Journal: Economic Theory
"Approximate optimality and the risk/reward tradeoff given repeated gambles"
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ONLINE APPENDIX

Lemma: Our CLT, Proposition 6, is valid also if $\underline{\sigma} = 0$.
Proof: As in the proof of Proposition 6, it suffices to take $u \in C_b^\infty(\mathbb{R}^2)$.

Given $\underline{\sigma} = 0$, we add a perturbation to the random returns of the K arms. For any $1 \leq k \leq K$ and $n \geq 1$, let $X_{k,n}^\epsilon = X_{k,n} + \epsilon\zeta_n$, where $\epsilon > 0$ is a fixed small constant and $\{\zeta_n\}$ is a sequence of i.i.d. standard normal random variables, independent with $\{X_{k,n}\}$. Then, for any $\theta \in \Theta$ and $n \geq 1$, the corresponding reward is denoted by $Z_n^{\theta,\epsilon} = Z_n^\theta + \epsilon\zeta_n$, and the corresponding set of mean-variance pairs is denoted by

$$\mathcal{A}_\epsilon = \{(\mu_{k,\epsilon}, \sigma_{k,\epsilon}^2) : 1 \leq k \leq K\},$$

where $\mu_{k,\epsilon} = \mu_k$ and $\sigma_{k,\epsilon}^2 = \sigma_k^2 + \epsilon^2$. The corresponding bounds are $\overline{\mu}_\epsilon, \underline{\mu}_\epsilon, \overline{\sigma}_\epsilon^2$, and $\underline{\sigma}_\epsilon^2 > 0$.

Define

$$V_n^\epsilon = \sup_{\theta\in\Theta} E_P\left[u\left(\frac{\sum_{i=1}^n Z_i^{\theta,\epsilon}}{n}, \frac{\sum_{i=1}^n (Z_i^{\theta,\epsilon} - E_P[Z_i^{\theta,\epsilon}|\mathcal{H}_{i-1}^\theta])}{\sqrt{n}}\right)\right]$$

By Proposition 6 for $\{Z_n^{\theta,\epsilon}\}$,

$$\lim_{n\to\infty} V_n^\epsilon = \sup_{a\in[\mathcal{A}_\epsilon](0,1)} E_P\left[u\left(\int_0^1 a_s^{(1)}ds, \int_0^1 a_s^{(2)}dB_s^{(2)}\right)\right] = v_\epsilon(0,0,0), \qquad (43)$$

where $v_\epsilon(t,x,y)$ is the solution of PDE (20) with function G_ϵ instead of G,

$$G_\epsilon(p,q) = \sup_{(\mu,\sigma^2)\in\mathcal{A}_\epsilon}\left[\mu p + \frac{1}{2}\sigma^2 q\right], \quad (p,q) \in \mathbb{R}^2. \qquad (44)$$

By Yong and Zhou (1999, Propn. 5.10, Ch. 4), $\exists C' > 0$ such that

$$|v_\epsilon(t,x,y) - v(t,x,y)| \leq C'\sqrt{\epsilon}, \quad \forall (t,x,y) \in [0,1) \times \mathbb{R}^2.$$

We also have

$$|V_n - V_n^\epsilon|^2 \leq C\epsilon^2 E_P\left[\left|\frac{\sum_{i=1}^n \zeta_i}{n}\right|^2 + \left|\frac{\sum_{i=1}^n \zeta_i}{\sqrt{n}}\right|^2\right] \leq 2C\epsilon^2,$$

where the constant C depends only on the bounds of $\partial_x u$ and $\partial_y u$.

Letting as $\epsilon \to 0$ in (43), the CLT (25) is proven for $\underline{\sigma} = 0$. Similar arguments show that (26) is also valid. ■

Lemma: Our CLT, Proposition 6, is valid also if u is continuous and, for some $g \geq 1$ and $c > 0$, $|u(x,y)| \leq c(1 + ||(x.y)||^{g-1})$ and $\sup_{1 \leq k \leq K} E_P[|X_k|^g] < \infty$.

Proof: We prove that (25) remains valid. Refer to it as "the CLT."

