r_{kt} = \alpha K_t^{\alpha-1} - \delta = \alpha\left(N_t + Q_t N_t + B_t\right)^{\alpha-1} - \delta$.

Next we derive the bubbleless equilibrium system. Setting $B_t = 0$ in (22), we obtain

$$\begin{aligned} rQ_t N_t^j dt &= \max_{dC_t^j, D_t^j} \left\{ dC_t^j + Q_t \left[r_{kt} \left(N_t^j + D_t^j \right) dt - dC_t^j - rD_t^j dt \right] \right\} + N_t^j dQ_t \\ &+ (1-\pi)\,\theta dt \max_{dC_t^j, D_t^j} \left\{ dC_t^j + Q_t \left[\left(N_t^j + D_t^j - dC_t^j \right) \mu - D_t^j \right] - Q_t N_t^j \right\} \\ &+ \pi \theta dt \left(N_t^j - Q_t N_t^j \right). \end{aligned}$$

Guess that $Q_t > 1/\mu > 1$ and $r_{kt} > r$ in a neighborhood of the bubbleless steady state. It follows that for the first maximization problem, $dC_t^j = 0$ and the borrowing constraint (25) binds so that $D_t^j = Q_t N_t^j$. But, for the second maximization problem, $dC_t^j = 0$ and $D_t^j = 0$. Substituting this decision rule into the Bellman equation yields

$$\begin{aligned} & rQ_t N_t^j dt = Q_t \left[r_{kt} N_t^j + (r_{kt} - r)\, Q_t N_t^j dt \right] + N_t^j dQ_t \\ & + (1-\pi)\,\theta dt \left(Q_t N_t^j \mu - Q_t N_t^j \right) + \pi \theta dt \left(N_t^j - Q_t N_t^j \right). \end{aligned}$$

Matching coefficients of N_t^j yields the differential equation for Q_t as in the proposition.

$$\begin{aligned} N_{t+dt} &= (1-\theta dt) \int \left[N_t^j e^{r_{kt} dt} + \left(e^{r_{kt} dt} - e^{rdt} \right) Q_t N_t^j \right] dj \\ &\quad + (1-\pi)\,\theta dt \int \mu N_t^j e^{r_{kt} dt} dj + \pi \theta dt \int \left(N_t^j e^{r_{kt} dt} - N_t^j \right) dj. \end{aligned}$$

Taking the limit as $dt \to 0$, we obtain the differential equation for N_t in the proposition. Aggregating (11) yields aggregate capital $K_t = Q_t N_t + N_t$. Thus $r_{kt} = \alpha K_t^{\alpha-1} - \delta = \alpha \left(N_t + Q_t N_t \right)^{\alpha-1} - \delta$.

Finally, we use the bubbly and bubbleless equilibrium systems to derive the steady states given in the proposition. Given the assumption in the proposition, we can easily check that $Q^* > 1/\mu > 1$ and $r_k^* > r$ in the bubbleless steady state and $1 < Q_b < 1/\mu$ and $r_{kb} > r$ in the bubbly steady state. This verifies that our previous guess in the neighborhood of a steady state is valid. We can also use the law of motion for the aggregate net worth to compute the steady-state size of the bubble as

$$\frac{B}{N} = \frac{r_k^b - \theta}{r_k^b - r} - \xi Q^b = \frac{\theta^2 - 2r\theta - r^2}{r\,(\theta - r)},$$

which is positive by assumption. Q.E.D.

The intuition behind this proposition is similar to that behind Proposition 4. So we omit a discussion. Note that when $\pi = 1$, Proposition 5 reduces to Propositions 3 and 4 with $\xi = 1$.

