Journal of Economic Behavior & Organization 140 (2017) 336–353

Contents lists available at ScienceDirect

Journal of Economic Behavior & Organization

journal homepage: www.elsevier.com/locate/jebo

Quality and competition between public and private firms

Liisa T. Laine[1], Ching-to Albert Ma*

Department of Economics, Boston University, 270 Bay State Road, Boston, MA 02215, USA

ARTICLE INFO

Article history:
Received 27 July 2016
Received in revised form 9 May 2017
Accepted 11 May 2017
Available online 12 May 2017

JEL classification:
D4
L1
L2
L3

Keywords:
Price-quality competition
Quality
Public firm
Private firm
Mixed oligopoly

ABSTRACT

We study a multistage, quality-then-price game between a public firm and a private firm. The market consists of a set of consumers who have different quality valuations. The public firm aims to maximize social surplus, whereas the private firm maximizes profit. In the first stage, both firms simultaneously choose qualities. In the second stage, both firms simultaneously choose prices. Consumers' quality valuations are drawn from a general distribution. Each firm's unit production cost is an increasing and convex function of quality. There are multiple equilibria. In some, the public firm chooses a low quality, and the private firm chooses a high quality. In others, the opposite is true. We characterize subgame-perfect equilibria. Equilibrium qualities are often inefficient, but under some conditions on consumer valuation distribution, equilibrium qualities are first best. Various policy implications are drawn.

1. Introduction

Public and private firms compete in many markets. In many countries, general education, health care services and transportation are provided by public and private firms in various degrees. In higher education, most universities in Europe and Asia are public, but in the United States the market is a mixed oligopoly. Furthermore, in different markets in the U.S., quality segmentation varies. The best universities in the U.S. Northeast are private, but many public universities in California and the western states have higher quality than private colleges (see Deming and Goldin, 2012). In health care, again, many European markets are dominated by public firms, but in many countries the private market is very active. Again, quality segmentation differs. For example, in the U.S., according to the U.S. News ranking in 2016–2017, four out of the five best U.S. hospitals were private. However, it has been well documented that U.S. public nursing homes have higher quality than private nursing homes (see Comondore et al., 2009).

Quality is a major concern in these markets. The interest in quality stems from a fundamental point made by Spence (1975). Because a good's quality benefits all buyers, the social benefit of quality is the sum of consumers' valuations. At a social optimum, the average consumer quality valuation should be equal to the quality marginal cost. Yet, a profit- maximizing firm is only concerned with the consumer who is indifferent between buying and not. A firm's choice of quality will be one that maximizes the surplus of this marginal consumer. The classic Spence (1975) result says that even when products

* Corresponding author.
E-mail addresses: liisa.t.laine@jyu.fi (L.T. Laine), ma@bu.edu (C.-t.A. Ma).

[1] Permanent address: School of Business and Economics, P.O. Box 35, 40014 University of Jyväskylä, Finland.

http://dx.doi.org/10.1016/j.jebo.2017.05.012
0167-2681/

are priced at marginal costs, their qualities will be inefficient. We show that a mixed oligopoly may be a mechanism for remedying this inefficiency.

We use a standard model of vertical product differentiation. In the first stage, two firms simultaneously choose product qualities. In the second stage, firms simultaneously choose product prices. Consumers' quality valuations are drawn from a general distribution. The two firms have access to the same technology. The only difference from the textbook setup is that one is a social-surplus maximizing public firm, whereas the other remains a profit- maximizing private firm. Surprisingly, this single difference has many implications.

First, the model exhibits multiple equilibria: in some equilibria, the public firm's product quality is higher than the private firm's, but in others, the opposite is true. These multiple equilibria illustrate the variety of quality segmentations in the markets mentioned above. Second, and more important, we present general conditions on consumers' quality-valuation distribution under which qualities in low-public-quality equilibria are efficient, as well as general conditions under which qualities in high-public-quality equilibria are efficient. When equilibrium qualities are inefficient, deviations from the first best go in tandem: either qualities in public and private firms are both below the corresponding first-best levels, or they are both above. Equilibrium qualities form a rich set, and we have constructed examples with many configurations.

Our analysis proceeds in the standard way. Given a subgame defined by a pair of qualities, we find the equilibrium prices. Then we solve for equilibrium qualities, letting firms anticipate that their quality choices lead to continuation equilibrium prices. In the pricing subgame, qualities are given. The public firm's objective is to maximize social surplus, so its price best response must achieve the efficient allocation of consumers across the two firms. This requires that consumers fully internalize the cost difference between high and low qualities. The public firm sets its price in order that the difference in prices is exactly the difference in quality costs. The private firm's best response is the typical inverse demand elasticity rule.

When firms choose qualities, they anticipate equilibrium prices in the next stage. Given the private firm's quality, the public firm chooses its quality to maximize social surplus, anticipating the equilibrium consumer assignment among firms in the next stage. The private firm, however, will try to manipulate the equilibrium prices through its quality. Without any price response from the public firm, the private firm would have chosen the quality that would be optimal for the marginal consumer, as in Spence (1975). A larger quality difference, however, would be preferred because that would raise the private firm's price. Because of the price manipulation, the private firm's equilibrium quality is one that maximizes the utility of an inframarginal consumer, not the utility of the marginal consumer.

In the first best, the socially efficient qualities are determined by equating average consumer valuations and marginal cost of quality. The surprise is that in contrast to private duopoly, the private firm's equilibrium quality choice may coincide with the first-best quality. In other words, the inframarginal consumer whose utility is being maximized by the private firm happens to have the average valuation among the private firm's customers.

The (sufficient) conditions for first-best equilibria refer to the consumers' quality-valuation distribution. In equilibria where the public firm produces at a low quality, equilibrium qualities are first best when the valuation distribution has a linear inverse hazard rate.[2] In equilibria where the public firm produces at a high quality, equilibrium qualities are first best when the valuation distribution has a linear inverse reverse hazard rate. The linear inverse hazard and inverse reverse hazard rate conditions are equivalent to the private firm's marginal revenue function being linear in consumer valuation. Nevertheless, linear inverse hazard and inverse reverse hazard rates are special. A generic valuation distribution violates linearity. In an inefficient equilibrium, both firms' qualities are either too high or too low relative to the first best. This is in sharp contrast to the private duopoly in which excessive quality differentiation is used to relax price competition.

We draw various policy implications from our results. First, if a public firm is to take over a private one in a duopoly, should it enter in the high-quality or low- quality segment of the market? High-public-quality equilibria and low-public-quality equilibria generate different social surpluses. Second, our use of a social-welfare objective function for the public firm can be regarded as making a normative point. If the public firm aims to maximize only consumer surplus, it will subscribe to marginal-cost pricing. Because the private firm never prices at marginal cost, equilibrium-price difference between firms will never be equal to the quality-cost difference, so consumer assignments across firms will never be efficient. A social-welfare objective does mean that the public firm tolerates high prices. However, our policy recommendation is that undesirable effects from high prices should be remedied by a tax credit or subsidy to consumers regardless of where they purchase from.

Our research contributes to the literature of mixed oligopolies. We use the classical model of quality-price competition in Gabszewicz and Thisse (1979, 1986) and Shaked and Sutton (1982, 1983). However, the mixed oligopoly literature revolves around the theme that a public firm may improve welfare. Grilo (1994) studies a mixed duopoly in the vertical differentiation framework. In her model, consumers' valuations of qualities follow a uniform distribution. The unit cost of production may be convex or concave in quality. The paper derives first-best equilibria. In a Hotelling, horizontal differentiation model with quadratic transportation cost, Cremer et al. (1991) show that a public firm improves welfare when the total number of firms is either two, or more than six. Also using a Hotelling model, Matsumura and Matsushima (2004) show that mixed oligopoly gives some cost-reduction incentives. In a Cournot model, Cremer et al. (1989) show the disciplinary effect of replacing some private firms by public enterprises. Comparing Cournot and Bertrand models in mixed market, Ghosh and Mitra (2010) show that the results from private Cournot-Bertrand comparisons do not hold when a private firm is replaced

[2] See Lemmas 3 and 6 below. If F denotes the distribution, and f the density, then the inverse hazard rate is $\frac{1-F}{f}$, and the inverse reverse hazard rate is $\frac{F}{f}$.

by a welfare- maximizing public firm. Our paper is consistent with these results. However, we use a general consumer valuation distribution and cost function, and present multiple equilibria, which have not been the focus in the literature.

For profit-maximizing firms, Cremer and Thisse (1991) show that, under very mild conditions on transportation costs, horizontal differentiation models are actually a special case of vertical product differentiation (see also Champsaur and Rochet, 1989). The isomorphism can be transferred to mixed duopolies. The key in the Cremer-Thisse (1991) proof is that demands in horizontal models can be translated into equivalent demands in vertical models. Firms' objectives are unimportant. Hence, results in horizontal mixed oligopolies do relate to vertical mixed oligopolies. In most horizontal differentiation models, consumers are assumed to be uniformly distributed on the product space, and the transportation or mismatch costs are quadratic. These assumptions translate to a uniform distribution of consumer quality valuations and a quadratic quality cost function in vertical differentiation models.

The first-best results in Grilo (1994) are related to the efficient equilibria in the two-firm case in Cremer et al. (1991) because both papers use the uniform distribution for consumer valuations. By contrast, we use a general distribution for consumer valuation. Our results simultaneously reveal the limitation of the uniform distribution and which properties of the uniform distribution (linear inverse hazard and linear inverse reverse hazard rates) have been the driver of earlier results. Furthermore, when consumer valuations follow a uniform distribution, the issue of multiple equilibria is moot for a duopoly. By contrast, we show that multiple equilibria are important for general distributions. Moreover, our equilibrium qualities translate to equilibrium locations under general consumer distributions on the Hotelling line.

For private firms, Anderson et al. (1997) give the first characterization for a general location distribution with quadratic transportation costs. Our techniques are consistent with those in Anderson et al. (1997), but we use a general cost function. A recent paper by Benassi et al. (2006) uses a symmetric trapezoid valuation distribution and explores consumers' nonpurchase options. Yurko (2011) works with lognormal distributions. Our monotone inverse hazard and inverse reverse hazard rate assumptions are valid under the trapezoid distribution, but invalid under lognormal distributions.

Qualities in mixed provisions are often discussed in the education and health sectors. However, perspectives such as political economy, taxation, and income redistribution are incorporated, so public firms typically are assumed to have objective functions different from social welfare. Brunello and Rocco (2008) combine consumers voting and quality choices by public and private schools, and let the public school be a Stackelberg leader. Epple and Romano (1998) consider vouchers and peer effects but use a competitive model for interaction between public and private schools. (For recent surveys on education and health care, see Urquiola, 2016; Barros and Siciliani, 2012.) Grassi and Ma (2011, 2012) present models of publicly rationed supply and private firm price responses under public commitment and noncommitment. Our results here indicate that commitment may not be necessary, and imperfectly competitive markets may sometimes yield efficient qualities.

Privatization has been a policy topic in mixed oligopolies. Ishibashi and Kaneko (2008) set up a mixed duopoly with price and quality competition. The model has both horizontal and vertical differentiation. However, all consumers have the same valuation on quality, and are uniformly distributed on the horizontal product space (as in Ma and Burgess, 1993). They show that the government should manipulate the objective of the public firm so that it maximizes a weighted sum of profit and social welfare, a form of partial privatization. (Using a Cournot model, Matsumura, 1998 earlier demonstrates that partial privatization is a valuable policy.) Our model is richer on the vertical dimension, but consists of no horizontal differentiation. Our policy implication has a privatization component to it, but a simple social welfare objective for the public firm is sufficient.

