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INTERTEMPORAL ASSET PRICING UNDER KNIGHTIAN UNCERTAINTY

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In conformity with the Savage model of decision-making, modern asset pricing theory assumes that agents' beliefs about the likelihoods of future states of the world may be represented by a probability measure. As a result, no meaningful distinction is allowed between risk, where probabilities are available to guide choice, and uncertainty, where information is too imprecise to be summarized adequately by probabilities. In contrast, Knight and Keynes emphasized the distinction between risk and uncertainty and argued that uncertainty is more common in economic decision-making. Moreover, the Savage model is contradicted by evidence, such as the Ellsberg Paradox, that people prefer to act on known rather than unknown or vague probabilities. This paper provides a formal model of asset price determination in which Knightian uncertainty plays a role. Specifically, we extend the Lucas (1978) general equilibrium pure exchange economy by suitably generalizing the representation of beliefs along the lines suggested by Gilboa and Schmeidler. Two principal results are the proof of existence of equilibrium and the characterization of equilibrium prices by an "Euler inequality." A noteworthy feature of the model is that uncertainty may lead to equilibria that are indeterminate, that is, there may exist a continuum of equilibria for given fundamentals. That leaves the determination of a particular equilibrium price process to "animal spirits" and sizable volatility may result. Finally, it is argued that empirical investigation of our model is potentially fruitful.

KEYWORDS: Uncertainty, ambiguity, vagueness, multiple prior, asset pricing, price indeterminacy, price volatility, recursive utility.

1. INTRODUCTION

MODERN ASSET PRICING THEORY typically adopts strong assumptions about agents' beliefs. According to the rational expectations hypothesis, for example, there exists an objective probability law describing the state process, and it is assumed that agents know this probability law precisely. More generally, even if existence of the latter is not assumed, each agent's beliefs about the likelihoods of future states of the world are represented by a subjective probability measure or Bayesian prior, in conformity with the Bayesian model of decision-making and, more particularly, with the Savage (1954) axioms. As a result, no meaningful distinction is allowed between risk, where probabilities are available to guide choice, and uncertainty, where information is too imprecise to be summarized adequately by probabilities. In contrast, Knight (1921) emphasized the distinction between risk and uncertainty and argued that uncertainty is more common in economic decision-making.[2] Particularly in the context of asset prices, Keynes emphasized the importance of "animal spirits" when, because of Knightian uncertainty, individuals cannot estimate probabilities reliably and so cannot

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[2] See, however, LeRoy and Singell (1987), for a different interpretation of Knight.

make a good calculation of expected values. (See Keynes (1936) and (1921, Chapter 6); see also Koppl (1991) for discussion and additional references.)

This paper provides a formal model of asset price determination in which Knightian uncertainty plays a role. Specifically, we extend the Lucas (1978) general equilibrium pure exchange economy by suitably generalizing the representation of beliefs. Two principal results are the proof of existence of equilibrium and the characterization of equilibrium prices by an "Euler inequality." The latter represents the appropriate generalization of the standard Euler equation to the context of uncertainty. A noteworthy feature of our model is that uncertainty may lead to equilibria that are indeterminate; that is, there may exist a continuum of equilibria for given fundamentals. That leaves the determination of a particular equilibrium price process to "animal spirits" and sizable volatility may result. Overall our model conforms closely to Keynes' (1936, p. 154) description of the consequences of uncertainty:

> "A conventional valuation which is established as the outcome of the mass psychology of a large number of ignorant individuals is liable to change violently as a result of a sudden fluctuation of opinion due to factors which do not really make much difference to the prospective yield; since there will be no strong roots of conviction to hold it steady."

Besides the motivation provided by the intuitively appealing ideas of Knight and Keynes, our paper is motivated also by evidence that people prefer to act on known rather than unknown or vague probabilities. For example, they typically prefer to bet on drawing a red ball from an urn containing 50 red and black balls each, than from an urn containing 100 red and black balls in undisclosed proportions. The best known such evidence is the Ellsberg Paradox (Ellsberg (1961)); the large body of empirical evidence inspired by this paradox, both experimental and market-based, is surveyed by Camerer and Weber (1992). Behavior such as that exhibited in the context of the Ellsberg Paradox contradicts the Bayesian paradigm, that is the existence of any prior underlying choices. Intuitively, the reason is that a probability measure cannot adequately represent both the relative likelihoods of events and the amount, type, and reliability of the information underlying those likelihoods. On the other hand, in a multiperiod setting such as ours, one may wonder whether "vagueness" might disappear asymptotically as a result of learning by the agent, at least if the environment is stationary. Learning in the presence of uncertainty has not yet been studied sufficiently well to provide a definitive theoretical answer to this question. In any event, it would seem that economic processes are typically too complicated or unstable to be modeled in detail and understood precisely. Thus we would not presume uncertainty to be strictly a short-run phenomenon. See Walley (1991, Chapter 5) for further arguments about the general prevalence of imprecision and Zarnowitz (1992, pp. 61–63) for cogent arguments in a business cycle setting. In an asset pricing context, Barsky and DeLong (1992) argue that there is substantial uncertainty about the structure of the aggregate dividend process in the U.S. over the last century, even on the part of current analysts who have the benefit of hindsight. In addition, many processes of interest are presumably physically indeterminate; Papamarcou and Fine (1991) describe an empirical process that generates relative frequencies that can be modeled by a set of probability measures, but not by any single probability measure.

Ultimately, our objective is to investigate whether the noted shortcoming of the Bayesian paradigm is at all responsible for any of the empirical failures of the consumption-based asset pricing model derived from Lucas (1978). While serious empirical analysis is beyond the scope of this paper, we will address the empirical content of our model informally. We do so first in Section 3.4 where we indicate the potential usefulness of our model for resolving the excess volatility puzzle (Shiller (1981) and Cochrane (1991), for example). Further discussion of empirical content is provided in Section 4.

There are now available a number of extensions of the Bayesian model that admit a distinction between risk and uncertainty. One, due to Bewley (1986), drops Savage's assumption that preferences are complete and adds a model of the "status quo." An alternative direction, due to Gilboa and Schmeidler (1989), is to weaken Savage's Sure-Thing Principle. The consequence for the representation of preferences and beliefs is that Savage's single prior is replaced by a set of priors. In this paper, we take this multiple-priors model as our starting point.[3] Then, since our framework is intertemporal and since the Gilboa-Schmeidler (1989) framework is atemporal and deals exclusively with one-shot choice, we extend their model (nonaxiomatically) to an intertemporal, infinite-horizon setting. Moreover, this is done in a way that delivers two attractive properties of the standard expected additive utility model that dominates economics and finance—dynamic consistency and tractability. Since such an extension is potentially useful for addressing issues other than asset pricing where uncertainty may be important, we view it as a separate contribution of the paper.

While the rational expectations hypothesis has considerable a priori appeal for economists, it has come under scrutiny in recent years because of apparently contradictory empirical evidence. We have already mentioned the asset pricing anomalies that indicate rejection of a collection of joint hypotheses that includes rational expectations. In addition, where it has been tested separately by means of survey data, the rational expectations hypothesis has generally been rejected (for example, see Cragg and Malkiel (1982), Zarnowitz (1984), Ito (1990), Frankel and Froot (1990)). As a result, models with "irrational expectations" have been developed, involving "fads" (Shiller (1991), Barsky and DeLong (1992)) or "noise traders" (DeLong et al. (1990)). A focus on beliefs is shared by the model proposed in this paper, though, in a sense, we deviate much less from the standard Lucas-style model. One can interpret our model as differing from Lucas' only by replacing the Sure-Thing Principle and its implied Bayesian prior, by the Gilboa-Schmeidler set of axioms, suitably adapted to the intertemporal framework, and the resulting set of priors.

[3] A closely related model, due to Schmeidler (1989) and Gilboa (1987), replaces the Bayesian prior by a nonadditive probability measure or capacity. In an earlier working paper, the capacity-based model was adopted as a starting point and virtually the identical results were obtained.

We proceed as follows: Section 2 describes our model of intertemporal utility, including beliefs. Equilibrium asset pricing is studied in Section 3. We conclude in Section 4 with some comments on the empirical content of our model. Technical details are collected in appendices.

2. INTERTEMPORAL UTILITY

2.1. *Background*

The standard specification of utility over infinite horizon consumption processes is given by

$$(2.1.1) \quad U(c) = E\left(\sum_{1}^{\infty} \beta^t u(c_t)\right),$$

or in recursive form

$$(2.1.2) \quad U(c) = u(c_1) + \beta EU(c_2, c_3, \ldots).$$

Here E denotes the expectation operator conditional on available information; other notation is standard and will shortly be defined precisely in any event, as will the underlying stochastic environment. Beliefs about the likelihoods of future underlying states of the world are represented by a conditional probability measure π.

In the rational expectations paradigm, π is an objective probability law that governs the evolution of states of the world and is assumed known to the decision-maker. An alternative justification for π is the Savage representation theorem according to which π is a subjective probability measure; an objective probability law need not exist in principle. In either approach, a role for Knightian uncertainty or imprecise information is excluded a priori, either because information is assumed to be precise, or, in the second approach, because the Savage axioms imply that imprecision is a matter of indifference to the decision-maker (as discussed further below).

Our objective is to investigate the implications of imprecise information and thus we need to adopt a more general representation for beliefs. In order to focus more sharply on our objective, we consider otherwise "minimal" variations of (2.1.1) and (2.1.2).

2.2. *The Environment and Beliefs*

The set of states is Ω, a compact metric space with Borel σ-algebra $\mathscr{B}(\Omega)$. Under the weak convergence topology, $\mathscr{M}(\Omega)$, the space of all Borel probability measures, is also a compact metric space. At time t the decision-maker observes some realization $\omega_t \in \Omega$. Beliefs about the evolution of the process $\{\omega_t\}$ conform to a time-homogeneous Markov structure. In standard models, this would involve a Markov probability kernel giving conditional probabilities. Here we assume that beliefs conditional on ω_t are too vague to be represented by a probability measure and are represented instead by a set of probability measures. Thus we model beliefs by a *probability kernel correspondence* $\mathscr{P}$, which is a (nonempty valued) correspondence $\mathscr{P}:\Omega\to\mathscr{M}(\Omega)$, assumed to be continuous, compact-valued, and convex-valued. For each $\omega\in\Omega$, we think of $\mathscr{P}(\omega)$ as the set of probability measures representing beliefs about next period's state. However, the rigorous interpretation of $\mathscr{P}$ is as a component of the representation of the preference ordering over consumption processes as described in the next subsection.

Anticipating somewhat the noted representation of preferences, adapt common terminology and refer to the multivalued nature of $\mathscr{P}$ as reflecting *uncertainty aversion* of preferences (see Schmeidler (1989, Proposition, p. 582)). In fact, the multivaluedness of $\mathscr{P}$ reflects *both* the presence of uncertainty *and* the agent's aversion to uncertainty; for our purposes, there is no need to attempt to define a meaningful distinction between the "absence of uncertainty" on the one hand, and the presence of uncertainty accompanied by indifference to it on the other. If $\mathscr{P}$ is singleton-valued, then $\mathscr{P}=\{\pi\}$, where π is a *probability kernel*, that is, a continuous map from Ω into $\mathscr{M}(\Omega)$. Since this Bayesian representation of beliefs excludes any role for uncertainty, we refer to uncertainty neutrality or indifference in this case.

