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Source: *Econometrica*, Vol. 59, No. 1 (Jan., 1991), pp. 139-163
Published by: Econometric Society
Stable URL: http://www.jstor.org/stable/2938244
Accessed: 20-12-2015 16:20 UTC

Econometrica, Vol. 59, No. 1 (January, 1991), 139–163

MIXTURE SYMMETRY AND QUADRATIC UTILITY[1]

By S. H. Chew, L. G. Epstein, and U. Segal

The independence axiom of expected utility theory has recently been weakened to the betweenness axiom. In this paper an even weaker axiom, called mixture symmetry, is presented. The corresponding functional structure is such that utility is a betweenness functional on part of its domain and quadratic in probabilities elsewhere. The experimental evidence against betweenness provides one motivation for the more general theory presented here. Another advantage of the mixture symmetric class of utility functions is that it is sufficiently flexible to permit the disentangling of attitudes towards risk and towards randomization.

Keywords: Uncertainty, betweenness, quadratic utility, mixture symmetry.

1. INTRODUCTION

Experimental studies have revealed widespread and systematic violations of expected utility theory and more particularly of its cornerstone, the independence axiom (Machina (1982)). Consequently, several recent studies have proposed weaker axioms which define theories of choice under uncertainty that are compatible with the experimental evidence. One such axiom, called betweenness, is the basis for the theories described in Chew (1983, 1989), Fishburn (1983), Nakamura (1983), and Dekel (1986). The betweenness axiom—if two probability distributions are indifferent, then any probability mixture of them is equally as good—implies that indifference curves in the probability simplex are straight lines in the three-outcome probability simplex (and more generally, are hyperplanes in higher dimensional simplices). But there also exists evidence which contradicts betweenness. Thus in this paper we present a weaker axiom, called mixture symmetry, which is both simple and tractable. Mixture symmetry permits indifference curves in the simplex to be nonlinear; moreover, the deviations from linearity which it admits accord well with the available empirical evidence. In addition, the corresponding utility functions have a convenient functional structure.

Consider two indifferent lotteries represented by their cumulative distribution functions F and G. Mixture symmetry requires that any probability mixture $\alpha F+(1-\alpha)G$, with $0<\alpha<\frac{1}{2}$, be indifferent to a mixture $\beta F+(1-\beta)G$ for some $\beta\in(\frac{1}{2},1)$ (though not necessarily indifferent to F and G). In the remainder of this introduction we explain the structure of the implied utility functions and elaborate on the motivation for the paper.

Since mixture symmetry is weaker than betweenness, the betweenness-conforming utility functionals described in the papers cited in the opening para-

[1] We are grateful to J. Aczèl for discussion of the functional equation aspects of the problem solved in the paper and to Mark Machina, the editor, and two referees for helpful suggestions. A referee pointed out an error in an earlier version. Chew and Epstein received financial support from the National Science Foundation and Segal wishes to acknowledge the financial support of the Social Sciences and Humanities Research Council of Canada.

graph satisfy mixture symmetry. The latter is also satisfied by all utility functionals that are "quadratic in probabilities." In fact, these two examples exhaust the utility functionals satisfying mixture symmetry (and some auxiliary hypotheses) in the sense that every such functional is betweenness-conforming on part of its domain and quadratic elsewhere. This representation, described in Theorems 4 and 6, is the major result of the paper. Note that the use of quadratic utility functionals as (part of) a generalization of expected utility is intuitive (apart from the appeal of the mixture symmetry axiom) when viewed from the perspective of standard calculus, since expected utility is linear in probabilities.

While the betweenness based theories can explain much of the experimental evidence against expected utility theory such as the Allais paradox, there exists some evidence, surveyed in Machina (1985, p. 579), contradicting betweenness. Given betweenness, utility is both quasiconcave and quasiconvex in the space of cumulative distribution functions. Assuming the axiom of reduction of compound lotteries, quasiconcavity (quasiconvexity) implies an affinity for (aversion to) randomization between indifferent lotteries. Thus evidence of randomized choice, which has been found by some studies, implies quasiconcavity and, except for knife-edge cases, rules out linearity. By an alternative approach, Becker et al. (1963) are able to reject linearity, even though their experimental design is only capable of detecting violations in the direction of quasiconcavity. Coombs and Huang (1976) find a significant proportion (45%) of violations of betweenness, with 59% of the violations being in the direction of strict quasiconcavity and 41% consistent with strict quasiconvexity. Chew and Waller (1986) embed a test of betweenness along with the standard Allais paradox and the common ratio effect within a single design. They also find some evidence against betweenness (see Section 5 below). Overall, the evidence shows significant violations of betweenness but does not provide justification for ruling out either quasiconcavity or quasiconvexity.[2] In Section 5 we describe some evidence, taken from the Chew and Waller study, which suggests a systematic nature to violations of betweenness.

Our objective is to develop an axiomatic theory of preference which can account for the prevalent empirical evidence against the independence axiom as well as the more limited evidence against betweenness. In addition, the axiomatic theory described below is sufficiently flexible so that either quasiconcavity or quasiconvexity can be accommodated, in conjunction with the usual hypothesis of risk aversion. Such flexibility is also appealing on theoretical grounds since attitudes towards risk and attitudes towards randomization between indifferent probability distributions represent two conceptually distinct aspects of preference. The ability to separate attitudes towards risk from attitudes towards randomization is particularly important in game theory where attitudes towards randomization are critical. Thus, for example, the quasiconcavity of utility is needed for the proof of existence of a Nash equilibrium, since

[2] For theoretical arguments in support of quasiconvexity, see Machina (1984) and Green (1987). We consider the latter in Section 5.

if preferences are strictly quasiconvex the agent will be averse to the randomization of strategies which an equilibrium may require. (Recently, however, Crawford (1990) has proposed an alternative notion of equilibrium which exists even if utility is strictly quasiconvex.)

Of course, one can also achieve a separation between risk aversion and attitudes towards randomization by positing a general preference functional that is Fréchet differentiable (Machina (1982)). We adopt the view, implicit in the literature on axiomatic generalizations of expected utility, that good theory development involves guarded departures from expected utility. Moreover, the existence of an axiomatic basis contributes to such a development, both in judging the extent of the deviation of the model from expected utility and in elucidating the empirical implications of the adopted utility specification, thereby facilitating the efficient explanation of empirical evidence.

There exists an alternative axiomatic generalization of expected utility theory, called rank-dependent or anticipated utility theory, which has been proposed in order to explain Allais-type behavior (see Quiggin (1982), Yaari (1987), Segal (1989), and Chew (1985)). However, in the latter paper it is shown that rank-dependent utility exhibits risk aversion if and only if it is quasiconvex. Thus strict quasiconcavity and risk aversion cannot be jointly accommodated within this theory. Generalizations of rank-dependent theory which retain a central role for the rank ordering of outcomes are described in Segal (1989), Green and Jullien (1988), and Chew and Epstein (1990). It is straightforward to show that these generalizations suffer from similar inflexibility.

The next section presents our axioms. Functional forms are described in Section 3 and representation theorems are presented in Section 4. Section 5 concludes with a comparison of the systematic violations of betweenness admitted by our framework and some empirical and theoretical evidence regarding the nature of such violations. Most proofs are relegated to appendices.

2. AXIOMS

We consider a complete and transitive preference ordering $\gtrsim$ on $D(X)$, the set of cumulative distribution functions (c.d.f.'s) on the compact set $X \subset R^1$. Endow $D(X)$ with the topology of weak convergence. The following axioms are imposed on $\gtrsim$:

CONTINUITY: *For each* $F \in D(X)$, $\{G \in D(X): G \gtrsim F\}$ *and* $\{G \in D(X): F \gtrsim G\}$ *are closed.*

MONOTONICITY: $\gtrsim$ *is increasing in the sense of first degree stochastic dominance.*[3]

By Debreu (1964), there is no loss of generality in assuming that $\gtrsim$ can be represented by a utility functional $V: D(X) \to R^1$.

[3] Throughout the paper, "increasing" is intended in the strict sense.

For completeness, we write the independence axiom.[4]

INDEPENDENCE: *For every* F, G *and* $H \in D(X)$ *and* $\alpha \in [0,1]$,

$$F \sim G \Rightarrow \alpha F + (1-\alpha)H \sim \alpha G + (1-\alpha)H.$$

The nature of independence can be understood by reference to the probability simplex in the case of lotteries having three possible outcomes. The axiom implies that indifference curves in the simplex are straight and parallel. The bulk of the empirical evidence discussed by Machina (1982) is inconsistent with parallelism. This evidence motivated the development of betweenness-conforming theories which have straight but nonparallel indifference curves in the simplex. The betweenness axiom is stated below.

BETWEENNESS: *For every* F *and* $G \in D(X)$ *and* $\alpha \in [0,1]$,

$$F \sim G \Rightarrow \alpha F + (1-\alpha)G \sim F.$$

Given continuity and monotonicity (see footnote 4), betweenness is the conjunction of quasiconcavity and quasiconvexity, which we state below.

QUASICONCAVITY: *For each* F *and* $G \in D(X)$ *and* $\alpha \in (0,1)$,

$$F \sim G \Rightarrow \alpha F + (1-\alpha)G \gtrsim F.$$

QUASICONVEXITY: *For each* F *and* $G \in D(X)$ *and* $\alpha \in (0,1)$,

$$F \sim G \Rightarrow F \gtrsim \alpha F + (1-\alpha)G.$$

The strict forms of these axioms are defined in the obvious way.

It will be useful to define a notion which is intermediate between quasiconcavity and strict quasiconcavity. First, say that an indifference set $I(F) = \{G \in D(X)\colon G \sim F\}$ is planar if it is convex and is not equal to the singleton $\{F\}$. Say that $\gtrsim$ satisfies proper quasiconcavity if it is quasiconcave and if it contains no planar indifference sets. Proper quasiconvexity is defined similarly.

In the introduction we cited evidence contradicting betweenness. Thus we propose a further weakening of independence, which is compatible with either quasiconcavity or quasiconvexity but does not imply either. Our central axiom is as follows:

MIXTURE SYMMETRY: *For every* F *and* G *in* $D(X)$, $F \sim G \Rightarrow \forall \alpha \in (0, \frac{1}{2})$ $\exists \beta \in (\frac{1}{2}, 1)$ *such that* $\alpha F + (1-\alpha)G \sim \beta F + (1-\beta)G$.

