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A Simple Dynamic General Equilibrium Model*

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We analyse a single sector economy with $H > 1$ infinitely-lived agents that operate in a continuous-time framework. Utility functions are recursive but not additive. Both efficient and perfect foresight competitive equilibrium allocations are considered. The existence and stability of such allocations are investigated locally, i.e., in a neighbourhood of steady-state allocations. The model is shown to be useful for explaining the distribution of wealth and consumption across agents, and for analysing the way in which wealth redistribution can affect the dynamics of aggregate economic variables. *Journal of Economic Literature* Classification Numbers: 021, 022, 111. © 1987 Academic Press, Inc.

1. INTRODUCTION

We analyse an economy with $H > 1$ infinitely-lived agents that operate in a continuous-time framework. The major simplification in the model is the assumption that there is only a single good, as in the Koopmans–Cass growth model. Both efficient and perfect foresight competitive equilibrium allocations are considered. The existence and stability of such allocations are investigated locally, i.e., in a neighbourhood of steady-state allocations. The local stability analysis provides information regarding the mode of convergence (cyclical or noncyclical) and the speed of convergence. Finally, the model is shown to be amenable to qualitative comparative dynamics analysis.

The paper achieves two broad objectives. First, by investigating stability in a multiple-agent economy, the paper provides some perspective on the existing turnpike literature. In particular, some insight is provided into the importance for stability of the common assumption of the existence of a representative consumer. The single good framework is maintained in order

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to focus on the consequences of the extension of the Koopmans–Cass model to many consumers. Of course, stability theorems are important because of the information they provide regarding the dynamics of efficient or competitive economies, and because they justify comparative statics analysis of steady states.

The Koopmans–Cass model has been applied fruitfully in a variety of fields in economics, but the representative agent assumption limits its use in problems where distribution "matters." The second objective of this paper is to provide a minimal extension of the Koopmans–Cass model that will be useful in such contexts. In particular, it is important that the model be tractable from the point of view of comparative dynamics analysis.

Distributional concerns may enter in either of the following ways: First, one may be interested in the distribution of welfare or assets across agents.[1] The model "explains" the distribution of assets, particularly in the long run—individual asset holdings converge to unique steady-state values. The limiting distribution is independent of initial conditions and depends in a simple and intuitive fashion [9, pp. 628–629] on how rates of time preference differ across agents. Specifically, individuals who are more patient in an appropriate sense, have larger asset holdings in the steady state. It is argued that the simplicity and tractability of this model of wealth distribution distinguish it from the more direct multiple-agent extension of the Koopmans–Cass model that is investigated in [5].

Second, it may be important to recognize the effects of wealth distribution on aggregate demands. In static general equilibrium theory the consequences of differing income effects are well known. But the only dynamic models which explicitly consider the effect of distribution on aggregate savings are the two-class models in the early growth literature, where the savings propensities of two groups of agents are assumed to differ. Of course, these models are not based on optimizing behaviour. In the present paper such distributional considerations are integrated into a dynamic general equilibrium model in which all agents optimize. Moreover, the qualitative comparative dynamics consequences of wealth redistribution may be analysed.

Table I summarizes the highlights of existing turnpike literature as it relates to the present paper. Particularly noteworthy is that in this paper utility functions are assumed to be recursive [18, 28] and not additive. As a result the rate of time preference is not constrained to be constant, but

[1] Given the dynastic view implicit in the assumption of infinitely lived agents, the model can address interdynastic (but not intergenerational) distribution. For questions of intergenerational distribution, the overlapping generations model can be applied. Of course, agents need not literally live forever in order that the dynastic approach be of interest. It can provide a useful approximation if agents have finite lifetimes but bequests are important [2].

TABLE I

Literature Survey

	Cass, Koopmans	McKenzie, Magill, Scheinkman	Bewley, Yano	Beals and Koopmans	Lucas and Stokey	Epstein
No. of Goods	1	>1	>1	1	>1	1
No. of Agents	1	1	>1	1	>1	>1
Utility Specification	additive	additive	additive	recursive-minimal extension of additive specification for which rate of time preference depends on consumption path.	recursive (discrete time)	recursive (continuous time)
Main Results and Features	globally stable steady state tractable framework for comparative dynamics	local-global stability given "small" discount rates and/or "sufficient" curvature assumptions	strive for generality of GE analysis, but to obtain nondegenerate steady states they must restrict preference heterogeneity and impose common discount rates; difficulty is due to the maintained constancy of the discount rate. comparative statics and dynamics are difficult because steady state depends on initial conditions	Cass–Koopmans results are confirmed if "increasing marginal impatience" is satisfied	GE level of generality but stability proven only for 2 person, 1 good exchange model with no storage	steady state is independent of initial conditions local stability given "increasing marginal impatience" with no restriction on the level of discounting tractable framework for comparative dynamics analysis

rather can vary with the consumption path where it is evaluated. This flexibility eliminates the unappealing feature of the additive specification whereby more than one agent can own capital in a steady state only if those agents share a common discount rate [4, 5, 30].

The additive utility specification has another important disadvantage—even if the discount rates of all consumers are equal, the long run distribution of consumption in a multiple agent economy depends on the initial distribution of capital stocks across agents, and in a complicated fashion [5, 30].[2] Thus even steady-state analysis requires solution of the full dynamic model and is *not* elementary as it is in a model where the turnpike property applies. In contrast, in the present model with the recursive utility specification, the steady state is unique and is independent of initial conditions. Thus the analysis of steady states is elementary. Moreover, since local stability is proven, a straightforward procedure for comparative dynamics is available: Linearize the dynamic system about the steady state. This yields a constant coefficient, linear differential equation system which can be solved explicitly. The explicit solution faithfully reflects the qualitative dynamics of the original nonlinear system in a neighbourhood of the steady state. It is worth emphasizing that this common procedure is generally *not* available in the additive utility model.

The continuous-time framework of this paper is appealing because of the sharp distinction between stocks and flows that it admits. But the formulation of recursive utility functionals is more difficult in continuous time and the Uzawa functionals are the only ones that have been defined in the literature. The Uzawa specification, with minor modifications, is adopted here. A rationale for this specification is provided in [8] where it is shown that the Uzawa class, broadly defined, is precisely that subset of recursive functionals which retain a recursive structure in a stochastic framework, when the utility functionals are taken to be von Neumann–Morgenstern utility indices. Since the extension of the analysis to a stochastic framework is a logical future step in the research agenda, the Uzawa specification is natural. Its tractability is another appealing feature.

This paper proceeds as follows: Utility functionals are described in Section 2. Efficient allocations are analysed in Section 3 and Section 4 considers equilibrium allocations. Some qualitative properties of the equilibrium model are described in Section 5 and compared with those of the Koopmans–Cass model. Many proofs are omitted. (They may be found in the original working paper version which is available from the author upon request.) But brief outlines of some proofs and assorted technical details are collected in an Appendix.

[2] The long run consumption allocation is readily determined given knowledge of each agent's marginal utility of wealth in equilibrium, but the latter can be obtained only upon analysis of the complete dynamic equilibrium.

2. Recursive Utility

There is a single consumption good available at each instant in an infinite horizon. A consumption path is denoted C, and maps $[0, \infty)$ into $(0, \infty)$. The positivity of consumption is imposed from the start since only interior paths and solutions will be considered. For present purposes there is no loss of generality in defining the consumption set to be $\Lambda \equiv \{C\colon C \text{ is a continuously differentiable map of } [0, \infty) \text{ into } (0, \infty)\}$. The tth period consumption level corresponding to C is denoted $c(t)$, and for each $T \geqslant 0$ ${}_TC$ denotes the appropriate "tail" of C, i.e., ${}_TC$ is the path having tth period consumption equal to $c(T+t)$.

In this paper U will be said to be *recursive* if

$$U(C) = \int_0^\infty v(c)\, e^{-\int_0^t u(c)\,d\tau}\, dt, \qquad C \in \Lambda, \tag{1}$$

where U, u, and v satisfy the assumptions which follow. (See [8] for an axiomatization of the corresponding functional in discrete time.)