Step 1: Prove the CLT for any $u \in C_b(\mathbb{R}^2)$ with compact support (constant outside a compact subset of $\mathbb{R}^2$). In this case, $\forall \epsilon > 0\ \exists \hat{u} \in C_{b,Lip}(\mathbb{R}^2)$ such that $\sup_{z \in \mathbb{R}^2} |u(z) - \hat{u}(z)| \leq \frac{\epsilon}{2}$. Then

$$\left| \sup_{\theta \in \Theta} E_P \left[u \left(\frac{S_n^\theta}{n}, \frac{\overline{S}_n^\theta}{\sqrt{n}} \right) \right] - \sup_{a \in [\mathcal{A}](0,1)} E_P[u(\int_0^1 a_s^{(1)} ds, \int_0^1 a_s^{(2)} dB_s^{(2)})] \right|$$
$$\leq \epsilon + \left| \sup_{\theta \in \Theta} E_P \left[\hat{u} \left(\frac{S_n^\theta}{n}, \frac{\overline{S}_n^\theta}{\sqrt{n}} \right) \right] - \sup_{a \in [\mathcal{A}](0,1)} E_P[\hat{u}(\int_0^1 a_s^{(1)} ds, \int_0^1 a_s^{(2)} dB_s^{(2)})] \right|$$

Therefore,

$$\limsup_{n \to \infty} \left| \sup_{\theta \in \Theta} E_P \left[u \left(\frac{S_n^\theta}{n}, \frac{\overline{S}_n^\theta}{\sqrt{n}} \right) \right] - \sup_{a \in [\mathcal{A}](0,1)} E_P[u(\int_0^1 a_s^{(1)} ds, \int_0^1 a_s^{(2)} dB_s^{(2)})] \right| \leq \epsilon,$$

which proves the CLT since ϵ is arbitrary.

Step 2: Let $u \in C(\mathbb{R}^2)$ satisfy the growth condition $|u(z)| \leq c(1 + |z|^{g-1})$ for $g \geq 1$. For any $N > 0$, $\exists u_1, u_2 \in C(\mathbb{R}^2)$ such that $u = u_1 + u_2$, where u_1 has a compact support and $u_2(z) = 0$ for $|z| \leq N$, and $|u_2(z)| \leq |u(z)|$ for all z. Then

$$|u_2(z)| \leq \frac{2c(1 + |z|^g)}{N}, \quad \forall z \in \mathbb{R}^2,$$

and

$$\left| \sup_{\theta \in \Theta} E_P \left[u \left(\frac{S_n^\theta}{n}, \frac{\overline{S}_n^\theta}{\sqrt{n}} \right) \right] - \sup_{a \in [\mathcal{A}](0,1)} E_P[u(\int_0^1 a_s^{(1)} ds, \int_0^1 a_s^{(2)} dB_s^{(2)})] \right|$$
$$\leq \left| \sup_{\theta \in \Theta} E_P \left[u_1 \left(\frac{S_n^\theta}{n}, \frac{\overline{S}_n^\theta}{\sqrt{n}} \right) \right] - \sup_{a \in [\mathcal{A}](0,1)} E_P[u_1(\int_0^1 a_s^{(1)} ds, \int_0^1 a_s^{(2)} dB_s^{(2)})] \right|$$
$$+ \sup_{\theta \in \Theta} E_P \left[\left| u_2 \left(\frac{S_n^\theta}{n}, \frac{\overline{S}_n^\theta}{\sqrt{n}} \right) \right| \right] + \sup_{a \in [\mathcal{A}](0,1)} E_P[|u_2(\int_0^1 a_s^{(1)} ds, \int_0^1 a_s^{(2)} dB_s^{(2)})|]$$
$$\leq \left| \sup_{\theta \in \Theta} E_P \left[u_1 \left(\frac{S_n^\theta}{n}, \frac{\overline{S}_n^\theta}{\sqrt{n}} \right) \right] - \sup_{a \in [\mathcal{A}](0,1)} E_P[u_1(\int_0^1 a_s^{(1)} ds, \int_0^1 a_s^{(2)} dB_s^{(2)})] \right|$$
$$+ \frac{2c}{N} \left(2 + \sup_{\theta \in \Theta} E_P \left[\left| \frac{S_n^\theta}{n} \right|^g + \left| \frac{\overline{S}_n^\theta}{\sqrt{n}} \right|^g \right] + \sup_{a \in [\mathcal{A}](0,1)} E_P \left[\left| \int_0^1 a_s^{(1)} ds \right|^g + \left| \int_0^1 a_s^{(2)} dB_s^{(2)} \right|^g \right] \right)$$