[4] Given continuity and monotonicity, this form of independence is equivalent to the more common formulation involving weak preference rather than indifference. Similarly, the betweenness axiom below is equivalent, given continuity and monotonicity, to the form of betweenness that appears in Chew (1983) and Dekel (1986). They assume that the "better than" and "worse than" sets are both convex in the mixture sense.

The axiom requires that given $F \sim G$, any probability mixture which places strictly less weight on F, be indifferent to some other mixture in which F receives more weight than G. Clearly, mixture symmetry is implied by betweenness since given the latter all probability mixtures are indifferent to F; however, the converse is false as demonstrated amply below.

Our objective is to formulate a positive theory of preference which can be consistent with empirical evidence, but the normative case for mixture symmetry is still worth noting. Since the latter is weaker than the independence axiom, the well known normative argument for the latter applies also to mixture symmetry. Mixture symmetry might, however, be acceptable even if independence and betweenness are rejected. To see this, interpret probability mixtures in the usual way (e.g., Raiffa (1970, p. 82)) as two-stage lotteries. The additional uncertainty introduced through the first stage experiment may render the probability mixture $\alpha F + (1-\alpha)G$ sufficiently distinct from the component single-stage lotteries F and G that, contrary to the prescriptions of independence and betweenness, $\alpha F + (1-\alpha)G$ may not be indifferent to F. (Such a distinction between two-stage and single-stage lotteries is emphasized by Segal (1990).) On the other hand, if α and β are both in the open interval $(0,1)$, then $\alpha F + (1-\alpha)G$ and $\beta F + (1-\beta)G$ both involve some uncertainty at the first stage and they could plausibly be indifferent for suitable choices of α and β. For example, it might be the case that $\beta = 1-\alpha$ satisfies the requirement of mixture symmetry. As a concrete illustration of this latter case, consider an experiment in which a ball is drawn from an urn containing red and blue balls in proportions α and $(1-\alpha)$. The color drawn determines whether F or G is received in the second stage. One might be indifferent as to whether red leads to F or to G, while at the same time not being indifferent between F and this two-stage lottery.

The above discussion draws attention to a stronger form of mixture symmetry in which the requisite indifference is necessarily satisfied by the choice $\beta = 1-\alpha$, as illustrated in Figure 1. We make the following formal definition:

Strong Mixture Symmetry: *For every F and $G \in D(X)$ and $\alpha \in [0,1]$,*

$$F \sim G \Rightarrow \alpha F + (1-\alpha)G \sim (1-\alpha)F + \alpha G.$$

In fact, given continuity and monotonicity, strong mixture symmetry is equivalent to mixture symmetry, as described in the following theorem (proven in Appendix 1).

Theorem 1: *Let $\succsim$ satisfy continuity and monotonicity on $D(X)$. Then $\succsim$ satisfies mixture symmetry if and only if it satisfies strong mixture symmetry.*

3. FUNCTIONAL FORMS

Now turn to functional forms for utility functionals. First we will be interested in functionals W which satisfy betweenness. As shown by Chew (1989) and

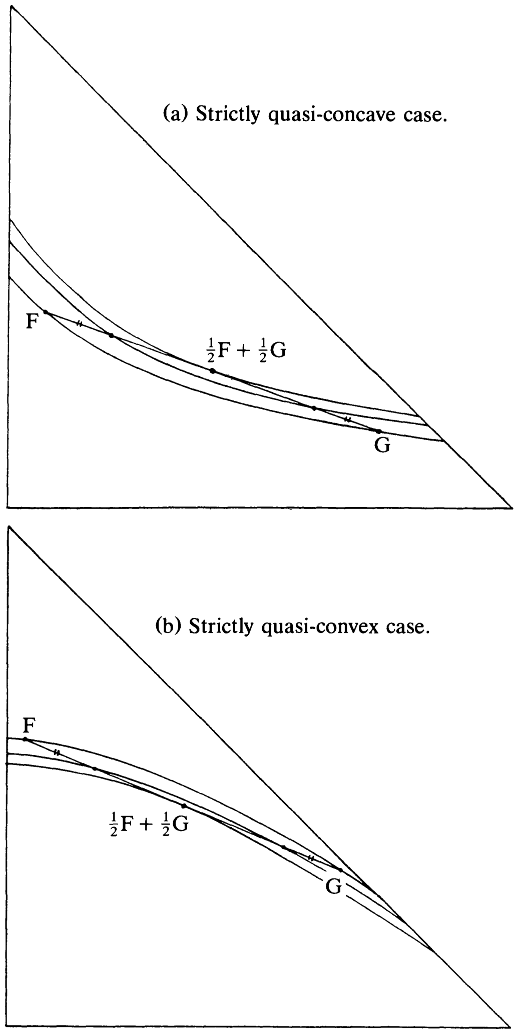

FIGURE 1.

Dekel (1986), they are defined implicitly by an equation of the form

$$(1) \qquad \int h(x, W(F))\, dF(x) = 0$$

for some function h with suitable properties. If $h(x, s)$ is of the form $h(x, s) = w(x)[v(x) - s]$, then W is a weighted utility function (Chew (1983)) which reduces to expected utility if w is constant. If h is continuous, if $h(x, \cdot)$ is decreasing, and if $h(\cdot, z)$ is increasing and concave for all z, then W satisfies continuity, monotonicity, and risk aversion in the sense of aversion to mean preserving spreads. For details, the reader is referred to the cited papers.

Of primary interest here is the class of quadratic functions. A utility functional V is said to be quadratic in probabilities if it can be expressed in the form

$$(2) \qquad V(F) = \iint \phi(x, y)\, dF(x)\, dF(y), \qquad F \in D(X),$$

for some symmetric function ϕ: $X \times X \to R^1$. There is no loss of generality in restricting ϕ to be symmetric, since an arbitrary $\phi(x, y)$ can always be replaced by $(\phi(x, y) + \phi(y, x))/2$. For c.d.f.'s F having finite support $\{x_1, \ldots, x_n\}$ and corresponding probabilities $p_1, \ldots, p_n$, $V(F)$ takes the form

$$(3) \qquad V(F) = \sum_{i=1}^{n} \sum_{j=1}^{n} \phi(x_i, x_j) p_i p_j.$$

The general quadratic functional form in (2) appears in a footnote in Machina (1982, p. 295). In the text of his paper, Machina discusses the special case corresponding to

$$\phi(x, y) = v(x)v(y) + (w(x) + w(y))/2,$$

which leads to

$$(4) \qquad V(F) = \left(\int v(x)\, dF(x)\right)^2 + \int w(x)\, dF(x).$$

A similar example has $\phi(x, y) = [v(x)w(y) + v(y)w(x)]/2$, and

$$(5) \qquad V(F)\left(\int v(x)\, dF(x)\right)\left(\int w(x)\, dF(x)\right),$$

i.e., the product of two expected utility functionals.

Two additional examples will clarify the scope of the structure in (2). First, if $\phi(x, y) = (v(x) + v(y))/2$, then

$$(6) \qquad V(F) = \int v(x)\, dF(x).$$

Thus, expected utility (linearity in probabilities) is a special case of (2). Finally, let

$$(7) \qquad \phi(x, y) = \max(v(x), v(y)).$$

Then[5]

$$(8) \qquad V(F) = \int v(x)\, d[F^2(x)],$$

[5] To establish (8), verify it first for c.d.f.'s which have finite support and in which all outcomes are equally likely. If the possible outcomes are $x_1 < x_2 < \cdots < x_n$, then

$$\iint \phi\, dF\, dF = \sum \frac{(2i-1)}{n^2} v(x_i) = \int v(x)\, dF^2(x).$$

The above class of c.d.f.'s is dense in $D(X)$ and thus (8) may be extended to all of $D(X)$. Similarly if $\phi(x, y) = \min(v(x), v(y))$, then $\iint \phi\, dF\, dF = \int v(x)\, d[1 - (1 - F(x))^2]$, which is the special case of rank dependent utility theory for which $g(p) = 1 - (1 - p)^2$.

which is the special case of rank dependent utility theory ($V(F = \int v(x)\, d[g(F(x))]$, where g: $[0,1] \to [0,1]$ is increasing and onto) in which the probability transformation function g is quadratic, $g(p) = p^2$.

In the remainder of this section we explore some properties of the quadratic functional form. First it is natural to wonder about the uniqueness class of ϕ. One might conjecture that ϕ^i, $i = a, b$, represent the same preference ordering if and only if they are related by a positive linear transformation. But that is readily disproven since $\phi^a(x, y) \equiv v(x)v(y)$ and $\phi^b(x, y) \equiv (v(x) + v(y))/2$ define the same expected utility ordering if $v > 0$. The conjecture is true, however, if the expected utility, or linear in probabilities, case is excluded. It is convenient, therefore, to introduce the following terminology: say that a quadratic function V on $D(X)$ is proper if it is not ordinally equivalent to an expected utility function. The uniqueness class of ϕ can now be described as follows.

THEOREM 2: *Let V^a and V^b be quadratic functionals of the form in* (2) *and corresponding to ϕ^a and ϕ^b respectively. Then V^a and V^b are ordinally equivalent if and only if either of the following conditions is satisfied*:

(i) *V^a and V^b are ordinally equivalent to expected utility functions in which case $\exists u$: $X \to R$ and constants A^i, B^i, and C^i such that*

$$\phi^i(x, y) = A^i u(x)u(y) + B^i(u(x) + u(y)) + C^i, \qquad i = a, b, \tag{9}$$

where the functions $A^i x^2 + 2B^i x + C^i$, $i = a, b$, are increasing on the range of u; or

(ii) *V^a and V^b are proper quadratic in which case $\exists \alpha$ and β, $\beta > 0$, such that*

$$\phi^a(x, y) = \alpha + \beta\phi^b(x, y) \qquad \forall (x, y) \in X^2.$$

PROOF: (ii) Assume ordinal equivalence of V^a and V^b. Evaluate the utility of the gamble with outcomes $x < y < z$ and the corresponding probabilities $1 - p - q$, p, and q. If ϕ^i is used, the utility is

$$V^i = \phi^i(x, x)(1 - p - q)^2 + \phi^i(y, y)p^2 + \phi^i(z, z)q^2 + 2\phi^i(y, z)pq + 2\phi^i(x, y)p(1 - p - q) + 2\phi^i(x, z)q(1 - p - q).$$

Fix x, y, and z and view V^i as a (quadratic) function of p and q. By hypothesis, V^a and V^b are ordinally equivalent. The desired conclusion now follows from Lemma A3.1 in the Appendix which shows that V^a and V^b must be cardinally equivalent. The converse is trivial.