Section 2 presents the model. Section 3 studies equilibria in which the public firm's quality is lower than the private firm's, and Section 4 studies the opposite case. In each section, we first derive subgame-perfect equilibrium prices, and then equilibrium qualities. We present a characterization of equilibrium qualities, and conditions for equilibrium qualities to be first best. Section 5 considers policies, various robustness issues, and existence of equilibria. We consider alternative preferences for the public firm. We also let cost functions of the firms be different. Then we let consumers have outside options, and introduce multiple private firms. Finally we consider existence of equilibria. The last section presents some concluding remarks. Proofs are collected in the Appendix. Details of numerical computation are in the Supplement.

2. The model

2.1. Consumers

There is a set of consumers with total mass normalized at 1. Each consumer would like to receive one unit of a good or service. In our context, it is helpful to think of such goods and services as education, transportation, and health care including child care, medical, and nursing home services. The public sector often participates actively in these markets.

A good has a quality, denoted by q, which is assumed to be positive. Each consumer has a valuation of quality v. This valuation varies among consumers. We let v be a random variable defined on the positive support $[\underline{v}, \bar{v}]$ with distribution F and strictly positive density f. We also assume that f is continuously differentiable.

We will use two properties of the distribution, namely $[1-F]/f \equiv h$, and $F/f \equiv k$. We assume that h is decreasing, and that k is increasing, so $h'(v) < 0$ and $k'(v) > 0$. The assumptions ensure that profit functions, to be defined below, are quasi-concave, and are implied by f being logconcave (Anderson et al., 1997). These monotonicity assumptions are satisfied by many common distributions such as the uniform, the exponential, the beta, etc. (Bagnoli and Bergstrom, 2004). We call h the inverse hazard rate (because $1/h$ is the hazard), and k the inverse reverse hazard rate (because $1/k$ is the reverse hazard).

Valuation variations among consumers have the usual interpretation of preference diversity due to wealth, taste, or cultural differences. We may call a consumer with valuation v a type-v consumer, or simply consumer v. If a type-v consumer purchases a good with quality q at price p, his utility is $vq - p$. The quasi-linear utility function is commonly adopted in the literature (see, for example, the standard texts Anderson et al., 1992 and Tirole, 1988).

We assume that each consumer will buy a unit of the good. This can be made explicit by postulating that each good offers a sufficiently high benefit which is independent of v, or that the minimum valuation $\underline{v}$ is sufficiently high. The full-market coverage assumption is commonly used in the extant literature of product differentiation (either horizontal or vertical), but Delbono et al. (1996) and Benassi et al. (2016) have explored the implications of consumer outside options, and we defer to Subsection 5.4 for more discussions. Relatedly, the introduction of a public firm may be a policy for market expansion. We ignore this consideration by the full-market coverage assumption.

2.2. Public and private firms

There are two firms, Firm 1 and Firm 2, and they have the same technology. Production requires a fixed cost. The implicit assumption is that the fixed cost is so high that entries by many firms cannot be sustained. We focus on the case of a mixed oligopoly so we do not consider the rather trivial case of two public firms. Often a mixed oligopoly is motivated by a more efficient private sector, so in Subsection 5.3 we let firms have different technologies, and will explain how our results remain robust.

The variable, unit production cost of the good at quality q is $c(q)$, where $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing and strictly convex function. A higher quality requires a higher unit cost, which increases at an increasing rate. We also assume that c is twice differentiable, and that it satisfies the usual Inada conditions: $\lim_{q\to 0^+} c(q) = \lim_{q\to 0^+} c'(q) = 0$, so both firms always will be active.

Firm 1 is a public firm, and its objective is to maximize social surplus; the discussion of a general objective function for the public firm is deferred until Section 5.2. Firm 2 is a profit-maximizing private firm. Each firm chooses its product quality and price. We let p_1 and q_1 denote Firm 1's price and quality; similarly, p_2 and q_2 denote Firm 2's price and quality. Given these prices and qualities, each consumer buys from the firm that offers the higher utility. A consumer chooses a firm with a probability equal to a half if he is indifferent between them.

Consider any (p_1, q_1) and (p_2, q_2), and define $\hat{v}$ by $\hat{v}q_1 - p_1 = \hat{v}q_2 - p_2$. Consumer $\hat{v}$ is just indifferent between purchasing from Firm 1 and Firm 2. If $\hat{v} \in [\underline{v}, \overline{v}]$, then the demands for the two firms are as follows:

$$\begin{array}{ccll} \text{Demand for Firm 1} & \text{Demand for Firm 2} & & \\ F(\hat{v}) & 1 - F(\hat{v}) & \text{if} & q_1 < q_2 \\ 1 - F(\hat{v}) & F(\hat{v}) & \text{if} & q_1 > q_2 \\ 1/2 & 1/2 & \text{if} & q_1 = q_2 \end{array} \qquad (1)$$

We sometimes call consumer $\hat{v}$ the indifferent or marginal consumer. (Otherwise, if $\hat{v} \notin [\underline{v}, \overline{v}]$, or fails to exist, one firm will be unable to sell to any consumer.)

If Firm 1's product quality is lower than Firm 2's, its demand is $F(\hat{v})$ when its price is sufficiently lower than Firm 2's price. Conversely, if Firm 2's price is not too high, then its demand is $1 - F(\hat{v})$. If the two firms' product qualities are identical, then they must charge the same price if both have positive demands. In this case, all consumers are indifferent between them, and each firm receives half of the market.

2.3. Allocation, social surplus, and first best

An allocation consists of a pair of product qualities, one at each firm, and an assignment of consumers across the firms. The social surplus from an allocation is

$$\int_{\underline{v}}^{v} [xq_\ell - c(q_\ell)] f(x) \mathrm{d}x + \int_{v}^{\overline{v}} [xq_h - c(q_h)] f(x) \mathrm{d}x. \qquad (2)$$

Here, the qualities at the two firms are q_ℓ and q_h, $q_\ell < q_h$. Those consumers with valuations between $\underline{v}$ and v get the good with quality q_ℓ, whereas those with valuations between v and $\overline{v}$ get the good with quality q_h. The first best is (q_ℓ^*, q_h^*, v^*) that maximizes (2), and is characterized by the following:

$$\frac{\int_{\underline{v}}^{v^*} xf(x)\mathrm{d}x}{F(v^*)} = c'(q_\ell^*) \tag{3}$$

$$\frac{\int_{v^*}^{\overline{v}} xf(x)\mathrm{d}x}{1-F(v^*)} = c'(q_h^*) \tag{4}$$

$$v^*q_\ell^* - c(q_\ell^*) = v^*q_h^* - c(q_h^*). \tag{5}$$

In the characterization of the first best in (3), (4), and (5), those consumers with lower valuations should consume the good at a low quality (q_ℓ^*), and those with higher valuations should consume at a high quality (q_h^*). For the first best, divide consumers into two groups: those with $v \in [\underline{v}, v^*]$ and those with $v \in [v^*, \overline{v}]$. The (conditional) average valuation of consumers in $[\underline{v}, v^*]$ is in the left-hand side of (3) , and, in the first best, this is equal to the marginal cost of the lower first-best quality, the right-hand side of (3). A similar interpretation applies to (4) for those consumers with higher valuations. Finally, the division of consumers into the two groups is achieved by identifying consumer v^* who enjoys the same surplus from both qualities, and this yields (5).

As Spence (1975) has shown, quality is like a public good, so the total social benefit is the aggregate consumer benefit, and in the first best, the average valuation should be equal to the marginal cost of quality. As a result the indifferent consumer v^* actually receives too little surplus from q_ℓ because $v^* > c'(q_\ell)$, but too much from q_h because $v^* < c'(q_h)$. In a private duopoly, firms will choose qualities to relax price competition, so one firm's equilibrium quality will be lower than the first best, whereas the other firm's equilibrium quality will be higher. Section 6 of Laine and Ma (2016), the working paper version, contains this result.[3]

2.4. Extensive form

We study subgame-perfect equilibria of the following game.

Stage 0: Nature draws consumers' valuations v and these are known to consumers only.

Stage 1: Firm 1 chooses a quality q_1; simultaneously, Firm 2 chooses a quality q_2.

Stage 2: Qualities in Stage 1 are common knowledge. Firm 1 chooses a price p_1; simultaneously, Firm 2 chooses a price p_2. Consumers then observe price-quality offers and pick a firm for purchase.

An outcome of this game consists of firms' prices and qualities, (p_1, q_1) and (p_2, q_2), and the allocations of consumers across the two firms. Subgames at Stage 2 are defined by the firms' quality pair (q_1, q_2). Subgame-perfect equilibrium prices in Stage 2 are those that are best responses in subgames defined by (q_1, q_2). Finally, equilibrium qualities in Stage 1 are those that are best responses given that prices are given by a subgame-perfect equilibrium in Stage 2.

There are multiple equilibria. In one class of equilibria, in Stage 1, the public firm chooses low quality, whereas the private firm chooses high quality, and in Stage 2, the public firm sets a low price, and the private firm chooses a high price. In the other class, the roles of the firms, in terms of their ranking of qualities and prices, are reversed. However, because the two firms have different objectives, equilibria in these two classes yield different allocations.[4]

3. Equilibria with low quality at public firm

3.1. Subgame-perfect equilibrium prices

Consider subgames in Stage 2 defined by (q_1, q_2) with $q_1 < q_2$. According to (1), each firm will have a positive demand only if $p_1 < p_2$, and there is $\tilde{v} \in [\underline{v}, \overline{v}]$ with

$$\tilde{v}q_1 - p_1 = \tilde{v}q_2 - p_2 \quad \text{or} \quad \tilde{v}(p_1, p_2; q_1.q_2) = \frac{p_2 - p_1}{q_2 - q_1}, \tag{6}$$

[3] Also, the multiple-quality duopoly with general valuation distributions and cost functions in Barigozzi and Ma (2016) can generate a special case for the single-quality duopoly with inefficient equilibrium qualities.

[4] These are all possible pure- strategy subgame-perfect equilibria. There is no equilibrium in which both firms choose the same quality. Indeed, the unique continuation equilibrium of subgames with identical qualities is firms setting price at the unit cost. Earning no profit, the private firm will deviate to another quality.

where we have emphasized that $\tilde{v}$, the consumer indifferent between buying from Firm 1 and Firm 2, depends on qualities and prices. Expression (6) characterizes firms' demand functions. Firm 1 and Firm 2s' payoffs are, respectively,

$$\int_{\underline{v}}^{\tilde{v}} [xq_1 - c(q_1)]f(x)\mathrm{d}x + \int_{\tilde{v}}^{\overline{v}} [xq_2 - c(q_2)]f(x)\mathrm{d}x \tag{7}$$

and

$$[1 - F(\tilde{v})][p_2 - c(q_2)]. \tag{8}$$

The expression in (7) is social surplus when consumers with valuations in $[\underline{v}, \tilde{v}]$ buy from Firm 1, whereas others buy from Firm 2. The prices that consumers pay to firms are transfers, so do not affect social surplus. The expression in (8) is Firm 2's profit.