By the Burkholder-Davis-Gundy inequality (Mao (2008, Theorem 1.7.3)),

$$\limsup_{n\to\infty}\left|\sup_{\theta\in\Theta}E_P\left[u\left(\frac{S_n^\theta}{n},\frac{\overline{S}_n^\theta}{\sqrt{n}}\right)\right]-\sup_{a\in[\mathcal{A}](0,1)}E_P[u(\int_0^1 a_s^{(1)}ds,\int_0^1 a_s^{(2)}dB_s^{(2)})]\right|$$
$$\leq\frac{2c}{N}\left(2+\max\{|\overline{\mu}|^g,|\underline{\mu}|^g\}+\overline{\sigma}^g+\sup_n\sup_{\theta\in\Theta}E_P\left[\left|\frac{S_n^\theta}{n}\right|^g+\left|\frac{\overline{S}_n^\theta}{\sqrt{n}}\right|^g\right]\right).$$

Since N can be arbitrarily large, it suffices to prove

$$\sup_n\sup_{\theta\in\Theta}E_P\left[\left|\frac{S_n^\theta}{n}\right|^g+\left|\frac{\overline{S}_n^\theta}{\sqrt{n}}\right|^g\right]<\infty$$

Step 3: Prove the preceding inequality. For any n,

$$\sup_{\theta\in\Theta}E_P\left[\left|\frac{S_n^\theta}{n}\right|^g\right]\leq\sup_{\theta\in\Theta}E_P\left[\frac{n^{g-1}}{n^g}\sum_{i=1}^n|Z_i^\theta|^g\right]\leq K\sup_{1\leq k\leq K}E_P[|X_k|^g].$$

For $1\leq g\leq 2$,

$$\begin{aligned}\left(\sup_{\theta\in\Theta}E_P\left[\left|\frac{\overline{S}_n^\theta}{\sqrt{n}}\right|^g\right]\right)^{\frac{2}{g}}&\leq\sup_{\theta\in\Theta}E_P\left[\left(\frac{\overline{S}_n^\theta}{\sqrt{n}}\right)^2\right]\\&=\frac{1}{n}\sup_{\theta\in\Theta}E_P\left[\left(\overline{S}_{n-1}^\theta\right)^2+2\overline{S}_{n-1}^\theta\overline{Z}_n^\theta+(\overline{Z}_n^\theta)^2\right]\\&\leq\frac{1}{n}\sup_{\theta\in\Theta}E_P\left[\left(\overline{S}_{n-1}^\theta\right)^2+\overline{\sigma}^2\right]\leq\overline{\sigma}^2.\end{aligned}$$

For $g>2$,

$$|x+y|^g\leq 2^g g^2|x|^g+|y|^g+gx|y|^{g-1}sgn(y)+2^g g^2x^2|y|^{g-2},\ \forall x,y\in\mathbb{R}.$$

Let $T_k^\theta=\max\{\overline{S}_k^\theta,\overline{S}_k^\theta-\overline{S}_1^\theta,\cdots,\overline{S}_k^\theta-\overline{S}_{k-1}^\theta\}$. Then $T_k^\theta=\overline{Z}_k^\theta+(T_{k-1}^\theta)^+$ and

$$\begin{aligned}&\sup_{\theta\in\Theta}E_P[|T_k^\theta|^g]\\&\leq 2^g g^2\sup_{\theta\in\Theta}E_P[|\overline{Z}_k^\theta|^g]+\sup_{\theta\in\Theta}E_P[|(T_{k-1}^\theta)^+|^g]\\&\quad+g\sup_{\theta\in\Theta}E_P[\overline{Z}_k^\theta|(T_{k-1}^\theta)^+|^{g-1}]+2^g g^2\sup_{\theta\in\Theta}E_P[(\overline{Z}_k^\theta)^2|(T_{k-1}^\theta)^+|^{g-2}]\\&\leq 2^g g^2\sum_{i=1}^k\sup_{\theta\in\Theta}E_P[|\overline{Z}_i^\theta|^g]+2^g g^2\sum_{i=2}^k\sup_{\theta\in\Theta}E_P[(\overline{Z}_i^\theta)^2|(T_{i-1}^\theta)^+|^{g-2}]\\&\leq 2^g g^2\sum_{i=1}^n\sup_{\theta\in\Theta}E_P[|\overline{Z}_i^\theta|^g]+2^g g^2\overline{\sigma}^2\sum_{i=1}^n\left(\sup_{\theta\in\Theta}E_P[|(T_i^\theta)^+|^g]\right)^{\frac{g-2}{g}}.\end{aligned}$$