It will be convenient to adopt the following notation: for any bounded, Borel-measurable $f:\Omega\to\mathbb{R}$ and for any set $\boldsymbol{P}\subset\mathscr{M}(\Omega)$,

$$(2.2.1)\qquad \int_\Omega f\,d\boldsymbol{P}\equiv\inf\left\{\int_\Omega f\,dm : m\in\boldsymbol{P}\right\}$$

and accordingly,

$$(2.2.2)\qquad \boldsymbol{P}(A)\equiv\inf\{m(A):m\in\boldsymbol{P}\},\qquad A\in\mathscr{B}(\Omega).$$

In particular, if $\boldsymbol{P}=\mathscr{P}(\omega)$ for some ω, then

$$(2.2.3)\qquad \mathscr{P}(\omega,A)\equiv\inf\{m(A):m\in\mathscr{P}(\omega)\},$$

and for any continuous f,

$$(2.2.4)\qquad \int f(\cdot)\,d\mathscr{P}(\omega,\cdot)\equiv\int f\,d\mathscr{P}(\omega)\equiv\min\left\{\int f\,dm : m\in\mathscr{P}(\omega)\right\}.$$

Note that the latter minimum exists since $\mathscr{P}(\omega)$ is compact and the map $m\mapsto\int f\,dm$ is continuous by the nature of the weak convergence topology. We stretch common terminology and refer to the expressions on the left sides of (2.2.1) and (2.2.4) as "integrals" or "expected values."[4]

[4] Under suitable additional restrictions on $\boldsymbol{P}$, the map $A\mapsto\boldsymbol{P}(A)$ defines a capacity and $\int f\,d\boldsymbol{P}$ equals the associated Choquet integral, that is central to the capacity-based model of Schmeidler (1989) and Gilboa (1987). (See Appendix C.) With this link in mind, we add the following remark concerning the above definition of uncertainty aversion: Not every uncertainty averse $\mathscr{P}$ can accommodate Ellsberg-type behavior; the latter is inconsistent with the "small" subclass of correspondences $\mathscr{P}$ for which the capacity $\mathscr{P}(\omega,\cdot)$, for some ω, defines a qualitative probability (Schmeidler (1989, p. 585)).

On occasion, we will want to impose a link between beliefs and "reality," at least with respect to which events are null or impossible. Suppose therefore that objectively null events are defined in the obvious way by the probability kernel π^*. It is not necessary to assume that the $\{\omega_t\}$ process evolves according to π^* or any other probability kernel. Say that $\mathscr{P}$ is *absolutely continuous* with respect to π^* if

$$(2.2.5) \quad \forall \omega \in \Omega, \forall A \in \mathscr{B}(\Omega), \qquad \pi^*(\omega, A) = 0 \Rightarrow m(A) = 0 \quad \forall m \in \mathscr{P}(\omega);$$

that is, the objective nullity of $A(\pi^*(\omega, A) = 0)$ implies the subjective nullity of A, with "conditioning on ω" understood throughout.

For other purposes, it is useful to have the following property satisfied for all $\omega \in \Omega$ and all continuous functions $f, g : \Omega \to \mathbb{R}_+$,

$$(2.2.6) \qquad f \geqslant g, \qquad \pi^*(\omega, \{\omega' : f(\omega') > g(\omega')\}) > 0$$

$$\Rightarrow \int f d\mathscr{P}(\omega) > \int g d\mathscr{P}(\omega).$$

With this in mind, we assume where explicitly stated that $\mathscr{P}$ has *full support*, that is, $m(A) > 0$ for all $\omega \in \Omega$, $m \in \mathscr{P}(\omega)$ and nonempty open subsets $A \subset \Omega$. Given this assumption, the indicated strict inequality for the integrals holds if $f \geqslant g$ and $f \neq g$. Thus we avoid the expositional and notational clutter associated with qualifications of the form "a.e. $[\pi^*(\omega, \cdot)]$" in various definitions and statements of theorems below. Note that $\mathscr{P}(\omega, A) > 0 \Rightarrow \mathscr{P}(\omega, \Omega \setminus A) < 1$. Therefore, the assumption of full support limits the class of subjectively null events and guarantees that, at least for open sets A, $\mathscr{P}(\omega, \Omega \setminus A) = 1 \Rightarrow A = \varnothing \Rightarrow \pi^*(\omega, A) = 0$, which is the converse of the implication in (2.2.5).

2.3. *Examples of Probability Kernel Correspondences*

Many natural and useful specifications of sets of priors have been studied in the statistics literatures (see, for example, Wasserman (1990), Wasserman and Kadane (1990), and Walley (1991)) and many of these are readily extended to probability kernel correspondences. Here we describe two such examples.

EXAMPLE 1 (ε-Contamination): Fix a probability kernel π^* and a continuous function $\varepsilon : \Omega \to [0,1]$. Let $\mathscr{P}$ be defined by

$$(2.3.1) \quad \mathscr{P}(\omega) \equiv \{(1 - \varepsilon(\omega))\pi^*(\omega) + \varepsilon(\omega)m : m \in \mathscr{M}(\Omega)\}.$$

Then the associated integrals (2.2.2) take the form

$$(2.3.2) \quad \int_\Omega f d\mathscr{P}(\omega) = (1 - \varepsilon(\omega)) \int_\Omega f(\omega')\pi^*(\omega, d\omega') + \varepsilon(\omega) \cdot \inf_\Omega f,$$

and for each $B \in \mathscr{B}(\Omega)$,

$$(2.3.3) \quad \mathscr{P}(\omega, B) = \begin{cases} (1-\varepsilon(\omega))\pi^*(\omega, B) & \text{if } B \neq \Omega, \\ 1 & \text{if } B = \Omega. \end{cases}$$

The correspondence $\mathscr{P}$ has full support if $\varepsilon < 1$ and $\operatorname{supp} \pi^*(\omega) = \Omega$ for all $\omega \in \Omega$. It reduces to the probability kernel π^* if $\varepsilon \equiv 0$. The other extreme, called *complete ignorance*, has $\varepsilon \equiv 1$ and $\mathscr{P}(\cdot) \equiv \mathscr{M}(\Omega)$, in which case

$$\int f \, d\mathscr{P}(\omega) = \inf_{\Omega} f.$$

The set $\mathscr{P}(\omega)$ includes all perturbations of $\pi^*(\omega, \cdot)$, where $\varepsilon(\omega)$ reflects the amount of error deemed possible. Accordingly, a possible rationale for the above specification of $\mathscr{P}$ is that π^* represents the "true" probability law on Ω that the agent knows only imprecisely.[5] Other forms of perturbations are also possible as suggested by the examples in the references cited above. An attractive feature of the particular perturbation represented by (2.3.1) is the explicit formula (2.3.2) available for associated integrals.

Example 2 (Belief Function Kernels): The set of states Ω is assumed to be exhaustive and therefore is presumably large and complex. Consequently, the law of motion on Ω may be too complicated to be understood precisely, or alternatively may not be representable by a probability kernel. Suppose, however, that the agent observes N statistics, each a function of the current state, and that the probability law governing the dynamics of these statistics is known. More precisely, let

$$(2.3.4) \quad G: \Omega \to \mathbb{R}^N$$

and let p be a probability kernel that describes the evolution of $\{G(\omega_t)\}$ as a time-homogeneous Markov process; that is, $p(\cdot \mid y)$ is a conditional probability measure on $G(\Omega)$ that varies continuously with $y \in G(\Omega)$. We assume both that G is continuous and that the inverse $y \to G^{-1}(y)$ is a continuous correspondence. Since, as described below, pay-off relevant variables, such as consumption and dividends, vary with ω_t rather than $G(\omega_t)$, assessment of likelihoods over Ω is essential to the agent. It is important to note that p and G do not imply a probability kernel over Ω unless G is one-to-one. However, a representation of likelihoods in terms of a probability kernel correspondence may be constructed for arbitrary G in the following intuitively plausible fashion: For any $\omega \in \Omega$ and $B \in \mathscr{B}(\Omega)$, let

$$(2.3.5) \quad \mu(\omega, B) \equiv p\left(\{y \in \mathbb{R}^N : G^{-1}(y) \subseteq B\} \mid G(\omega)\right),$$

the probability according to p of those realizations for the statistics that imply B

[5] Another possible rationale for (2.3.1) is based on the hypothesis that the set of states Ω is not exhaustive. See Epstein and Wang (1992) for elaboration and for another class of examples motivated by "missing states."

conditional on the values of the statistics at ω. Now define $\mathscr{P}_G$ by

$$\mathscr{P}_G(\omega) \equiv \{m \in \mathscr{M}(\Omega) : m(B) \geqslant \mu(\omega, B), \forall B \in \mathscr{B}(\Omega)\}. \tag{2.3.6}$$

Then $\mathscr{P}_G$ is continuous and therefore is a probability kernel correspondence.[6]

To elucidate (2.3.6), note that for each $\omega \in \Omega$, $\mu(\omega, \cdot)$ is a special capacity, called a belief function (Dempster (1967)) and $\mathscr{P}_G(\omega)$ is the "core" of $\mu(\omega, \cdot)$; see also Wasserman (1988, 1990), Jaffray (1992), and Schmeidler (1989).

Examination of the integration formulae implied by (2.3.6) provides further insight into the nature of $\mathscr{P}_G$. The analogues of (2.2.3)–(2.2.4) are

$$\mathscr{P}_G(\omega, A) = \mu(\omega, A), \qquad \omega \in \Omega, \qquad A \in \mathscr{B}(\Omega),$$

and

$$\int f \, d\mathscr{P}_G(\omega) \equiv \int f^*(y) \, dp(y | G(\omega)), \tag{2.3.7}$$

where $f^* : G(\Omega) \to \mathbb{R}$ is defined by $f^*(y) \equiv \min\{f(\omega') : G(\omega') = y\}$. Since f^* is defined as the indicated minimum, the integral on the right side of (2.3.7) reflects the agent's ignorance on each level set $\{\omega' : G(\omega') = y\}$. Thus $\mathscr{P}_G$ models the situation where the law of motion p for the statistics G is the *only* information available regarding the law of motion on Ω.

2.4. *Utility*

This subsection completes the description of the utility function over consumption processes, the first component of which is the probability kernel correspondence $\mathscr{P}$.

To define the domain of consumption processes, we need some notation and terminology. The measurable space underlying all random processes is (Ω^∞, the product Borel σ-algebra $\mathscr{B}(\Omega^\infty)$). For $\omega \in \Omega^\infty$ and $t \geqslant 1$, $\omega^t \equiv (\omega_1, \ldots, \omega_t)$; Ω^t is the collection of all such points. Let $\mathscr{B}(\Omega^t)$ be the product Borel σ-algebra and embed it in the usual fashion in $\mathscr{B}(\Omega^\infty)$. A process $\{X_t\}$, $X_t : \Omega^\infty \to \mathbb{R}^n$ for each t, is *adapted* if X_t is $\mathscr{B}(\Omega^t)$-measurable for all t. Given such measurability, we can identify X_t with a map from $\Omega^t \to \mathbb{R}^n$. If each such map is also continuous, refer to the process $\{X_t\}$ as a continuous process. The process is real-valued if $n = 1$.

Consumption processes lie in the complete normed space

$$\mathscr{D} \equiv \Big\{X = \{X_t\} : \{X_t\} \text{ is an adapted and continuous real-valued process, } X_t(\omega^t) \geqslant 0 \text{ for all } t \geqslant 1 \text{ and } \omega^t \in \Omega^t, \text{ and } \|X\| \equiv \sup_t \sup_{\omega^t} |X_t(\omega^t)|/b^t < \infty\Big\},$$

[6] The continuity of $\mathscr{P}_G$ follows from Epstein and Wang (1992, Proposition A.2.1). A closely related form of continuity is apparent from (2.3.7) below. Under our assumptions on p and G, the integrals there vary continuously with ω since f^* is continuous by the Maximum Theorem.

where $b \geqslant 1$ is a fixed real number that provides an upper bound for the average rate of growth of consumption. The restriction to adapted processes is natural; consumption at time t can depend only on information available then. The assumption of continuity is undoubtedly less natural. Nevertheless, it affords considerable analytical simplification and is important for the analysis of equilibrium asset pricing and therefore seems appropriate in our attempt to balance mathematical generality with economic significance and accessibility.[7] Consumption processes are typically denoted by $c = \{c_t\}$. Since $\mathscr{D}$ will also be the ambient space for utility and price processes, the "neutral" dummy variable X is used above. An element X in $\mathscr{D}$ is Markovian if for each t and $\omega_t \in \Omega$, $X_t(\cdot, \omega_t)$ is constant on Ω^{t-1} and time-homogeneous if in addition X_t does not vary with t.