(i) The sufficiency of (9) is clear, since it implies that

$$V^i(F) = h^i\left(\int u(z)\, dF(z)\right), \qquad \text{where} \qquad h^i(x) = A^i x^2 + 2B^i x + C^i.$$

To prove its necessity we may assume that for each i,

$$\iint \phi^i(x, y)\, dF(x)\, dF(y) = h^i\left(\int u(x)\, dF(x)\right),$$

for some u and increasing h^i. When F is the c.d.f. for the binary gamble $(x, p; y, 1-p)$, we obtain

$$(10)\qquad p^2\phi^i(x,x) + (1-p)^2\phi^i(y,y) + 2p(1-p)\phi^i(x,y) = h^i(pu(x) + (1-p)u(y)).$$

Then h^i must be quadratic as in the statement of the proposition.

For the degenerate gamble which yields x with certainty, (10) implies $\phi^i(x,x) = h^i(u(x))$. Substitution into (10) yields (9). *Q.E.D.*

Frequently we wish V (or the underlying preference order) to satisfy properties such as continuity, monotonicity, and risk aversion, the latter in the sense of aversion to mean preserving spreads. The restrictions on ϕ corresponding to these properties for V are described below.

THEOREM 3: *Let $\gtrsim$ be represented by the function V in* (2). *Then $\gtrsim$ is continuous if and only if ϕ is jointly continuous on X^2. Moreover, given continuity, the ordering is* (*i*) *monotonic if and only if* (*i'*) *$\phi(\cdot, y)$ is nondecreasing and $\phi(x,x) > \phi(y,y)$ whenever $y < x, (x,y) \in X^2$; and the ordering is* (*ii*) *risk averse if and only if* (*ii'*) *$\phi(\cdot, y)$ is concave $\forall y \in X$.*

These conditions on ϕ are readily imposed in the context of the above examples, with the single exception that (7) cannot satisfy the concavity requirement. Thus the rank dependent functional (8) is not always averse to mean preserving spreads, an observation which is consistent with Chew, Karni, and Safra (1987). Note that V defined in (4) is both quasiconvex (indeed convex) and risk averse if $v > 0$ and v and w are both concave. Also, the functional defined in (5) is both quasiconcave (since $\log V(\cdot)$ is concave) and risk averse if both v and w are concave and positive. This substantiates the claim in the introduction regarding the flexibility of the theory developed here with respect to the separation of risk aversion and attitudes towards randomization.[6]

The proof of Theorem 3 is facilitated by consideration of the differentiability properties of quadratic functionals. Since these properties are also of independent interest we examine them briefly. First, it can be shown as in Chew, Karni, and Safra (1987) that the rank dependent functional (8) is not Fréchet differentiable. Thus the quadratic is generally not differentiable in that sense.[7] But that paper also shows that much of the machinery developed by Machina (1982) under the assumption of Fréchet differentiability can be adapted to the more general framework of Gateaux differentiability. In particular, one can define local utility functions which play a similar role to that in Machina's work.

[6] A sufficient condition on ϕ for the quasiconcavity or quasiconvexity of utility can be derived as follows: For c.d.f.'s with finite support, $V(F)$ is a quadratic form in the probabilities with coefficient matrix $(\phi(x_i, x_j))_{i,j}$. Thus V is quasiconcave on $D(X)$ if $(\phi(x_i, x_j))_{i,j}$ is negative semi-definite for all $x_1, \ldots, x_n$ and n. But this property is not readily verified given a specification for ϕ.

[7] The existence and continuity of the cross partial derivative $\phi_{12}(x, y)$ is sufficient for V to be Fréchet differentiable. The sufficiency of more general conditions is established in Appendix 5.

Moreover, quadratic functionals are always Gateaux differentiable since

$$(11)\qquad \frac{d}{dt}V((1-t)F+tG)\bigg|_{t=0^+} = \int u(x;F)\,d(G(x)-F(x)), \qquad \text{where}$$

$$u(x;F) = 2\int \phi(x,y)\,dF(y).$$

The function $u(\cdot;F)$ is the local utility function at F.

By the above noted extension of Machina's analysis, risk aversion of V is equivalent to the concavity of $u(\cdot;F)$ $\forall F$. Take F to be the degenerate distribution concentrated at y and conclude that the concavity of $\phi(\cdot,y)$ $\forall y \in X$ is necessary for risk aversion. Since it is clearly sufficient for the concavity of $u(\cdot;F)$ and hence also for risk aversion, we have proven that (ii) $\Leftrightarrow$ (ii′) in Theorem 3. The remainder of the proof is provided below.

PROOF OF THEOREM 3: Assume V is continuous and let $(x_n, y_n) \to (x, y)$. Denote by δ_z the degenerate c.d.f. which assigns unit mass to z. Then

$$\delta_{x_n} \to \delta_x \Rightarrow V(\delta_{x_n}) \to V(\delta_x) \Rightarrow \phi(x_n, x_n) \to \phi(x, x).$$

Similarly, $\phi(y_n, y_n)$ converges to $\phi(y,y)$. Also, $\forall p \in (0,1)$,

$$\begin{aligned} &V(p\delta_{x_n} + (1-p)\delta_{y_n}) \\ &\quad \to V(p\delta_x + (1-p)\delta_y) \\ &\quad \Rightarrow \{p^2\phi(x_n,x_n) + 2p(1-p)\phi(x_n,y_n) + (1-p)^2\phi(y_n,y_n)\} \\ &\quad \to \{p^2\phi(x,x) + 2p(1-p)\phi(x,y) + (1-p)^2\phi(y,y)\}, \end{aligned}$$

which implies $\phi(x_n, y_n) \to \phi(x,y)$. Thus ϕ is continuous. The converse is immediate since ϕ is continuous and bounded on $X \times X$.

(i) $\Leftrightarrow$ (i′): The monotonicity of $\gtrsim$ implies that $u(\cdot;F)$ is nondecreasing and $\phi(x,x) = V(\delta_x) > V(\delta_y) = \phi(y,y)$ if $x > y$. (Recall that the monotonicity axiom is strict.) For the converse, assume (i′). It is enough to show that $V(\Sigma_{i=1}^n p_i \delta_{x_i})$ is increasing in each x_j for which $p_j > 0$, which is true since

$$V\left(\sum_1^n p_i \delta_{x_i}\right) = \sum_{i \neq j}\sum_{k \neq j} \phi(x_i, x_k) p_i p_k + p_j^2 \phi(x_j, x_j) + 2\sum_{i \neq j} \phi(x_i, x_j) p_i p_j.$$

The equivalence of (ii) and (ii′) was proven above. *Q.E.D.*

Finally, note that the proper quadratic functional form can "explain" the behavioral evidence against expected utility theory. In particular it can satisfy Machina's Hypothesis II (Machina (1982, pp. 310–311)) and thus can explain the Allais paradox, the common consequence effect, the common ratio effect, and other behavioral evidence. Of course, betweenness functionals can also resolve these behavioral paradoxes (Chew (1983)). Moreover, it is readily shown that proper quadratic functionals and betweenness functionals define disjoint classes.

4. REPRESENTATION THEOREMS

In this section the preceding axioms and functional forms are related. In particular, the consequences of mixture symmetry are derived.

It is readily verified that the quadratic functional form satisfies mixture symmetry. Indeed, it satisfies strong mixture symmetry since

$$(12) \qquad V(\alpha F+(1-\alpha)G)-V((1-\alpha)F+\alpha G) \\ =[\alpha-(1-\alpha)][V(F)-V(G)].$$

On the other hand, since betweenness-conforming functionals satisfy mixture symmetry but are not necessarily quadratic, it is clear that mixture symmetry does not characterize quadratic functionals. However, it does (in conjunction with some of the other axioms) imply that the utility functional is quadratic on part of its domain and betweenness-conforming on the complementary region. This characterization is described precisely in the theorems to follow.

THEOREM 4: *Let the preference ordering $\succsim$ on $D(X)$ satisfy continuity, monotonicity, and quasiconcavity (quasiconvexity). Then $\succsim$ satisfies mixture symmetry if and only if it can be represented numerically by a utility function V which has the following form: there exists $F_0 \in D(X)$ such that*

$$(13) \qquad V(F)=\begin{cases} Q(F), & \forall F \succsim (\precsim) F_0, \\ W(F), & \forall F \precsim (\succsim) F_0, \end{cases}$$

where W satisfies betweenness and where Q is a proper quadratic function.

The betweenness functional W has the structure described in (1) and Q is a proper quadratic specialization of (2). Thus (13) provides a complete description of the functional structure of V, which structure is illustrated in the 3-outcome probability simplex in Figure 2. In that figure, region III (I) is void if the ordering is quasiconvex (quasiconcave). It is evident that any V which combines a quadratic and betweenness functional as in (13) satisfies mixture symmetry. The necessity of (13) is nontrivial, however, and is proven in Appendices 2–4. Appendix 2 treats the case of c.d.f.'s corresponding to three outcome gambles. This proof is accomplished by establishing a link between mixture symmetry and a characteristic property of conics called the projection property. The extension to finite outcome gambles is considered in Appendix 3 and the proof is completed in Appendix 4.

Consider the consequence for Theorem 4 of strengthening quasiconcavity or quasiconvexity to their proper forms (see Section 2). Clearly, the betweenness region is thereby eliminated and a globally quadratic utility function is implied. Thus we immediately obtain the following result.

THEOREM 5: *Let $\succsim$ satisfy continuity, monotonicity, and proper quasiconcavity (quasiconvexity). Then $\succsim$ satisfies mixture symmetry if and only if it can be represented numerically by a proper quadratic utility function of the form* (2).