2 More on the Gertler-Karadi-Kiyotaki Approach

We first introduce the Gertler-Karadi-Kiyotaki-type borrowing constraint (20) into the baseline model of Section 2 in the paper. We shall show that a bubble can exist, but it affects prices only without affecting real allocation in the steady state. Given the conjectured value function of the bank in

$$V_t\left(N_t\right) = Q_t N_t + B_t, \tag{26}$$

we can rewrite (20) as

$$D_t \leq \frac{Q_t N_t + B_t}{\xi} - N_t. \tag{27}$$

From the Bellman equation

$$r\left(Q_t N_t + B_t\right) = \max_{C_t^b, D_t} \quad C_t^b + Q_t\left(r_{kt} N_t + \left(r_{kt} - r\right) D_t - C_t^b\right) + N_t \dot{Q}_t + \dot{B}_t, \tag{28}$$

we deduce that if $Q_t > 1$ and $r_{kt} > r$, then the constraints (19) and (27) bind so that $C_t^b = \theta N_t$ and

$$D_t = \frac{Q_t N_t + B_t}{\xi} - N_t.$$

Substituting these decision rules into (28) and matching coefficients, we obtain the following system of differential equations:

$$\begin{aligned}
rQ_t &= Q_t\left[r_{kt} + \left(r_{kt} - r\right)\left(Q_t/\xi - 1\right)\right] + \theta(1 - Q_t) + \dot{Q}_t, \\
rB_t &= Q_t\left(r_{kt} - r\right) B_t/\xi + \dot{B}_t, \\
\dot{N}_t &= \left[r_{kt} - \theta + \left(r_{kt} - r\right)\left(Q_t/\xi - 1\right)\right] N_t + \left(r_{kt} - r\right) B_t/\xi
\end{aligned}$$

We can solve for the bubbleless steady state in which $B_t = 0$ for all t, and

$$Q^* = \frac{\theta}{r}, \; r_k^* - r = \frac{\theta - r}{\theta} r\xi.$$

We can also solve for the bubbly steady state in which,

$$Q^b = \frac{\theta}{\theta - r},\ r_k^b - r = \frac{r\xi}{Q} = \frac{\theta - r}{\theta} r\xi,\ \frac{B}{N} = \frac{\theta}{r} - \frac{\theta}{\theta - r}.$$

Under the assumption $\theta > 2r$, the bubbly and bubbleless steady states coexist. In addition, $Q^* > Q^b > 1$, $r_k^b > r$ and $r_k^* > r$. Thus these relations also hold in a neighborhood of a steady state, verifying our previous guess. We also find that $r_k^b = r_k^*$. This implies that both the bubbly and bubbleless equilibria deliver the same steady-state capital stock and hence the same steady-state real allocation.

Next we introduce the borrowing constraint (20) into the model of Section 1.1.1, which adopts the modeling of dividend policy in Gertler and Karadi (2011) and Gertler and Kiyotaki (2010). Suppose that there is no new equity issuance so that $C_t^j \geq 0$. We shall show that a bubble cannot exist in that model. Given the conjectured value function $V\left(N_t^j, Q_t, B_t\right) = Q_t N_t^j + B_t$, we can rewrite (20) as

$$D_t^j \leq \frac{Q_t N_t^j + B_t}{\xi} - N_t^j. \tag{29}$$

Bank j's Bellman equation is given by (4) subject to $C_t^j \geq 0$ and the above borrowing constraint. Under the assumption $Q_t > 1$ and $r_{kt} > r$, $C_t^j = 0$ and $D_t^j = (Q_t/\xi - 1)N_t^j + B_t/\xi$.[7] Substituting these decision rules into the Bellman equation and matching coefficients, we obtain

$$\begin{aligned}
rQ_t &= r_{kt}Q_t + Q_t\left(r_{kt} - r\right)\left(Q_t/\xi - 1\right) + \dot{Q}_t + \rho\left(1 - Q_t\right), \\
\left(r + \rho\right) B_t &= \left(r_{kt} - r\right) Q_t B_t/\xi + \dot{B}_t, \\
\dot{N}_t &= r_{kt}N_t + \left(r_{kt} - r\right)\left[(Q_t/\xi - 1)N_t + B_t/\xi\right] - \rho(1 - \omega)N_t.
\end{aligned}$$