Let $A_n = \sup_{k\le n}\sup_{\theta\in\Theta} E_P[|T_k^\theta|^g]$. Then, by Young's inequality (Peng (2019, Lemma 1.4.1)),[13]

$$A_n \le 2^g g^2 \sum_{i=1}^n \sup_{\theta\in\Theta} E_P[|\overline{Z}_i^\theta|^g] + 2^g g^2 \overline{\sigma}^2 n A_n^{\frac{g-2}{g}}$$
$$\le 2^g g^2 \sum_{i=1}^n \sup_{\theta\in\Theta} E_P[|\overline{Z}_i^\theta|^g] + \frac{2}{g}(2^g g^2 \overline{\sigma}^2 n)^{\frac{g}{2}} + \frac{g-2}{g} A_n.$$

Therefore,

$$A_n \le C_{g,1} \sum_{i=1}^n \sup_{\theta\in\Theta} E_P[|\overline{Z}_i^\theta|^g] + C_{g,2} n^{\frac{g}{2}}$$
$$\le C_{g,1} \sum_{i=1}^n \sup_{\theta\in\Theta} E_P[|Z_i^\theta|^g + \max\{|\overline{\mu}|^g, |\underline{\mu}|^g\}] + C_{g,2} n^{\frac{g}{2}}$$
$$\le C_{g,1} nK \sup_{1\le k\le K} E_P[|X_k|^g] + C_{g,1} n \max\{|\overline{\mu}|^g, |\underline{\mu}|^g\} + C_{g,2} n^{\frac{g}{2}}.$$

Finally,

$$\sup_{\theta\in\Theta} E_P\left[\left|\frac{\overline{S}_n^\theta}{\sqrt{n}}\right|^g\right] \le n^{-\frac{g}{2}} A_n$$
$$\le C_{g,1} n^{1-\frac{g}{2}} K \sup_{1\le k\le K} E_P[|X_k|^g] + C_{g,1} n^{1-\frac{g}{2}} \max\{|\overline{\mu}|^g, |\underline{\mu}|^g\} + C_{g,2}.$$

Since $\sup_{1\le k\le K} E_P[|X_k|^g] < \infty$, Step 3 is complete and the Lemma is proven. ■

Proof of Corollary 7: The preceding Lemma proves the extension for Proposition 6.

To prove (27), define

$$v(t,x,y) = \sup_{a\in[\mathcal{A}](t,1)} E_P\left[u\left(x + \int_t^1 a_s^{(1)} ds, y + \int_t^1 a_s^{(2)} dB_s^{(2)}\right)\right], \quad (x,y)\in\mathbb{R}^2.$$

As in the proof of Lemma 8(1), for $u \in C_{b,Lip}(\mathbb{R}^2)$, it can be checked that (Yong and Zhou (1999, Theorem 5.2 in Chapter 4)) v is the unique viscosity solution of the HJB-equation (20) with function G given in (28). Then we have

$$V = \sup_{a\in[\mathcal{A}](0,1)} E_P\left[u\left(\int_t^1 a_s^{(1)} ds, \int_t^1 a_s^{(2)} dB_s^{(2)}\right)\right] = v(0,0,0).$$

For $u \in C(\mathbb{R}^2)$ with growth condition, the value function is still the unique viscosity solution of the PDE (20) with function G given in (28). Supporting

[13] $| ab | \le p^{-1} | a |^p + q^{-1} | a |^q$ if $1 < p, q < \infty$ and $p^{-1} + q^{-1} = 1$.

details can be found in Pham (2009, p.66) or Aivaliotis and Palczewski (2010, Corollary 4.7). ■