Assumption 1. *u and v are real valued and twice continuously differentiable on $(0, \infty)$ and U is real valued on Λ.*

An essential feature of U is the rate of time preference implicit in its structure. To analyse this issue it is necessary to define marginal utilities and marginal rates of substitution. In continuous time this may be accomplished by making use of the concept of a Volterra derivative. (See [29]. Heal and Ryder [17] make a similar application of the concept.) Denote by $U_T(C)$ the marginal utility of U with respect to a small increment in consumption along the path C and at times near T, in the sense made precise by the Volterra derivative. For the specification (1), $U_T(C)$ is given by

$$U_T(C) = e^{\int_0^T u(c)\,dt}\, [v_c(c(T)) - u_c(c(T)) \cdot U({}_TC)], \qquad C \in \Lambda. \tag{2}$$

The next assumption is a form of strong monotonicity for U.

Assumption 2. *For all $T \geqslant 0$ and $C \in \Lambda$, $U_T(C) > 0$.*

It is consistent with discrete time analysis to define the rate of time preference ρ as the negative of the logarithmic rate of change of marginal utility along a locally constant path. Precisely,

$$\rho \equiv -\frac{d}{dT} \log U_T(C)|_{\dot{c}(T)=0}. \tag{3}$$

For the specification (1), ρ depends on the particular path C through the consumption level $c(T)$ at T and through aggregate future utility $U({}_TC)$. Thus, $\rho = \rho(c(T), U({}_TC))$, where

$$\rho(c, \phi) \equiv [v(c)\, u_c(c) - u(c)\, v_c(c)]/[\phi u_c(c) - v_c(c)],\ c > 0,$$
$$\phi \in U(A). \quad (4)$$

$\rho(\cdot, \cdot)$ defined by (4) is the *rate of time preference function* for U. Note that, because of (3), it describes properties of the preference ordering underlying U, rather than simply the particular numerical representation U of that ordering.

ASSUMPTION 3. *For all $c > 0$ and $\phi \in U(A)$, $\rho(c, \phi) \geq \rho_{\min} > 0$, for some constant $\rho_{\min}$.*

The rate of time preference function is positive and bounded away from zero on its domain.

Along constant paths, the rate of time preference function simplifies considerably. Denote by G_c the path that is constant at the level c. Then from (1) and (4) it follows that

$$\rho(c, U(G_c)) = u(c), \qquad c > 0. \quad (5)$$

Thus the function u defines the rate of time preference along constant paths.

ASSUMPTION 4. *$u_c(c) > 0$ for all $c > 0$.*

This assumption is critical for the analysis below and thus requires some comment, particularly since it is often supposed that the rate of time preference varies inversely with the stationary level of consumption [10]. Friedman [11, p. 30] criticizes this latter hypothesis and argues that it is no more compelling than $u_c > 0$. Assumption 4 is adopted by Uzawa, and Lucas and Stokey, who refer to it as "increasing marginal impatience of preferences."

Three arguments are offered here in support of Assumption 4. First, it follows from [3] and [23] that local stability of steady states may fail in standard environments, even in single agent models, if $u_c < 0$. Thus Assumption 4, at least in the weak form $u_c \geq 0$, appears *necessary* to generate the appealing dynamics described below, though its empirical validity remains to be investigated. (This necessity is addressed further following Theorem 1.) This is the only apparent justification for Assumption 4 offered in [20] and [28].

Secondly, note that $u_c > 0$ is *implied* by the hypothesis

$$\rho_\phi(c, \phi) > 0 \qquad \text{for all} \quad c > 0 \quad \text{and} \quad \phi \in U(A),$$

and this hypothesis is plausible on introspective grounds, i.e., an increase in ϕ indicates an increase in future consumption so that present consumption is given more weight.[3] To prove the implication, differentiate (4) and evaluate along a constant path G_c, where $\phi = U(G_c) = v(c)/u(c)$, to obtain

$$\rho_\phi(c, \phi)|_{\phi = U(G_c)} = u_c(c)/[v_c(c) - v(c)\,u_c(c)/u(c)].$$

The denominator is positive by (3) and Assumption 2. Thus

$$\rho_\phi(c, \phi)|_{\phi = U(G_c)} > 0 \Rightarrow u_c(c) > 0.$$

Finally, it was noted in the introduction that the extension of the present analysis to a stochastic framework is desirable and that the specification of U as a von Neumann Morgenstern index is natural. Thus consider the properties of U as such an index. In a corresponding discrete time model, in [8, p. 140], it is shown that $u_c > 0$ is *equivalent* to the implied preference ordering over random consumption paths exhibiting an aversion to correlation in the consumption levels of any two periods. Since such an aversion is plausible on introspective grounds, this observation provides further support for Assumption 4.

To interpret the next assumption it is useful to derive the following generalization of (3):

$$-\frac{d}{dT} \log U_T(C) = \rho(c(T), U({}_TC)) - \dot{c}(T) \cdot \alpha(c(T), U({}_TC)), \tag{6}$$

where $\alpha(c, \phi) \equiv [v_{cc}(c) - \phi u_{cc}(c)]/[v_c(c) - \phi u_c(c)]$. The proportional rate of change of marginal utility is nonzero because of the systematic undervaluation of future consumption ($\rho > 0$), and because of the growth in consumption. This decomposition corresponds to Böhm–Bawerk's two grounds for the existence of interest. (Frisch [12] provides such an interpretation for a similar equation.) Böhm–Bawerk's hypothesis was that consumption growth reduces the marginal felicity of consumption. The next assumption captures the spirit of this hypothesis.

[3] Friedman's (p. 30) discussion relates implicitly to the sign of the *total* derivative $(d/dc)\,\rho(c, U(G_c))$, where an increase in c changes both current and future consumption and thus has no clear effect on the rate of time preference. In contrast, the argument here is based on the sign of the *partial* derivative $\rho_\phi(c, \phi)|_{\phi = U(G_c)}$ which can be argued to be positive, and consequently a clearer case emerges for the sign of u_c.

ASSUMPTION 5. *If C and $T \geqslant 0$ are such that $\dot{c}(T) > 0$, then*

$$-\frac{d}{dT}\log U_T(C) > \rho(c(T), U({}_TC)). \tag{7}$$

This assumption, like its two immediate predecessors, is a statement about the ordinal properties of U.

In the case of additive utility, i.e., u constant in (1), Assumption 5 is equivalent to the positivity of $-v_{cc}/v_c$ on its domain. Since v_c would be positive by Assumption 2, the (strong) concavity of v would be implied; and that would suffice to prove that the usual first order conditions, including transversality conditions, are sufficient for an optimum in standard planning problems. But in the more general case of recursive utility an additional assumption appears to be necessary to prove *sufficiency* of first order conditions in the optimization problems considered below. The assumption takes the following form:

ASSUMPTION 6. *For all C, $C^* \in A$ such that $\lim_{t\to\infty} c_t^*$ exists and is positive,*

$$\int_0^\infty U_t(C^*)\cdot(c(t)-c^*(t))\,dt \leqslant 0 \Rightarrow U(C) \leqslant U(C^*),$$

with a strict inequality if $C \neq C^*$.

The analogous condition for a finite horizon, discrete time model is equivalent to the (strict) quasiconcavity of the utility function. In the present infinite dimensional setting it has not been shown that Assumption 6 is equivalent to the convexity of U's upper contour sets. But it is evidently a statement about the underlying preference ordering, since Assumption 6 is true for U if and only if it is true for any (differentiable) monotonic transformation of U. The same applies to Assumptions 2–5 also. In contrast, the earlier literature employing recursive utility has not succeeded to the same degree in maintaining only ordinal assumptions. (For example, see [3, p. 1014; 20, p. 142; 28, p. 489].)

To show that Assumptions 1–6 are consistent with a large class of functionals, Lemma 1 describes a set of sufficient conditions, expressed in terms of u and v, in order that these assumptions are satisfied.

LEMMA 1. *On $(0, \infty)$ let u and v be twice continuously differentiable, $\inf_{c>0} u(c) > 0$, $u_c > 0$, $u_{cc} \leqslant 0$, $-\infty < \inf_{c>0} v(c) \leqslant \sup_{c>0} v(c) < 0$, $v_c \geqslant 0$ and $\log(-v)$ convex. Then U defined by (1) satisfies Assumptions 1–6.*

Note that (1) reduces to a standard additive functional if $u(c) = \delta > 0$ for all $c > 0$. In that case Assumptions 1–3 and 5 are standard and Assumption 6 is implied. Moreover, the expression $-f_{cc}/f_c$ reduces to $-v_{cc}/v_c$, the common measure of concavity of the utility index v. But the strict inequality in Assumption 4 is violated, and a weak inequality is *not* sufficient for the local stability analysis that follows. Thus the strict form of Assumption 4 is critical.