Suppose that a bubbly steady state exists, in which $B_t = B > 0$ for all t. It follows from the preceding differential equations that in the bubbly steady state,

$$\begin{aligned}
r + \rho &= \left(r_k - r\right) Q/\xi, \\
0 &= Q\left(r_k - r\right) Q/\xi + \rho\left(1 - Q\right).
\end{aligned}$$

[7] Gertler and Karadi (2011) and Gertler and Kiyotaki (2011) assume that banker j consumes N_t^j only when he dies so that $C_t^j = 0$ before his death.

Thus $Q = -\rho/r < 0$, which is impossible in equilibrium.

3 Discount Window Policy

We now show that the central bank can stabilize the economy by a specific discount window policy to achieve the deterministic bubbly equilibrium. We borrow the idea from Benhabib, Miao and Wang (2014). Suppose that the central bank targets the interest rate $r_{TG} = r_k^b$ at the bubbly steady-state rate. To achieve this target, the discount window provides liquidity to bank j according to the rule

$$M_t^j = \max(K^b - K_t, 0) = M_t, \tag{30}$$

where K^b satisfies $r_k^b = \alpha \left(K^b\right)^{\alpha-1}$. We will show that such a rule will generate a unique bubbly steady state equilibrium. We assume that the central bank lends the money to commercial banks at a rate $r_w \in \left(r, r^b\right)$. We can show that the Bellman function (28) for the bank becomes

$$\begin{aligned} r\left(Q_t N_t + B_t\right) \quad = \quad & Q_t\left[r_{kt} N_t + \left(r_{kt} - r\right)\left(N_t \xi Q_t + B_t\right) + (r_{kt} - r_w) M_t\right] + \left(1 - Q_t\right)\theta N_t \\ & + N_t \dot{Q}_t + \dot{B}_t. \end{aligned} \tag{31}$$

Matching coefficients yields

$$rQ_t = Q_t\left[r_{kt} + \left(r_{kt} - r\right)\xi Q_t\right] + \theta(1 - Q_t) + \dot{Q}_t,$$

$$rB_t = Q_t[\left(r_{kt} - r\right)B_t + (r_{kt} - r_w)M_t] + \dot{B}_t. \tag{32}$$

From the flow-of-funds constraint, we can show that

$$\dot{N}_t = \left(r_{kt} - \theta + \left(r_{kt} - r\right)\xi Q_t\right)N_t + \left(r_{kt} - r\right)B_t + (r_{kt} - r_w)M_t. \tag{33}$$

In equilibrium, $r_{kt} = \alpha K_t^{\alpha-1}$ and $K_t = M_t + (1 + \xi Q_t)N_t + B_t$.

We can verify that the bubbly steady state $\{Q^b, B, N^b, r_k^b,\ K^b\}$ satisfies the preceding equilibrium conditions, in which $M_t = 0$. Thus it is an equilibrium under the discount window policy. However, the bubbleless steady state is not an equilibrium. If it were an equilibrium, equation (30) would imply that $M = \max\left(K^b - K^*, 0\right) > 0$. But the conditions that $B = 0$, $r_k^* > r_k^b > r_w$, and $M > 0$ are not compatible with (32). Since the bubbleless steady state is no long an equilibrium, such a policy also eliminates the stochastic bubbly equilibrium.

Note that in the bubbly steady state equilibrium, $M_t = 0$ for all t and hence the central bank does not need actually to intervene. But during the crisis period when $K_t < K^b$ or the credit spread $r_{kt} - r$ exceeds $r_k^b - r$, the central bank automatically generates a liquidity injection dynamics according to (30) with high front loading.