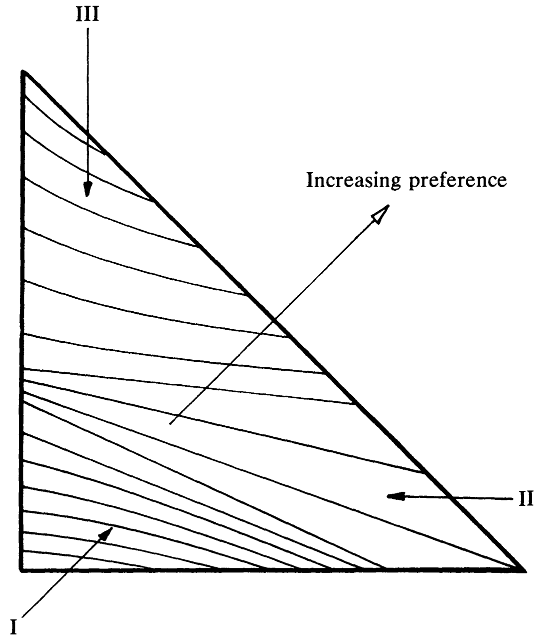

FIGURE 2.

The consequences for functional structure if the quasiconcavity and quasiconvexity axioms are deleted are unspecified above. In the 3-outcome probability simplex, it follows from Appendix 2 that V must be defined by three regions as in Figure 2: it is proper quadratic and quasiconcave (quasiconvex) in the upper (lower) region and betweenness-conforming in the intermediate region. But more complicated structures are possible on higher dimensional simplices. For example, consider the quadratic utility function given by

$$V(F) = \left(\int u(x)\, dF(x)\right)^2 + \left(\int v(x)\, dF(x)\right)\left(\int w(x)\, dF(x)\right),$$

where u, v, and w are continuous, positive, increasing, and concave functions on X. On the hyperplane $\{F\colon \int u(x)\, dF(x) = K\}$, V resembles the example in (5) and can be shown to be quasiconcave. On the other hand, on the hyperplane $\{F\colon \int w(x)\, dF(x) = K'\}$, V resembles the example in (4) and can be shown to be quasiconvex. It follows that V is neither quasiconcave nor quasiconvex on any open convex subset of $D(X)$. Indeed, even in the 4-outcome probability simplex there exist regions where V is neither quasiconcave nor quasiconvex.

We can establish the general structure portrayed in Figure 2, i.e., where both regions I and III may be nonempty, under the following circumstances. Recall that an indifference set $I(F) = \{G \in D(X)\colon G \sim F\}$ is planar if it is convex and is not equal to the singleton $\{F\}$. Two possibilities exist—either $\exists F^0 \in D(X)$ such that the corresponding indifference set $I(F^0)$ is planar, or there does not. The latter case is illustrated by the example given above. In the former case, the desired structure may be established in the following way.

THEOREM 6: *Let $\gtrsim$ satisfy continuity and monotonicity on $D(X)$. Suppose further that $\exists F^0 \in D(X)$ such that the indifference set $I(F^0)$ is planar. Then $\gtrsim$ satisfies mixture symmetry if and only if it can be represented by a utility function V which has the following form*:

$$V(F) = \begin{cases} Q_2(F), & F^2 \lesssim F, \\ W(F), & F^1 \lesssim F \lesssim F^2, \\ Q_1(F), & F \lesssim F^1, \end{cases} \tag{14}$$

for some F^1 and $F^2 \in D(X)$, $F^1 \lesssim F^0 \lesssim F^2$, where W satisfies betweenness and where Q_1 and Q_2 are proper quadratic functionals.

PROOF: Since $I(F^0)$ is not a singleton by assumption, F^0 cannot be the worst or best element in $D(X)$. Let $x_{\min} = \min\{x : x \in X\}$. Choose $F \sim G > F^0$ and $H_1 = \delta_{x_{\min}}$, H_2, and H_3 in $D(X)$ such that (i) F and G lie in the interior of the simplex $\Delta(H_1, H_2, H_3)$ consisting of all probability mixtures of H_1, H_2, and H_3; and (ii) the portion $\mathscr{T}$ of the indifference curve containing F and G which lies between them is connected in the interior of $\Delta(H_1, H_2, H_3)$. The ranking of $(1-p-q)\delta_{x_{\min}} + pH_2 + qH_3$ is increasing in p and q. Therefore, Appendix 2 and the structure portrayed in Figure 1 apply to a neighborhood of $\mathscr{T}$ in $\Delta(H_1, H_2, H_3)$. It follows, using Lemma A1.2, that $\gtrsim$ is quasiconcave on that part of $D(X)$ which lies above $I(F^0)$, i.e., on $\{F \in D(X)\colon F \gtrsim F^0\}$. Similarly, it must be quasiconvex on the region below F^0, $\{F \in D(X)\colon F \lesssim F^0\}$. Moreover, if there are two distinct planar indifference sets in $D(X)$, then all indifference sets between them must be planar.

Let $B = \cup\{I(F)\colon F \in D(X), I(F) \text{ is planar}\}$. Let $F^1(F^2)$ be a worst (best) element in the closure of B. Clearly, $I(F^1)$ and $I(F^2)$ are planar. Moreover, the ordering $\gtrsim$ satisfies proper quasiconcavity on the region strictly above F^2 and thus Theorem 5, restricted to this subdomain of $D(X)$, may be applied to yield the desired quadratic utility representation there. Similarly for the region strictly below F^1. The desired utility representation on the betweenness region follows from Chew (1989) and Dekel (1986). *Q.E.D.*

In Section 3 we provided examples of utility functionals satisfying our axioms. To conclude this section we describe an example of a continuous and monotonic functional which is betweenness-conforming on part of its domain and quadratic on the remainder. Thus the "schizophrenic" functional structure (13) or (14) cannot be improved upon given the axioms in Theorems 4 and 6.

EXAMPLE: Let $X=[0,1]$,

$$\phi(x,y) \equiv 4xy - 2x^2y - 2xy^2 + x^2 + y^2 - \tfrac{1}{2} \qquad \text{and}$$

$$V(F) = \begin{cases} \iint \phi(x,y)\,dF(x)\,dF(y), & \mu(F) \geqslant \tfrac{1}{2}, \\ \mu(F), & \mu(F) \leqslant \tfrac{1}{2}, \end{cases}$$

where $\mu(F)$ denotes the mean of F. Then ϕ is increasing and continuous. Moreover, V is well-defined and continuous, since

$$\begin{aligned} \iint \phi(x,y)\,dF(x)\,dF(y) &= 4\mu^2(F) - 2\mu(F)\int x^2\,dF(x) \\ &\quad - 2\mu(F)\int y^2\,dF(y) \\ &\quad + \int x^2\,dF(x) + \int y^2\,dF(y) - \tfrac{1}{2} \\ &= \tfrac{1}{2} \qquad \text{if} \qquad \mu(F) = \tfrac{1}{2}. \end{aligned}$$

Finally, V clearly satisfies betweenness on the region where $\mu(F) \leqslant \frac{1}{2}$, but not elsewhere. For example, $V(F) = V(G) = 9/8 \neq V(\frac{1}{2}F + \frac{1}{2}G)$, where F corresponds to the gamble with equiprobable payoffs $\frac{1}{2}$ and 1 while G yields payoffs 0 and 1 with probabilities $3/16$ and $13/16$ respectively.

5. DISCUSSION

In the introduction we cited some evidence against betweenness. Here we describe some preliminary evidence, taken from Chew and Waller (1986), regarding the nature of observed violations of betweenness. Then we comment upon our representation results in light of this evidence. We also relate our discussion to Green's (1987) theoretical argument against quasiconcavity.

Chew and Waller investigated the nature of indifference curves in the three-outcome probability simplex. The outcome parameters used, as well as the resulting frequencies of nonbetweenness observations, are described in Table I. While the overall frequencies of betweenness violations (32% for Experiment 1 and 22% for Experiment 2) are not significantly greater than the chance hit rate of $1/2$, it is noteworthy that the nature of the betweenness violations displays a systematic dependence on the outcome parameters. Quasiconcave (quasiconvex) behavior is most pronounced when the outcomes are all positive (negative).

Such behavior accords well with the result in Theorem 6, whereby quasiconcavity prevails in the "upper" part of the domain and quasiconvexity in the lower "portion." In fact, we can apply Theorem 6 to each of the outcome sets $X = \{-100, -40, 0, 40, 60, 100\}$ and $X = \{-5{,}000, 0, 10{,}000, 15{,}000, 20{,}000, 27{,}000, 30{,}000\}$, and deduce that preference is more likely to be quasiconcave for the more attractive comparisons (e.g., contexts 1a, 2a, 2b) and quasiconvex for the less attractive comparisons (e.g., contexts 1c, 2c).

TABLE I[a]

		Experiment 1 a	Experiment 1 b	Experiment 1 c	Experiment 2 a	Experiment 2 b	Experiment 2 c
	x_1	0	−40	−100	10,000	0	−5,000
	x_2	40	0	−40	27,000	10,000	0
Outcomes	x_3	100	60	0	30,000	20,000	15,000
% quasiconvexity patterns		0	11	21	4	3	11
% quasiconcavity patterns		32	21	11	19	22	5
% nonbetweenness patterns		32	32	32	23	25	16

[a]Total number of observations = 56.

Green (1987) argues in support of quasiconvexity by demonstrating that a quasiconcave agent may be manipulated into making a series of choices that leave him stochastically dominated by his initial position. One possible "explanation" of quasiconcavity is that it reflects the "utility of gambling," or the "consumption" benefits of randomization. Viewed in this light, Theorem 6 is appealing. It suggests that a preference for randomization can be satiated in a losing streak. In particular the potential losses that can be traced to quasiconcavity are bounded and represent a "price" paid for the enjoyment of gambling.

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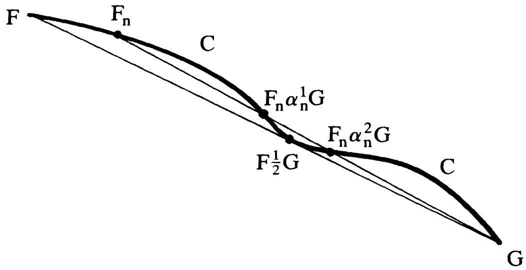
Manuscript received August, 1988; final revision received November, 1989.