Finally, there is an alternative representation of the functional (1) which will be useful. Define the function f by

$$f(c, \theta) \equiv v(c) - \theta u(c), \qquad c > 0 \quad \text{and} \quad \theta \in U(\Lambda). \tag{8}$$

Then for each given $C \in \Lambda$, $U(C) = \phi(0)$, where $\phi(\cdot)$ is any solution to the differential equation system

$$\dot{\phi}(t) = -f(c(t), \phi(t)) \qquad \text{and} \qquad \phi(T)\, e^{-\int_0^T u(c)\,dt} \to 0, \quad \text{as} \quad T \to \infty. \tag{9}$$

Straightforward integration of (9) shows that $\phi(t) \equiv U({}_tC)$, $\forall t$ defines the unique solution to (9). Thus $\phi(t)$ has the interpretation as the utility associated with the t-th period tail of the given C.

Equation (9) defines $U(C)$ via the solution to a differential equation. In discrete time recursive utility is usually defined by means of a difference equation of the form

$$U(c_0, c_1, c_2, \ldots) = F[c_0, U(c_1, c_2, \ldots)],$$

where the obvious change in notation has been adopted. F is called an aggregator function [18] and it plays an important role in the analysis of optimal behaviour ([3] and [20]). The function f from (8) will play an important role in the present continuous time analysis. In particular, note that (because of Eqs. (2) and (6))

$$\text{Assumptions 2 and 5 are jointly } \textit{equivalent} \text{ to "} f_c > 0 \text{ and } f_{cc} < 0 \text{ on the domain of } f\text{."} \tag{10}$$

Moreover, the function f_{cc}/f_c plays a role in the stability analysis below. Since $f_{cc}/f_c = \alpha$ in (6), that equation provides an interpretation for f_{cc}/f_c; i.e., the latter function measures the effect of consumption growth on the proportional rate of change of "current valued" marginal utility $e^{\int_0^T \rho\,dt}\, U_T(C)$. Thus it provides a measure of the desire to smooth consumption given U.

3. Pareto Optimal Allocations

There are $H \geqslant 1$ consumers each of which has a utility function that satisfies Assumptions 1–6. Functions or variables that belong to consumer h will have a superscripted h. Each consumer supplies one unit of labour services inelastically at each instant.

The technology is simple. The production process uses labour inputs, which are fixed in total supply at H and suppressed in the notation, and capital services which are proportional to the capital stock x. Given x, the flow of output net of depreciation is $g(x)$, where $g(0)=0$, $g_x > 0$ and $g_{xx} < 0$ on the positive real line. (If $g_x(x)$ is negative for x sufficiently large, say for $x > \mathbf{x}$, then the analysis to follow is valid as long as the initial capital stock is less than $\mathbf{x}$.) Initial capital stock is $x_0 > 0$.

An *allocation* is a vector $(C^1, \ldots, C^H)$ such that $C^h \in A$ for $h = 1, \ldots, H$. An allocation is *feasible* (relative to an initial endowment $x_0 > 0$) if it lies in the set $\hat{Y}(x_0)$, where $\hat{Y}(x_0)$ is the disposable hull of $Y(x_0)$, and

$$Y(x_0) \equiv \Big\{ (C^1, \ldots, C^H) \colon C^h \in A,\ h = 1, \ldots, H \quad \text{and} \quad x(t) > 0 \quad \text{for all} \quad t, \\ \text{where } x(\cdot) \quad \text{solves} \quad \dot{x}(t) = g(x(t)) - \sum_{h=1}^{H} c^h(t),\ x(0) = x_0 \Big\}.$$

An allocation $(C^{1*}, \ldots, C^{H*})$ is *efficient* if it is feasible and if there does not exist another feasible allocation $(C^1, \ldots, C^H)$ for which $U^h(C^h) \geqslant U^h(C^{h*})$ for all $h = 1, \ldots, H$ with strict inequality for at least one h.

The *utility possibility set* is $S(x_0) \equiv \{ (U^1(C^1), \ldots, U^H(C^H)) \colon (C^1, \ldots, C^H) \in \hat{Y}(x_0) \}$. $S^1(x_0) \equiv \{ (\gamma^2, \ldots, \gamma^H) \colon \exists \gamma^1,\ (\gamma^1, \ldots, \gamma^H) \in S(x_0) \}$ is a projection of this set. Efficient allocations are precisely those that solve a problem of the following type:

$$\max_{C^1, \ldots, C^H} \{ U^1(C^1) \colon (C^1, \ldots, C^H) \in Y(x_0),\ U^h(C^h) = \phi_0^h,\ h = 2, \ldots, H \},$$

where $(\phi_0^2, \ldots, \phi_0^H) \in S^1(x_0)$ is given. (The existence of the maximum, or equivalently of efficient allocations, is proven below.)

Thus efficient allocations may be found by solving the following optimal control problem with integral constraints:

$$\max \int_0^\infty v^1(c^1)\, e^{-z^1}\, dt \tag{11}$$

subject to

$$\dot{z}^h = u^h(c^h) \quad \text{and} \quad z^h(0) = 0, \quad h = 1, \dots, H,$$

$$\dot{x} = g(x) - \sum_1^H c^h, \; x(t) > 0, \qquad \forall t \quad \text{and} \quad x(0) = x_0,$$

$$c^h(t) > 0, \; \forall t \quad \text{and} \quad h = 1, \dots, H,$$

and

$$\int_0^\infty v^h(c^h)\, e^{-z^h}\, dt = \phi_0^h, \; h = 2, \dots, H.$$

$(\phi_0^2, \dots, \phi_0^H)$ is a typical point in $S^1(x_0)$. The control variables in this problem are the c^h's, while the state variables are x and z^h, $h = 1, \dots, H$. The latter are artificial variables introduced into the problem in order to make it conform to an optimal control setting. Uzawa [28, p. 490] employs this trick, but his subsequent solution procedure for his single agent optimization problem differs from the procedure followed here in that he uses z as the independent variable in place of time.)

The optimization problem (11) is a problem of Hestenes [27, pp. 657–659] and the necessary conditions are readily derived. It is convenient to use r as a state variable rather than x, where $r \equiv g_x(x)$ is the implicit real interest rate. Let $r_0 = g_x(x_0)$. Then, after application of (9), the following set of necessary conditions is obtained for efficient paths:

$$\begin{aligned} \dot{c}^h &= \frac{f_c^h(c^h, \phi^h)}{f_{cc}^h(c^h, \phi^h)} \cdot [\rho^h(c^h, \phi^h) - r], && h = 1, \dots, H, \\ \dot{\phi}^h &= -f^h(c^h, \phi^h), && h = 1, \dots, H, \\ \dot{r} &= g_{xx}(g_x^{-1}(r)) \cdot \left[g(g_x^{-1}(r)) - \sum_1^H c^h \right], \\ r(0) &= r_0, \qquad \phi^h(0) = \phi_0^h, \quad h = 2, \dots, H, \end{aligned} \tag{12}$$

and

$$\phi^h(t)\, e^{-\int_0^t u^h(c^h)\, d\tau} \to 0 \qquad \text{as} \quad t \to \infty \quad \text{for} \quad h = 1, \dots, H.$$

This is a $(2H+1)$-dimensional first order differential equation system with H initial conditions and H boundary constraints at ∞. In general additional constraints (transversality conditions) are required to determine a unique solution to (12). Below these conditions will take the form of convergence to a steady state.

Note that $\phi^1(0)$ is unrestricted. In fact it is determined as part of the solution to (12) and the imminent transversality conditions. By relating the optimal $\phi^1(0)$ to each given $(\phi_0^2,..., \phi_0^H) \in S^1(x_0)$, the *utility possibility frontier* for the economy is determined.