Firm 1 chooses its price p_1 to maximize (7) given the demand (6) and price p_2. Firm 2 chooses price p_2 to maximize (8) given the demand (6) and price p_1. Equilibrium prices, $(\hat{p}_1, \hat{p}_2)$, are best responses against each other.

Lemma 1. *In subgames (q_1, q_2) with $q_1 < q_2$, and $\underline{v} < \frac{c(q_2)-c(q_1)}{q_2-q_1} < \overline{v}$, equilibrium prices $(\hat{p}_1, \hat{p}_2)$ are:*

$$\hat{p}_1 - c(q_1) = \hat{p}_2 - c(q_2) = (q_2 - q_1)\frac{1 - F(\hat{v})}{f(\hat{v})} \equiv (q_2 - q_1)h(\hat{v}), \tag{9}$$

$$\text{where } \hat{v} = \frac{c(q_2) - c(q_1)}{q_2 - q_1}. \tag{10}$$

In Lemma 1 the equilibrium price difference across firms is the same as the cost difference: $\hat{p}_2 - \hat{p}_1 = c(q_2) - c(q_1)$. Also, Firm 2 makes a profit, and its price-cost margin is proportional to the quality differential and the inverse hazard rate h.

We explain the result as follows. Firm 1's payoff is social surplus, so it seeks the consumer assignment to the two firms, $\tilde{v}$, to maximize social surplus (7). This is achieved by getting consumers to fully internalize the cost difference between the high and low qualities. Therefore, given $\hat{p}_2$, Firm 1 sets $\hat{p}_1$ so that the price differential $\hat{p}_2 - \hat{p}_1$ is equal to the cost differential $c(q_2) - c\,(q_1)$. In equilibrium, the indifferent consumer is given by $\hat{v}q_1 - c(q_1) = \hat{v}q_2 - c(q_2)$, which indicates an efficient allocation in the quality subgame (q_1, q_2).

Firm 2 seeks to maximize profit. Given Firm 1's price $\hat{p}_1$, Firm 2's optimal price follows the marginal-revenue-marginal-cost calculus. For a unit increase in p_2, the marginal loss is $[p_2 - c(q_2)]f(\tilde{v})/(q_2 - q_1)$, whereas the marginal gain is $[1 - F(\tilde{v})]$. Therefore, profit maximization yields $\hat{p}_2 - c(q_2) = (q_2 - q_1)\frac{1-F(\hat{v})}{f(\hat{v})}$, the inverse elasticity rule for Firm 2's price-cost margin.[5] Lemma 1 follows from these best responses.

The key point in Lemma 1 is that equilibrium market shares and prices can be determined separately. Once qualities are given, Firm 1 will aim for the socially efficient allocation, and it adjusts its price, given Firm 2's price, to achieve that. Firm 2, on the other hand, aims to maximize profit so its best response depends on Firm 1's price as well as the elasticity of demand. Firm 1 does make a profit, and we will return to this issue in Subsection 5.2.

To complete the characterization of price equilibria, we consider subgames (q_1, q_2) with $q_1 < q_2$, and either $\frac{c(q_2)-c(q_1)}{q_2-q_1} < \underline{v}$ or $\overline{v} < \frac{c(q_2)-c(q_1)}{q_2-q_1}$. In the former case, Firm 1 would like to allocate all consumers to Firm 2, whereas in the other case, Firm 1 would like to allocate all consumers to itself. In both cases, there are multiple equilibrium prices. They take the form of high values of $\hat{p}_1$ when all consumers go to Firm 2, but low values of $\hat{p}_1$ in the other. However, equilibria in the game must have two active firms, so these subgames cannot arise in equilibrium.

The equilibrium prices $(\hat{p}_1, \hat{p}_2)$ in (9) and (10) formally establish three functional relationships, those that relate any qualities to equilibrium prices and allocation of consumers across firms. We can write them as $\hat{p}_1(q_1, q_2)$, $\hat{p}_2(q_1, q_2)$, and $\hat{v}(q_1, q_2) \equiv \tilde{v}(\hat{p}_1(q_1, q_2), \hat{p}_2(q_1, q_2); q_1, q_2)$. We differentiate (9) with $\hat{v}$ in (10) to determine how equilibrium prices and market share change with qualities. As it turns out, we will only need to use the information of how $\hat{p}_1(q_1, q_2)$ and $\hat{p}_2(q_1, q_2)$ change with q_2.

Lemma 2. *From the definition of $(\hat{p}_1, \hat{p}_2)$ and $\hat{v}$ in (9) and (10), we have $\hat{v}$ increasing in q_1 and q_2, and*

$$\frac{\partial \hat{p}_1(q_1, q_2)}{\partial q_2} = h(\hat{v}) + h'(\hat{v})[c'(q_2) - \hat{v}] \tag{11}$$

$$\frac{\partial \hat{p}_2(q_1, q_2)}{\partial q_2} = c'(q_2) + h(\hat{v}) + h'(\hat{v})[c'(q_2) - \hat{v}]. \tag{12}$$

Lemma 2 describes how the equilibrium indifferent consumer changes with qualities, and the strategic effect of Firm 2's quality on Firm 1's price. The marginal consumer $\hat{v}$ is defined by $\hat{v}q_1 - c(q_1) = \hat{v}q_2 - c(q_2)$. Because $q_1 < q_2$, if q_1 increases, consumer $\hat{v}$ strictly prefers to buy from Firm 1, as does consumer $\hat{v} + \epsilon$ for a small and positive ϵ. Next, suppose that q_2

[5] Firm 2's demand is $1 - F(\tilde{v})$. Hence, elasticity is $\frac{\mathrm{d}(1-F(\tilde{v}))}{\mathrm{d}p_2}\frac{p_2}{1-F(\tilde{v})} = -\frac{q_2-q_1}{h(\tilde{v})}p_2$.

increases, consumer $\hat{v}$ also strictly prefers to buy from Firm 1. The point is that quality q_1 is too low for consumer $\hat{v}$ but quality q_2 is too high. An increase in q_1 makes Firm 1 more attractive to consumer $\hat{v}$, and an increase in q_2 makes Firm 2 less attractive to him.

If Firm 2 increases its quality, it expects to lose market share. However, it does not mean that its profit must decrease. From (8), Firm 2's profit is increasing in Firm 1's price.[6] If in fact Firm 1 raises its price against a higher q_2, Firm 2 may earn a higher profit. In any case, because h is decreasing, and $c'(q_2) > \hat{v}$, according to Lemma 2, an increase in q_2 may result in higher or lower equilibrium prices. The point is simply that Firm 2 can influence Firm 1's price response. Also, from the difference between (12) and (11), Firm 2's equilibrium price always increases at a higher rate than Firm 1's: $\partial\hat{p}_2/\partial q_2 - \partial\hat{p}_1/\partial q_2 = c'(q_2)$.

3.2. Subgame-perfect equilibrium qualities

At qualities q_1 and q_2, the continuation equilibrium payoffs for Firms 1 and 2 are, respectively,

$$\int_{\underline{v}}^{\hat{v}(q_1,q_2)} [xq_1 - c(q_1)]f(x)\mathrm{d}x + \int_{\hat{v}(q_1,q_2)}^{\bar{v}} [xq_2 - c(q_2)]f(x)\mathrm{d}x, \quad \text{and} \tag{13}$$

$$[1 - F(\hat{v}(q_1, q_2))][\hat{p}_2(q_1, q_2) - c(q_2)], \tag{14}$$

where $\hat{p}_2$ is Firm 2's equilibrium price and $\hat{v}$ is the indifferent consumer from Lemma 1. Let $(\hat{q}_1, \hat{q}_2)$ be the equilibrium qualities. They are mutual best responses, given continuation equilibrium prices:

$$\hat{q}_1 = \mathrm{argmax}_{q_1} \int_{\underline{v}}^{\hat{v}(q_1,\hat{q}_2)} [xq_1 - c(q_1)]f(x)\mathrm{d}x + \int_{\hat{v}(q_1,\hat{q}_2)}^{\bar{v}} [x\hat{q}_2 - c(\hat{q}_2)]f(x)\mathrm{d}x \tag{15}$$

$$\hat{q}_2 = \mathrm{argmax}_{q_2}[1 - F(\hat{v}(\hat{q}_1, q_2))][\hat{p}_2(\hat{q}_1, q_2) - c(q_2)]. \tag{16}$$

A change in quality q_1 has two effects on social surplus (13). First, it directly changes $vq_1 - c(q_1)$, the surplus of consumers who purchase the good at quality q_1. Second, it changes the equilibrium prices and the marginal consumer $\hat{v}$ (hence market shares) in Stage 2. This second effect is second order because the equilibrium prices in Stage 2 maximize social surplus. Hence, the first-order derivative of (13) with respect to q_1 is $\int_{\underline{v}}^{\hat{v}(q_1,q_2)}[x - c'(q_1)]f(x)\mathrm{d}x$ (although Firm 1's objective is to maximize social surplus of the entire market).

Similarly, a change in quality q_2 has two effects on Firm 2's profit. First, it directly changes the marginal consumer's surplus $\hat{v}q_2 - c(q_2)$. Second, it changes the equilibrium prices and the marginal consumer. We rewrite (16) as

$$[1 - F(\hat{v}(q_1, q_2))][\hat{v}(q_1, q_2)q_2 - c(q_2) - \hat{v}(q_1, q_2)q_1 + \hat{p}_1(q_1, q_2)] \tag{17}$$

because

$$\hat{v}(q_1, q_2) = \tilde{v}(\hat{p}_1(q_1, q_2), \hat{p}_2(q_1, q_2); q_1, q_2) \equiv \frac{\hat{p}_2(q_1, q_2) - \hat{p}_1(q_1, q_2)}{q_2 - q_1}, \tag{18}$$

which gives the channels for the influence of q_2 on prices. Firm 2's equilibrium price in Stage 2 maximizes profit, so the effect of q_2 on profit in (17) via $\hat{v}(q_1, q_2)$ has a second-order effect. Therefore, the first-order derivative of (17) with respect to quality q_2 is $\hat{v}(q_1, q_2) - c'(q_2) + \frac{\partial\hat{p}_1(q_1,q_2)}{\partial q_2}$ (where we have omitted the factor $[1 - F(\hat{v}(q_1, q_2))]$).

We set the first-order derivatives of social surplus with respect to q_1 and of profit with respect to q_2 to zero. Then we apply (11) in Lemma 2 to obtain the following.

Proposition 1. *Equilibrium qualities $(\hat{q}_1,\hat{q}_2)$, and the marginal consumer $\hat{v}$ solve the following three equations in q_1, q_2, and v*

$$\frac{\int_{\underline{v}}^{v} xf(x)\mathrm{d}x}{F(v)} = c'(q_1)$$

$$v + \frac{h(v)}{1 - h'(v)} = c'(q_2)$$

$$vq_2 - c(q_2) = vq_1 - c(q_1).$$

Firm 1's objective is to maximize social surplus. However, given Firm 2's quality and the continuation equilibrium prices, the assignment of consumers across firms will always be efficient. Therefore, Firm 1's return to quality q_1 consists of the

[6] The partial derivative of (8) with respect to p_1 is $\frac{f(\tilde{v})[p_2 - c(q_2)]}{q_2 - q_1} > 0$.

benefits of its own consumers. Hence $\hat{q}_1$ equates the conditional average valuation of consumers in $[\underline{v}, \hat{v}]$, $\frac{\int_{\underline{v}}^{\hat{v}} xf(x)\mathrm{d}x}{F(\hat{v})}$, and the marginal cost $c'(q_1)$. This is the first equation.