4 Another Type of Equilibria

In our paper we focus on symmetric equilibria in which a reorganized bank upon default is treated as the same bank before default. The only difference is that they have different levels of net worth. We now consider another type of equilibria in which the agents believe that the reorganized bank upon default loses a fraction $1 - \gamma \in (0, 1)$ of its bubble value in addition to losing a fraction of its net worth. The interpretation is that the bank upon default will lose its reputation and the bubble component of the bank value may reflect the bank's reputation. We then write the representative bank's Bellman equation as

$$\begin{aligned} rV\left(N_t, Q_t, B_t\right) &= \max_{C_t^b, D_t} \; C_t^b + V_N\left(N_t, Q_t, B_t\right)\dot{N}_t \\ &\quad + V_Q\left(N_t, Q_t, B_t\right)\dot{Q}_t + V_B\left(N_t, Q_t, B_t\right)\dot{B}_t \end{aligned}$$

subject to $D_t \geq 0$, $N_t \geq 0$,

$$\dot{N}_t = r_{kt}N_t + \left(r_{kt} - r\right)D_t - C_t^b,$$

$$C_t^b \geq \theta N_t,$$

$$D_t \leq V\left(\xi N_t, Q_t, \gamma B_t\right).$$

The last constraint is the new incentive constraint. We conjecture that the value function takes the form

$$V\left(N_t, Q_t, B_t\right) = Q_t N_t + B_t, \tag{34}$$

where $1 > \gamma > 0$. Following the same procedure as in the paper, we can derive the following differential equations

$$rQ_t = Q_t \left[r_{kt} + (r_{kt} - r)\, \xi Q_t \right] + \theta(1 - Q_t) + \dot{Q}_t, \tag{35}$$

$$rB_t = Q_t\, (r_{kt} - r)\, \gamma B_t + \dot{B}_t, \tag{36}$$

$$\dot{N}_t = (r_{kt} - \theta + (r_{kt} - r)\, \xi Q_t)\, N_t + (r_{kt} - r)\, \gamma B_t, \tag{37}$$

provided that $Q_t > 1$ and $r_{kt} > r$. Using the market-clearing condition, the firm's first-order condition, and the bank's balance-sheet condition, we can derive

$$r_{kt} = \alpha \left((\xi Q_t + 1)\, N_t + \gamma B_t \right)^{1-\alpha}. \tag{38}$$

The equilibrium system consists of four equations (35), (36), (37), and (38) for four variables $\{Q_t, N_t, B_t, r_{kt}\}$. We need to verify that $Q_t > 1$ and $r_{kt} > r$ hold in equilibrium. If $\gamma = 1$, then this system reduces to that in our baseline model. The bubbleless steady state is unchanged. We can derive the bubbly steady state as follows:

$$Q^b = \frac{r + \theta\gamma}{\theta\gamma - \xi r},$$

$$r_k^b - r = \frac{r\,(\theta\gamma - \xi r)}{\gamma(r + \theta\gamma)},$$

$$\frac{B}{N^b} = \frac{-\left(r_k^b - \theta + \left(r_k^b - r \right) \xi Q^b \right)}{(r_k^b - r)\gamma}.$$

Simple algebra shows that $B/N^b > 0$ if and only if

$$0 < \xi < \frac{\theta\gamma}{r} - \frac{r}{\theta} - \gamma. \tag{39}$$

This condition implies that $Q^b > 1$ and $r_k^b > r$ and thus $Q_t > 1$ and $r_{kt} > r$ hold in a neighborhood of the bubbly steady state. As a result, the equilibrium defined above exists. The condition above also implies that $\theta > r$ and thus is equivalent to

$$\frac{\xi + r/\theta}{\theta/r - 1} < \gamma \leq 1.$$

That is, as long as γ is sufficiently close to one, the bubbly equilibrium in which the reorganized bank upon default only keeps a fraction γ of the bubble value exists. Since (39) implies that

$$0 < \xi < \frac{\theta}{r} - \frac{r}{\theta} - 1,$$

which is condition (13) in our paper. This means that the symmetric bubbly equilibrium analyzed in our paper also exists.

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