A *steady state* solution to (12) is $(\bar{c}^1,..., \bar{c}^H, \bar{\phi}^1,..., \bar{\phi}^H, \bar{r})$ such that $\bar{c}^h > 0$ for all h and

$$u^h(\bar{c}^h) = \bar{r}, \qquad h = 1,..., H,$$
$$\sum_1^H \bar{c}^h = g(g_x^{-1}(\bar{r})), \tag{13}$$

and

$$\bar{\phi}^h = v^h(\bar{c}^h)/u^h(\bar{c}^h), \qquad h = 1,... H.$$

The existence of a steady state may be proven with some additional Inada-type assumptions as in Lucas and Stokey [20, pp. 160–163]. If it exists, the steady state is unique since, for example, $\bar{r}$ is the unique solution to the equation $\sum (u^h)^{-1}(\bar{r}) - g(g_x^{-1}(\bar{r})) = 0$, where the function on the left is strictly increasing in $\bar{r}$. Henceforth, existence of a steady state is assumed and its properties (i.e., optimality and local stability) investigated.

Though steady states solve (12), it does not follow from what has been established to this point that they are efficient. To show this, (12) must be expanded to a set of conditions that are *sufficient* for optimality in (11).

LEMMA 2. *Let C^h, ϕ^h, $h = 1,..., H$ and r solve* (12) *and suppose that for each h, $c^h(t) \to \bar{c}^h$, $\phi^h(t) \to \bar{\phi}^h$ and $r(t) \to \bar{r}$ as $t \to \infty$. Then $(C^1,..., C^H)$ is the unique optimal consumption profile in* (11) *and is an efficient allocation given the initial stock x_0, $g_x(x_0) = r_0$.*

As an immediate implication of the Lemma deduce that the stationary consumption paths in (13) define an efficient allocation if $x_0 = \bar{x} = g_x^{-1}(\bar{r})$. Thus a steady state allocation consistent with (13) is Pareto optimal if it is feasible.

Consider next the local stability of this allocation, i.e., do efficient allocations from initial stock x_0 necessarily converge to the steady state allocation if $|x_0 - \bar{x}|$ is sufficiently small? The answer is "yes" for those efficient allocations $(C^1,..., C^H)$ for which $|U^h(C^h) - \bar{\phi}^h|$ is sufficiently small for $h = 2,..., H$, as established in Theorem 1.

Local stability analysis proceeds in the usual fashion by linearization of (12) about the steady state. Thus consider the following linear system

$$\begin{bmatrix} \dot{c} \\ \dot{\phi} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} O_{H \times H} & \widehat{(\bar{r}u_c/f_{cc})} & -f_c/f_{cc} \\ -\hat{f}_c & \bar{r}1_{H \times H} & O_{H \times 1} \\ -g_{xx}e_{1 \times H} & O_{1 \times H} & \bar{r} \end{bmatrix} \begin{bmatrix} c - \bar{c} \\ \phi - \bar{\phi} \\ r - \bar{r} \end{bmatrix}, \tag{14}$$

where: c and ϕ are H-dimensional vectors with typical components c^h and ϕ^h, respectively; $\bar{c}$ and $\bar{\phi}$ are defined in the natural way; $e_{1\times H}$ is an H-dimensional row vector consisting of 1's;

$$-f_c/f_{cc} \equiv \begin{bmatrix} -f_c^1/f_{cc}^1 \\ \vdots \\ -f_c^H/f_{cc}^H \end{bmatrix}, \qquad \widehat{(\bar{r}u_c/f_{cc})} \equiv \begin{bmatrix} \bar{r}u_c^1/f_{cc}^1 & & 0 \\ & \ddots & \\ 0 & & \bar{r}u_c^H/f_{cc}^H \end{bmatrix},$$

$$\hat{f}_c \equiv \begin{bmatrix} f_c^1 & & 0 \\ & \ddots & \\ 0 & & f_c^H \end{bmatrix};$$

and where all functions are evaluated at the steady state. (To derive (14) note that $\rho_c(c, \phi) = 0$ and $\rho_\phi(c, \phi) = uu_c/f_c$ along a constant path, i.e., if $\phi = U(G_c)$. This, in turn, may be proven by differentiating (4).) Henceforth, denote the coefficient matrix in (14) by A.

LEMMA 3. *There exists $\varepsilon > 0$ such that if in* (12) *it is the case that* $|r_0 - \bar{r}| \leqslant \varepsilon$ *and* $|\phi_0^h - \bar{\phi}^h| \leqslant \varepsilon$, $h = 2,..., H$, *then there exist unique initial values* $c^1(0),..., c^H(0)$ *and* $\phi^1(0)$ *such that the trajectory defined by* (12) *converges to the steady state.*

It is shown in the Appendix that (i) A has H (real) negative eigenvalues and $(H+1)$ positive (real) eigenvalues. Thus there is an H-dimensional linear stable manifold MF_L passing through the steady state. It is also shown that (ii) the projection of MF_L onto the subspace defined by the $\phi^2,..., \phi^H$ and r coordinates coincides with that subspace. Therefore ([16, pp. 242–244] and [15, pp. 111–112]) the nonlinear system has a stable manifold MF_{NL} which is tangent to MF_L at the steady state. This proves Lemma 3.

Lemmas 2 and 3 combine to prove the following central result of this section:

THEOREM 1. *There exists $\varepsilon > 0$ such that the following statements are valid: If* $|r_0 - \bar{r}| \leqslant \varepsilon$ *and* $|\phi_0^h - \bar{\phi}^h| \leqslant \varepsilon$, $h = 2,..., H$, *then there exists a unique efficient allocation* $(C^1,..., C^H)$ *for the initial stock* $x_0 = g_x^{-1}(r_0)$, *which satisfies* $U^h(C^h) = \phi_0^h$, $h = 2,..., H$. *Moreover, the allocation converges to the steady state, i.e.,* $c^h(t) \to \bar{c}^h$, $U^h({}_tC^h) \to \bar{\phi}^h$, $h = 1,..., H$, *and* $r(t) \to \bar{r}$, *where r is the corresponding (implicit) interest rate.*

Note that the theorem establishes not only stability, but also existence of efficient allocations. Of course the Theorem does *not* prove that *all* efficient allocations for x_0 near $\bar{x}$ converge to the steady state. Stability is ensured only if the associated lifetime utilities of agents 2 through H are near their steady state values. A global analysis would be required to determine if this qualification could be dropped. But the presence of the qualification is not surprising since for recursive (nonadditive) utility, the future utilities ϕ^h constitute relevant state variables.

Remark. Since A in (14) has only real eigenvalues, efficient paths are *noncyclical* near the steady state. Thus the presence of many agents alone, in a single good model, does not lead to the possibility of cycles. This is in contrast to single agent models with additive utility and many goods. (See [22].)

Remark. The maintained assumptions in Theorem 1 do not restrict individual discount rates to be small. This is in contrast to the n-good single agent model where local stability is often proven under the assumption of a small discount rate ([26, Part II] and [21]). The stability propositions in the multiple good-multiple agent models in [5] and [30] also maintain small discount rates.

Remark. Theorem 1 shows that the maintained assumptions on preferences and technology are *sufficient* for local stability. Consider briefly whether they are *necessary*. Epstein and Hynes [9, pp. 621–622] show by example that (weakly) diminishing marginal productivity is *not* necessary for stability, as long as impatience is sufficiently increasing. Lucas and Stokey [20, pp. 168–169] suggest that convergence to an interior stationary point should be possible if some consumers' preferences fail to exhibit $u^h_c > 0$, provided the others' had it in a strong enough way to be offsetting. But in fact, local stability fails if there exist two individuals for which $u^h_c < 0$. (See the Appendix.) Thus increasing marginal impatience for all but at most one consumer is *necessary* for stability.

Given that the convergence of efficient allocations has been established, it is natural to consider the speed of convergence and its determinants. Such an investigation concludes this section.

Denote by $\xi_1, \ldots, \xi_H$ the H negative eigenvalues of A, repeated according to multiplicity. Say that convergent solutions of (14) converge at the rate $-\xi^{\text{sp}}$ if $\xi^{\text{sp}} = \max_{1 \leqslant i \leqslant H} \xi_i$, i.e., if ξ^{sp} is the least negative eigenvalue.

First, suppose there is only a single consumer. Then

$$\xi^{\text{sp}} = \xi_1 = \tfrac{1}{2}[\bar{r} - \sqrt{\bar{r}^2 - 4(f^1_c/f^1_{cc})(\bar{r}u^1_c - g_{xx})}].$$

Thus the convergence speed is greater the larger is u^1_c or $(-g_{xx})$. Both increasing impatience and diminishing marginal productivity contribute to stability. Also, the convergence speed depends inversely on $-f^1_{cc}/f^1_c$, which was interpretated in the last section. The inverse relationship is intuitive—the larger is $(-f^1_{cc}/f^1_c)$ the stronger is the preference for a smooth consumption profile and hence the smaller is the rate of adjustment towards the steady state. In the case of additive utility, $-f^1_{cc}/f^1_c = -v^1_{cc}/v^1_c$, and the above inverse relationship is well known.