Utilities over $\mathscr{D}$ are defined by three primitives: a probability kernel correspondence $\mathscr{P}$, a discount factor $\beta \in (0,1)$, and an instantaneous utility or felicity function $u : \mathbb{R}_+ \to \mathbb{R}_+$ assumed to be continuous, increasing, concave, and normalized to satisfy $u(0) = 0$.

For each given c in $\mathscr{D}$ we define a utility process $\{V_t(c)\}_1^\infty$ as the unique element of $\mathscr{D}$ satisfying the following recursive relation: for all $t \geqslant 1$ and ω^t in Ω^t,

$$(2.4.1) \quad V_t(c;\omega^t) = u\big(c_t(\omega^t)\big) + \beta \int V_{t+1}(c;\omega^t,\cdot)\, d\mathscr{P}(\omega_t,\cdot),$$

where $V_t(c;\omega^t)$ denotes $V_t(c)(\omega^t)$. Think of $V_t(c;\omega^t)$ as the utility of the continuation consumption process ${}^tc \equiv (c_t, c_{t+1}, \dots)$ conditional on the history ω^t. The (initial) utility of the entire path c is $V_1(c;\omega_1)$.

The interpretation of (2.4.1) is clear. Given the history ω^t at time t, the individual evaluates the consumption process for the remaining future in two stages. First, the future from $(t+1)$ onward is evaluated by means of the "expected value" of V_{t+1} with respect to beliefs $\mathscr{P}(\omega_t)$. This summary index of future is then combined with the instantaneous utility of time t consumption to define the utility of the consumption process from t onward. If each $\mathscr{P}(\omega_t)$ is a singleton probability measure, then (2.4.1) reduces to the standard model (2.1.2). Uncertainty aversion is incorporated into preferences in the general case by permitting $\mathscr{P}(\omega_t)$ to be multivalued.[8]

[7] The class of functions that are analytic in the sense of analytic set theory (see Dellacherie and Meyer (1982)) is the appropriate one for Choquet integration, which is closely related to the integration notion (2.2.1) employed here. Therefore, future extensions of $\mathscr{D}$ may need to go beyond the space of adapted processes to include processes for which each X_t is analytic.

[8] Epstein and Zin (1989) study recursive relations of the form $V_t(c;\omega^t) = W(c(\omega^t), m(V_{t+1}(c;\omega^t,\cdot)))$, where m is a generalized certainty equivalent operator. Relation (2.4.1) is the special case in which $W(c,z) = u(c) + \beta z$ and m is the generalized expected value operator (2.2.4). Several of our results can be extended considerably beyond the specification (2.4.1) by applying and adapting available results on recursive utility, but such extensions would detract from the main focus of this paper. Note also that the presence of uncertainty aversion introduces an important technical difference, namely a lack of Gâteaux differentiability, relative to the analysis in Epstein and Zin. See Sections 2.5 and 3 for elaboration and for the economic significance of the nondifferentiability.

By routine arguments based on the contraction mapping theorem, we show in Appendix A that utilities are well-defined by (2.4.1). To state our theorem, adopt the notation ${}^t c|\omega^{t-1} \equiv \{c_\tau(\omega^{t-1}, \cdot)\}_{\tau=t}^{\infty} \in \mathscr{D}$, for the continuation of c given the history ω^{t-1} preceding t. Also, if c and c' are elements of $\mathscr{D}$, write $c' > c$ if $c' \neq c$ and $c'_t \geqslant c_t$ for all t; $c' \gg c$ if $c' > c$ and there exists t such that $c'_t(\omega^t) > c_t(\omega^t)$ for all $\omega^t \in \Omega^t$. Finally, if $U: \mathscr{D} \to \mathbb{R}_+$, say that U is (*strictly*) *increasing* if $(c' > c)$ $c' \gg c$ implies $U(c') > U(c)$.

THEOREM 1 (Existence of Utility): *Suppose that* $\beta b < 1$. *Then for each* $c \in \mathscr{D}$, *there exists a unique* $V(c) \in \mathscr{D}$ *satisfying* (2.4.1). *Moreover, for all* c, $t \geqslant 1$ *and* ω^t,

$$(2.4.2) \quad V_t(c;\omega^t) = V_1({}^t c|\omega^{t-1};\omega_t).$$

For each $\omega \in \Omega$, $V_1(\cdot;\omega)$ *is increasing and concave on* $\mathscr{D}$; *it is strictly increasing if* $\mathscr{P}$ *has full support. Finally, if* u *satisfies a growth condition, that is, if there exist constants* k_1 *and* $k_2 > 0$ *such that* $u(x) \leqslant k_1 + k_2 x$ *for all* $x \in \mathbb{R}_+$, *then* $V_1(c;\omega)$ *is jointly continuous on* $\mathscr{D} \times \Omega$.

Condition (2.4.2) asserts that time t utility equals a time-invariant function of the continuation consumption path ${}^t c|\omega^{t-1}$ and the current state ω^t. This follows from the time-homogeneous, first-order Markov structure assumed for beliefs. Since the time 1 designation is irrelevant, we denote $V_1(c;\omega)$ simply by $V(c;\omega)$ and refer to V as *the utility function* defined by (2.4.1). By the last part of the theorem, V possesses some standard regularity conditions. Note that the assumption $\beta b < 1$ is adopted throughout.

Another important property of V, or at least of the entire utility process, is dynamic consistency. The recursive construction of utility via (2.4.1) suggests that dynamic consistency (suitably defined) will be satisfied. To be more precise, each $V_t(\cdot;\omega^t)$ is a utility function over $\mathscr{D}$; denote by $\{V_t\}$ the corresponding process of utility functions. Say that $\{V_t\}$ is *dynamically consistent* if for all $\omega_1 \in \Omega$, c' and c in $\mathscr{D}$ and $T \geqslant 1$, $V_1(c';\omega_1) > V_1(c;\omega_1)$ if:

(i) $c'_t = c_t$ for $t = 1, \ldots, T-1$,

(ii) $V_T(c';\omega_1, \cdot) \neq V_T(c;\omega_1, \cdot)$, and

(iii) $V_T(c';\omega_1, \cdot) \geqslant V_T(c;\omega_1, \cdot)$ on Ω^{T-1}.

Say that $\{V_t\}$ is *weakly dynamically consistent* if (i)–(iii) imply only that $V_1(c';\omega_1) \geqslant V_1(c;\omega_1)$. The stronger notion of dynamic consistency is the counterpart for our framework of the usual definition (Epstein and Zin (1989), Duffie and Epstein (1992), for example). Only the weaker notion is satisfied in general by the process $\{V_t\}$, since the set of histories $\omega_2, \ldots, \omega_{T-1}$ where $V_T(c';\omega_1, \cdot) > V_T(c;\omega_1, \cdot)$ could be "null" from the perspective of time 1 and the beliefs prevailing there and thus $V_1(c';\omega_1) = V_1(c;\omega_1)$ is possible. That possibility is ruled out if $\mathscr{P}$ has full support, in which case dynamic consistency holds (see Appendix A). However, even if only weak dynamic consistency obtains, we show that our asset pricing model of Section 3 has an equilibrium along which optimal plans are carried out.

Risk aversion for V is not mentioned above since it is well-defined only given the existence of probabilities that can be used to define actuarial fairness. For that purpose, suppose there exist events in $\mathscr{B}(\Omega)$ that can be assigned probabilities; that is, suppose $\mathscr{B}^*$ is a sub-sigma algebra of $\mathscr{B}(\Omega)$ such that for each ω, any two measures in $\mathscr{P}(\omega)$ agree when restricted to $\mathscr{B}^*$. Then V has the form (2.1.1) on the subdomain of consumption processes defined by $\mathscr{B}^*$-measurability, and so is clearly risk averse there.

Finally, in this subsection we relate our recursive model of utility to Gilboa and Schmeidler (1989) and argue that (2.4.1) represents a sensible extension of their atemporal model to an intertemporal framework. An alternative extension has the following form: there exists a correspondence $\mathscr{K}: \Omega \to \mathscr{M}(\Omega^\infty)$, with the set of measures $\mathscr{K}(\omega_t)$ representing beliefs at (t, ω_t) about the entire future, such that intertemporal utility U_t is given by

$$(2.4.3)\quad U_t(c;\omega^t) = \int_{\Omega^\infty}\left[\sum_t^\infty \beta^{i-t}u(c_i)\right] d\mathscr{K}(\omega_t)$$
$$\equiv \inf\left\{\int_{\Omega^\infty}\left[\sum_t^\infty \beta^{i-t}u(c_i)\right] dm : m \in \mathscr{K}(\omega_t)\right\}.$$

In comparing (2.4.1) and (2.4.3), note first that they coincide under uncertainty neutrality but not more generally. In particular, if $\mathscr{P}$ is a probability kernel, then, given ω_t, it determines a unique probability measure $p(\omega_t)$ on Ω^∞ and (2.4.3) is derived with $\mathscr{K}(\omega_t)$ equal to the singleton $\{p(\omega_t)\}$. However, such a derivation of (2.4.3) from (2.4.1) fails more generally since the additivity property of Lebesgue integration with respect to a probability measure is not satisfied by "integration" with respect to a set of probability measures.

Given that (2.4.1) and (2.4.3) represent distinct models of intertemporal utility, one is left wondering which is more attractive. A definitive judgement would presumably require an examination of the axiomatic underpinnings of each model.[9] While such an examination is beyond the scope of this paper, we can point to an axiomatic difference between the two models that is important and supportive of (2.4.1) at least when "time" is taken seriously. That feature is simply that $\{U_t\}$ is generally weakly dynamically inconsistent. Therefore, in the absence of an explanation of how dynamic inconsistency is resolved, the model (2.4.3) does not deliver predictions about choice behavior. In an important sense, therefore, the model (2.4.3) is incomplete; in particular, it is not clear how it should be applied to describe consumption/savings behavior and asset

[9] Though we have not axiomatized our model, there is reason to believe that it can be provided with a respectable axiomatic basis. That is because in the literature on preferences under risk, the corresponding question of how to extend atemporal theories has been thoroughly examined and the recursive approach has been provided with respectable axiomatic credentials (see Kreps and Porteus (1978), Chew and Epstein (1991), and, for an overview, see Epstein (1993)). In addition, Skiadas (1992) axiomatizes recursive utility in a Savage-style framework where conditional subjective probabilities are derived.

price determination in the model economy of Section 3. A game-theoretic resolution of dynamic inconsistency has been examined in related models, but the tractability of such an approach is a serious concern in the setting of Section 3.

There is a closely related observation concerning (2.4.3) that also merits mention. One might think of adopting the specification (2.4.3) at $t = 1$ and then suitably updating the set of priors $\mathscr{K}(\omega_t)$ as time proceeds. However, any updating rule will invariably imply the weak dynamic inconsistency of preferences, excluding a "small" number of arguably uninteresting specifications for $\mathscr{K}(\omega_t)$, one of which is that $\mathscr{K}(\omega_t)$ is a single probability measure (see Epstein and LeBreton (1993)). This difficulty reflects the problematic nature of rules for updating vague beliefs that is now well recognized (see Walley (1991, pp. 279–281), Gilboa and Schmeidler (1993), Jaffray (1992), Epstein and LeBreton (1993)).