APPENDIX 1: PROOF OF THEOREM 1

The proof is accomplished via a sequence of lemmas. Continuity, monotonicity, and mixture symmetry are assumed throughout. Also, $F\alpha G$ denotes $\alpha F + (1-\alpha)G$. The set of mixtures of F_1, F_2, and F_3 is denoted $\Delta(F_1, F_2, F_3)$, or simply Δ with the F_i's suppressed. The set of mixtures of F_1 and F_2 is denoted $[F_1, F_2]$.

LEMMA A1.1: *Let $F \sim G$ where F and G lie in the interior of $D(X)$. If there exists $\alpha \in (0,1)$ such that $F \sim F\alpha G$, then for every $\alpha \in (0,1)$, $F \sim F\alpha G$.*

PROOF: There exist F' and G' in $D(X)$ and $x_{\min} \equiv \min\{x : x \in X\}$ such that F and G lie in the interior of $\Delta(\delta_{x_{\min}}, F', G')$. Moreover, the preference relation is monotonic on this set in the sense that the ranking of $pF' + qG' + (1-p-q)\delta_{x_{\min}}$ is increasing in p and q. This monotonicity, which is weaker than consistency with first degree stochastic dominance, is used below.



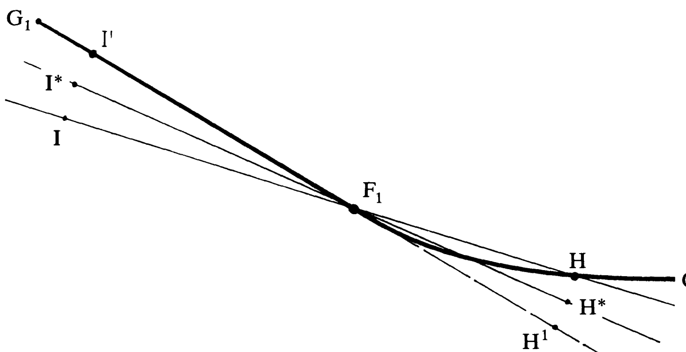


FIGURE 3a.

If the set $\{\alpha\colon F\alpha G \sim F\}$ is dense in $[0,1]$, the lemma follows by continuity. Otherwise, there is a segment $[\beta_1, \beta_2]$ such that for $\alpha \in (\beta_1, \beta_2)$, $F\alpha G \nsim F$, but for $\alpha \in \{\beta_1, \beta_2\}$, $F\alpha G \sim F$. By hypothesis, it is impossible that $\beta_1 = 0$ and $\beta_2 = 1$. Suppose $\beta_2 < 1$. (The argument for $\beta_1 > 0$ is similar.) For $\beta_3 > \beta_2$ such that $\beta_3 - \beta_2 < \beta_2 - \beta_1$ it follows that $F\beta_3 G \nsim F$, otherwise $F\beta_2 G$, $F\beta_1 G$ and $F\beta_3 G$ imply a violation of mixture symmetry. Since $F1G = F$, it follows that $2\beta_2 - \beta_1 \leqslant 1$. Now let $\beta_3 \equiv \min\{\beta \in [2\beta_2 - \beta_1, 1]\colon F\beta G \sim F\}$. By continuity, the minimum exists and $F\beta_3 G \sim F$. If $\beta_3 - \beta_2 > \beta_2 - \beta_1$, then $F\beta_2 G$, $F\beta_1 G$ and $F\beta_3 G$ violate mixture symmetry. Thus $\beta_3 - \beta_2 = \beta_2 - \beta_1$.

By repeating the above arguments we obtain scalars $0 = \gamma_0 < \cdots < \gamma_i < \gamma_{i+1} < \cdots < \gamma_{n+1} = 1$ such that $n \geqslant 1$ and for $i = 1, \dots, n$,

$$F\gamma_i G \sim F, \qquad \gamma_i = (\gamma_{i+1} + \gamma_{i-1})/2,$$

and $F\alpha G \nsim F$, $\forall \alpha \in \bigcup_0^n (\gamma_i, \gamma_{i+1})$. Refer to $\{F\gamma_i G\colon i = 0, \dots, n+1\}$ as a uniform partition of $[F, G]$ with partition length $\gamma_{i+1} - \gamma_i$. We show that such a partition is impossible. It suffices to consider the case $n = 1$ above and to rule out the following: $F \sim G$, $F\frac{1}{2}G \sim F$, and for every other $\alpha \in (0,1)$, $F\alpha G \nsim F$. By mixture symmetry this implies

(i) $\qquad F\alpha G \gtrsim F, \qquad \forall \alpha \in (0,1)$, or

(ii) $\qquad F\alpha G \lesssim F, \qquad \forall \alpha \in (0,1)$,

with strict preference in either case when $\alpha \notin \{0, 1/2, 1\}$.

Suppose (ii) applies. (The argument for (i) is similar.) Let C be the indifference curve through F and G. Let $F_n \to F$, $F_n \sim F$, and $\forall n F_n$ is between F and G (see Figure 3a). Since $F > F\alpha G$, for sufficiently large n the chord joining F_n and G intersects C at least once between F_n and $F\frac{1}{2}G$ and at least once between $F\frac{1}{2}G$ and G. Thus $\exists \alpha_n^1 > \alpha_n^2$ such that $F_n\alpha_n^1 G \sim F_n\alpha_n^2 G \sim F$, $\alpha_n^1 \to 1/2$ and $\alpha_n^2 \to 1/2$. By the above discussion applied to $[F_n, G]$ rather than $[F, G]$, we deduce the existence of a uniform partition of $[F_n, G]$ of length no greater than $\alpha_n^1 - \alpha_n^2$, which approaches 0 as $n \to \infty$. By continuity, therefore, $F\alpha G \sim F$, $\forall \alpha \in (0,1)$, which yields the desired contradiction. *Q.E.D.*

CONCLUSION 1: *If $F \sim G$, then $\gtrsim$ is either quasiconcave or quasiconvex on $[F, G]$.*

Say that $[F_1, F_2]$ is an isopreference if all of its elements are indifferent to one another and if $F_1 \neq F_2$.

LEMMA A1.2: *Let F and $G \in \mathrm{int}(D(X))$.*
(a) If $[F_1, G_1] \subset [F, G]$ is an isopreference, then so is $[F, G]$.
(b) If $F \sim G$, then $F\frac{1}{2}G$ is a best or worst element in $[F, G]$. Moreover, if $F\alpha^* G \sim F\frac{1}{2}G$ for some $\alpha^* \neq \frac{1}{2}$, then $[F, G]$ is an isopreference.

PROOF: As in the proof of the previous lemma $\exists \Delta < D(X)$, containing F and G, where the ordering is monotonic.

(a) Refer to Figure 3b, where it is assumed that the indifference curve C containing $[F_1, G_1]$ is not linear. There is no loss of generality in supposing that $[F_1, G_1]$ is a maximal isopreference in $[F, G]$. Suppose also that the quasiconcave case applies. (The other case may be treated similarly.) Choose H, I, H' and I' such that $F_1 = H\alpha I = H'\alpha I'$ with $\alpha > \frac{1}{2}$, $H > I$, and $I' > H'$. By continuity, there exists H^* and I^* such that $F_1 = H^*\alpha I^*$, $H^* \sim I^*$, and for $\beta < \alpha$, $F_1 > H^*\beta I^*$, a violation of mixture symmetry.

(b) Suppose that $F\frac{1}{2}G \gtrsim F$. Let $F\alpha^* G > F\frac{1}{2}G$ for some $\alpha^* \neq \frac{1}{2}$, say $\alpha^* < \frac{1}{2}$. By mixture symmetry $\exists \beta^* > \frac{1}{2}$ for which $F\alpha^* G \sim F\beta^* G$. By continuity $\exists \alpha_0 \in (0, \alpha^*)$, $F\frac{1}{2}G < F\alpha_0 G < F\alpha^* G$. Let $\beta_0 > \frac{1}{2}$ be a corresponding probability weight provided by mixture symmetry. Then $F_0 \equiv F\alpha_0 G$ and $G_0 \equiv F\beta_0 G$ violate Conclusion 1. Thus $F\frac{1}{2}G \gtrsim F\alpha G$, $\forall \alpha \in [0,1]$.

If $F\frac{1}{2}G \sim F$, then $[F, G]$ is an isopreference by Lemma A1.1. Thus it remains only to consider the case $F^*G \sim F\frac{1}{2}G > F$ for some $\alpha^* \neq \frac{1}{2}$. Let

$$\underline{\alpha} \equiv \min\{\alpha \in [0,1]: F\alpha G \sim F\tfrac{1}{2}G\} \quad \text{and}$$

$$\bar{\alpha} \equiv \max\{\alpha \in [0,1]: F\alpha G \sim F\tfrac{1}{2}G\}.$$

Then $\underline{\alpha} < \frac{1}{2} < \bar{\alpha}$ by mixture symmetry. By Lemma A1.1, $[F\underline{\alpha}G, F\bar{\alpha}G]$ is an isopreference. By part (a) the same is true of $[F, G]$. *Q.E.D.*

PROOF OF THEOREM 1: Let F and G lie in the interior of $D(X)$. Suppose $F\alpha G \sim F\beta G$ and $\beta \neq 1 - \alpha$. Then $F\frac{1}{2}G$ is a best or worst point in $[F\alpha G, F\beta G]$ but yet is not a midpoint. By Lemma A1.2, therefore, $[F, G]$ is an isopreference and so $F\alpha G \sim F(1-\alpha)G$ trivially.