Similar qualitative relationships between ξ^{sp} and the underlying preference and technology parameters prevail in the case of many consumers.

THEOREM 2. *Let A be the coefficient matrix in* (14) *and suppose that A is perturbed by any number of changes of the following type:*

(i) *u^h_c is increased for some h, $h = 1,\ldots, H$;*

(ii) *$(-g_{xx})$ is increased;*

(iii) *$-f^h_c/f^h_{cc}$ is increased for some h, $h = 1,\ldots, H$. (Of course, it is the values of these functions at the given steady state which are to be increased.) Then the speed of convergence for the new matrix is at least as large as that of A, i.e.,*

$$-\xi^{\mathrm{sp}} \geqslant -\xi^{\mathrm{sp}},$$

where ξ^{sp} denotes the maximum negative eigenvalue of the new matrix.

Remark. As a special case of the theorem compare an economy in which $g_{xx}(\bar{x}) < 0$ with one in which $g_{xx}(\bar{x}) = 0$. In the latter case the H negative eigenvalues of A are

$$s^h = \tfrac{1}{2}\{\bar{r} - \sqrt{\bar{r}^2 - 4v^h}\}, \qquad v^h \equiv \bar{r}u^h_c f^h_c/f^h_{cc}, \; h = 1,\ldots, H. \tag{15}$$

The speed of convergence in such an economy is $-\max_{1 \leqslant h \leqslant H} s^h$, and the theorem implies that

$$\xi^{\mathrm{sp}} \leqslant \max_{1 \leqslant h \leqslant H} s^h, \tag{16}$$

where ξ^{sp} is the largest negative eigenvalue of A when $g_{xx} < 0$. The right side of (16) reflects properties of preferences only (at least for given $\bar{r}$), whereas ξ^{sp} includes the effect of diminishing marginal productivity. Thus the inequality is consistent with the anticipated stabilizing influence of $g_{xx} < 0$.

The preference parameters s^h will play an important role in the comparative dynamics analysis of Section 5. To interpret them note that in a life cycle framework where the interest rate is constant and exogenous at the level $\bar{r}$, $-s^h$ gives the speed at which consumer h optimally adjusts his consumption to its steady state value. Thus call $-s^h$ the *partial equilibrium adjustment speed for person h.*

Remark. In the single consumer case, the convergence speed is strictly monotonic in the noted preference and technology parameters. But only the weak inequality in Theorem 2 is valid in general.

Changes in $-g_{xx}$ provide an intriguing demonstration of the need for a weak inequality in Theorem 2. In the Appendix it is proven that if there are at least two individuals, such that

$$s^h = s^k = \max_{1 \leqslant i \leqslant H} s^i, \tag{17}$$

then

$$\xi^{\mathrm{sp}} = s^h = \max_{1 \leqslant i \leqslant H} s^i. \tag{18}$$

The significance of (18) is that the right side is independent of the value of g_{xx}. Thus the speed of convergence of the economy is *independent* of the rate of declining marginal productivity. This is true if (and only if) there exist two agents that share the smallest partial equilibrium adjustment speed $\max_{1 \leqslant h \leqslant H}(-s^h)$.

Some rough intuition may be provided. For simplicity suppose that $s^h = s$, $\forall h$. Planning decisions may be divided conceptually into two stages:

(i) the determination of aggregate consumption and capital accumulation profiles, and

(ii) the distribution of the chosen total consumption across consumers at each instant.

When all consumers share a common s^h these planning functions can be separated; in particular, distributional concerns can be ignored in (i) because there exists a representative consumer for the economy. (This is demonstrated in the analysis of decentralized economies below.) Moreover, the magnitude of $-g_{xx}$ affects the adjustment speed of the activities in (i) but not those in (ii). Thus if the convergence of the distribution activities is slower, then it will determine the overall speed of adjustment $-\xi^{\mathrm{sp}}$, which will be independent of $-g_{xx}$. Finally, note that the adjustment in (ii) *is* slower essentially because of (16).

4. Equilibrium Allocations

Preferences and the technology satisfy the same assumptions as maintained in the last section. Consumers and a representative firm interact in a continuum of spot markets for consumption, labour, capital and bonds.The firm does not incur adjustment costs and simply maximizes profits at each instant. Consumer h solves

$$\max_{C^h \in A} U^h(C^h) \qquad \text{subject to } x_0^h = \int_0^\infty e^{-\int_0^t r(\tau)\,d\tau}(c^h(t) - w(t))\,dt, \tag{19}$$

where x_0^h is his initial endowment of capital, $r(\cdot)$ is the real interest rate profile for bonds that he expects and $w(\cdot)$ is the expected wage rate profile. (For simplicity it is maintained that consumers possess identical labour endowments and face identical wage rates.) The budget constraint indicates that borrowing and lending are allowed. Also, x_0^h may be negative for some h's.

Given $x_0^1,..., x_0^H$ and $x_0 \equiv \sum_1^H x_0^h > 0$, a *competitive equilibrium* is $(C^1,..., C^H, r(\cdot), w(\cdot))$ such that

$$C^h \quad \text{solves} \quad (19) \quad \text{given} \quad r(\cdot), w(\cdot) \quad \text{and} \quad x_0^h, h = 1,..., H, \tag{20a}$$

$$r(t) = g_x(x(t)) \text{ and } w(t) = \frac{1}{H}[g(x(t)) - x(t)\,g_x(x(t))], \ \forall t \geq 0, \tag{20b}$$

where $x(\cdot) > 0$ solves

$$\dot{x}(t) = g(x(t)) - \sum_1^H c^h(t), \qquad t \geq 0 \qquad \text{and} \qquad x(0) = x_0. \tag{20c}$$

Equations (20b) are profit maximization conditions. In conjunction with (20c) they guarantee that interest rate and wage rate expectations are fulfilled in an equilibrium. Given an equilibrium as above, refer to $(C^1,..., C^H)$ as an *equilibrium allocation.*

A competitive equilibrium is *stationary* if the consumption, interest rate and wage rate profiles are stationary. The corresponding consumption profiles are referred to as a stationary equilibrium allocation.

Of interest are both the existence and asymptotic properties of a competitive equilibrium. To pursue these questions, the utility maximization problems must be analysed more closely. First, note that (19) can be transformed into the following equivalent problem:

$$\max \int_0^\infty v^h(c^h)\,e^{-z^h}\,dt, \qquad \text{subject to}$$

$$\dot{z}^h = u^h(c^h), \qquad\qquad z^h(0) = 0$$

$$\dot{x}^h = rx^h + w - c^h, \qquad\qquad x^h(0) = x_0^h$$

and

$$x^h(t)\exp\left[-\int_0^t r\,d\tau\right]\to 0, \qquad \text{as} \quad t\to\infty. \tag{21}$$

Necessary first order conditions for this problem can be derived as in the analysis of [11] in the last section. Combine these conditions for all consumers to obtain the following *necessary* conditions for a competitive equilibrium:

$$\begin{aligned}
&\dot{c}^h = \frac{f_c^h(c^h,\phi^h)}{f_{cc}^h(c^h,\phi^h)}\,[\rho^h(c^h,\phi^h)-r], && h=1,\dots,H\\
&\dot{\phi}^h = -f^h(c^h,\phi^h), && h=1,\dots,H\\
&\dot{x}^h = rx^h + w - c^h, && h=1,\dots,H\\
&x^h(0)=x_0^h \quad \text{and} \quad x^h\exp\left[-\int_0^t r\,d\tau\right]\to 0, && h=1,\dots,H,\\
&\sum x^h(t)>0, && \forall t,
\end{aligned}$$

where

$$r=g_x\left(\sum_1^H x^h\right) \qquad \text{and} \qquad w=\frac{1}{H}\left[g\left(\sum x^h\right)-r\sum x^h\right]. \tag{22}$$

These conditions are also *sufficient* for a competitive equilibrium if all the variables involved converge to their steady-state values.

An immediate corollary is that a steady state solution of (22) (assumed to exist[4] and denoted by bars over the variables) defines a stationary equilibrium. A second corollary is a form of the second theorem of welfare economics.