In contrast, our model of utility delivers weak dynamic consistency. By adopting "conditional belief," represented by $\mathscr{P}$, as the primitive, we obviate the need for an updating rule. Moreover, we feel that the recursive framework has some psychological plausibility because of the algorithmic appeal of backward induction.

2.5. *Utility Supergradients*

Since we will be concerned below with the (shadow) pricing of securities, we are led naturally to an examination of the supergradients, suitably defined, of our utility function V. A novel feature of V relative to utility functions that have been generally applied previously in the macro/finance literature is that $V(\cdot, \omega)$ is "frequently" nondifferentiable in the Gâteaux sense unless $\mathscr{P}$ is a probability kernel. However, since $V(\cdot; \omega)$ is concave, it possesses one-sided Gâteaux derivatives. Here we derive representations for these one-sided derivatives and the associated supergradients. These representations are applied to the security valuation problem in Section 3.2.

Let $e \in \mathscr{D}$ represent a base consumption process that is everywhere strictly positive and consider the effect on utility of perturbations in specified directions. It will suffice to consider perturbations in "today's" and "next period's" consumption only, that is, consider the change from e to $e + \xi h$ where $\xi \in \mathbb{R}^1$ and $h = \{h_t\}_1^\infty$ is a continuous real-valued process, such that $h_t \equiv 0$ for $t \neq 1, 2$, $h_1 \in \mathbb{R}$, and $h_2 \in C(\Omega)$. Note that $e + \xi h \in \mathscr{D}$ for sufficiently small ξ. Therefore, $V(e + \xi h; \omega)$ is defined for such ξ. Since V is defined via a minimum over a set of probability measures as in (2.2.4) and (2.4.1), one-sided directional derivatives of V may be derived by an appropriate "envelope theorem." The one-sided derivatives are described in the following important lemma, which is a special case of the envelope theorem result in Aubin (1979, p. 118). For simplicity, the lemma deals with the case where e is Markovian and time-homogeneous.

LEMMA 1: *Let $e \in \mathscr{D}$ be a positive, Markovian and time-homogeneous consumption process with $e_t(\omega^t) = e^*(\omega_t)$. Let $h = \{h_t\}_1^\infty$ with $h_t = 0$ for $t \neq 1,2$, $h_1 \in \mathbb{R}$, and $h_2 \in C(\Omega)$. Define the convex-valued and compact-valued correspondence $Q : \Omega \to \mathscr{M}(\Omega)$ by*

$$(2.5.1)\quad Q(\omega) = \left\{ m \in \mathscr{P}(\omega) : \int V^* \, dm = \int V^* \, d\mathscr{P}(\omega) \right\},$$

where

$$(2.5.2)\quad V^*(\omega) \equiv V(e;\omega), \qquad \omega \in \Omega.$$

Then the one-sided Gâteaux derivatives of $V(\cdot, \omega)$ at e and in the direction h, are given by

$$(2.5.3)\quad \frac{d}{d\xi} V(e + \xi h; \omega)\bigg|_{0^+} = u'(e^*(\omega))h_1 + \beta \min\left\{ \int u'(e^*) h_2 \, dm : m \in Q(\omega) \right\},$$

$$\frac{d}{d\xi} V(e + \xi h; \omega)\bigg|_{0^-} = u'(e^*(\omega))h_1 + \beta \max\left\{ \int u'(e^*) h_2 \, dm : m \in Q(\omega) \right\}.$$

The Lemma suggests, and this will be confirmed by examples below, that utility is not Gateaux differentiable in general. The "origin" of this nondifferentiability is clear since utility is defined via a pointwise minimum, namely the "integral" on the right side of (2.4.1), corresponding to the Gilboa-Schmeidler (1989) way of modeling uncertainty aversion, and a pointwise minimum of functions is not differentiable in general.

The particular representation for one-sided derivatives provided in (2.5.3) is also intuitive for "envelope theorem" reasons. To elaborate and in order to pave the way for its role in our study of asset prices, we spell out the following interpretation for $Q : m \in Q(\omega)$ if and only if m is (i) "compatible" with beliefs, in the sense of lying in $\mathscr{P}(\omega)$, and (ii) "equivalent" to $\mathscr{P}(\omega)$, in the sense of the calculation of expected future utility for the given base process e. Since any single prior reflects the absence of (or indifference to) uncertainty, the relation between $\mathscr{P}(\omega)$ and each $m \in Q(\omega)$ is akin to that between a random payoff and its certainty equivalent familiar in the case of risk. Accordingly, refer to $Q(\omega)$ as the set of *uncertainty adjusted* probability measures corresponding to $\mathscr{P}(\omega)$, for the given e.

A critical question for our purposes is whether the nondifferentiability suggested by (2.5.3) is likely to be sufficiently frequent to be "significant." We postpone discussion of this question until Section 3.4, at which point the relevance of (2.4.4) for asset pricing will have been described.

Finally, in this subsection, we provide an alternative formulation of Lemma 1. Let e and h be as above. Refer to s as a (one-period ahead) supergradient of $V(\cdot,\omega)$ at e if s is a continuous linear functional on $\mathbb{R}\times C(\Omega)$ satisfying

(2.5.4) $V(e+h;\omega)-V(e;\omega)\leqslant s(h_1,h_2)$

for all (h_1,h_2) such that $e+h\in\mathscr{D}$. Denote by $\mathscr{M}^+(\Omega)$ the space of positive countably additive measures on Ω endowed with the weak topology induced by $C(\Omega)$. By the Riesz Representation Theorem, each s can be identified with an element $(s_1,p)\in\mathbb{R}_+\times\mathscr{M}^+(\Omega)$ in the sense that

$$s(h_1,h_2)=s_1h_1+\beta\int_\Omega h_2\,dp,\qquad (h_1,h_2)\in\mathbb{R}\times C(\Omega).$$

Lemma 1 shows that $s_1=u'(e^*(\omega))$ and $dp=u'(e^*)\,dm$ for some $m\in Q(\omega)$. Therefore the set of supergradients of $V(\cdot;\omega)$ at $e,\partial V(e;\omega)$, viewed as a subset of $\mathbb{R}_+\times\mathscr{M}^+(\Omega)$, is given by

(2.5.5) $$\partial V(e;\omega)=\{(u'(e^*(\omega)),p):p\in\mathscr{M}^+(\Omega),\exists m\in Q(\omega),\ dp=u'(e^*)\,dm\}.$$

The continuity of the correspondence $\omega\mapsto\partial V(e;\omega)$ is important in the proof of existence of an equilibrium in the economies to which we now turn.

3. EQUILIBRIUM ASSET PRICING

3.1. *The Economy*

We consider an extension of the Lucas (1978) pure exchange economy, having a representative agent, or equivalently a number of agents with identical preferences and endowments. Preferences are as above, with the exception that we add the assumptions that the felicity function u is strictly increasing and continuously differentiable, with $u'(0)=\infty$ admissable. Such a "minimal" variation of the Lucas model seems appropriate given our focus on the effects of uncertainty aversion.

There is a single perishable consumption good with the total supply available at any time and state described by the endowment process $e=\{e_t\}\in\mathscr{D}$. For simplicity, assume that the endowment process has a time-homogeneous Markov structure, in the sense that for some function e^*,

(3.1.1) $e_t(\omega^t)=e^*(\omega_t)\qquad t\geqslant 1,\qquad \omega^t\in\Omega^t,$

and that endowments are positive, that is,

(3.1.2) $e^*(\omega)>0\qquad$ on Ω.

There are n securities, where the ith provides the dividend process $d_i=\{d_{i,t}\}\in\mathscr{D}$. In each period, the securities are traded in a competitive market at prices $q_i=\{q_{i,t}\}\in\mathscr{D}$, $i=1,\ldots,n$, with consumption in each period serving as numeraire.[10] Write $q_t\equiv(q_{1,t},\ldots,q_{n,t})$ and $q=\{q_t\}\in\mathscr{D}^n$. Without loss of gen-

[10] In particular, we assume that for each security buying and selling prices coincide. In fact, the presence of uncertainty aversion can "explain" bid-ask spreads even in the absence of transactions costs. We leave this extension of our model to a separate paper.

erality, that is, by redefining e if necessary, we can assume that each asset is available in zero net supply at all times and states.

At the beginning of each period, the consumer plans consumption and portfolio holdings for the current period and all future periods in order to maximize intertemporal utility. Plans are represented by a pair (c, θ), where $c \in \mathscr{D}$ and $\theta = \{\theta_t\}$ is a continuous process with $\theta_t = (\theta_{1,t}, \ldots, \theta_{n,t})$ representing the portfolio plan for period t. Consider a time-history pair (t, ω^t). Refer to (c, θ) as *(t, ω^t)-feasible* if for all $\tau \geqslant t$

$$q_\tau \cdot \theta_\tau + c_\tau = \theta_{\tau-1} \cdot [q_\tau + d_\tau] + e_\tau, \qquad \theta_{t-1}(\omega^{t-1}) \equiv 0, \qquad \text{and}$$

$$\inf_{i, \tau, \omega^\tau} \theta_{i,\tau}(\omega^\tau) > -\infty,$$

where the latter is a weak restriction on short sales and $\theta_0 \equiv 0$.[11] Say that the (t, ω^t)-feasible plan (c, θ) is *(t, ω^t)-optimal* if $V({}^tc|\omega^{t-1}; \omega^t) \geqslant V({}^tc'|\omega^{t-1}; \omega^t)$ for all other plans (c', θ') that are (t, ω^t)-feasible.

An *equilibrium* is a price process $\{q_t\}_1^\infty \in \mathscr{D}^n$ such that $\{(e_\tau, 0)\}_1^\infty$ is a (t, ω^t)-optimal plan for all $t \geqslant 1$ and $\omega^t \in \Omega^t$. In an equilibrium, spot asset and consumption good markets clear at any (t, ω^t) when the agent optimizes given expectations regarding future prices described by $\{q_\tau\}_{t+1}^\infty$; and subsequently, those expectations are fulfilled in that they clear later spot markets. Note that the consumer is dynamically consistent in equilibrium in the sense that the given (t, ω^t)-optimal plan remains optimal from the perspective of all later times and histories. A weaker notion of equilibrium would require only that $\{(e_t, 0)\}_1^\infty$ be $(1, \omega_1)$-optimal. The relation between these two equilibrium notions is clarified in Theorem 2. The term "equilibrium" is reserved for the first definition.

3.2. *Euler Inequalities*

In this section we derive necessary conditions for an equilibrium from the first-order conditions for the agent's optimization problem. These conditions generalize the standard Euler equations; they take the form of inequalities, rather than equalities, because $V(\cdot; \omega)$ is generally nondifferentiable in the Gâteaux sense (see Section 2.5) unless $\mathscr{P}$ is a probability kernel.