Firm 2's quality will affect Firm 1's price in Stage 2. If this were not the case (imagine that $\partial\hat{p}_1/\partial q_2$ were 0), the profit-maximizing quality would be the optimal level for the marginal consumer: $\hat{v} = c'(q)$, reminiscent of the basic property of quality in Spence (1975). By raising quality from one satisfying $\hat{v} = c'(q)$, Firm 2 may also raise Firm 1's price, hence its own profit. This is a first-order gain. The optimal tradeoff is now given by $\hat{v} + \frac{\partial\hat{p}_1(\hat{q}_1,\hat{q}_2)}{\partial q_2} = c'(\hat{q}_2)$. We use (11) to simplify, and show that Firm 2 sets its quality to be efficient for a consumer with valuation $\hat{v} + \frac{h(\hat{v})}{1-h'(\hat{v})}$. This is the second equation.

Proposition 1 presents remarkably simple equilibrium characterizations. The only difference between equilibrium qualities and those in the first best stems from how Firm 2 chooses its quality. Firm 2's consumers have average valuation $\frac{\int_{\hat{v}}^{\bar{v}} xf(x)\mathrm{d}x}{1-F(\hat{v})}$, which should be set to the marginal cost of Firm 2's quality for social efficiency. However, Firm 2's profit-maximization objective leads it to set quality so that the marginal cost is equal to $\hat{v} + \frac{h(\hat{v})}{1-h'(\hat{v})}$. Our next result gives a class of valuation distributions for which the answer is affirmative. First, we present a mathematical lemma, which, through a simple application of integration by parts, allows us to write the conditional expectation of valuations in terms of inverse hazard rate and the density.

Lemma 3. *For any distribution F (and its corresponding density f and inverse hazard rate $h \equiv (1 - F)/f$),*

$$\frac{\int_v^{\bar{v}} xf(x)\mathrm{d}x}{1-F(v)} \equiv v + \frac{\int_v^{\bar{v}} f(x)h(x)\mathrm{d}x}{f(v)h(v)}. \tag{19}$$

Proposition 2. *Suppose that the inverse hazard rate h is linear; that is, $h(x) = \alpha - \beta x$, $x \in [\underline{v}, \bar{v}]$, for some α and $\beta \geq 0$. Then for any v*

$$v + \frac{h(v)}{1-h'(v)} = \frac{\int_v^{\bar{v}} xf(x)\mathrm{d}x}{1-F(v)} \equiv v + \frac{\int_v^{\bar{v}} f(x)h(x)\mathrm{d}x}{f(v)h(v)}. \tag{20}$$

Equilibrium qualities and market shares are first best.

Proposition 2 exhibits a set of consumer valuation distributions for which the quality-price competition game yields first-best equilibrium qualities. We have managed to write the conditional average in terms of the inverse hazard rate in Lemma 3, and this is $\hat{v} + \frac{\int_{\hat{v}}^{\bar{v}} f(x)h(x)\mathrm{d}x}{f(\hat{v})h(\hat{v})}$. When the inverse hazard rate is linear, $\frac{h(\hat{v})}{1-h'(\hat{v})} \equiv \frac{\int_{\hat{v}}^{\bar{v}} f(x)h(x)\mathrm{d}x}{f(\hat{v})h(\hat{v})}$, Firm 2's profit-maximization incentive aligns with the social incentive. The following remark gives the economic interpretation for the linear inverse hazard rate.

Remark 1. When Firm 2 sells to high-valuation consumers, its marginal revenue is linear in consumer valuation if and only if $h(v)$ is linear.

The hazard rate has figured prominently in information economics and auction theory (see, for instance, Krishna, 2009; Laffont and Tirole, 1993; Myerson, 1997), and measures information rent, or virtual valuation. Here, in quality-price competition, its role is in how a private firm's quality changes the rival public firm's continuation equilibrium price. When the inverse hazard rate is linear, in auction and bargaining theory, strategies become linear and tractability is available (see Chatterjee and Samuelson, 1983; Gresik, 1991; Satterthwaite and Williams, 1989). Here, linear inverse hazard rate implies efficiency in equilibrium qualities.

We can use the differential equation $[1 - F(v)]/f(v) = \alpha - \beta v$ to solve for the valuation density.

Remark 2. Suppose that $h(x) = \alpha - \beta x$. Then if $\beta = 0$, f is the exponential distribution $f(x) = \frac{A}{\alpha}\exp(-\frac{x}{\alpha})$, with $\bar{v} = \infty$, and $A = \exp(\frac{\underline{v}}{\alpha})$, so when $\underline{v} = 0$, $f(x) = \frac{1}{\alpha}\exp(-\frac{x}{\alpha})$ for $x \in \mathbb{R}_+$. If $\beta > 0$, then $f(x) = \left[\frac{(\alpha-\beta x)^{(1-\beta)}}{(\alpha-\beta\underline{v})}\right]^{\frac{1}{\beta}}$, with $\alpha - \beta\bar{v} = 0$. For the uniform distribution, we have $h(x) = \bar{v} - x$ (so $\alpha = \bar{v}$, and $\beta = 1$).

Although equilibrium qualities are efficient when the inverse hazard rate is linear, Remark 2 shows that the set of valuation densities with linear inverse hazard rate is quite special—even among the set of two-parameter densities. The inverse hazard rate is unlikely to be linear for a randomly chosen distribution: the efficiency result in Proposition 2 may not be generic. What happens to qualities when they are inefficient? Our next result addresses that.

Proposition 3. *Let an equilibrium be written as $(\hat{q}_1, \hat{q}_2, \hat{v})$, corresponding to Firm 1's quality, Firm 2's quality, and the marginal consumer. If the equilibrium is not first best, either*

$$(\hat{q}_1, \hat{q}_2, \hat{v}) < (q_\ell^*, q_h^*, v^*) \quad \text{or} \quad (\hat{q}_1, \hat{q}_2, \hat{v}) > (q_\ell^*, q_h^*, v^*).$$

That is, when equilibrium qualities are not first best, either both firms have equilibrium qualities lower than the corresponding first-best levels, or both have equilibrium qualities correspondingly higher.

The proposition can be explained as follows. Firm 1 aims to maximize social surplus. If Firm 2 chooses $q_2 = q_h^*$, Firm 1's best response is to pick $q_1 = q_\ell^*$. Next, Firm 1's best response is increasing in q_2. This stems from the properties of $\hat{v}(q_1, q_2)$, the efficient allocation of consumers across the two firms. Quality q_1 is too low for consumer $\hat{v}$, whereas quality q_2 is too high. If q_2 increases, consumer $\hat{v}$ would become worse off buying from Firm 2, so actually $\hat{v}$ increases. This also means that Firm 1 should raise its quality because it now serves consumers with higher valuations. In other words, if Firm 2 raises its quality, Firm 1's best response is to raise quality. Therefore, Firm 1's quality is higher than the first best q_ℓ^* if and only if Firm 2's quality is higher than the first best q_h^*.

We can rewrite the equation for Firm 2's equilibrium quality choice as follows:

$$v + \left[\frac{\int_v^{\overline{v}} f(x)h(x)\mathrm{d}x}{f(v)h(v)} - \frac{\int_v^{\overline{v}} f(x)h(x)\mathrm{d}x}{f(v)h(v)} \right] + \frac{h(v)}{1-h'(v)} = c'(q_2)$$

$$\left\{ \frac{h(v)}{1-h'(v)} - \frac{\int_v^{\overline{v}} f(x)h(x)\mathrm{d}x}{f(v)h(v)} \right\} + \frac{\int_v^{\overline{v}} xf(x)\mathrm{d}x}{1-F(v)} = c'(q_2).$$

The term inside the curly brackets is the discrepancy in the characterization of the first-best high quality and Firm 2's equilibrium quality. We have provided a condition for this term to be zero in Proposition 2, but this cannot be expected to hold for most distributions. The property of this term will then determine the distortion described in Proposition 3.

We have constructed a number of examples to verify that equilibrium qualities can be either below or above the first best. However, it is more effective if we discuss these examples after we have presented the other class of equilibria in which the public firm chooses a higher quality than the private firm. The examples are presented in Section 4.3. Also, we will defer robustness and policy discussions until after we have presented the other class of equilibria, in Sections 5.1 and 5.2.

4. Equilibria with high quality at public firm

Because the two firms have different objectives, equilibria in this class are not isomorphic to those in the previous section. However, the logic of the analysis is similar to the previous subsections, so we will omit proofs (but some can be found in Laine and Ma, 2016).

4.1. Subgame-perfect equilibrium prices

When $q_1 > q_2$, the firms have positive demand only if $p_1 > p_2$. Now, consumers with high valuations buy from the public firm. We now write the definition of the indifferent consumer $\tilde{v}$ as:

$$\tilde{v}q_1 - p_1 = \tilde{v}q_2 - p_2 \quad \text{or} \quad \tilde{v}(p_1, p_2; q_1, q_2) = \frac{p_1 - p_2}{q_1 - q_2}. \tag{21}$$

Firm 1 and 2's payoffs are, respectively, social surplus and profit:

$$\int_{\underline{v}}^{\tilde{v}} [xq_2 - c(q_2)]f(x)\mathrm{d}x + \int_{\tilde{v}}^{\overline{v}} [xq_1 - c(q_1)]f(x)\mathrm{d}x \quad \text{and} \quad F(\tilde{v})[p_2 - c(q_2)].$$

Equilibrium prices, $(\hat{p}_1, \hat{p}_2)$, are best responses against each other, and characterized in the following lemma.

Lemma 4. *In subgames (q_1, q_2) with $q_1 > q_2$, and $\underline{v} < \frac{c(q_1)-c(q_2)}{q_1-q_2} < \overline{v}$, equilibrium prices $(\hat{p}_1, \hat{p}_2)$ are:*

$$\hat{p}_1 - c(q_1) = \hat{p}_2 - c(q_2) = (q_1 - q_2)\frac{F(\hat{v})}{f(\hat{v})} \equiv (q_1 - q_2)k(\hat{v}), \tag{22}$$

$$\text{where } \hat{v} = \frac{c(q_1) - c(q_2)}{q_1 - q_2}. \tag{23}$$

Firm 1 implements the socially efficient consumer allocation by setting a price differential equal to the cost differential, whereas Firm 2's profit maximization follows the usual marginal-revenue-marginal-cost tradeoff, which is now related to the inverse reverse hazard rate, $k = F/f$. Equilibrium prices, $\hat{p}_1(q_1, q_2)$, $\hat{p}_2(q_1, q_2)$ change with qualities in the following way.

Lemma 5. *From the definition of $(\hat{p}_1, \hat{p}_2)$ in (22) and (23), we have $\hat{v}$ increasing in q_1 and q_2,*

$$\frac{\partial \hat{p}_1(q_1,q_2)}{\partial q_2} = -k(\hat{v}) + k'(\hat{v})[\hat{v} - c'(q_2)], \tag{24}$$

$$\frac{\partial \hat{p}_2(q_1,q_2)}{\partial q_2} = c'(q_2) - k(\hat{v}) + k'(\hat{v})[\hat{v} - c'(q_2)]. \tag{25}$$

Unlike subgames where Firm 2's quality is higher than Firm 1's, Firm 2's market share increases with both q_1 and q_2. However, the effect of a higher quality q_2 on prices may be ambiguous, but the effect of q_2 on $\hat{p}_2$ is larger than that on $\hat{p}_1$ by $c'(q_2)$.