COROLLARY. *Let $(C^1,\dots,C^H)$ be an efficient allocation, given $x_0>0$, which converges to a stationary efficient allocation as described in Section 3. Let $r(\cdot)$ and $w(\cdot)$ be the interest and wage profiles implicit in this allocation, and define $x_0^h\equiv\int_0^\infty \exp[-\int_0^t r\,d\tau]\cdot(c^h-w)\,dt$, $h=1,\dots,H$. Then $(C^1,\dots,C^H$, $r(\cdot), w((\cdot)))$ is a competitive equilibrium given $x_0^1,\dots,x_0^H$.*

This corollary and the analysis of efficient allocations in Section 3 provide some insight into the nature of competitive equilibria from the *constructed*

[4] The equations defining the steady state values $\bar{c}^1,\dots,\bar{c}^H$, $\bar{r}$ and $\bar{w}$ are identical to those encountered in the analysis of efficient paths. Once these variables are determined, then $\bar{x}^h=(\bar{c}^h-\bar{w})/\bar{r}$, $h=1,\dots,H$, completes the specification of the steady state.

initial endowments $\{x_0^h\}$. But one is typically more interested in analysing equilibrium for *given* initial endowments as is done in the remainder of this section.

If existence of an equilibrium for given initial endowments is granted, the problem of local stability of equilibrium paths could be approached as follows: First, note that equilibrium allocations are efficient, i.e., the first welfare theorem is valid, as in [7]. Second, apply the stability result of Section 3 for efficient allocations. But the local stability result Theorem 1 can be applied only if it is first shown that the map taking individual initial endowments into lifetime utilities along an equilibrium path is continuous in a neighborhood of steady-state endowments. Unfortunately, it is not evident that this map is continuous or even that the continuity constitutes a more elementary proposition than the desired local stability result. (Araujo and Scheinkman [1] show that there exists a close relationship between certain smoothness [continuity and differentiability] properties of optimal paths and turnpike properties.) Thus a direct approach to the analysis of local stability, based on the system (22), is adopted here. This approach has the advantage of providing an existence result for equilibrium paths, as well as a straightforward procedure for local comparative dynamics analysis.

Therefore, consider the linearization of (22) around a steady state. As in the analysis of efficient paths, it is convenient to use the rate of interest as a state variable. Thus use r, $x_2, \ldots$ and x_H as state variables, instead of $x_1, x_2, \ldots, x_H$, where $r \equiv g_x(\sum_1^H x_h)$. The following $3H$-dimensional linear system is obtained:

$$\begin{bmatrix} \dot{c} \\ \dot{\phi} \\ \dot{r} \\ {}_2\dot{x} \end{bmatrix} = \left[\begin{array}{ccc|c} & A & & 0_{(2H+1)\times(H-1)} \\ \hline D & 0_{(H-1)\times H} & d & \bar{r}1_{(H-1)\times(H-1)} \end{array}\right] \begin{bmatrix} c-\bar{c} \\ \phi-\bar{\phi} \\ r-\bar{r} \\ {}_2x - {}_2\bar{x} \end{bmatrix}, \tag{23}$$

where A is the coefficient matrix from (14), other notation is also adopted from (14),

$${}_2x \equiv \begin{bmatrix} x^2 \\ \vdots \\ x^H \end{bmatrix},$$

${}_2\bar{x}$ is the corresponding steady state vector, D is the

$$(H-1)\times H \text{ matrix} \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & 0 & -1 \end{bmatrix},$$

and

$$d \equiv \begin{bmatrix} d^2 \\ \vdots \\ d^H \end{bmatrix},$$

where for each h, $d^h = \bar{x}^h - \sum_{i=1}^{H} \bar{x}^i/H$ measures the deviation of individual h's capital stock from the average holdings in the steady state.

Denote by B the coefficient matrix in (23). It's eigenvalues are readily determined. (In computing the characteristic polynomial, note that because of the block partitioned structure of B, $\det(B-\lambda 1_{3H\times 3H}) = \det(A-\lambda 1_{(2H+1)\times(2H+1)})\cdot(\bar{r}-\lambda)^{H-1}$.) In particular the eigenvalues of B are $\bar{r}$ and the eigenvalues of A. Thus B has H (real) negative eigenvalues and $2H$ positive eigenvalues. This provides precisely the correct number of negative eigenvalues for stability, since there are H predetermined variables in (26). In particular, there exists an H-dimensional linear stable manifold MF_L passing through the steady state.

But another condition [26, p. 20] must be satisfied to prove stability by this line of argument. Here that condition takes the following form:

REGULARITY. The projection of MF_L onto the subspace of R^{3H} defined by the r, $x^2,\ldots, x^H$ co-ordinates coincides with that subspace.

B depends on the preference and technology parameters that enter into A and also on the distribution of steady state stocks as reflected by d. Thus the regularity condition is a joint requirement on all of these parameters. In a sense made precise in the Appendix, Regularity fails only for a "small and insignificant set of economies." But since there exist economies where it does fail, they must be ruled out in the following theorem:

THEOREM 3. *If Regularity is satisfied, then there exists* $\varepsilon > 0$ *such that the following is true: If* $|x_0^h - \bar{x}^h| \leqslant \varepsilon$, $h = 1,\ldots, H$, *then there exists a unique competitive equilibrium* $(C^1,\ldots, C^H, r(\cdot), w(\cdot))$ *which converges to the steady* *state equilibrium, i.e.,* $c^h(t) \to \bar{c}^h$, $r(t) \to \bar{r}$, $w(t) \to \bar{w}$ *and* $x^h(t) \to \bar{x}^h$ *as* $t \to \infty$, *for* $h = 1, \ldots, H$.

Remark. Since the matrices A and B (see (23)) share the same negative eigenvalues, the results regarding the speed of convergence of efficient allocations translate directly to the present context of equilibrium allocations.

5. SOME QUALITATIVE PREDICTIONS[5]

A model with more than one consumer is appealing on the basis of realism, and is necessary for addressing distributional issues. But if only aggregate variables are of interest, then perhaps not much is lost in predictive accuracy by using a single agent model. The preceding two sections have shown that local turnpike propositions are largely robust to our extension of the Cass–Koopmans model to many consumers. The only exceptions arise in the case of many consumers of a single type when the economy's speed of convergence to the steady state may be independent of the concavity of the production function, and possibly in decentralized economies which violate Regularity.

In this section the robustness of other well-known qualitative results of the Koopmans–Cass model are investigated. Of critical importance are the s^h parameters, the partial equilibrium adjustment speeds defined in Section 3, and more precisely whether or not they differ across consumers. Their significance in this section stems from their relationship with the income effect on the demand for consumption. More precisely, imagine consumer h in a life cycle framework facing an exogenous and constant interest rate $\bar{r}$. Let $c^h(0)$ denote his consumption demand at $t = 0$ and let x_0^h be his initial stock of wealth. Then, when the derivative is evaluated at the steady state $x_0^h = \bar{x}^h$, it is the case that

$$\partial c^h(0)/\partial x_0^h \doteq \bar{r} - s^h.$$

Thus the marginal propensity to consume (m.p.c.) is greater the larger is $-s^h$, i.e., the larger the speed of adjustment.

For simplicity, a model with two consumers is employed since that suffices to make the desired points. When m.p.c.'s are identical for both individuals ($s^1 = s^2$), the predictions of the Koopmans–Cass model for

[5] The results in this section are based on the explicit solution of (23) and hence on the structure of the eigen vectors for B. The block structure of A is important here. Detailed proofs are available from the author. Regularity is maintained throughout. The restriction this imposes on the parameters of the economy is described in the Appendix.

aggregate variables are confirmed. In fact a representative consumer exists. If m.p.c.'s differ, then the dynamics of aggregate variables are affected also by distribution and predictions based on a single agent model can be misleading. The consequences of redistribution for prices and welfare are examined. In particular, the two-person model is used to demonstrate the possibility of the transfer paradox occurring in an infinite horizon model with the turnpike property.