Suppose $\{q_t\}$ is an equilibrium. At any given (t, ω^t), consider a variation (c, θ) of the optimal policy such that $c_\tau = e_\tau$ and $\theta_\tau = 0$ for $\tau \neq t$, $t+1$, $c_t = e_t - \xi(\Delta \cdot q_t)$, $\theta_t = \xi\Delta$, $\theta_{t+1} = 0$, and $c_{t+1} = e_{t+1} + \xi\Delta \cdot (q_{t+1} + d_{t+1})$, where $\Delta \in \mathbb{R}^n$ represents the direction in which the period t portfolio is perturbed and $\xi \in \mathbb{R}$ represents the "size" of the perturbation. Any such perturbation must leave the

[11] By c_τ, we mean the function $c_\tau(\omega^t, \cdot)$ on $\Omega^{\tau-t}$ and similarly for q_τ, θ_τ and so on. The indicated equality and inequality are intended at the level of functions and so apply throughout $\Omega^{\tau-t}$. Similar simplifying notation is adopted throughout the paper. Finally, note that restrictions on short sales are commonly assumed in the literature in order to guarantee existence of planning optima and equilibria.

agent worse-off. In other words, if

$$h_t \equiv -\Delta \cdot q_t \qquad \text{and} \qquad h_{t+1} \equiv \Delta \cdot (q_{t+1} + d_{t+1}),$$

then in the obvious notation

$$(3.2.1) \quad 0 \in \underset{\xi}{\operatorname{argmax}} V(e + \xi(h_t, h_{t+1}, 0, \ldots); \omega_t).$$

By Lemma 1, the first-order conditions for this problem take the form[12]

$$\beta \min_{m \in Q(\omega_t)} \int u'(e^*) \Delta \cdot (q_{t+1} + d_{t+1})\, dm$$

$$\leqslant u'(e_t) \Delta \cdot q_t \leqslant \beta \max_{m \in Q(\omega_t)} \int u'(e^*) \Delta \cdot (q_{t+1} + d_{t+1})\, dm.$$

We can rewrite these inequalities in the more compact and equivalent form

$$(3.2.2) \quad \min_{m \in Q(\omega_t)} \left\{ \beta E_m \left[\frac{u'(e_{t+1})}{u'(e_t)} \Delta \cdot (q_{t+1} + d_{t+1}) \right] - \Delta \cdot q_t \right\} \leqslant 0 \qquad \forall \Delta \in \mathbb{R}^n,$$

where E_m denotes integration with respect to the probability measure m.

We wish to express this infinite collection of inequalities in a more efficient and useful way. In usual formulations, where differentiability obtains, there is no loss in restricting Δ to the coordinate directions. Such equivalence fails here, however, since the expression in (3.2.2) is not linear in Δ, or equivalently, the one-sided Gâteaux derivatives of V, described in Lemma 1, are not linear in the perturbation. Therefore, a slightly more elaborate procedure is required.

First, rewrite (3.2.2) in the more manageable form

$$\sup_{\Delta} \min_{m \in Q(\omega_t)} F(m, \Delta) \leqslant 0.$$

Since $F(m, \cdot)$ is linearly homogeneous, this inequality is equivalent to

$$\max_{\Delta \in \gamma} \min_{m \in Q(\omega_t)} F(m, \Delta) \leqslant 0,$$

where γ is the convex hull of $\{\pm i\text{th unit coordinate vector: } i = 1, \ldots, n\}$. By Fan's Theorem (see Appendix B), the latter inequality is equivalent to

$$\min_{m \in Q(\omega_t)} \max_{\Delta \in \gamma} F(m, \Delta) \leqslant 0.$$

By the Maximum Theorem, there exists $m^* \in Q(\omega_t)$ for which the minimum over $Q(\omega_t)$ equals $\max_{\Delta \in \gamma} F(m^*, \Delta) \leqslant 0$. By the linear homogeneity of $F(m^*, \cdot)$

[12] The objective function in (3.2.1) is concave in ξ and therefore is almost everywhere differentiable in ξ, for given e, d, q, and Δ. It is incorrect, however, to interpret this fact as implying that the price indeterminacy discussed below is "infrequent." Only differentiability at $\xi = 0$ is relevant to price determinacy. Thus the relevant question is whether for "many" specifications of e, d, q, and Δ, the objective function in (3.2.1) is nondifferentiable in ξ at $\xi = 0$. The frequency of price indeterminacy is examined in Section 3.4.

and the fact that $\Delta \in \gamma \Leftrightarrow -\Delta \in \gamma$, we conclude that for all $\Delta' \in \gamma$,

$$(3.2.3)\quad F(m^*, \Delta') = \min_{m \in Q(\omega_t)} \max_{\Delta \in \gamma} F(m, \Delta) = 0.$$

Since for each m, $F(m, \cdot)$ is linear, $\max\{F(m, \Delta) : \Delta \in \gamma\}$ is attained on the set of extreme points of γ. We arrive finally at the following system of *Euler inequalities* that must be satisfied in equilibrium: for all (t, ω^t),

$$(3.2.4)\quad \min_{m \in Q(\omega_t)} \max_i \left\{ \left| \beta E_m \left[\frac{u'(e_{t+1})}{u'(e_t)} (q_{i,t+1} + d_{i,t+1}) \right] - q_{i,t} \right| \right\} = 0.$$

The presence of the minimization over $Q(\omega_t)$ on the left side of (3.2.4) justifies our use of the term "inequalities" to refer to (3.2.4), in spite of the equality with zero. The inequality nature of (3.2.4) is highlighted in the single asset case ($n = 1$) where it reduces, in the obvious notation, to

$$(3.2.5)\quad \min_{m \in Q(\omega_t)} \beta E_m \left[\frac{u'(e_{t+1})}{u'(e_t)} (q_{t+1} + d_{t+1}) \right] \leqslant q_t \leqslant \max_{m \in Q(\omega_t)} \beta E_m \left[\frac{u'(e_{t+1})}{u'(e_t)} (q_{t+1} + d_{t+1}) \right].$$

When $n > 1$, (3.2.4) implies an inequality analogous to (3.2.5) for each asset, but this collection of n inequalities is not exhaustive for the reasons given above concerning the nonlinearity of one-sided Gâteaux derivatives.

Of course, if $\mathscr{P}$ is a probability kernel, then both $\mathscr{P}(\omega_t)$ and $Q(\omega_t)$ are singletons and (3.2.4) reduces to the standard Euler equation

$$q_{i,t} = \beta E_{\mu(\omega_t, \cdot)} \left[\frac{u'(e_{t+1})}{u'(e_t)} (q_{i,t+1} + d_{i,t+1}) \right], \qquad \text{for all } i.$$

3.3. *Equilibrium*

The Euler inequalities are not only necessary, but they are also sufficient for an equilibrium, that is, any price process $\{q_t\}$ satisfying (3.2.4) is an equilibrium, as we show shortly. To establish the existence of solutions to (3.2.4), and therefore of equilibria, we need to restrict the probability kernel correspondence $\mathscr{P}$. To formulate the added assumption, define the correspondence Q_f from Ω into $\mathscr{M}(\Omega)$, for any given $f \in C(\Omega)$, by

$$(3.3.1)\quad Q_f(\omega) \equiv \operatorname{argmin} \left\{ \int f\, dm : m \in \mathscr{P}(\omega) \right\}.$$

Assumption (Strict Feller Property for $\mathscr{P}$): *Q_f is a continuous correspondence for each $f \in C(\Omega)$.*

If $\mathscr{P}$ is a probability kernel, then Q_f is continuous since $Q_f = \mathscr{P}$. Another trivial case, termed *i.i.d. beliefs*, has $\mathscr{P}(\omega)$ independent of ω; then Q_f is constant and a fortiori continuous. The continuity of Q_f is trivial also if Ω is finite and endowed with the discrete topology. More generally, we can infer from the continuity of $\mathscr{P}$ and the Maximum Theorem only that Q_f is upper semi-continuous.

The interpretation of the strict Feller property is facilitated by reference to Section 2.5. From (2.5.5), we see that it implies that the set of supergradients of $V(\cdot;\omega)$ varies continuously with ω. From the perspective of the question of the existence of equilibria, such "continuous superdifferentiability" is the essential content of the added assumption. Note that for the proof of existence in the economy corresponding to the specific endowment process e, it suffices that Q_{V^*} be continuous (see Lemma 1 and note that $Q_{V^*} = Q$). In particular, existence is guaranteed if e^* is constant, since the V^* is constant and thus $Q = \mathscr{P}$.

The proof of existence of solutions to the Euler inequalities now proceeds as follows.[13] Since Q is compact and convex-valued and continuous, it admits a continuous selection, that is, there exists a sequence of probability kernels $\{\pi_t\}$ such that $\pi_t(\omega_t, \cdot) \in Q(\omega_t)$ for all t and $\omega_t \in \Omega$. Now consider the equations

$$(3.3.2) \qquad q_{i,t} = \beta E_{\pi_t(\omega_t,\cdot)}\left\{\frac{u'(e_{t+1})}{u'(e_t)}(q_{i,t+1} + d_{i,t+1})\right\}$$

for all t, ω_t, and i. By contraction mapping arguments (as extended in Lemma A.1), one can prove the existence of a unique (given $\{\pi_t\}$) price process satisfying (3.3.2). For that solution q, the Euler inequalities follow immediately.

The above arguments lead us to the following central theorem, the proof of which is completed in Appendix B:

THEOREM 2 (Existence and Characterization of Equilibria):
(a) *The set of equilibria coincides with the set of price processes satisfying* (3.2.4).
(b) *If $\mathscr{P}$ satisfies the strict Feller property, there exist equilibria.*
(c) *If $\mathscr{P}$ has full support, then $q \in \mathscr{D}^n$ is an equilibrium if and only if $\{(e_\tau, 0)\}_1^\infty$ is $(1, \omega_1)$-optimal for all $\omega_1 \in \Omega$.*

Part (c) shows that the two equilibrium notions described earlier coincide if $\mathscr{P}$ has full support. This is not surprising in light of the dynamic consistency property of the utility process implied by the full support assumption, as discussed in Section 2.4.

There exists an equilibrium for each sequence of selections $\{\pi_t\}$, used as in (3.3.2), implying that there may be many equilibria in our economy. This nonuniqueness is related to the findings of Dow and Werlang (1992), who show in a static model with one risky and one riskless asset, that there exists a set of asset prices that support the optimal choice of a riskless portfolio. Here we extend their analysis to an infinite-horizon, multiple-asset framework and we

[13] We continue to write Q rather than Q_{V^*}.

show that the nonuniqueness of supporting prices is not restricted to riskless positions. Simonsen and Werlang (1991) also observe the potential nonuniqueness of supporting prices under uncertainty aversion in a static setting. Note also that the nonuniqueness of prices and its "origin" in the multiplicity of underlying priors accord well with Keynes' intuition. He writes (1936, p. 152) that the "existing market valuation... cannot be uniquely correct, since our existing knowledge does not provide a sufficient basis for a calculated mathematical expectation."

In order to discuss further the nonuniqueness or indeterminacy of equilibrium prices, adopt the following notation and terminology: Denote by $\mathscr{E}$ the set of all equilibria. Say that the price of the ith security is determinate if for all q and q' in $\mathscr{E}$, $\{q_{i,t}\}_1^\infty = \{q'_{i,t}\}_1^\infty$.