4.2. Subgame-perfect equilibrium qualities

Equilibrium qualities $(\hat{q}_1, \hat{q}_2)$ are mutual best responses, given continuation equilibrium prices:

$$\hat{q}_1 = \text{argmax}_{q_1} \int_{\underline{v}}^{\hat{v}(q_1,\hat{q}_2)} [x\hat{q}_2 - c(\hat{q}_2)]f(x)\text{d}x + \int_{\hat{v}(q_1,\hat{q}_2)}^{\bar{v}} [xq_1 - c(q_1)]f(x)\text{d}x \tag{26}$$

$$\hat{q}_2 = \text{argmax}_{q_2} F(\hat{v}(\hat{q}_1, q_2))[\hat{p}_2(\hat{q}_1, q_2) - c(q_2)], \tag{27}$$

where $\hat{p}_2$ is Firm 2's equilibrium price and $\hat{v}$ is the equilibrium indifferent consumer (see Lemma 4).

We apply the same method to characterize equilibrium qualities. Changing q_1 in Firm 1's payoff in (26) only affects the second integral there because the effect via the first integral is second order by the Envelope Theorem. Changing q_2 has only two effects: the direct effect on the surplus of the marginal consumer $\hat{v}q - c(q_2)$, and the effect on Firm 1's equilibrium price, because any effect on the marginal consumer is second order according to the Envelope Theorem. We obtain the first-order conditions

$$\int_{\hat{v}(q_1,q_2)}^{\bar{v}} [x - c'(q_1)]f(x)\text{d}x = 0 \text{ and } \hat{v}(\hat{q}_1, q_2) - c'(q_2) + \frac{\partial \hat{p}_1(\hat{q}_1, q_2)}{\partial q_2} = 0.$$

After applying Lemma 5 to the last first-order condition, we obtain the following.

Proposition 4. *Equilibrium qualities $(\hat{q}_1,\hat{q}_2)$, and the marginal consumer $\hat{v}$ solve the following three equations in q_1, q_2, and v*

$$\frac{\int_v^{\bar{v}} xf(x)\text{d}x}{1 - F(v)} = c'(q_1)$$

$$v - \frac{k(v)}{1 + k'(v)} = c'(q_2)$$

$$vq_2 - c(q_2) = vq_1 - c(q_1).$$

Proposition 4 shares the same intuition behind Proposition 1. Firm 1 chooses q_1 to maximize the surplus of those consumers with valuations higher than $\hat{v}$. Firm 2 chooses the quality that is efficient for a type lower than the marginal consumer, at valuation $\hat{v} - \frac{k(\hat{v})}{1+k'(\hat{v})}$. Firm 2's lower quality serves to use product differentiation to create a bigger cost differential, and hence a bigger price differential between the two firms.

We can identify a class of distributions for which Firm 2's profit incentive aligns with the social incentive. An intermediate result is the following.

Lemma 6. *For any distribution F (and its corresponding density f and inverse reverse hazard rate $k \equiv F/f$),*

$$\frac{\int_{\underline{v}}^{v} xf(x)\text{d}x}{F(v)} \equiv v - \frac{\int_{\underline{v}}^{v} f(x)k(x)\text{d}x}{f(v)k(v)}. \tag{28}$$

Proposition 5. *Suppose that the inverse reverse hazard rate k is linear; that is, $k(x) = \gamma + \delta x$, $x \in [\underline{v}, \bar{v}]$, for some γ and $\delta \geq 0$. Then for any v*

$$v - \frac{k(v)}{1 + k'(v)} = \frac{\int_{\underline{v}}^{v} xf(x)\text{d}x}{F(v)} \equiv v - \frac{\int_{\underline{v}}^{v} f(x)k(x)\text{d}x}{f(v)k(v)}. \tag{29}$$

Equilibrium qualities and market shares are first best.

The following two remarks, respectively, relate the linear inverse reverse hazard rate to the private firm's marginal revenue, and present the corresponding densities.

Remark 3. When Firm 2 sells to low-valuation consumers, its marginal revenue is linear in consumer valuation if and only if $k(v)$ is linear.

Remark 4. Suppose that $k(x) = \gamma + \delta x$. Then $\delta > 0$, and $f(x) = \left[\frac{(\gamma+\delta x)^{1-\delta}}{(\gamma+\delta \bar{v})}\right]^{\frac{1}{\delta}}$ with $\gamma + \delta \underline{v} = 0$. For the uniform distribution, $\gamma = -\underline{v}$ and $\delta = 1$.

Again, the above shows that densities that have linear reverse hazard rates constitute a small class. Generically, inefficient equilibrium qualities can be expected. When the equilibrium is not first best, the distortion in equilibria with higher public qualities exhibits the same pattern as in equilibria with lower public qualities: Proposition 3 holds verbatim for the class of high-public-quality equilibria: either both firms produce qualities higher than first best, or both produce qualities lower than first best.

4.3. Examples and comparisons between equilibrium and first-best qualities

Propositions 2, 3, and 5 point to a rich set of equilibrium qualities, which are often inefficient. Here, we construct a number of illustrative examples. We assume a quadratic cost function $c(q) = \frac{1}{2}q^2$. We consider six valuation distributions: two for each of triangular, truncated exponential, and beta distributions. For each distribution, we look at low-public-quality and high-public-quality equilibria. Diagrams 1–3 present the equilibrium qualities and social surpluses. (In each diagram, we mark the equilibrium and first-best qualities on a line, and write down the corresponding social surpluses to the right of the qualities.) Formulas of the inverse hazard and inverse reverse hazard rates and *Mathematica* programs are in the Supplement.

Example 1. A triangular distribution $f(v) = 2v$, and its reverse $f(v) = 2(1 - v)$, $v \in [0, 1]$.

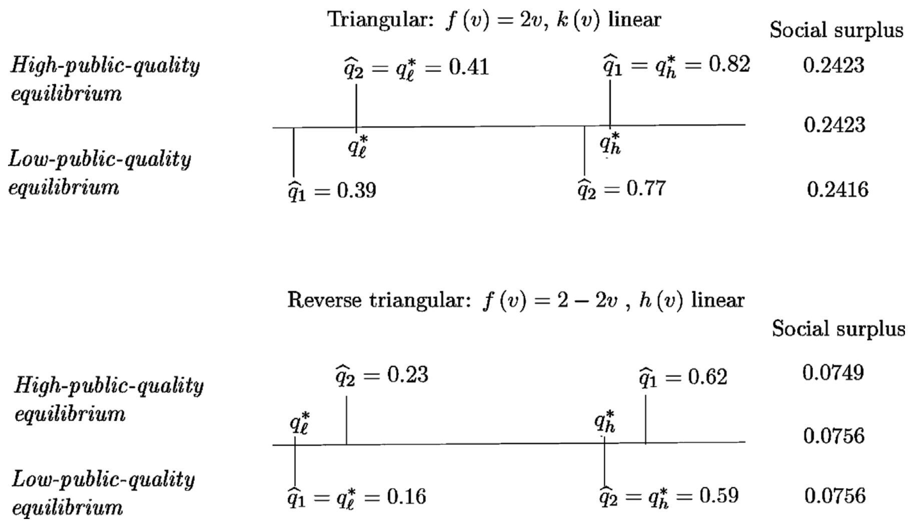

Diagram 1. Equilibria for triangular valuation distributions.

Example 1 shows the possibility of the first best. However, where equilibria are inefficient, qualities may be higher or lower than first best.

Example 2. A truncated exponential distribution $f(v) = \frac{[\exp(-v/\alpha)]/\alpha}{1-\exp(-\bar{v}/\alpha)}$, and its reverse $f(v) = \frac{[\exp(-(\bar{v}-v)/\alpha)]/\alpha}{1-\exp(-\bar{v}/\alpha)}$, $\alpha = 20$, and $v \in [0, \bar{v}] = [0, 100]$.

Example 2 shows that for the exponential distribution, equilibrium qualities must always be higher than first best, but for the reverse exponential distribution, equilibrium qualities must always be lower. The low-public-quality equilibrium yields a higher social surplus in the exponential distribution, but the reverse is true with the reverse exponential distribution.

Example 3. Two beta distributions: $f(v) = \frac{v^{(\alpha-1)}(1-v)^{(\beta-1)}}{\int_0^1 x^{(\alpha-1)}(1-x)^{(\beta-1)}\mathrm{d}x}$, $v \in [0, 1]$, $(\alpha, \beta) = (2, 5)$ and $(\alpha, \beta) = (5, 2)$.

In Example 3, for each beta distribution, qualities are higher than first best in one equilibrium, but lower in the other. For the beta(2,5) distribution, the low- public-quality equilibrium yields a higher social surplus, but the revese is true for the beta(5,2) distribution.

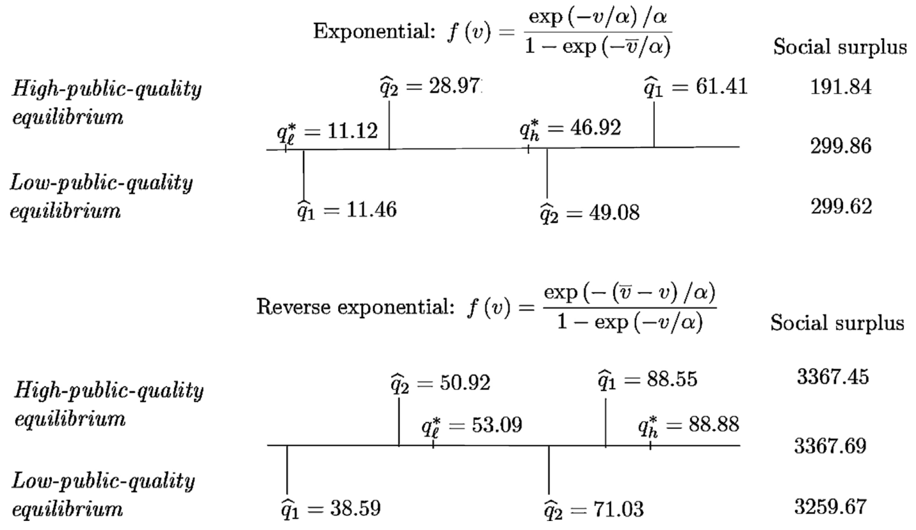

Diagram 2. Equilibria for truncated exponential valuation distributions.

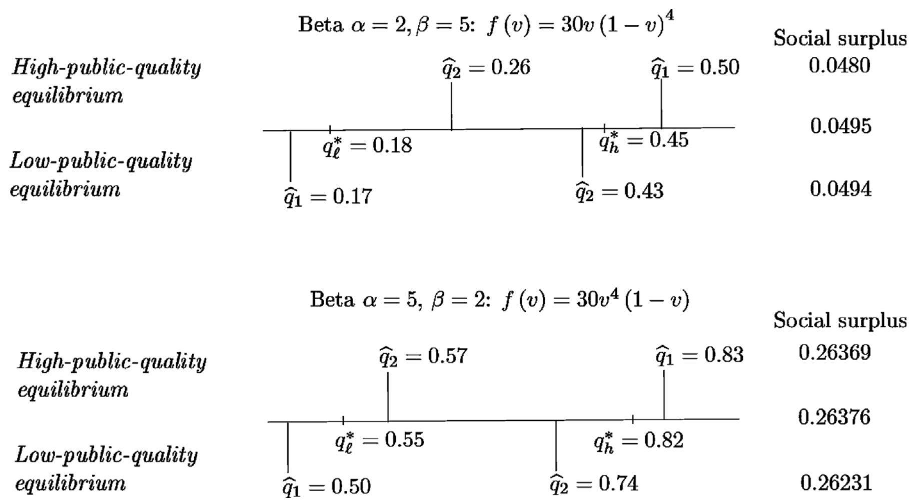

Diagram 3. Equilibria for beta valuation distributions.