The qualitative dynamics of the economy near the steady state are faithfully reflected by the linear system (23).[6] Therefore, for present purposes the latter can be taken to represent the dynamics of the economy. It is a 6-dimensional first order system with predetermined variables r and x^2 having initial values r_0 an x_0^2, respectively. An explicit solution for (23) is possible and implies the following dynamics for the interest rate

$$\dot{r}(t) = m_1(r(t) - \bar{r}) + m_2(x^2(t) - \bar{x}^2), \tag{24}$$

where the m_i's are adjustment coefficients. In general, $m_2 \neq 0$ and the dynamics of the interest rate depend on both aggregate capital stock $(x(t) = x^1(t) + x^2(t) = g_x^{-1}(r(t)))$ and on the distribution of that stock (as defined by $x^2(t)$). The latter distributional effect is the source of all the novel results to be described shortly. This effect vanishes $(m_2 = 0)$ *if and only if* $s^1 = s^2$. In that case, the dynamics of r (and aggregate consumption) are determined by the total capital stock only and are qualitatively identical to the dynamics predicted by the single agent Koopmans–Cass model.

Now consider in turn three predictions of the Koopmans–Cass model.

PREDICTION 1. *The rate of interest is monotonic along an equilibrium path.*

In particular, in the Koopmans–Cass model $\dot{r}(t)$ and $\bar{r} - r(t)$ have the same signs for all t. It is evident from (24) that this need not be the case in a two-consumer model as long as the distribution of capital affects the dynamics of r $(m_2 \neq 0)$. For example, if $m_2 > 0$ then $\dot{r}(0) > 0$ even though $r_0 - \bar{r} > 0$, as long as $(x_0^2 - \bar{x}^2))/(r_0 - \bar{r})$ is sufficiently large. (Of course,

[6] To be precise, the variational derivatives of solutions to (22), evaluated along a steady state path, are equal to the corresponding variational derivatives of the linear system (23). Thus statements which follow are based on the values of derivatives such as

$$\frac{\partial r}{\partial x_0^2}(0; x_0^2, r_0)\,\Big|_{x_0^2 = \bar{x}^2, r_0 = \bar{r}}, \qquad \text{or} \qquad \frac{\partial \sum c^h}{\partial r_0}(0; x_0^2, r_0)\,\Big|_{x_0^2 = \bar{x}^2, r_0 = \bar{r}},$$

where $r(\cdot; x_0^2, r_0)$ and $c^h(\cdot; x_0^2, r_0)$, $h = 1, 2$, solve the appropriate version of (22) given the initial values x_0^2 and r_0 for the state variables.

$x_0^2 - \bar{x}^2$ and $r_0 - \bar{r}$ can still be as small as necessary to make (23) a good approximation.)

But note that even when $s^1 \neq s^2$ the Koopmans–Cass prediction *is* valid asymptotically, in the sense that $\lim_{t \to \infty} \dot{r}(t)/(r(t) - \bar{r})$ exists and is negative.

PREDICTION 2. *The time rate of change of aggregate consumption is positive (negative) if aggregate capital stock is being accumulated (decumulated).*

This is generally false when distribution "matters" and the dynamics involve x^2. In particular, for "*almost all*" economies, as specified by steady state preference and technology parameters, there exist initial conditions r_0 and x_0^2 (with $|r_0 - \bar{r}|$ and $|x_0^2 - \bar{x}^2|$ arbitrarily small) such that $\sum \dot{c}^h(0)$ and $\dot{r}(0)$ have identical signs.

But even when $s^1 \neq s^2$ Prediction 2 *is* valid asymptotically in the sense that $\lim_{t \to \infty} \sum_1^2 \dot{c}^h(t)/\dot{r}(t)$ exists and is negative.

PREDICTION 3. *Aggregate consumption is an increasing function of the existing aggregate stock of capital.*

Once again, this prediction is generally false in an economy with more than one consumer.[7] One might be tempted to explain this difference in predictions by suggesting that the recursive specification (1) permits consumption to be an inferior good, while of course, for additive utility consumption is normal. But in fact consumption is a normal good also for (1) given Assumptions 1–6. (See [8, Theorem 5] for a proof in a corresponding discrete time model.) Moreover, the prediction is valid in a single agent economy with recursive utility. The proper explanation is the following: An increase in the capital endowment of one consumer will increase the aggregate demand for current consumption at current interest rates. To restore equilibrium interest rates for $t \in (0, \infty)$ may increase, and this to such an extent that aggregate consumption will be lower in the new equilibrium.

To conclude, consider some price and welfare effects of wealth redistribution within the two consumer model. (Of course these questions cannot be addressed in the Koopman-Cass model.) In particular, consider a transfer of wealth from consumer 1 to consumer 2. Since aggregate stock is kept fixed, the effects of such a transfer may be determined by partial dif-

[7] With many consumers there are many ways of increasing the total initial stock depending upon how the increase is distributed across consumers. In the particular experiment referred to here all of the increment is given to one consumer.

ferentiation of the dynamic system (23) with respect to x_0^2, keeping r_0 fixed. It can be shown that

$$\frac{\partial \dot{r}(0)}{\partial x_0^2} = \left[\frac{a}{z^1 - z^2} - bd^2\right]^{-1}, \qquad \text{where} \quad z^h \equiv s^h(\bar{r} - s^h),\ h = 1, 2$$

$$a, b > 0 \text{ and independent of } d^2. \tag{25}$$

The sign of the derivative is ambiguous in general and depends on the signs of both $z^1 - z^2$ (equivalently $s^1 - s^2$) and d^2. For example, the interest rate is induced to rise along the initial portion of an equilibrium path if:

(i) $s^2 < s^1$, and

(ii) $d^2 \leqslant 0$.

The intuition behind (i) is clear-redistribution towards the high m.p.c. individual will increase aggregate consumption demand, reduce future capital stock and hence increase future interest rates. But intuition for the significance of (ii) has not yet been found.

A zero value for the square bracket in (25) corresponds to a violation of Regularity. As $d^2 \to a(z^1 - z^2)^{-1}/b$, $\partial \dot{r}(0)/\partial x_0^2 \to \pm\infty$. Thus derivatives of interest in comparative dynamics analysis are discontinuous in the economy's parameters at a "singular" economy, or else fail to exist there.

Finally, consider the welfare effects of a transfer of wealth from agent 1 to agent 2. It can be shown that $\partial\phi^1(0)/\partial x_0^2$ and $\partial\phi^2(0)/\partial x_0^2$ are oppositely signed, and

$$\partial\phi^2(0)/\partial x_0^2 = K \cdot [\partial \dot{r}(0)/\partial x_0^2]/(z^1 - z^2), \tag{26}$$

where $z^h \equiv s^h(\bar{r} - s^h)$, $h = 1, 2$ and $K > 0$. Thus the so-called transfer paradox, $\partial\phi^2(0)/\partial x_0^2 < 0$ and $\partial\phi^1(0)/\partial x_0^2 > 0$, occurs in this model if, for example, $\partial \dot{r}(0)/\partial x_0^2 < 0$ and $s^2 < s^1$, i.e., if aggregate consumption and interest rates fall as a result of a transfer to the high m.p.c. individual.[8]

[8] The transfer paradox was first observed by Leontief [19] in an exchange model. Galor and Polemarchakis [13] investigate its occurence in an overlapping generations model with production. In particular they consider the theoretical presumption, due to Samuelson [25], against the compatibility of the transfer paradox with Walrasian stability. The latter has not been determined in the present model with a continuum of goods (consumption at each t), but another form of stability, namely turnpike stability, does coexist with the transfer paradox in the present model.

Appendix

Some proofs are omitted and in many cases, only brief outlines of proofs are presented. A complete set of detailed proofs is available from the author upon request.

Proof of Lemma 2. Let $(\mathbf{C}^1,..., \mathbf{C}^H)$ be an allocation, with associated capital profile $\mathbf{x}(\cdot)$, that is feasible in problem (11) and show that $U^1(\mathbf{C}^1) \leqslant U^1(C^1)$. By Assumption 6, it suffices to prove that $\int_0^\infty U_t^1(C^1) \cdot (\mathbf{c}^1(t) - c^1(t))\, dt \leqslant 0$, which in turn is implied by

(i) $U_t^h(C^h) = U_0^h(C^h) \cdot e^{-\int_0^t r(\tau)\, d\tau}$, $t \geqslant 0, h = 1,..., H$,

(ii) $\int_0^\infty U_t^h(C^h) \cdot (\mathbf{c}^h(t) - c^h(t))\, dt \geqslant 0$, $h = 2,..., H$, and

(iii) $\int_0^\infty e^{-\int_0^t r(\tau)\, d\tau} \cdot (\sum_1^H \mathbf{c}^h(t) - \sum_1^H c^h(t))\, dt \leqslant 0$.