If F and G are on the boundary, take $F_n \to F$, $G_n \to G$, $F_n \sim G_n$, F_n and $G_n \in \mathrm{int}(\mathrm{D}(X))$. Then

$$F_n\alpha G_n \sim F_n(1-\alpha)G_n \ \forall n \Rightarrow F\alpha G \sim F(1-\alpha)G. \qquad Q.E.D.$$

APPENDIX 2: PROOF OF THEOREM 4 IN THE 3-OUTCOME CASE

This appendix deals exclusively with c.d.f.'s for three outcome gambles. Thus we consider $S^2 = \{(p, q) \in R_+^2 : p + q \leqslant 1\}$, where p and q are the probabilities associated with the intermediate and largest outcomes respectively, and we write $V(p, q)$. The continuity and monotonicity axioms imply that V is continuous and increasing on S^2. Note that indifference curves in S^2 are connected. (Otherwise, there would be distinct indifferent points along the edge joining $(1,0)$ and $(0,1)$ which would violate first degree stochastic dominance.) Of course, mixture symmetry (or equivalently, strong mixture symmetry) is maintained. Neither quasiconcavity nor quasiconvexity is assumed in this appendix.

LEMMA A2.1: *Each indifference curve lying in the interior of S^2 is either strictly convex, strictly concave, or a straight line.*

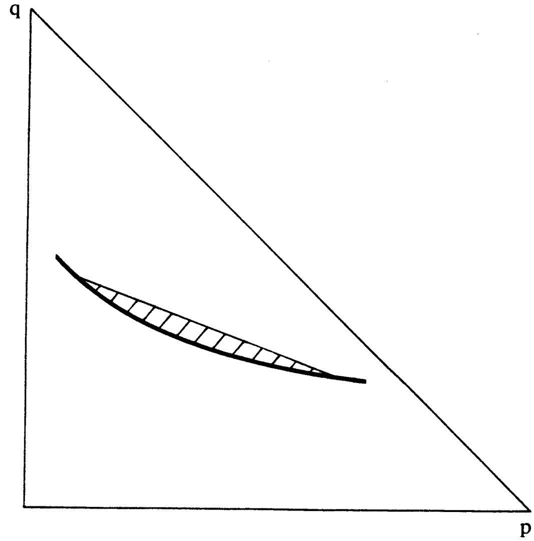


FIGURE 4.

PROOF: Follows from Lemmas A1.1 and A1.2. *Q.E.D.*

We show below that S^2 is divided into regions I, II, and III (some of which may be empty) such that indifference curves are strictly concave, linear, and strictly convex in regions I, II, and III respectively (see Figure 2). Region I is below and to the left of II, the latter is below and to the left of III, and the boundaries between I and II and between III are linear indifference curves.

We will show that the preference ordering can be represented by a proper quadratic function on III. (A similar argument applies to I while the desired representation on II follows from Chew (1989) and Dekel (1986).) This is done by establishing the representation on each small open rectangle in III. Such open rectangles exist if III is nonempty, since given a strictly convex indifference curve as in Figure 4, all indifference curves in the shaded region above it must also be strictly convex. We can write region III $= \bigcup_1^\infty O_i$, where each O_i is an open rectangle having a quadratic representation V_i and where $O_i \cap O_{i+1} \neq \varnothing \; \forall i \geqslant 1$. V_i and V_{i+1} are ordinally equivalent on $O_i \cap O_{i+1}$ and thus also cardinally equivalent there by Lemma A3.1. We can redefine V_{i+1} if necessary to guarantee that $V_{i+1} = V_i$ on $O_i \cap O_{i+1}$. By starting this argument at $i = 1$ we can construct a quadratic function V on region III which represents $\gtrsim$ on each O_i. Indifference curves are connected subsets of region III. Thus it is straightforward to show that V is constant along indifference curves and subsequently that it represents $\gtrsim$ on region III.

The arguments to follow should be understood to apply to a rectangle which lies wholly in III. The (income) expansion paths (e.p.) of the indifference map play a central role. Two points a and b lie on the same expansion path if there is a common subgradient to the indifference curves at a and b.

The proof presented below may be outlined as follows: we show that (i) all e.p.'s are linear, and (ii) they are perspective, i.e., they have a common point of intersection, which could be at infinity if they are parallel lines (Lemmas A2.3–5). By strong mixture symmetry and the quasiconcavity which prevails in the subdomain upon which we focus, the optimal mix between two indifferent bundles is the midpoint between them. Thus each e.p. bisects the chords of an indifference curve which have absolute slope equal to the price ratio underlying the e.p. We conclude that indifference curves in S^2 possess the following projection property: the loci of midpoints of parallel chords are perspective. It is known (Coxeter (1974)) that conics have the projection property. We prove the converse (Theorem A2.1) to establish that indifference curves are conics, from which the desired representation for V follows.

LEMMA A2.2: *All expansion paths are straight lines. Moreover, each expansion path is a locus of midpoints of parallel chords of an indifference curve.*

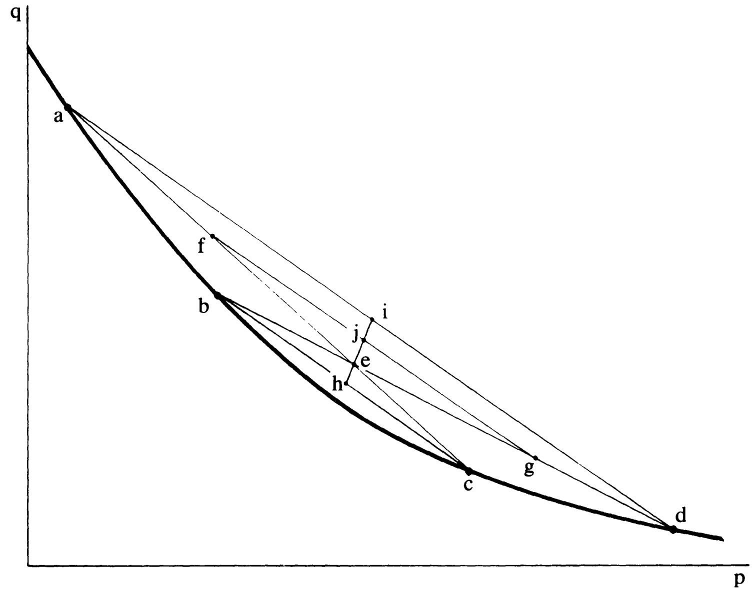


FIGURE 5.

PROOF: Refer to Figure 5. Start with h and i, points on an e.p. Draw the chord through h with absolute slope equal to a price ratio underlying the e.p. In this way points b and c are defined. By drawing a parallel chord through i, the points a and d may be constructed. Next draw the lines ac and bd with intersection point e.

Strong mixture symmetry and quasiconcavity $\Rightarrow h$ and i are the midpoints of bc and ad respectively. It follows by plane geometry that e is on the line hi. By construction it is not the midpoint of ac or bd. (Otherwise ad and bc would have the same length.) Let f be the point on ac that is symmetric to e, in the sense of being the mirror image of e in a reflection through the midpoint of ac. Construct g similarly on bd. By strong mixture symmetry, $f \sim e \sim g$. Let j be the midpoint of fg. Then by plane geometry j is on hi and, by strong mixture symmetry and quasiconcavity, it is on the given e.p.

This proves that given any two points h and i on the e.p., there exists a third point on the e.p. that is also on the line segment hi. Thus the e.p. must be linear. Note that h, i, and j are midpoints of the respective parallel chords. *Q.E.D.*

LEMMA A2.3: *If two e.p.'s intersect, then there is another e.p. lying between them such that the three paths have a common intersection point.*

PROOF: Refer to Figure 6. Start with the indifference curve and with the two e.p.'s A and B, with the slopes of bd and ce for A and B respectively. The points c and e can be chosen so that $cd \| be$. By strong mixture symmetry, $bh = hd$ and $ci = ie$. Construct $ae \| bd$ with midpoint g and $bf \| ce$ with midpoint j. Let l and m be the midpoints of be and cd respectively. Then these points are on the same straight line C, C is an e.p., and it intersects A and B at their common intersection point. *Q.E.D.*

We assert, without proof, that the limit of a sequence of *e.p.*'s is itself an *e.p.* More precisely, the following is true:

LEMMA A2.4: *Let* $(p^n, q^n) \to (p^0, q^0)$ *where all points lie on a given indifference curve. Let* $q = r_n p + s_n$ *be corresponding e.p.'s going through these points,* $n = 0, 1, \ldots$. *Then* $r_n \to r_0$ *and* $s_n \to s_0$.

LEMMA A2.5: *Either all e.p.'s are parallel straight lines or they have a common intersection point.*

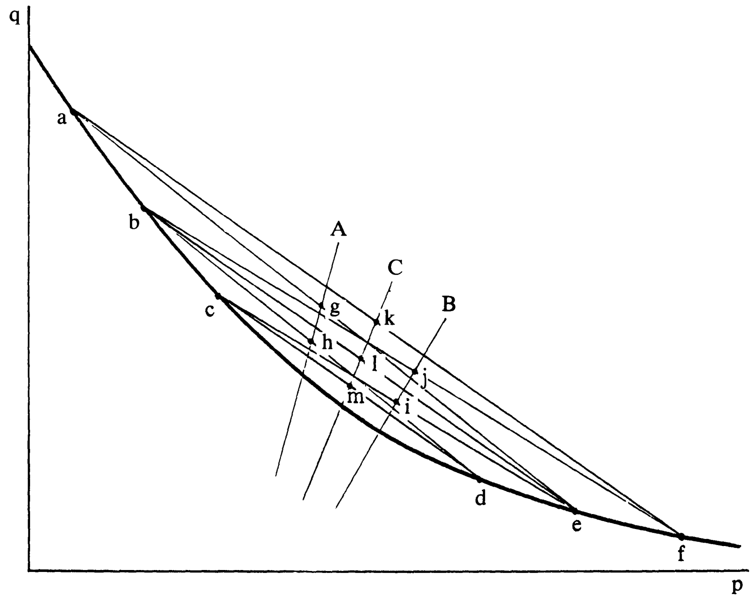


FIGURE 6.