5. Policies, robustness, and existence of equilibria

5.1. Competition policy

Suppose that the market initially consists of two private firms, so equilibrium qualities are inefficient. Qualities improve when one private firm is taken over by a public firm. Example 1 shows that for triangular and reverse triangular distributions, full efficiency can be restored if the public firm enters at the correct quality segment. Examples 2 and 3 show that generally low-public-quality and high-public-quality equilibria yield different social surpluses. Hence, entry by the public firm at the correct market segment is important. Our characterizations in Propositions 1 and 4 provide guidance.

Commitment by the public firm has been a common assumption in the previous literature. Equilibrium qualities are first best in the simultaneous-move games if and only if they are first best in the Stackelberg game, one in which the public firm can commit to quality or price. The reason is this. Suppose that Stackelberg equilibrium qualities are first best. Because the public firm's payoff is social surplus, the (first-best) low quality is a best response against the private firm's (first-best) high quality, so commitment is unnecessary. The converse is trivially true.

From Proposition 3, the improvement in welfare from a Stackelberg game comes from the public firm choosing a quality closer to the first best. For example, if in an equilibrium, qualities are lower than the first best (as in the reverse truncated exponential distribution case in Example 2), a higher public quality leads to a higher best response by the private firm, so both qualities will become closer to the first best.

5.2. General objective for the public firm and subsidies

So far our focus has been on quality efficiency. The public firm's objective function has been social welfare. Prices are transfers between consumers and firms, so do not affect social welfare. A more general objective function for a public firm can be a weighted sum of consumer surplus, and profits, also a common assumption in the literature. In this case, we can rewrite Firm 1's objective function as

$$\theta\left\{\int_{\underline{v}}^{\tilde{v}}[xq_1-p_1]f(x)\mathrm{d}x+\int_{\tilde{v}}^{\bar{v}}[xq_2-p_2]f(x)\mathrm{d}x\right\}+(1-\theta)\left\{F(\tilde{v})[p_1-c(q_1)]+[1-F(\tilde{v})][p_2-c(q_2)]\right\}. \quad (30)$$

Here, consumers are paying for the lower quality q_1 at price p_1, and the higher quality q_2 at price p_2. The weight on consumer surplus is $\theta>\frac{1}{2}$, whereas the weight on profits is $1-\theta$, so profits are unattractive from a social perspective. We can rewrite (30) as

$$\theta\left\{\int_{\underline{v}}^{\tilde{v}}[xq_1-c(q_1)]f(x)\mathrm{d}x+\int_{\tilde{v}}^{\bar{v}}[xq_2-c(q_2)]f(x)\mathrm{d}x\right\}-(2\theta-1)\left\{F(\tilde{v})[p_1-c(q_1)]+[1-F(\tilde{v})][p_2-c(q_2)]\right\},$$

which always decreases in Firm 1's price. If we impose a balanced-budget constraint, then the public firm must set price p_1 at marginal cost $c(q_1)$ to break even.

Lemmas 1 and 4 can no longer be valid. The first best cannot be an equilibrium because consumers do not bear the full incremental cost between high and low qualities. Suppose that $q_2>q_1$. The public firm will reduce price p_1 to marginal cost $c(q_1)$. However, in any price equilibrium, Firm 2's profit-maximizing price-cost margin has $p_2-c(q_2)>0$, so we have $p_2-p_1>c(q_2)-c(q_1)$. Fewer consumers will use the high-quality private firm.

We can regard the public firm's social-surplus preferences as a normative recommendation; otherwise, the distribution of consumers among firms will be inefficient. The concern for distribution should be addressed by a subsidy. Firms earn profits, according to Lemmas 1 and 4. Consider a low-public-quality equilibrium. Let equilibrium prices be $\hat{p}_1$ and $\hat{p}_2$. Impose taxes on Firms 1 and 2, respectively, at $F(\hat{v})[\hat{p}_1-c(q_1)]$ and $[1-F(\hat{v})][\hat{p}_2-c(q_2)]$, where $\hat{v}$ is in (10). The total tax revenue can be used as a consumer subsidy. For example, it can be equally distributed to all consumers, or be set up as a voucher for buying from either firm, or paid to consumers according to other criteria (say consumers with lower valuations get more). The only requirement is that the subsidy does not alter the difference of firms' prices, so that $p_2-p_1=c(q_2)-c(q_1)$, a necessary condition for the first best.

5.3. Different cost functions for public and private firms

We now let firms have different cost functions. Let $c_1(q)$ and $c_2(q)$ be Firm 1's and Firm 2's unit cost at product quality q, and these functions are increasing and convex.[7] The analysis in Sections 3 and 4 remains exactly the same. Simply replace every $c(q_1)$ by $c_1(q_1)$ and every $c(q_2)$ by $c_2(q_2)$. In the price subgame, the equilibrium still has a price difference equal to cost difference: $p_2-p_1=c_2(q_2)-c_1(q_1)$. The equilibrium qualities continue to satisfy their respective conditions after first-order conditions are simplified.

Propositions 2 and 5 have to be adjusted. This is because the first best in Section 2.3 has to be redefined. There are now two ways to assign technology. In one, low-valuation consumers pay the cost $c_1(q_\ell)$ for the low quality q_ℓ, and high-valuation consumers incur the cost $c_2(q_h)$ for the high quality q_h. In the other, it is the opposite. One of these technology assignments will yield a higher social welfare. However, our abstract model does not allow us to determine which technology should be used for low quality.[8]

The likelihood that the first best is achieved by the public firm taking over a private firm is small, again because linear inverse hazard and reverse hazard rates are nongeneric. The relevant question is whether the public firm should enter the low-quality segment or high-quality segement. Our examples for the case of identical cost functions show that the answer depends on the model specifics. This conclusion for competition policy should remain valid when costs are different.

5.4. Consumer outside option and many private firms

The consumer having an outside option is the same as introducing a fictitious firm offering a product at zero quality and zero price. In the first best, some consumers with very low valuations may not consume. The public firm's price affects

[7] Often the public firm is assumed to be less efficient. For example, we can let $c_1(q)>c_2(q)$ and $c_1'(q)>c_2'(q)$, so both unit and marginal unit costs are higher at the public firm. Our formal model, however, does not require this particular comparative advantage.

[8] As an illustration, let $c_1(q)=(1+s)c(q)$, and $c_2(q)=(1-s)c(q)$. The social welfare from using c_1 to produce the low quality is $\int_{\underline{v}}^{v}[xq_\ell-(1+s)c(q_\ell)]f(x)\mathrm{d}x+\int_{v}^{\bar{v}}[xq_h-(1-s)c(q_h)]f(x)\mathrm{d}x$. At $s=0$, this is the model in Subsection 2.1. From the Envelope Theorem, the derivative of the maximized welfare with respect to s, evaluated at $s=0$, is the partial derivative of welfare with respect to s: $-c(q_\ell^*)F(v^*)+c(q_h^*)[1-F(v^*)]$. Properties of q_ℓ^*, q_h^*, and v^* from (3), (4), and (5) do not indicate whether this derivative is positive or negative. It appears that the distribution F and the cost functions may interact in many ways.

decisions of two marginal consumers: the one who choose between the low-quality good and the high-quality good, and the one who choose between the low-quality good and no consumption at all.

In fact, Delbono et al. (1996) show that under a uniform valuation distribution, the first best is not an equilibrium. Efficient allocation requires that all consumers face price differentials equal to cost differentials. Hence, if Firm 1 produces a low quality q_1 and Firm 2 produces a high quality q_2, then efficiency requires $p_2 - p_1 = c(q_2) - c(q_1)$. When $p_2 > c(q_2)$ due to Firm 2's market power, $p_1 > c(q_1)$. However, to induce consumers to make efficient nonpurchase decisions, p_1 should be set at $c(q_1)$.

The case of many private firms is formally very similar. When a public firm has to interact with, say, two private firms, it does not have enough instruments to induce efficient decisions. Suppose that there are three firms, and that the medium quality is produced by a public firm, whereas the private firms produce low and high qualities. The public firm cannot simultaneously use one price to induce two efficient margins.

5.5. Existence of equilibria

In the previous sections, we have assumed the existence of equilibria. We now write down conditions for the solutions in Propositions 1 and 4 to be mutual best responses. For this, we consider two types of deviations: i) a firm choosing a lower quality than the rival's, and ii) a firm choosing a higher quality than the rival's.

Let $\pi_L(q_1) \equiv \max_{q_2 \le q_1} F(\hat{v}(q_1, q_2))[\hat{p}_2(q_1, q_2) - c(q_2)]$, where $\hat{v}(q_1, q_2) = [c(q_1) - c(q_2)]/[q_1 - q_2]$. Here, Firm 2 gets low-valuation consumers and the continuation equilibrium profit $\pi_L(q_1)$. Using the Envelope Theorem, we can show that $\pi_L(q_1)$ is strictly increasing.[9] If Firm 2 must choose only qualities that are lower than q_1, it benefits more when q_1 is higher because it has a bigger choice set. Let $\pi_U(q_1) \equiv \max_{q_2 \ge q_1} [1 - F(\hat{v}(q_1, q_2))][\hat{p}_2(q_1, q_2) - c(q_2)]$. Now, Firm 2 gets the high-valuation consumers and the profit $\pi_U(q_1)$. Again, using the Envelope Theorem, we can show that $\pi_U(q_1)$ is strictly decreasing. Firm 2's maximum profits from a continuation equilibrium is the upper envelope of $\pi_L(q_1)$ and $\pi_U(q_1)$, $\max\{\pi_L(q_1), \pi_U(q_1)\}$. Define $\overline{q}_1$ by $\pi_L(\overline{q}_1) = \pi_U(\overline{q}_1)$. The critical value $\overline{q}_1$ exists and is unique. It is a best response for Firm 2 to choose a high quality if and only if Firm 1's quality is below the critical value $\overline{q}_1$.

Next, for Firm 1's best response, we let $s_L(q_2) \equiv \max_{q_1 \le q_2} \int_{\underline{v}}^{\hat{v}(q_1,q_2)} [xq_1 - c(q_1)]f(x)dx + \int_{\hat{v}(q_1,q_2)}^{\overline{v}} [xq_2 - c(q_2)]f(x)dx$. This is the maximum social surplus when Firm 1's quality is lower than Firm 2's. Similarly, let $s_U(q_2) \equiv \max_{q_1 \ge q_2} \int_{\underline{v}}^{\hat{v}(q_1,q_2)} [xq_2 - c(q_2)]f(x)dx + \int_{\hat{v}(q_1,q_2)}^{\overline{v}} [xq_1 - c(q_1)]f(x)dx$, the maximum social surplus when Firm 1's quality is higher than Firm 2's. Again, using the Envelope Theorem, we show that $s_L(q_2)$ is strictly increasing, and $s_H(q_2)$ is strictly decreasing. Define $\overline{q}_2$ by $s_L(\overline{q}_2) = s_U(\overline{q}_2)$. It is a best response for Firm 1 to choose a low quality if and only if Firm 2's quality is above the critical value $\overline{q}_2$.