THEOREM 3 (Structure of Set of Equilibria): *If $\mathscr{P}$ satisfies the strict Feller property, then*:

(a) *$\mathscr{E}$ is a closed and connected subset of $\mathscr{D}^n$.*

(b) *For each i, the equations*

$$(3.3.3)\quad \bar{q}_{i,t} = \beta \max_{m \in Q(\omega_t)} E_m\left\{ \frac{u'(e_{t+1})}{u'(e_t)} (\bar{q}_{i,t+1} + d_{i,t+1}) \right\} \quad \text{and}$$

$$\underline{q}_{i,t} = \beta \min_{m \in Q(\omega_t)} E_m\left\{ \frac{u'(e_{t+1})}{u'(e_t)} (\underline{q}_{i,t+1} + d_{i,t+1}) \right\}$$

have unique solutions in $\mathscr{D}$, denoted $\bar{q}_i$ and $\underline{q}_i$ respectively. These solutions satisfy the condition that for any $q \in \mathscr{E}$ and for any i and t,

$$(3.3.4)\quad \underline{q}_{i,t} \leqslant q_{i,t} \leqslant \bar{q}_{i,t} \quad \text{on } \Omega^t.$$

Moreover, given i, t, and any $\varepsilon > 0$, there exist q^1 and q^2 in $\mathscr{E}$ such that

$$(3.3.5)\quad q^1_{i,t} \leqslant \underline{q}_{i,t} + \varepsilon \quad \text{and} \quad q^2_{i,t} \geqslant \bar{q}_{i,t} - \varepsilon \quad \text{on } \Omega^t.$$

Finally, the ith security price is indeterminate if and only if for some t

$$(3.3.6)\quad \underline{q}_{i,t} \neq \bar{q}_{i,t},$$

in which case $\{q_i : q \in \mathscr{E}\}$ is an uncountably infinite set.[14]

Part (a) provides some information regarding the size of $\mathscr{E}$. Since $\mathscr{E}$ is a connected complete metric space, it follows from the Baire category theorem (Royden (1988, p. 159)), that if the equilibrium is not unique, then there exists an uncountable infinity of equilibria. This is confirmed by part (b). The latter first provides, via (3.3.4), bounds for the equilibrium price of any security and

[14] It is common in the literature to assume a time-homogeneous Markov structure for dividends and to restrict attention to price processes that are time-homogeneous and Markovian. Therefore, we point out that under the above assumption, Theorems 2 and 3 remain valid if price processes are defined to be elements of $\mathscr{D}$ that are time-homogeneous and Markovian.

then shows that these bounds are tight, in the natural sense of (3.3.5). Finally, (3.3.6) provides necessary and sufficient conditions for price indeterminacy. In special circumstances, those conditions assume a simpler form. For instance, the condition

$$\min_{m \in Q(\omega_t)} E_m\left\{\frac{u'(e_{t+1})}{u'(e_t)}\right\} \neq \max_{m \in Q(\omega_t)} E_m\left\{\frac{u'(e_{t+1})}{u'(e_t)}\right\}$$

characterizes the indeterminacy of the price of a one-period discount bond issued at (t, ω^t) and paying one unit of consumption at $(t+1)$.

Intuitively, we would expect a link between indeterminacy of asset prices and intertemporal price volatility. This intuition can be confirmed in the special case of "i.i.d. beliefs," that is, where $\mathscr{P}(\omega)$ is independent of ω, in which case the correspondence Q is also constant. Hence, for a security with time-homogeneous dividend process, if the price of the security is determinate, then it must be constant (across time and states). Consequently, any fluctuation in price is a reflection of indeterminacy. More generally, the link between indeterminacy and volatility can be thought of in the usual way in terms of the existence of "sunspot equilibria." That is, if the selection $\{\pi_t\}$ from Q (see (3.3.2)) is made to depend on a "sunspot" or "extrinsic" variable, then the corresponding equilibrium price process will also depend on that variable.[15]

The discussion to this point has assumed implicitly that price indeterminacy is a significant feature of our model in the sense of occurring on a "nonnegligible" set of economies. That this assumption is warranted is most easily demonstrated in the context of specific examples of probability kernel correspondences and so we defer further discussion to the next section.

The final result of this section provides a further characterization of equilibria. Let q be an equilibrium and reconsider (3.2.3). For the given t, we will now consider ω^t to be variable and thus the dependence of F on ω^t (through q_t and q_{t+1}) is made explicit by writing $F(m, \Delta, \omega^t)$. From (3.2.3) and the linearity of $F(m, \cdot, \omega^t)$ we derive

$$\min_{m \in Q(\omega_t)} g(m, \omega^t) = 0,$$

where $g(m, \omega^t) \equiv \max\{F(m, \Delta, \omega^t) : \Delta \text{ an extreme point of } \gamma\}$. By the Maximum Theorem, g is continuous and the correspondence of minimizers above is upper semicontinuous. Therefore, it admits a measurable selection (Klein and Thomson (1984, Theorem 4.2.1)), that is, there exists for each t

$$(3.3.7) \quad \xi_t : \Omega^t \to \mathscr{M}(\Omega) \text{ measurable}, \qquad \xi_t(\omega^t, \cdot) \in Q(\omega_t) \qquad \forall \omega^t \in \Omega^t$$

[15] It is well known that sunspot equilibria may exist, even in infinitely lived representative agent models, given financial constraints, externalities, nonconvexities, or other sources of market imperfection that lead to inefficient equilibrium allocations. See Guesnerie and Woodford (1993) for a survey. In contrast, in our model no such imperfections exist and the equilibrium allocation is trivially efficient, but quantities do not vary with the extrinsic state.

and $g(\xi_t(\omega^t, \cdot), \omega^t) \equiv 0$. Substitution of the appropriate expressions for g and F establishes the nontrivial portion of the following result.

THEOREM 4 (Further Characterization of Equilibria): *q is in $\mathscr{E}$ if and only if q is in $\mathscr{D}^n$ and for some $\{\xi_t\}$ as in* (3.3.7), *q satisfies*

$$(3.3.8) \quad q_{i,t} = \beta E_{\xi_t(\omega^t, \cdot)} \left\{ \frac{u'(e_{t+1})}{u'(e_t)} (q_{i,t+1} + d_{i,t+1}) \right\},$$

for all t and i.[16]

The characterization provided by Theorem 4 is helpful in placing our model of asset price determination in the context of the literature. In order to proceed, adopt the standard assumption that the actual evolution of $\{\omega_t\}$ is described by a probability kernel π^*. In place of the rational expectations hypothesis that π^* is known precisely by the agent, assume instead that Q is absolutely continuous with respect to π^*, for which it suffices that the probability kernel correspondence $\mathscr{P}$ be absolutely continuous (recall (2.2.5)). Such absolute continuity is assured if Ω is finite and $\pi^*(\omega, \omega') > 0$ for all ω and ω' in Ω. Denote by $z_{t+1}(\omega^t, \cdot): \Omega \to \mathbb{R}_+$ the Radon-Nikodym derivative of $\xi_t(\omega^t, \cdot)$. Then (3.3.8) has the form

$$(3.3.9) \quad q_{i,t} = \beta E_{\pi^*(\omega_t, \cdot)} \left\{ \frac{u'(e_{t+1})}{u'(e_t)} z_{t+1}(q_{i,t+1} + d_{i,t+1}) \right\}.$$

By construction, $\{z_{t+1}\}$ is restricted by $z_{t+1} \geqslant 0$,

$$(3.3.10) \quad \int_\Omega z_{t+1} \, d\pi^*(\omega_t, \cdot) \equiv 1 \quad \text{and} \quad \xi_t(\omega^t, \cdot) \in Q(\omega_t),$$

$$d\xi_t(\omega^t, \cdot) \equiv z_{t+1}(\omega^t, \cdot) \, d\pi^*(\omega_t, \cdot).$$

The relations (3.3.9), without (3.3.10) or other restrictions on $\{z_{t+1}\}$, can be established under fairly general considerations and contain commonly considered models as special cases (see Hansen and Richard (1987) and Hansen and Jagannathan (1991)). Generally, (3.3.9) is rewritten in terms of the "stochastic discount factors" $\gamma_{t+1} \equiv \beta z_{t+1} u'(e_{t+1})/u'(e_t)$ in the form

$$(3.3.11) \quad q_{i,t} = E_{\pi^*(\omega_t, \cdot)}[\gamma_{t+1}(q_{i,t+1} + d_{i,t+1})], \qquad i = 1, \ldots, n.$$

[16] Note the difference between (3.3.2) and (3.3.8). The former is sufficient for q to be an equilibrium since the selection $\{\pi_t\}$ is assumed to be continuous and hence the solution q to (3.3.2) *must* lie in $\mathscr{D}^n$. On the other hand, as just shown, the existence of a measurable selection $\{\xi_t\}$ as in (3.3.7)–(3.3.8) is a necessary condition for q to be an equilibrium. It is also sufficient only if, as in Theorem 4, we *assume* that the solution q to (3.3.8) lies in $\mathscr{D}^n$.

Since one can always find some $\{\gamma_{t+1}\}$ so that (3.3.11) is satisfied, the empirical content of any particular model of asset prices is represented by the restrictions it imposes on the discount factors $\{\gamma_{t+1}\}$ or equivalently, on $\{z_{t+1}\}$. For our model, those restrictions are represented by (3.3.10). The standard Lucas based rational expectations model imposes the stronger restriction $z_{t+1} \equiv 1$. See Cochrane and Hansen (1992) for examples of other restrictions on stochastic discount factors that have been studied in the literature.

3.4. *Examples*

We illustrate and elaborate upon our analysis of asset price determination in the context of the two examples of probability kernel correspondences of Section 2.3. Then, in order to lend indirect support to our "explanation" of price indeterminacy, we examine another model where indeterminacy can occur —a Lucas-style model where the felicity function u is not differentiable. Finally, we consider briefly an example of an economy where agents are uncertainty averse and *heterogeneous* so that trade may occur.

ε-Contamination: It follows from (2.3.2) that for any $f \in C(\Omega)$,

$$Q_f(\omega) = \left\{(1-\varepsilon(\omega))\pi^*(\omega) + \varepsilon(\omega)m : m \in \mathscr{M}\left(\underset{\Omega}{\operatorname{argmin}} f\right)\right\}.$$

Therefore, the strict Feller property is satisfied. In the particular case $f = V^*$ (see Lemma 1),

$$(3.4.1) \quad Q(\omega) = (1-\varepsilon(\omega))\pi^*(\omega) + \varepsilon(\omega)\mathscr{M}(\Omega_m), \qquad \Omega_m \equiv \underset{\Omega}{\operatorname{argmin}}\, V^*.$$

For simplicity, assume henceforth that dividend processes are Markovian and time-homogeneous,[17]

$$d_{i,t}(\omega^t) = d_i^*(\omega_t), \qquad \text{for all } i, t, \text{ and } \omega^t.$$

Then it follows from Theorem 4 and (3.4.1) that the price of security i is indeterminate if and only if

$$(3.4.2) \quad u'(e^*)\, d_i^* \text{ is nonconstant on } \Omega_m.$$

The essential economic (as opposed to mathematical) content of this restriction is that knowledge of the level of intertemporal utility V^* is not sufficient to

[17] It is more common to assume a time-homogeneous Markov structure for growth rates rather than levels. Our analysis is readily modified accordingly with no effects on our qualitative results.

infer the weighted dividend $u'(e^*)\,d_i^*$, or more precisely

(3.4.3) $u'(e^*)\,d_i^*$ is not V^*-measurable.[18]

The conditions for indeterminacy simplify if we consider the i.i.d. case where $\varepsilon(\omega)$ and $\pi^*(\omega,\cdot)$, and hence also $\mathscr{P}(\omega)$ and $Q(\omega)$, are independent of ω. Then

$$V^*(\cdot) = u(e^*(\cdot)) + \text{constant}.$$

Therefore, by (3.4.2), the ith price is indeterminate if and only if

(3.4.4) d_i^* is nonconstant on $\underset{\Omega}{\operatorname{argmin}}\, e^*$

which in the sense explained in the preceding footnote is tantamount to[19]

(3.4.5) d_i^* is not e^*-measurable.