Thus it remains only to prove (i)–(iii). First, (i) is obtained by integrating (12). Feasibility in (11) requires $U^h(\mathbf{C}^h) = U^h(C^h)$, $\forall h \geqslant 2$. Thus (ii) is implied by Assumption 6. Finally, (iii) is implied by the capital accumulation equation in (11), the concavity of g and the boundedness of $x(\cdot)$. ∎

Theorem 1 is implied by Lemmas 2 and 3. Lemma 3, in turn is implied by the following two lemmas whose proofs are omitted.

Lemma A.1. *Let $M \equiv A(\bar{r}1 - A)$, where A is the coefficient matrix in (14) and 1 is the identity matrix of the appropriate dimension. Then:*

(a)

$$M = \begin{bmatrix} M_1 & 0_{H \times (H+1)} \\ 0_{(H+1) \times H} & M_2 \end{bmatrix}$$

for some square matrices M_1 and M_2.

(b) $M_1 = DP$, *where*

$$D = \begin{bmatrix} -f_c^1/f_{cc}^1 & & 0 \\ & \ddots & \\ 0 & & -f_c^H/f_{cc}^H \end{bmatrix}$$

and

$$P = \begin{bmatrix} (g_{xx} - \bar{r}u_c^1) & g_{xx} \cdots & g_{xx} \\ g_{xx} & \ddots & \vdots \\ \vdots & & g_{xx} \\ g_{xx} & \cdots & (g_{xx} - \bar{r}u_c^H) \end{bmatrix}$$

(c) *M_1 can be diagonalized and all its eigenvalues are real and negative.*

LEMMA A.2. (a) *$\bar{r}$ is an eigenvalue of A with multiplicity* 1.

(b) *Let $v_1,\ldots, v_S$ be the distinct eigenvalues of M_1 and define ξ_i^+ and ξ_i^- by*

$$\xi_i^+ = \tfrac{1}{2}\{\bar{r} + \sqrt{\bar{r}^2 - 4v_i}\}, \qquad \xi_i^- = \tfrac{1}{2}\{\bar{r} - \sqrt{\bar{r}^2 - 4v_i}\}, \quad i = 1,\ldots, S.$$

Then each ξ_i^+ and ξ_i^- is an eigenvalue of A with common multiplicity equal to the multiplicity of v_i as an eigenvalue of M_1.

(c) *If eigenvalues are counted as many times as their multiplicity, then A has $(2H+1)$ real eigenvalues, of which H are negative and $(H+1)$ are positive.*

(d) *Let $V^-(A)$ be the linear subspace of $(2H+1)$-dimensional Euclidean space $R^{(2H+1)}$ spanned by the set of all eigenvectors that correspond to some ξ_i^-. Let π: $R^{(2H+1)} \to R^H$ be the projection that maps a typical vector $(y_1,\ldots, y_{2H+1})$ into $(y_{H+2},\ldots, y_{2H+1})$. Then $\pi V^-(A) = R^H$.*

Proof of Theorem 2. Let $v_1,\ldots, v_H$ be the eigenvalues of M_1 (see Lemmas A.1 and A.2) and denote by $\xi_1,\ldots, \xi_H$ the corresponding negative eigenvalues of A. Since

$$\xi_i = \tfrac{1}{2}\{\bar{r} - \sqrt{\bar{r}^2 - 4v_i}\}, \qquad i = 1,\ldots, H,$$

$\xi_i < \xi_j \Leftrightarrow v_i < v_j$. Thus it suffices to prove the inequality indicated in the statement of the theorem for the eigenvalues of M_1 rather than those of A.

Denote by $\hat{M}_1$ the matrix obtained from M_1 as a result of any of the perturbations described in the theorem. Then $\hat{M}_1 - M_1$ is negative semidefinite. That the largest eigenvalue of $\hat{M}_1$ cannot exceed $\max_{1 \leqslant i \leqslant H} v_i$ now follows from the extremal characterization of eigenvalues [14, Theorem 10, p. 319]. ∎

In a remark following Theorem 2 it is claimed that an increase in $(-g_{xx})$ may leave the speed of convergence unaffected. To see this, suppose that agents h and k have $s^h = s^k = s$. Then rows h and k of the matrix $M_1 - s(\bar{r} - s) \cdot 1_{H \times H}$ are scalar multiples of one another. Thus the matrix is singular, $v = s(\bar{r} - s)$ is an eigen value of M_1, and $s = \tfrac{1}{2}\{\bar{r} - \sqrt{\bar{r}^2 - 4v}\}$ is a negative eigenvalue of A. That implies that $\xi^{\text{sp}} \equiv \max\{\xi_i$: ξ_i is a negative eigenvalue of $A\} \geqslant s$. On the other hand, $\xi^{\text{sp}} \leqslant \max_{1 \leqslant i \leqslant H} s^i$ by (16). Thus $\xi^{\text{sp}} = s$ if $s = \max_{1 \leqslant i \leqslant H} s^i$, as required by (17), and this regardless of the value of g_{xx}.

These observations can be used to prove the remark following Theorem 1 regarding *necessary* conditions for local stability. Argue as follows: If

$u_c^1 < 0$ and $u_c^2 < 0$, then $s^1 > 0$ and $s^2 > 0$. Suppose first that $s^1 = s^2 = s > 0$. Then, by the above argument, M_1 has the positive eigenvalue $s(\bar{r} - s)$. Therefore, A has at most $(H-1)$ negative eigenvalues and local stability fails. More generally suppose that $0 < s^1 \leqslant s^2$. It is possible to perturb this economy by increasing appropriately both u_c^1 and u_c^2 until, in the new economy, $\mathbf{u}_c^1 f_c^1/f_{cc}^1 = \mathbf{u}_c^2 f_c^2/f_{cc}^2$ and so $0 \leqslant \mathbf{s}^1 = \mathbf{s}^2 = \mathbf{s}$. (Boldfaced symbols indicate values for the new economy.) By above $\mathbf{M}_1$ has a positive eigenvalue. By the proof of Theorem 2, therefore, M_1 also has a positive eigenvalue and local stability fails in the original economy.

One suspects that Regularity (defined in Sect. 4) is generic in an appropriate sense. To give some precision to this statement proceed as follows: Represent an economy by the steady state properties captured in the vector (e, d) where: d from (23) specifies the distribution of the steady state capital stock and $e \in R^{2H+2}$ is a vector of the form

$$e = (\bar{r}, g_{xx}, (u_c^h)_1^H, (f_c^h/f_{cc}^h)_1^H).$$

The components of e represent steady state values of the indicated functions, and they are assumed to reflect the maintained assumptions on preferences and technology—$\bar{r} > 0$, $g_{xx} < 0$, $u_c^h > 0$ and $f_c^h/f_{cc}^h < 0$. $\mathscr{E}$ denotes the set of economies. It inherits the induced Euclidean topology.

THEOREM A.1. *The set of economies which satisfy Regularity is an open dense subset of $\mathscr{E}$. In fact, $\forall (e, d) \in \mathscr{E}$ for which the partial equilibrium adjustment speeds $s^1, \ldots, s^H$ are all distinct, and $\forall \delta > 0$, $\exists \hat{d} \in R^{H-1}$, $|\hat{d} - d| < \delta$, such that the economy $(e, \hat{d})$ satisfies Regularity.*

For the two-person model of Section 5, Regularity is satisfied if and only if either $s^1 = s^2$, or $s^1 \neq s^2$ and

$$\frac{1}{g_{xx}}\left[\frac{v_1 - v_2}{z^1 - z^2}\right] + \frac{(\xi_1 - \xi_2)\, d^2}{(\xi_1 - \bar{r})(\xi_2 - \bar{r})} \neq 0,$$

where $z^h \equiv s^h(\bar{r} - s^h)$ and $v_i \equiv \xi_i(\bar{r} - \xi_i)$, $h, i = 1, 2$, and ξ_i, $i = 1, 2$, are the negative eigenvalues of A. A sufficient condition for Regularity is

$$d^2(s^1 - s^2) \leqslant 0.$$

This is true for example if $d^2 \geqslant 0$ $(\bar{x}^2 \geqslant \bar{x}^1)$ and $s^1 < s^2$ (consumer 1 has a larger partial equilibrium adjustment speed and m.p.c.).

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