Formally, the low-public-quality equilibrium exists when the equations in Proposition 1 yield a solution $(\hat{q}_1, \hat{q}_2)$ satisfying $\hat{q}_1 < \overline{q}_1$ and $\hat{q}_2 > \overline{q}_2$. Similarly, the high-public-quality equilibrium exists when the equations in Proposition 4 yield a solution $(\hat{q}_1, \hat{q}_2)$ satisfying $\hat{q}_1 > \overline{q}_1$ and $\hat{q}_2 < \overline{q}_2$. However, we are unaware of general conditions on f and c for these requirements.

To confirm the existence of particular equilibria, however, we only need to verify that candidate equilibrium qualities are mutual best responses. For the $f(v) = 2v$ triangular distribution example above, we have computed each firm's payoffs. Given the private firm's quality q_2 set at a (candidate) equilibrium level, we compute the public firm's payoffs from setting quality q_1 at levels below and above q_2. We do the same for the private firm given the public firm's (candidate) equilibrium quality. We have confirmed, indeed, that those qualities in the example form an equilibrium. The computation details are in the Supplement. (We have also done the same for a model with v on a uniform distribution [10, 11] and a quadratic cost function. The game has an equilibrium with qualities at 10.25 and 10.75.)

6. Concluding remarks

In this paper we have studied equilibria in a mixed duopoly. The public firm maximizes social surplus, and the private firm maximizes profit. We have used a general distribution for consumer's valuations and a general cost function for firms. We discuss two classes of equilibria. In one class, the public firm offers low quality and the private offers high quality. In the other class, the opposite is true. Whereas generically, equilibrium qualities are inefficient, when inverse hazard or inverse reverse hazard rates are linear, equilibrium qualities are first best. We have related our results to competition policies, and discussed various robustness issues.

Various directions for further research may be of interest. Clearly, duopoly is a limitation. However, a mixed oligopoly with an arbitrary number of firms is analytically very difficult. In the extant literature, models of product differentiation with many private firms typically impose very strong assumptions on either consumer valuation (equivalently location) distribution or production cost (equivalently mismatch disutility). The contribution here relies on our ability to identify the inverse hazard and inverse reverse hazard rates as the determining factors for properties of equilibrium qualities. It may

[9] The derivative of $\pi_L(q_1)$ is the partial derivative of the profit function with respect to q_1 evaluated at the profit-maximizing q_2. This is $f(\hat{v})[\hat{p}_2 - c(q_2)]\frac{\partial \hat{v}}{\partial q_1} + F(\hat{v})\frac{\partial \hat{p}_2}{\partial q_1}$. We obtain $\frac{\partial \hat{p}_2}{\partial q_1}$ from (22) in Lemma 4. We verify that both $\frac{\partial \hat{v}}{\partial q_1}$ and $\frac{\partial \hat{p}_2}{\partial q_1}$ are positive, and conclude that $\pi_L(q_1)$ is strictly increasing. The monotonicity of $\pi_H(q_1)$, $s_L(q_2)$, and $s_H(q_2)$ can be demonstrated by similar computation.

well be that they also turn out to be useful for a richer model. The unit cost being constant with respect to quantity is a common assumption in the literature. We have used the same “constant-return” approach. Scale effects may turn out to be important even for the mixed duopoly.

Acknowledgements

For their suggestions and comments, we thank Francesca Barigozzi, Giacomo Calzolari, Helmuth Cremer, Vincenzo Denicolo, Katharina Huesmann, Ari Hyytinen, Qihong Liu, and the seminar participants at Auburn University, University of Bologna, Boston University, Helsinki Center of Economic Research, University of Hong Kong, National Taiwan University, the 16th European Health Economics Workshop in Toulouse, and the 14th International Industrial Organization Conference in Philadelphia. We also thank coeditor Tom Gresik, an associate editor, and two reviewers for their guidance. Laine received financial support from the Yrjö Jahnsson Foundation and the KAUTE Foundation.

Appendix

Proof of Lemma 1. Consider $\hat{p}_2 = \operatorname{argmax}_{p_2}[1-F(\tilde{v})][p_2-c(q_2)]$, where $\tilde{v} = \frac{p_2-\hat{p}_1}{q_2-q_1}$ (see (6)). The first-order derivative of the profit function with respect to p_2 is

$$\begin{aligned} &[1-F(\tilde{v})]-f(\tilde{v})[p_2-c(q_2)]\frac{1}{q_2-q_1} \\ = \; & h(\tilde{v})-[p_2-c(q_2)]\frac{1}{q_2-q_1}, \end{aligned}$$

where we have used the partial derivative of $\tilde{v}$ with respect to p_2, namely $1/(q_2-q_1)$. From the assumption that h is decreasing, the second-order derivative is negative, so the first-order condition is sufficient. Therefore, $\hat{p}_2$ is given by $\hat{p}_2-c(q_2)=(q_2-q_1)h(\tilde{v})$.

Next, consider Firm 1 choosing p_1 to maximize (7) where $\tilde{v}=\frac{\hat{p}_2-p_1}{q_2-q_1}$ (see (6)). Because (7) is independent of p_1, we can choose $\tilde{v}$ to maximize (7) ignoring (6). The optimal value $\hat{v}$ is given by setting to zero the first-order derivative of (7) with respect to $\tilde{v}$: $\hat{v}q_1-c(q_1)=\hat{v}q_2-c(q_2)$. Then we simply choose $\hat{p}_1$ to satisfy (6) such that $\hat{v}=\frac{\hat{p}_2-\hat{p}_1}{q_2-q_1}=\frac{c(q_2)-c(q_1)}{q_2-q_1}$. We have shown that $\hat{p}_1$ and $\hat{p}_2$ in (9) and (10) are mutual best responses.

Proof of Lemma 2. First, from (10), we obtain $(q_2-q_1)\mathrm{d}\hat{v}+\hat{v}(\mathrm{d}q_2-\mathrm{d}q_1)=c'(q_2)\mathrm{d}q_2-c'(q_1)\mathrm{d}q_1$, which, together with the convexity of c, yields

$$\frac{\partial\hat{v}}{\partial q_1}=\frac{\hat{v}-c'(q_1)}{q_2-q_1}=\frac{1}{q_2-q_1}\left[\frac{c(q_2)-c(q_1)}{q_2-q_1}-c'(q_1)\right]>0 \tag{31}$$

$$\frac{\partial\hat{v}}{\partial q_2}=\frac{c'(q_2)-\hat{v}}{q_2-q_1}=\frac{1}{q_2-q_1}\left[c'(q_2)-\frac{c(q_2)-c(q_1)}{q_2-q_1}\right]>0. \tag{32}$$

Next, from (9), we obtain

$$\begin{aligned} \mathrm{d}\hat{p}_1-c'(q_1)\mathrm{d}q_1 &= (\mathrm{d}q_2-\mathrm{d}q_1)h(\hat{v})+(q_2-q_1)h'(\hat{v})\left(\frac{\partial\hat{v}}{\partial q_2}\mathrm{d}q_2-\frac{\partial\hat{v}}{\partial q_1}\mathrm{d}q_1\right) \\ \mathrm{d}\hat{p}_2-c'(q_2)\mathrm{d}q_2 &= (\mathrm{d}q_2-\mathrm{d}q_1)h(\hat{v})+(q_2-q_1)h'(\hat{v})\left(\frac{\partial\hat{v}}{\partial q_2}\mathrm{d}q_2-\frac{\partial\hat{v}}{\partial q_1}\mathrm{d}q_1\right). \end{aligned}$$

We then use (31) and (32) to simplify these, and obtain

$$\begin{aligned} \frac{\partial\hat{p}_1(q_1,q_2)}{\partial q_2} &= h(\hat{v})+h'(\hat{v})\left[c'(q_2)-\hat{v}\right] \\ \frac{\partial\hat{p}_2(q_1,q_2)}{\partial q_2} &= c'(q_2)+h(\hat{v})+h'(\hat{v})\left[c'(q_2)-\hat{v}\right], \end{aligned}$$

which are the expressions in the lemma.

Proof of Proposition 1. The first-order derivative of (13) with respect to q_1 is

$$\int_{\underline{v}}^{\hat{v}(q_1,q_2)}[x-c'(q_1)]f(x)\mathrm{d}x+\left\{[\hat{v}(q_1,q_2)q_1-c(q_1)]-[\hat{v}(q_1,q_2)q_2-c(q_2)]\right\}f(\hat{v}(q_1,q_2))\frac{\partial\hat{v}}{\partial q_1}.$$

By Lemma 1, the term inside the curly brackets is zero. By putting this first-order derivative to zero, we obtain the first equation in the Proposition. Also, because equilibrium prices $\hat{p}_1(q_1, q_2)$ and $\hat{p}_2(q_1, q_2)$ must follow Lemma 1, we have

$$\hat{v}(q_1, q_2) = \frac{c(q_2) - c(q_1)}{q_2 - q_1},$$

which is the last equation in the Proposition.

Next, we use (17) to obtain the first-order derivative of Firm 2's profit with respect to q_2:

$$[1 - F(\hat{v}(q_1, q_2))] \left[\hat{v}(q_1, q_2) - c'(q_2) + \frac{\partial \hat{p}_1(q_1, q_2)}{\partial q_2}\right] +$$
$$\left\{-f(\hat{v}(q_1, q_2))[\hat{p}_2(\hat{q}_1, q_2) - c(q_2)] + [1 - F(\hat{v}(q_1, q_2))](q_2 - q_1)\right\} \frac{\partial \hat{v}(q_1, q_2)}{\partial q_2}.$$

Again, by Lemma 1, the term inside the curly brackets is zero. After setting the first-order derivative to 0, we obtain

$$\hat{v}(q_1, q_2) - c'(q_2) + \frac{\partial \hat{p}_1(q_1, q_2)}{\partial q_2} = 0.$$

We then use (11) in Lemma 2 to substitute for $\frac{\partial \hat{p}_1(q_1, q_2)}{\partial q_2}$, and write the first- order condition as

$$\hat{v} - c'(q_2) + h(\hat{v}) + h'(\hat{v}) \left[c'(q_2) - \hat{v}\right] = 0,$$

which simplifies to

$$\hat{v} + \frac{h(\hat{v})}{1 - h'(\hat{v})} = c'(q_2),$$

the second equation in the Proposition.

Proof of Lemma 3. By definition, $f(x)h(x) = (1 - F(x))$. We have

$$\begin{aligned}
&\frac{\int_v^{\bar{v}} x f(x) \mathrm{d}x}{1 - F(v)} \\
= \; & -\frac{\int_v^{\bar{v}} x \mathrm{d}(1 - F(x))}{f(v)h(v)} \\
= \; & \frac{v(1 - F(v))}{f(v)h(v)} + \frac{\int_v^{\bar{v}} (1 - F(x)) \mathrm{d}x}{f(v)h(v)} \\
= \; & v + \frac{\int_v^{\bar{v}} f(x)h(x) \mathrm{d}x}{f(v)h(v)},
\end{aligned}$$

where the second equality is due to integration by parts.