This will be the case, for example, if there exist state variables affecting dividends that do not influence consumption. Since that is a plausible hypothesis, we conclude that our model predicts price indeterminacy for a "broad" or at least economically interesting class of dividend and endowment processes. Moreover, note that the model delivers predictions regarding the cross-sectional (across asset) variation of the degree of indeterminacy. That is, referring to Theorem 4, we see that

$$\underline{q}_{j,t} \leqslant \underline{q}_{i,t} \leqslant \bar{q}_{i,t} \leqslant \bar{q}_{j,t} \qquad \text{if}$$

$$\min_{\Omega_m} d_j^* \leqslant \min_{\Omega_m} d_i^* \leqslant \max_{\Omega_m} d_i^* \leqslant \max_{\Omega_m} d_j^*,$$

where, given the i.i.d. assumption, $\Omega_m = \operatorname{argmin}_\Omega e^*$. Therefore, asset j features a "large" degree of indeterminacy in its price if $[\min_{\Omega_m} d_j^*, \max_{\Omega_m} d_j^*]$ is large, which interval provides a measure of the extent to which d_j^* is "unpredictable" given consumption.[20]

Finally, consider the counterpart of (3.3.9)–(3.3.10), under the assumptions that the contamination function ε is constant, Ω is finite, and $\pi^*(\omega,\omega') > 0$ for all ω and ω' in Ω, ensuring thereby the absolute continuity of $\mathscr{P}$ with respect

[18] That is, $u'(e^*)d_i^*$ is not measurable with respect to the σ-algebra on Ω generated by the mapping $V^*\colon \Omega \to \mathbb{R}$. Note that (3.4.3) is weaker than (3.4.2) since the latter requires only that one not be able to infer the magnitude of $u'(e^*)d_i^*$ from knowledge that $V^* = \min_\Omega V^*$, while the former rules out the possibility of such inference given $V^* = k$ for some k. This difference does not appear to us to be economically significant and thus we will not differentiate between (3.4.2) and (3.4.3).

[19] If Ω consists of only two states, then (3.4.4) and (3.4.5) are each equivalent to: $(*)$ e^* is constant and d_i^* is not constant (on Ω). In particular, for this i.i.d. case, indeterminacy can occur only if consumption is certain. This conclusion that asset price indeterminacy is limited to riskless initial positions is also apparent from examination of the indifference curves of a Gilboa-Schmeidler utility in the state preference diagram for a static setting (see Simonsen and Werlang (1991), for example). However, one must be cautious in extrapolating to more general state spaces, where (3.4.4) implies not $(*)$, but rather that the conditions specified there apply on $\operatorname{arg\,min}_\Omega e^*$.

[20] Note the loose parallel with the case of risk ($\varepsilon = 0$ and $\mathscr{P}$ a probability kernel) where our model reduces to the consumption-based CAPM according to which the risk premium for asset j depends on the covariation of d_i^* and consumption.

to π^*. Then (3.3.10) is equivalent to[21]

$$(3.4.6)\quad \int_\Omega z_{t+1}\, d\pi^*(\omega_t, \cdot) = 1 \quad \text{and} \quad z_{t+1} \geqslant 1-\varepsilon,$$

$$z_{t+1}(\omega^t, \cdot) = 1 - \varepsilon \quad \text{on} \quad \Omega \setminus \Omega_m,$$

and the associated Euler equations (3.3.9) take the form

$$(3.4.7)\quad \beta^{-1} = E_{\pi^*(\omega_t, \cdot)}\left[\frac{u'(e_{t+1})}{u'(e_t)} z_{t+1} R_{i,t+1}\right], \qquad i = 1, \dots, n,$$

where $R_{i,t+1} \equiv [q_{i,t+1} + d_{i,t+1}]/q_{i,t}$.

The potential empirical significance of (3.4.6)–(3.4.7) can be illustrated through the analysis of stochastic discount factors in Hansen and Jagannathan (1991), for example. They infer from asset price and aggregate consumption data for the U.S. that stochastic discount factors that rationalize the data in the sense of (3.3.11) must have a large variance. The indicated variance is often considered too extreme to be compatible with any "reasonable" model of fundamentals and is occasionally interpreted as evidence for "fads" (Porterba and Summers (1988)). In particular, the consumption-based model having $z_{t+1} \equiv 1$, is rejected in this way because consumption is too smooth. It is interesting, therefore, to examine whether our specific model of discount factors (3.4.6) is compatible with a large variance. To highlight the role of uncertainty, we make the challenge facing our model of discount factors as difficult as possible and assume the extreme case of "smooth consumption," e^* constant. We then compute mvar(ε), the maximum variance of limiting distributions corresponding to some $\{z_{t+1}\}$ satisfying (3.4.6) and ergodicity. (Ergodicity justifies the approximation of moments of the limiting distribution by appropriate sample moments.) Assuming that $\{\omega_t\}$ under π^* is ergodic with limiting distribution described by $p \in \mathscr{M}(\Omega)$, we find that[22]

$$\text{mvar}(\varepsilon) = \varepsilon^2\Big(1 - \min_{\omega\in\Omega} p(\omega)\Big) \Big/ \min_{\omega\in\Omega} p(\omega).$$

A consideration in evaluating the implications of this expression is that the underlying state space Ω and therefore also π^*, may not be observable to the analyst even if the probability distributions induced by π^* on dividends and rates of return are observable or estimable. Note accordingly that for any given $\varepsilon > 0$, mvar$(\varepsilon) \to \infty$ as $\min_{\omega\in\Omega} p(\omega) \to 0$. It follows that, unless the analyst

[21] Under the stated assumptions, (3.4.1) implies that any $\xi_{t+1}(\omega^t, \cdot) \in Q(\omega_t)$ has Radon-Nikodym derivative of the form $z_{t+1}(\omega^t, \cdot) = 1 - \varepsilon + \varepsilon g_{t+1}(\omega^t, \cdot)$, for some $g_{t+1} \geqslant 0$ satisfying $\int_\Omega g_{t+1}\, d\pi^*(\omega_t, \cdot) \equiv 1$ and $g_{t+1} = 0$ on $\Omega \setminus .\Omega_m$. These restrictions on z_{t+1} are equivalent to (3.4.6).

[22] Specifically, if $\Omega = \{\omega_1, \dots, \omega_n\}$, then

$$\text{m var}(\varepsilon) \equiv \max\Big\{\sum p(\omega_i) z_i^2 - 1 : z \in \mathbb{R}^N, z_i \geqslant 1 - \varepsilon\ \forall i, \sum p(\omega_i) z_i = 1\Big\},$$

and the maximum is attained at one of the N extreme points of the constraint set $\{z^j\}_1^N$, where $z_j^j = 1 - \varepsilon + \varepsilon/p(\omega_j)$ and $z_i^j = 1 - \varepsilon$ if $i \neq j$. Finally, note that e^* constant implies that $\Omega_m = \Omega$.

insists on maintaining assumptions on Ω and π^* that are themselves arguably irrefutable, our model does not restrict the variance of discount factors. Moreover, the above is true for any fixed $\varepsilon > 0$, even arbitrarily small. This suggests, therefore, that some heretofore anomalous features of asset return data can be accommodated if we introduce a "small" amount of uncertainty aversion into the standard model.

The above is not to suggest that other important empirical puzzles are similarly resolvable or that the model (3.4.6)–(3.4.7) is irrefutable. Indeed, in other dimensions the empirical restrictiveness of the generalization (3.4.1) diminishes "continuously" as ε increases from 0, the standard model, to the extreme of complete ignorance, $\varepsilon = 1$. For example, assuming for simplicity that e^* is constant, it follows from (3.4.6)–(3.4.7) that the return to a one-period pure discount bond equals β^{-1} and that

$$E_{\pi^*(\omega_t,\cdot)}R_{t+1} - \beta^{-1} \leqslant \varepsilon\Big(E_{\pi^*(\omega_t,\cdot)}R_{t+1} - \min_{\Omega} R_{t+1}\Big),$$

where $R_{t+1} \equiv (q_{t+1} + d_{t+1})/q_t$. Consequently, the largest admissible equity premium is small if ε is small.[23]

Belief Function Kernels: Let $f \in C(\Omega)$ and define $\psi_f(y) \equiv \operatorname{argmin}\{f(\omega): G(\omega) = y\}$. From (2.3.7) (see also Wasserman (1990, Theorem 2.1)), it follows that for any given $f \in C(\Omega)$,

$$(3.4.8)\quad Q_f(\omega) = \Big\{m \in \mathscr{M}(\Omega): m(\cdot) = \int_{G(\Omega)} r(y)(\cdot)\, dp(y|G(\omega)) \text{ for some function } r: G(\Omega) \to \mathscr{M}(\Omega) \text{ such that } r(y)\big(\psi_f(y)\big) = 1 \text{ for all } y\Big\}.$$

Therefore, Q_f is a continuous correspondence and $\mathscr{P}_G$ satisfies the strict Feller property if the mapping $y' \to p(\cdot\,|y')$ is continuous in the strong topology.

If f is set equal to V^* in (3.4.8), we obtain a representation for elements of Q as a suitable mixture of measures $\{r(y): y \in G(\Omega)\}$, where $r(y)$ has support on $\psi_{V^*}(y)$. Since V^* is constant on each $\psi_{V^*}(y)$, every $m \in Q(\omega)$ induces the identical probability distribution for V^*. Nevertheless, $Q(\omega)$ is a nonsingleton if

[23] Under the additional assumption that the price of equity is constant across time and states (such an equilibrium exists if beliefs are i.i.d.), the maximum equity premium equals, in terms of primitives of the model,

$$\varepsilon(1-\beta)\beta^{-1}\Big[E_{\pi^*(\omega_t,\cdot)}d_{t+1} - \min_{\Omega} d_{t+1}\Big] \Big/ \Big[(1-\varepsilon)E_{\pi^*(\omega_t,\cdot)}d_{t+1} + \varepsilon \min_{\Omega} d_{t+1}\Big].$$

The latter vanishes if $\varepsilon = 0$. Therefore, this expression represents a premium for the uncertainty associated with holding equity rather than for the bearing of risk. We leave to a separate paper consideration of the equity premium puzzle (Mehra and Prescott (1985)) unrestricted by the numerous simplifying assumptions of this section.

the set of minimizers $\psi_{V^*}(y)$ is a nonsingleton for "many" y values, since then there are many possible choices for the measure $r(y)$ supported on $\psi_{V^*}(y)$.

Arguing as in the preceding example, we can show that the essential economic characterization of indeterminacy for the ith security price is the condition

$$(3.4.9) \quad u'(e^*)\, d_i^* \text{ is not } (G, V^*)\text{-measurable};$$

that is, the level of the weighted dividend $u^*(e^*)\, d_i^*$ cannot be inferred from knowledge of the levels of the statistics G and intertemporal utility V^*. This can be expected to be the case in situations where the statistics G provide only a crude summary of the underlying state.

Nondifferentiable Lucas Model: Price indeterminacy can occur also in a Lucas-style model where the felicity function u is not necessarily differentiable. However, such an "explanation" of indeterminacy differs from ours in two important respects. First, it does not capture Keynes' intuition, in the citation above, regarding the link between uncertainty and indeterminacy. In our model, $V(c) = \Sigma_0^\infty \beta^t u(c_t)$ for deterministic consumption processes, that is, those for which each c_t is a constant function. Therefore, all the usual regularity properties, including the uniqueness of supporting prices, are satisfied in the domain of deterministic consumption processes, supporting our assertion that indeterminacy is due to uncertainty. In contrast, in the modified Lucas model, supporting prices are nonunique even for deterministic consumption processes. The second important difference concerns the robustness of the prediction of indeterminacy. Since u can fail to be differentiable only on a zero Lebesgue measure subset K of $\mathbb{R}$, security prices are determinate in the Lucas model as long as all conditional distributions assign zero probability to consumption lying in K. For example, if the endowment process is constant with $e^* \equiv \bar{e}$, then security prices are determinate for all $\bar{e} \notin K$. On the other hand, for the constant endowment case our model predicts indeterminacy for *all* values of $\bar{e}$ and all securities paying nonconstant dividends (see (3.4.2), for example, and note that $\Omega_m = \Omega$ if e^* is constant). More generally, we have argued above that in our model indeterminacy occurs in a "large" set of economies.

Heterogeneous Agents: Our "justification" for representative agent modeling is the usual one, namely that it provides a simple way to organize observations in terms of familiar microeconomic principles and notions. One may also take a more stringent view and ask whether such a model can be justified theoretically in the context of an economy with heterogeneous agents. Here we adopt such an approach and prove a complete-markets aggregation theorem along the lines of Constantinides (1982), thereby providing an additional "example" to which our representative agent analysis applies. The example serves also to suggest an alternative interpretation for our price indeterminacy result in a model with trade and to clarify the "real" consequences of Knightian uncertainty in our model.