PROOF: Suppose there exist intersecting e.p.'s A and B as in Figure 6. Denote by z their common intersection point and by λ_A and λ_B their intersections with the indifference curve shown. Use the preceding two lemmas to argue that $\exists$ a dense subset of the segment of the indifference curve lying between λ_A and λ_B, such that through each point in the set there is an e.p. which contains z. By Lemma A2.4, all e.p.'s that lie between A and B contain z. We can apply Lemma A2.3 repeatedly to prove that the qualifier "between A and B" may be dropped. *Q.E.D.*

We have now established the projection property described at the beginning of this appendix. The next step is to prove that this property implies that indifference curves are conics. A conic is uniquely determined by five distinct arbitrary points. (Note that there are five free coefficients in $ap^2 + bq^2 + cpq + dp + eq$.) When the center of projection O is given, or (in the case of a parabola) when the center is at infinity, if the common slope of expansion paths is also given, then the conic is uniquely determined by three distinct arbitrary points. We can use any two of the three points to define a chord and then generate a fourth point by drawing a parallel chord from the third point such that the straight line from 0 (or the line parallel to expansion paths when 0 is at infinity), which bisects the first chord, also bisects the second chord. One additional step, with a similar construction, leads to a fifth point.

THEOREM A2.1: *A curve in R^2 is a conic if and only if it has the projection property, i.e. if and only if the loci of midpoints of parallel chords are perspective.*

PROOF: For the necessity of the projection property see Coxeter (1974). We prove sufficiency in the case where the center of projection O is finite. The argument for the case where O is at infinity is similar.

Refer to Figure 7. Given any two parallel chords a_0a_0' and a_1a_1', let Q_1 be the unique conic through a_0, a_0' and a_1 with center O. Let OM_0 bisect a_0a_0'. By the projection property, it must also bisect a_1a_1'. Moreover, since the conic Q_1 satisfies the projection property, $a_1' \in Q_1$. We will prove that C coincides with Q_1 for points along Q_1 between a_0 and a_1, and between a_0' and a_1'.

Construct another parallel chord a_2a_2' between a_0a_0' and a_1a_1' such that $a_0a_2' \| a_2a_1'$. Let Q_2 be the unique conic through a_0, a_0' and a_2 with center 0. As before, $a_2' \in Q_2$. Let ON_0 bisect a_0a_2' and consequently also a_2a_1'. It follows from the noted projection property of conics that a_1' and $a_1 \in Q_2$. Thus $\{a_0, a_0', a_1\} \subset Q_1 \cap Q_2 \Rightarrow Q_1 = Q_2 \Rightarrow a_2' \in Q_1$.

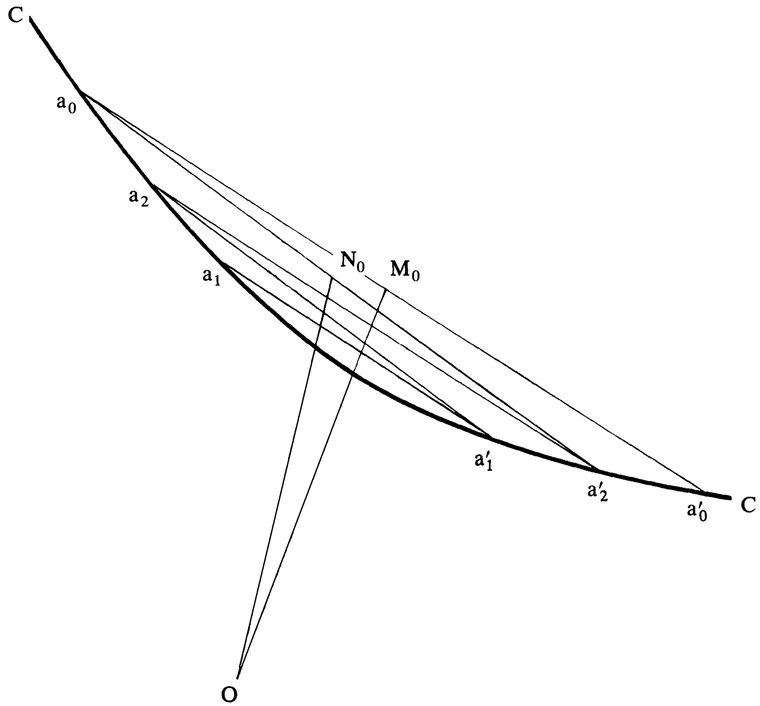


FIGURE 7.

Let $C[a_0, a_1]$ denote that portion of C lying between a_0 and a_1 and define $D \equiv C[a_0, a_1] \cap Q_1$. As above we can show that if a and b belong to D, then $\exists c \in D$ lying between a and b. Since D is a closed subset of $C[a_0, a_1]$, it follows that D is dense in $C[a_0, a_1]$ and hence that $D = C[a_0, a_1]$. Evidently, $C = Q_1$ between a_0 and a_1 and also between a_0' and a_1'. We finally conclude that $C = Q_1$ since the choice of $a_1 a_1'$ is arbitrary. *Q.E.D.*

The Theorem and Lemma A2.5 imply that each indifference curve is the graph of a quadratic function. Since e.p.'s are linear and have a common point of intersection, we can translate the coordinate system to a new origin such that the utility function V is homothetic in the new system. It follows that the same quadratic function applies to all indifference curves. Therefore, each one may be represented in the form

$$f(p,q) = K,$$

where f is a quadratic function (i.e., a second order polynomial function), and where K, but not f, varies across indifference curves. Thus the preference ordering may be represented by a proper quadratic utility function.

This establishes the desired representation on the shaded region in Figure 4. By working with overlapping regions, the representation can be extended to the entire region lying above any strictly convex indifference curve. That is because it is impossible, given the quadratic representation, to have indifference curves in Figure 2 flattening out and eventually becoming linear as one moves in the northeast direction.

APPENDIX 3: PROOF OF THEOREM 4 FOR CDF'S WITH FINITE SUPPORT

For 3-outcome gambles the desired representation follows from Appendix 2. Here we extend the representation result to the class of c.d.f.'s with finite support. For fixed outcomes $x_0 < \cdots < x_n$ in $X, n \geqslant 3$, utility depends on the corresponding probability vector. Thus the domain of V is taken to be $S^n \equiv \{(p_1, \ldots, p_n) \in R^n_+ : \Sigma_1^n p_i \leqslant 1\}$, where V is increasing, continuous, and satisfies quasiconcavity and mixture symmetry. The case of quasiconvexity can be treated similarly.

We will make use of the following lemma (used also in the proof of Theorem 2), which describes conditions under which the ordinal equivalence of two proper quadratic functions implies that they are cardinally equivalent.

LEMMA A.3.1: *Let f and g be proper quadratic functions defined on a convex subset of R^n such that*

$$f(\cdot) = \Theta(g(\cdot)) \tag{A3.1}$$

for some increasing Θ. Then Θ is linear on Rng g.

The proof follows readily by computing the Hessians of both sides of (A3.1). Details are omitted.

Say that $H \subset S^n$ is a plane of dimension k if $H = H' \cap S^n$, where H' is a k-dimensional plane in R^n. Denote by $V|H$ the restriction to H of V.

To show that V has the desired representation on S^n, we show by induction on k that $\forall k \leqslant n$, the following obtains:

$P(k)$: For each k-dimensional plane $H \subset S^n$, if C is an indifference set of $V|H$ which is not a singleton, then either

(i) C is a plane of dimension $(k-1)$ or k, or

(ii) for each $p^0 \in$ int $(S^n) \cap C$ there exists an open neighborhood N of p^0 in H and a proper quadratic function f: $N \to R^1$ such that for every indifference set C'

$$C' \cap N = \{p \in N: f(p) = K\}, \qquad \text{for some } K.$$

From $P(n)$ it then follows that each indifference set of V is planar or has local quadratic representations. As in the case of S^2 (Appendix 2), quadratic representations can be constructed so that they coincide on overlapping regions. Therefore, we deduce the existence of regions in S^n analogous to the (proper quasiconcave) quadratic and betweenness regions in Figure 2. That the latter must lie below the former follows from the corresponding fact for S^2, e.g., consider restrictions of V to 2-dimensional subspaces of S^n.

Turn to the proof of $P(k)$ and let $k = 2$. Appendix 2 proves $P(2)$ restricted to 2-dimensional planes H such that for some i_1 and i_2, $p \in H \Rightarrow p_i = 0$ $\forall i \neq i_1, i_2$. We now show that $P(2)$ is true, i.e., the desired property holds for all two-dimensional planes in S^n, provided the preference relation is quasiconcave.

Let H be a plane in S^n, and let p^0 lie in the relative interior of H. Assume first that p^0 is a minimum point of $\succsim$. Let $p^1, p^2 \in H$ such that $p^0 \in (p^1, p^2)$ the open line segment between p^1 and p^2 (the corresponding closed segment is denoted $[p^1, p^2]$). By quasiconcavity at least one of these two points, say p^1, satisfies $p^1 \sim p^0$. If $p^2 \sim p^0$, then by Lemmas A1.1 and A1.2 the entire line in H containing $[p^1, p^2]$ is an indifference set. If $p^2 > p^0$, then by quasiconcavity, the set $[p^1, p^0]$ is an indifference set, and so is the line containing this chord in H. This line separates H into two parts. By quasiconcavity, not both of them can contain points strictly preferred to p^0; hence by Lemma A1.2, H is an indifference set.

If p^0 is a maximum point, then by quasiconcavity and Appendix 1 either the indifference set containing it in H is linear, or p^0 is a unique maximum point. In the latter case, it is sufficient by continuity to prove $P(2)$ for all other points in the interior of H.

We proceed assuming that p^0 is neither a minimum nor a maximum point of $\succsim$ in H. Let p^* be a maximum point of $\succsim$ in H. By continuity, there are neighborhoods B^0 of p^0 and B^* of p^* such that $\forall p^2 \in B^0$ and $\forall p^3 \in B^*$, $p^3 > p^2$. Let $L(p^2, p^3)$ be the line containing p^2 and p^3. For $p^2 \in B^0$ and $p^3 \in B^*$ it follows by quasiconcavity that on $L(p^2, p^3) \cap B^0$, the order $\succsim$ is monotonic in the direction from p^2 to p^3.