Proof of Proposition 2. Suppose that $h(x) = \alpha - \beta x$. We have, $h'(x) = - \beta$, and

$$v + \frac{h(v)}{1 - h'(v)} = v + \frac{\alpha - \beta v}{1 + \beta} = \frac{v + \alpha}{1 + \beta}.$$

Then we compute

$$v+\frac{\displaystyle\int_v^{\bar v} f(x)h(x)\mathrm{d}x}{f(v)h(v)} = v+\frac{\displaystyle\int_v^{\bar v} f(x)(\alpha-\beta x)\mathrm{d}x}{f(v)h(v)}$$

$$= v+\frac{\alpha[1-F(v)]}{f(v)h(v)}-\beta\frac{\displaystyle\int_v^{\bar v} xf(x)\mathrm{d}x}{f(v)h(v)}$$

$$= v+\alpha-\beta\left\{v+\frac{\displaystyle\int_v^{\bar v} f(x)h(x)\mathrm{d}x}{f(v)h(v)}\right\},$$

where the expression in the curly brackets comes from the identity (19). Simplifying, we have

$$v+\frac{\displaystyle\int_v^{\bar v} f(x)h(x)\mathrm{d}x}{f(v)h(v)} = \frac{v+\alpha}{1+\beta}.$$

We have proved (20).

The three equations in Proposition 1 are now exactly those that define the first best in (3), (4), and (5). Equilibrium qualities and consumer allocation must be first best.

Proof of Remark 1. When Firm 2 sells to consumers with valuations above v at price p_2, its revenue is $[1-F(v)]p_2$, where $v=\frac{p_2-p_1}{q_2-q_1}$. If we express p_2 as a function of v, we have $p_2(v)=p_1+v(q_2-q_1)$. The marginal revenue is the derivative of revenue with respect to the firm's quantity, $[1-F(v)]$:

$$\begin{aligned}&\frac{\mathrm{d}[1-F(v)]p_2(v)}{\mathrm{d}[1-F(v)]}\\ =\;& p_2(v)+[1-F(v)]\frac{\mathrm{d}p_2(v)}{\mathrm{d}[1-F(v)]}=p_2(v)+[1-F(v)]\frac{\mathrm{d}p_2(v)/\mathrm{d}v}{\mathrm{d}[1-F(v)]/\mathrm{d}v}\\ =\;& p_2(v)-\frac{1-F(v)}{f(v)}\frac{\mathrm{d}p_2(v)}{\mathrm{d}v}=p_2(v)-h(v)(q_2-q_1).\end{aligned}$$

Because $p_2(v)$ is linear in v, marginal revenue is linear in v if and only if the inverse hazard rate $h(v)$ is linear.

Proof of Remark 2. Define $y\equiv 1-F$, so $y'=-f$. We have $h(x)=\alpha-\beta x$ equivalent to $\frac{y'}{y}=\frac{-1}{\alpha-\beta x}$. First, suppose that $\beta=0$. We have $\frac{y'}{y}=\frac{-1}{\alpha}$, so $y(v)=A\exp(-\frac{v}{\alpha})$, some A. Therefore, $F(v)=1-A\exp(-\frac{v}{\alpha})$. Because we have $F(\underline{v})=0$, we must have $A=\exp(\frac{\underline{v}}{\alpha})$. We also have $F(\bar v)=1$, which requires $\bar v=\infty$.

Second, suppose that $\beta>0$. We have $\frac{y'}{y}=\frac{-1}{\alpha-\beta v}$. Solving this differential equation, we have $y(v)=A(\alpha-\beta v)^{\frac{1}{\beta}}$, for some constant A. Hence, $F(v)=1-A(\alpha-\beta v)^{\frac{1}{\beta}}$, and we obtain the expression for f in the Remark by differentiation. Because $F(\underline{v})=0$, we have $A=(\alpha-\beta\underline{v})^{-\frac{1}{\beta}}$. Because $F(\bar v)=1$, we must have $\alpha-\beta\bar v=0$, so that α and β cannot be arbitrary.

Proof of Proposition 3. For any q_2 we consider Firm 1's best response function:

$$\tilde q_1(q_2)=\mathrm{argmax}_{q_1}\int_{\underline{v}}^{\hat v(q_1,q_2)}[xq_1-c(q_1)]f(x)\mathrm{d}x+\int_{\hat v(q_1,q_2)}^{\bar v}[xq_2-c(q_2)]f(x)\mathrm{d}x.$$

First, at $q_2=q_h^*$, we have $\tilde q_1(q_h^*)=q_\ell^*$. Clearly, if Firm 2 chooses q_h^*, from the definition of the first best, Firm 1's best response is $q_1=q_\ell^*$ because Firm 1 aims to maximize social surplus. It follows that the first best belongs to the graph of Firm 1's best response function.

Second, we establish that $\tilde q_1(q_2)$ is increasing in q_2. The sign of the derivative of $\tilde q_1(q_2)$ has the same sign of the cross partial derivative of Firm 1's objective function (13) evaluated at $q_1=\tilde q_1(q_2)$. The derivative of (13) with respect to q_1 is simply

$$\int_{\underline{v}}^{\hat v(q_1,q_2)}[x-c'(q_1)]f(x)dx$$

because the partial derivative with respect to $\hat{v}$ is zero. The cross partial is then obtained by differentiating the above with respect to q_2, and this gives

$$[\hat{v}(q_1, q_2) - c'(q_1)]f(\hat{v})\frac{\partial \hat{v}(q_1, q_2)}{\partial q_2} > 0,$$

where the inequality follows because at $q_1 = \tilde{q}_1(q_2)$, we have $\hat{v}(q_1, q_2) > c'(q_1)$ and $\frac{\partial \hat{v}}{\partial q_2} > 0$ by (32) in the proof of Lemma 2.

Appendix A. Supplementary Data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.jebo.2017.05.012.

References

Anderson, S., de Palma, A., Thisse, J.-F., 1992. Discrete Choice Theory of Product Differentiation. MIT Press.
Anderson, S., Goeree, J., Ramer, R., 1997. Location, location, location. J. Econ. Theory 77, 102–127.
Bagnoli, M., Bergstrom, T., 2004. Log-concave probability and its applications. Econ. Theory 26, 445–469.
Barigozzi, F., Ma, C.-T.A., 2016. Product Differentiation with Multiple Qualities. Boston University Economics Department working paper.
Barros, P., Siciliani, L., 2012. Public and private sector interface. In: Pauly, M., McGuire, T., Barros, P. (Eds.), Handbook of Health Economics 2. , pp. 927–1002.
Benassi, C., Chirco, A., Colombo, C., 2006. Vertical differentiation and the distribution of income. Bull. Econ. Res. 58, 345–367.
Benassi, C., Castellani, M., Mussoni, M., 2016. Price equilibrium and willingness to pay in a vertically differentiated mixed duopoly. J. Econ. Behav. Organization 125, 86–96.
Brunello, G., Rocco, L., 2008. Educational standards in private and public schools. Econ. J. 118, 1866–1887.
Champsaur, P., Rochet, J.-C., 1989. Multiproduct duopolists. Econometrica 57, 533–557.
Chatterjee, K., Samuelson, W., 1983. Bargaining under incomplete information. Oper. Res. 31, 835–851.
Comondore, V., Devereaux, P., Zhou, Q., Stone, S., Busse, J., Ravindran, N., 2009. Quality of care in for-profit and not-for-profit nursing homes: systematic review and meta-analysis. Br. Med. J. 339.
Cremer, H., Thisse, J.-F., 1991. Location models of horizontal differentiation: a special case of vertical differentiation models. J. Ind. Econ. 39, 383–390.
Cremer, H., Marchand, M., Thisse, J.-F., 1989. The public firm as an instrument for regulating an oligopolistic market. Oxford Economic Papers New Series 41, 283–301.
Cremer, H., Marchand, M., Thisse, J.-F., 1991. Mixed oligopoly with differentiated products. Int. J. Ind. Organization 9, 43–53.
Delbono, F., Denicolò, V., Scarpa, C., 1996. Quality choice in a vertically differentiated mixed duopoly. Economic Notes by Banca Monte dei Paschi di Siena SpA 25, 33–46.
Deming, D., Goldin, C., 2012. The for-profit postsecondary school sector: nimble critters or agile predators? J. Econ. Perspect. 26, 139–164.
Epple, D., Romano, R., 1998. Competition between private and public schools, vouchers, and peer-group effects. Am. Econ. Rev. 88, 33–62.
Gabszewicz, J., Thisse, J.-F., 1979. Price competition, quality and income disparities. J. Econ. Theory 20, 340–359.
Gabszewicz, J., Thisse, J.-F., 1986. On the nature of competition with differentiated products. Econ. J. 96, 160–172.
Ghosh, A., Mitra, M., 2010. Comparing Bertrand and Cournot in mixed markets. Econ. Lett. 109, 72–74.
Grassi, S., Ma, C.-T.A., 2011. Optimal public rationing and price response. J. Health Econ. 30, 1197–1206.
Grassi, S., Ma, C.-T.A., 2012. Public sector rationing and private sector selection. J. Public Econ. Theory 14, 1–34.
Gresik, T., 1991. The efficiency of linear equilibria of sealed-bid double auctions. J. Econ. Theory 53, 173–184.
Grilo, I., 1994. Mixed duopoly under vertical differentiation. In: Annales d'Économie et de Statistique, No. 33, Concurrence, entreprises publiques et coopératives /Competition and Non- Profit-Maximizing., pp. 91–112.
Ishibashi, K., Kaneko, T., 2008. Partial privatization in mixed duopoly with price and quality competition. J. Econ. 95, 213–231.
Krishna, V., 2009. Auction Theory, 2nd edition. Academic Press.
Laffont, J.-J., Tirole, J., 1993. A Theory of Incentives in Procurement and Regulation. MIT Press.
Laine, L., Ma, C.-T.A., 2016. Quality and Competition between Public and Private Firms. Boston University Economics Department working paper, http://www.bu.edu/econ/files/2016/11/Ma_qual-comp_July2016.pdf.
Ma, C.-T.A., Burgess, J., 1993. Quality competition, welfare, and regulation. J. Econ. 58, 153–173.
Matsumura, T., 1998. Partial privatization in mixed duopoly. J. Public Econ. 70, 473–483.
Matsumura, T., Matsushima, N., 2004. Endogenous cost differentials between public and private enterprises: a mixed duopoly approach. Economica 71, 671–688.
Myerson, R., 1997. Game Theory Analysis of Conflict. Harvard University Press.
Satterthwaite, M., Williams, S., 1989. Bilateral trade with the sealed bid k-double auction: existence and efficiency. J. Econ. Theory 48, 107–133.
Shaked, A., Sutton, J., 1982. Relaxing price competition through product differentiation. Rev. Econ. Stud. 49, 3–13.
Shaked, A., Sutton, J., 1983. Natural oligopolies. Econometrica 51, 1469–1483.
Spence, A., 1975. Monopoly, quality and regulation. Bell J. Econ. 6, 417–429.
Tirole, J., 1988. The Theory of Industrial Organization. MIT Press.
Urquiola, M., 2016. Competition among schools: traditional public and private schools. In: Hanushek, E., Machin, S., Woessmann, L. (Eds.), Handbook of the Economics of Education Vol. 5. , pp. 209–237.
Yurko, A., 2011. How does income inequality affect market outcomes in vertically differentiated markets? Int. J. Industrial Organization 29, 493–503.