The Krylov norm: We use the notation in Krylov (1987, Section 1.1); see also Peng (2019, Chapter 2.1). For $\Gamma \subset [0,\infty) \times \mathbb{R}^2$, $C(\Gamma)$ denotes the set of all real-valued functions v defined on Γ, continuous in the relative topology on Γ and having a finite norm,

$$\|v\|_{C(\Gamma)} = \sup_{(t,z)\in\Gamma} |v(t,z)|.$$

Similarly, given $\alpha, \beta \in (0,1)$,

$$\|v\|_{C^{\alpha,\beta}(\Gamma)} = \|v\|_{C(\Gamma)} + \sup_{(t,z),(t',z')\in\Gamma,(t,z)\neq(t',z')} \frac{|v(t,z) - v(t',z')|}{|t-t'|^{\alpha} + |z-z'|^{\beta}}$$

$$\|v\|_{C^{1+\alpha,1+\beta}(\Gamma)} = \|v\|_{C^{\alpha,\beta}(\Gamma)} + \|\partial_t v\|_{C^{\alpha,\beta}(\Gamma)} + \sum_{i=1}^{2} \|\partial_{z_i} v\|_{C^{\alpha,\beta}(\Gamma)}.$$

$$\|v\|_{C^{1+\alpha,2+\beta}(\Gamma)} = \|v\|_{C^{1+\alpha,1+\beta}(\Gamma)} + \sum_{i,j=1}^{2} \|\partial^2_{z_i z_j} v\|_{C^{\alpha,\beta}(\Gamma)}.$$

The corresponding subspaces of $C(\Gamma)$ in which the correspondent derivatives exist and the above norms are finite are denoted respectively by

$$C^{1+\alpha,1+\beta}(\Gamma) \text{ and } C^{1+\alpha,2+\beta}(\Gamma).$$

Therefore, the first and second derivatives $v(t,z)$ with respect to z exist and the related norms are finite. In particular, $\exists L > 0$ such that

$$\sup_{(t,z),(t,z')\in\Gamma, z\neq z'} \frac{|v(t,z) - v(t,z')|}{|z-z'|^{\beta}} < L.$$

In the proof of Lemma 8, we applied the preceding to $v(t,z) = H_t(z)$.

Completion of the proof of Theorem 3(iii): Show that specializing in arm 2 is not asymptotically optimal if $\alpha < \overline{\alpha}$.

Verify the inequality

$$\mu_1 \frac{\sigma_3}{\sigma_1+\sigma_3} + \mu_3 \frac{\sigma_1}{\sigma_1+\sigma_3} - \alpha \frac{\sigma_1\sigma_3^2}{\sigma_1+\sigma_3} > \mu_2 - \alpha \frac{\sigma_2^2}{2} = E_P\left[u\left(\mu_2, \sigma_2 B_1^{(2)}\right)\right],$$

and deduce that

$$\alpha < \frac{2(\mu_1-\mu_2)\left[(1-\lambda)\sigma_1 + \sqrt{(1-\lambda)\sigma_1^2 + \lambda\sigma_2^2}\right]}{2(1-\lambda)\sigma_1^3 + (2\lambda-1)\sigma_1\sigma_2^2 - \sigma_2^2\sqrt{(1-\lambda)\sigma_1^2+\lambda\sigma_2^2}} \equiv 2(\mu_1-\mu_2)g(\lambda).$$

It can be verified that, $g'(\lambda) > 0$ for $\lambda \in (0,1)$ and $\lim_{\lambda\to 1} g(\lambda) = \frac{1}{\sigma_2(\sigma_1-\sigma_2)}$.

Therefore, for any $\alpha < \overline{\alpha} = \frac{2(\mu_1-\mu_2)}{\sigma_2(\sigma_1-\sigma_2)}$, there exists $\lambda_1 \in (0,1)$ such that

$$\alpha < 2(\mu_1 - \mu_2)g(\lambda_1) < \frac{2(\mu_1 - \mu_2)}{\sigma_2(\sigma_1 - \sigma_2)}.$$

Choose $\lambda = \lambda_1$ in the definition (41) of $\hat{a} = (\hat{a}_s^{(1)}, \hat{a}_s^{(2)})$ and deduce that

$$\begin{aligned} V = &\sup_{a \in [\mathcal{A}](0,1)} E_P\left[u\left(\int_0^1 a_s^{(1)} ds, \int_0^1 a_s^{(2)} dB_s^{(2)}\right)\right] \\ \geq &E_P\left[u\left(\int_0^1 \hat{a}_s^{(1)} ds, \int_0^1 \hat{a}_s^{(2)} dB_s^{(2)}\right)\right] \\ > &E_P\left[u\left(\mu_2, \sigma_2 B_1^{(2)}\right)\right]. \end{aligned}$$

Therfore, specializing in arm 2 is NOT asymptotically optimal.

When $\sigma_2 = 0$, we can set $\overline{\alpha} = \infty$, and the above proof still holds. ∎