Let $p, p' \in B^*$ such that $p' \notin L(p^0, p)$. Let T: $H \to R^2$ be a linear transformation such that $T(p^0) = (0,0)$, $T(p) = (1,0)$, and $T(p') = (0,1)$. For $x, y \in T(H)$, define $x \succsim_T y \Leftrightarrow T^{-1}(x) \succsim T^{-1}(y)$. Around $T(p^0) = (0,0)$ the order $\succsim_T$ is monotonic. (To see this, let $x \geq y$, $x \neq y$ be close enough to $(0,0)$. By the nature of T, the slope of $L(T^{-1}(x), T^{-1}(y))$ is between that of $L(p^0, p)$ and $L(p^0, p')$. Therefore, if y is close enough to $(0,0)$, then $B^* \cap L(T^{-1}(x), T^{-1}(y)) \neq \varnothing$). By Appendix 2, $\succsim_T$ is quadratic around $(0,0)$. Hence $\succsim$ is quadratic around p^0.

Assume $P(k-1)$ and prove $P(k)$. Let $k = n$; the argument for general k is similar but notationally more cumbersome. Suppose C, an indifference set of V, is neither a singleton nor a plane. Let $p^0 \in C$ lie in the interior of S^n. By Lemma A1.2 $\exists p^1 \in C$ such that $(p^0, p^1) \cap C = \varnothing$.

Next we can find $p^2 \in C$ sufficiently close to p^1 such that $(p^0, p^2) \cap C = \varnothing$ and $p^2 - p^0$ is not a scalar multiple of $p^1 - p^0$. Proceeding inductively, we find $p^1, \dots, p^n$ such that $(p^0, p^i) \cap C = \varnothing \ \forall i$ and $R^n = \operatorname{span}\{p^i - p^0: i = 1, \dots, n\}$. Let H^i be the unique $(n-1)$-dimensional plane through p^0 spanned by $\{p^j - p^0: j = 1, \dots, n, \ j \neq i\}$. Then $C \cap H^i$ is not a plane since $\{p^0, p^j\} \subset C \cap H^i$ for $j \neq i$ but $[p^0, p^j]$ does not lie in C. Therefore, condition (ii) of $P(n-1)$ applies to $C \cap H^i$ and this is true for each i. Let $\{N^i\}$ be the open sets provided by (ii).

Without loss of generality (choose a suitable linear transformation), suppose that $p^i - p^0$ is e^i, the unit vector in the ith coordinate direction in R^n. Then $H^i = \{p \in S^n: \ p_i = p_i^0\}$ is the ith coordinate hyperplane through p^0. By above, V is ordinally equivalent to a proper quadratic function f^i on the open region $N^i \subset H^i$, for $i = 1, 2, \dots, n$. Redefine V so that it is proper quadratic on $N^1 (V \equiv f^1)$. Then on $N^1 \cap N^2$, V and f^2 are ordinally equivalent and proper quadratic. (They are clearly quadratic on $N^1 \cap N^2$. They are also proper quadratic functions there since $\{p^0, p^n\} \subset C \cap N^1 \cap N^2$, but $(p^0, p^n) \cap C = \varnothing$.) By Lemma A3.1, the transformation relating V and f^2 is linear and hence V is a proper quadratic on N^2. Similarly, V is a proper quadratic on each N^i.

By continuity, there exists an open neighborhood M of p^0 in S^n, such that V is proper quadratic on the intersection of each coordinate hyperplane with M. (To see this, let $\bar{H}^i$ be parallel to H^i and sufficiently close to it and let $\bar{N}^i \subset \bar{H}^i$ be the image of $N^i \subset H^i$ under this parallel translation. By continuity, we can assume that V does not satisfy betweenness on $\bar{N}^i$ and so it is ordinally quadratic there by the induction hypothesis. Since V is proper quadratic on $N^i \cap N^j$, $j \neq i$, we can assume that it is proper quadratic also on $\bar{N}^i \cap N^j$. As before, we can conclude that V is proper quadratic on each $\bar{N}^i$.)

It now follows that V is proper quadratic on an open region in S^n. To see this, consider the hyperplanes which fix the 1st and nth coordinates of p where the quadratic nature of V implies that (locally)

$$\begin{aligned} V(p_1, \dots, p_n) &= \sum_{i=1}^{n-1} \sum_{j=i}^{n-1} \phi_{ij}(p_n)\left(p_i - p_i^0\right)\left(p_j - p_j^0\right) + \sum_{i=1}^{n-1} \gamma_i(p_n)\left(p_i - p_i^0\right) \\ &= \sum_{i=2}^{n} \sum_{j=i}^{n} \Psi_{ij}(p_1)\left(p_i - p_i^0\right)\left(p_j - p_j^0\right) + \sum_{i=2}^{n} \delta_i(p_1)\left(p_i - p_i^0\right). \end{aligned}$$

(The normalization $V(p^0) = 0$ has been adopted.) From these two equations, V must be twice differentiable. Also, from the first equation for V, for each $i < n$, $V_{p_1 p_i}$ is independent of $p_1, \dots, p_{n-1}$. Thus, from the second equation for V it follows that $\delta_i(\cdot)$ is linear for $i < n$ and $\Psi_{ij}(\cdot)$ is constant for $i, j < n$. If we replace the nth coordinate hyperplane above by the kth, $k = 2, \dots, n-1$, we can conclude that $\delta_n(\cdot)$ is linear and that $\Psi_{ij}(\cdot)$ is constant even if i or $j = n$. Thus V is quadratic. It is proper quadratic because each of its coordinate hyperplane restrictions is.

APPENDIX 4: Completion of the Proof of Theorem 4

The proof of Theorem 4 is completed here. Two steps remain.

Step 1: We showed above that when restricted to gambles with $n+1$ outcomes, V is ordinally equivalent to a quadratic function in the probabilities $p_1, \dots, p_n$ of the largest n outcomes. We would like a "symmetric" representation that is quadratic in all the probabilities $p_0, p_1, \dots, p_n$. That is, given that

$$V(p_1, \dots, p_n) = \sum_{i,j=1}^{n} \alpha_{ij} p_i p_j + \sum_{i=1}^{n} \beta_i p_i + \gamma, \tag{A4.1}$$

we want to rewrite V in the form

$$V = \sum_{i=0}^{n} \sum_{j=0}^{n} \phi_{ij} p_i p_j, \qquad \text{where} \qquad \sum_{i=0}^{n} p_i = 1. \tag{A4.2}$$

The desired symmetric coefficients ϕ_{ij} are readily found by equating the expressions in (A4.1) and (A4.2) and exploiting $\Sigma_0^n p_i = 1$.

STEP 2: We need to extend the representation (A4.2) to all c.d.f.'s in $D(X)$. Let $Y=\{x_0, x_1, \dots\}$ be a countable dense subset of X. Denote by $\phi^n(x_i, x_j)$, $i, j = 0, \dots, n$, the coefficients corresponding-to (A4.2) of the quadratic representation for $D^n \equiv$ {c.d.f.'s having support in $\{x_0, x_1, \dots, x_n\}$} and let V^n be the quadratic utility function defined on D^n. Then V^n and V^{n+1} are ordinally equivalent on their common domain D^n. By the proof of Theorem 2(ii), therefore, $\exists \alpha, \beta$ such that

$$\phi^{n+1}(x_i, x_j) = \alpha + \beta\phi^n(x_i, x_j), \qquad 0 \leqslant i, j \leqslant n.$$

Thus we can continually redefine the ϕ^n's so that

$$\phi^{n+1}(x_i, x_j) = \phi^n(x_i, x_j), \qquad 0 \leqslant i, j \leqslant n.$$

Define ϕ on $Y \times Y$ as $\lim_{n\to\infty} \phi^n$ and extend ϕ to $X \times X$ by

$$\phi(x, x') \equiv \sup\left\{\phi(x_i, x_j) \colon (x_i, x_j) \in Y \times Y,\ x_i \leqslant x,\ x_j \leqslant x'\right\}.$$

This ϕ will do for the quadratic functional in (13).

APPENDIX 5: FRÉCHET DIFFERENTIABILITY OF QUADRATIC UTILITY

We describe sufficient conditions, expressed in terms of ϕ, for the quadratic functional (3) to be Fréchet differentiable. Let ϕ be continuous on X^2.

The sufficient condition is that $\exists K > 0$ such that

$$\text{(A5.1)} \qquad |\phi(x', y') - \phi(x', y) - \phi(x, y') + \phi(x, y)| \leqslant K|x' - x|\,|y' - y|$$

$\forall x, y, x'$ and y' in X. Note that (A5.1) is satisfied if $\phi_{12}(\cdot, \cdot)$ exists and is continuous (and hence bounded) on X^2. Then

$$\phi(x', y') - \phi(x', y) - \phi(x, y') + \phi(x, y) = \int_y^{y'} \int_x^{x'} \phi_{12}(s, t)\, ds\, dt \Rightarrow \text{(A5.1)} \quad \text{with}$$

$$K \equiv \max|\phi_{12}(\cdot)|.$$

We show that (A5.1) implies that V is Fréchet differentiable and in particular, that $|R|/\|G - F\|_1 \to 0$ as $\|G - F\|_1 \to 0$, where

$$R \equiv V(G) - V(F) - \int u(\cdot\,; F)\, d(G - F),$$

$\|\cdot\|_1$ denotes the L^1 norm, and $u(\cdot\,; F)$ is the local utility function defined in (11). It suffices to prove that

$$\text{(A5.2)} \qquad |R| \leqslant K \cdot (\|G - F\|_1)^2, \qquad \forall F, G \in D(X).$$

Moreover, (A5.2) is implied if the inequality there is proven for all c.d.f.'s corresponding to gambles with finitely many equally likely outcomes. So let

$$F = n^{-1} \sum_{i=1}^n \delta_{s_i} \qquad \text{and} \qquad G = n^{-1} \sum_{i=1}^n \delta_{t_i}.$$

Then

$$\begin{aligned} R &= V(G) - V(F) - \int u(\cdot\,; F)\, d(G - F) \\ &= \iint \phi(x, y)\, d(G(x) - F(x))\, d(G(y) - F(y)) \\ &= n^{-2} \sum_i \sum_j \left[\phi(s_i, s_j) + \phi(t_i, t_j) - \phi(s_i, t_j) - \phi(s_j, t_i)\right] \\ \Rightarrow |R| &\leqslant n^{-2} K \sum_i \sum_j |s_i - t_i|\,|s_j - t_j| \\ &= K\left(n^{-1} \sum |s_i - t_i|\right)^2 = K(\|G - F\|_1)^2. \end{aligned}$$

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