Expand the economy defined in Section 3.1 to admit H consumers, where consumer h has intertemporal utility function V^h corresponding to discount parameter β, belief kernel correspondence $\mathscr{P}$, and felicity function u^h, the only source of differences in consumer preferences. (Though restrictive, these assumptions are *weaker* than those in Constantinides (1982), where the standard single prior representation of beliefs is also imposed.) Specialize $\mathscr{P}$ further so that it implies i.i.d. beliefs ($\mathscr{P}(\omega)$ independent of ω), has full support and is based on the capacity representation of beliefs (Schmeidler (1989)); both the ε-contamination and belief function kernel examples fulfill the latter requirement. (See Appendix C for clarification and for a proof of the assertions below under more general assumptions.)

Though we will be interested in the competitive equilibria of a decentralized economy, it is useful first to characterize Pareto optimal allocations given the above preferences, an aggregate endowment process c (possibly different from e), and the initial state ω_0. For the usual reasons, it is enough to consider, for each vector $\alpha = (\alpha_h)_{h=1}^H$ of nonnegative utility weights, the planning problem

$$(3.4.10) \quad U^\alpha(c;\omega_0) \equiv \max\{\Sigma \alpha_h V^h(c^h;\omega_0) : c^h \in \mathscr{D}, \Sigma c^h = c\}, \qquad c \in \mathscr{D}.$$

This U^α is a candidate utility for the representative agent in the decentralized economy specified in the usual way (see Duffie (1992, Chapter 2), for example). Consumers begin with endowments $e^h \in \mathscr{D}$ of consumption and zero shares of each asset and then trade in complete asset markets. Focus on a (Pareto optimal) equilibrium allocation and denote by $q \in \mathscr{D}^n$ a corresponding equilibrium price and by α the utility weights corresponding to (3.4.10). Then, by suitable adaptations of Duffie (1992, pp. 9–11), q is also an equilibrium in the single-agent model with aggregate endowment e and intertemporal utility U^α.

The agent with utility U^α is "representative" if the intertemporal utility function U^α lies in the same recursive class defined in Section 2 containing the individual utilities. Under our assumptions, this is indeed the case: The standard risk-sharing rule, that is Pareto optimal in the expected utility framework of Constantinides, is to allocate the endowment x at any time t and state ω^t by solving

$$(3.4.11) \quad u^\alpha(x) \equiv \max\left\{\sum \alpha_h u^h(x_h) : \sum x_h = x\right\}.$$

Under our assumptions, this risk-sharing rule continues to be efficient given aversion to Knightian uncertainty, that is, the set of processes $\{\bar{c}^h\}_{h=1}^H$ solves (3.4.10) if and only if $\{\bar{c}_t^h(\omega^t)\}_{h=1}^H$ solves (3.4.11) for all t, ω^t and $x = c_t(\omega^t)$. It follows that U^α is the recursive intertemporal utility function corresponding, in the sense of our paper, to β, $\mathscr{P}$, and u^α.

This aggregation result "justifies" the application to aggregate data of our Euler inequalities (3.2.4) or the discount-factor model (3.3.9)–(3.3.10). However, interpretations of the indeterminacy of prices and its potential empirical relevance must be revised. That is because it is generally *not* the case that *every* equilibrium q for the representative agent economy with utility U^{α} is also a competitive equilibrium for the given initial endowments $\{e^h\}$. The situation is easily visualized in the context of an Edgeworth box where at an arbitrary point on the contract curve there exists a continuum of price lines that separate the better-than-sets for the two agents, but these lines do not all pass through the given initial endowment. Since not all selections from the set of representative agent equilibria are warranted, our earlier discussion of sunspots, animal spirits and price volatility seems wrong from this perspective. However, not all is lost if the analyst does not know the initial micro endowments. Indeed, if she knows nothing at all about them other than that they sum to e, then from her perspective all equilibria in the representative agent model have equal standing and the significance of price indeterminacy in the representative agent model is restored. More generally, one would expect there to remain a continuum of representative agent price equilibria that are consistent with the analyst's information about the micro endowments, and some potential for explaining price volatility would be retained. We emphasize that, according to this perspective, the "origin" of price indeterminacy and the associated price volatility lies in the *conjunction* of: (i) agents' aversion to Knightian uncertainty and (ii) incompleteness of a model formulated exclusively in terms of aggregate variables, or the *analyst's* incomplete information.

The preceding also clarifies the differing implications of our model for prices versus allocations. In the general representative agent model, prices may be indeterminate while consumption is exogenously specified and thus trivially determinate. This can "explain" greater volatility for prices than for consumption. These comparisons are more interesting in the heterogeneous agent model where the consumption side is nontrivial. Here we see the above confirmed in the sense that the prices supporting a given efficient allocation may be indeterminate. This is not to say, however, that Knightian uncertainty aversion has no real consequences, as it clearly influences the set of efficient and competitive allocations.

4. REMARKS ON EMPIRICAL CONTENT

Alternative models of irrational expectations, such as Shiller's model of "fads," have been criticized for not being well enough specified to produce rejectable implications (West (1988), Cochrane (1991), Leroy (1989)). Some readers may be sceptical also regarding the useful empirical content of our model. The discussions surrounding (3.3.9)–(3.3.10) and in Section 3.4 provided some indication of the potential usefulness of our model. Here we argue further that empirical investigation of our model is potentially fruitful. However, we caution the reader that the example just described may provide cause for suitably revising and weakening our arguments regarding empirical relevance.

One potential source of scepticism concerns Theorem 4. Equations (3.3.8) are the Euler equation implied by a Lucas style model in which $\{\xi_t\}$ represents beliefs. Note that ξ_t is not a probability kernel because it (i) depends on ω^t and not just ω_t and (ii) may not be continuous in ω_t. Nevertheless, the theorem raises concerns about whether our model is essentially observationally indistinguishable from a Lucas model, with the rational expectations hypothesis possibly deleted, but where beliefs are represented by probability measures and therefore uncertainty neutrality prevails. Observe, however, that to replicate an equilibrium q as an equilibrium of a Lucas style model and the associated Euler equations (3.3.8), the required "shadow" sequence of probability kernels $\{\xi_t\}$ may seem unnatural and contrived. (For convenience, we refer to the ξ_t's as probability kernels though they need not conform to our definition of the term.) For example, the ξ_t's will often depend on history or be time dependent for no "good" reason. Secondly, when some states are extrinsic (see the discussion of sunspot equilibria in Section 3.3), replication of a sunspot equilibrium q requires that "shadow" beliefs about intrinsic states, represented by $\{\xi_t\}$, depend upon extrinsic states. Therefore, acceptance of the Lucas model approximation requires that one revise the classification of "intrinsic" versus "extrinsic" states. Finally, we point out below that our model has some cross-sectional (across agent) implications. They can be delivered also by a Lucas style model with a larger number of agents, if each agent's beliefs are represented by some $\{\xi_t\}$, but the latter would have to vary across agents in an artificial way.

Another possible reason for scepticism is the feeling that our model "can explain anything" by a suitable specification of the capacity kernel representing beliefs, which are presumably unobservable. But similar remarks apply with respect to the specification of utility even if the Bayesian, rational expectations model of beliefs is adopted. That is, in principle, a wide range of specifications are possible for the intertemporal von Neumann-Morgenstern index $v(c_0, c_1, \ldots, c_t, \ldots)$. The strong predictive content of the Lucas asset pricing model derives in part from the parametric specialization of v to the additive form $\Sigma_0^\infty \beta^t u(c_t)$. This specialization is widely accepted, at least as a benchmark, both because of the tractability that it delivers and because we have some understanding of its plausibility, via its axiomatic underpinnings, for example. Analogy with the present context of modeling beliefs argues, not for scepticism, but rather for the need to study the properties of alternative specifications for $\mathscr{P}$. This paper points out some attractive features of the ε-contamination model (2.3.1) and of belief function kernels, but much more work in this direction is required.

In order to derive rejectable predictions for time series data, beliefs must be related to the actual evolution of the state process. One possible link is to posit that $\{\omega_t\}$ is governed by a probability kernel π^* and that beliefs incorporate some vagueness about π^* on the part of the investor. For reasons of robustness of empirical procedures, Lehmann (1992) suggests studying pricing equations for a range of discount factors, reflecting the analyst's imprecise information about the correct factors. It is at least as plausible to posit that investor's information is imprecise. Here such imprecision is incorporated into the theoretical framework and a "robust" theoretical model is delivered.

Finally, some may disagree with the presumption that beliefs are unobservable; for instance, a number of researchers, cited in the introduction, have used survey data as an independent measure of investors' expectations. Therefore, to conclude, suppose that such information is available for a cross-section of investors and consider some predictions of our model regarding expectations. We interpret our model as containing a number of agents with identical endowments and preferences, including the probability kernel correspondence $\mathscr{P}$. If surveys elicit entire correspondences, then agents will respond identically given our model. However, suppose that they are asked for a conditional probability distribution over next period's state variables, or for some summary moments and that they respond with an uncertainty adjusted conditional prior, that is, with an element of $Q(\omega)$. Then there is no reason to expect all investors to report the same element of $Q(\omega)$. Thus our model is consistent with heterogeneous measured forecasts, even though agents have common information in the form of $\mathscr{P}$. Moreover, the dispersion of forecasts should increase if $Q(\omega)$ increases in the sense of set inclusion.

Specialize to the ε-contamination model of beliefs (2.3.1) and suppose that $\varepsilon(\omega)$ is larger in those states ω where the "true" conditional probability measure $\pi^*(\omega)$ is riskier, e.g., has larger variance. Then a positive relation is indicated between dispersion of reported expectations of forecasters on the one hand and the poor performance of point forecasts, on the other. For a related prediction, recall our earlier discussion of a link between the indeterminacy and volatility of prices. Given such a link, our model suggests a positive relation between price volatility and the dispersion of reported expectations of forecasters. There is some supporting evidence for such a relation (Cragg and Malkiel (1982), Frankel and Froot (1990)).

One could derive a number of other predictions that would be testable given appropriate survey data. Needless to say, we are not asserting that such data are currently available (see, however, Zarnowitz and Lambros (1987)). The current paucity of suitable data is not damning of our model, however. After all, one of the roles of theory is to guide the collection of data.

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APPENDIX A: PROOF OF THEOREM 1

The following lemma is an adaptation to our space $\mathscr{D}$, consisting of sequences of real-valued functions, of the well-known Blackwell sufficient condition for a contraction mapping that applies to a space of real-valued functions.

LEMMA A.1: *Let $T: \mathscr{D} \to \mathscr{D}$ be an operator with the following properties*: (i) (*Monotonicity*): *if $f, g \in \mathscr{D}$ and $f \leqslant g$, that is, $f_t(\omega^t) \leqslant g_t(\omega^t)$ for all t and ω^t, then $Tf \leqslant Tg$*; (ii) (*Discounting*): *there exists a real constant β, $0 < b\beta < 1$, such that for any $f \in \mathscr{D}$ and sequence of constant functions $a = \{a_t\} \in \mathscr{D}$ with $a_t \in \mathbb{R}_+$, $(T(f+a))_t(\omega^t) \leqslant (Tf)_t(\omega^t) + a_{t+1}\beta$ for all t and ω^t. Then T has a unique fixed point.*

PROOF: Let f and $g \in \mathscr{D}$. Set $a_t = \|f_t - g_t\|$. Then $f \leqslant g + a$. By monotonicity and discounting, $(Tf)_t(\omega^t) \leqslant (T(g+a))_t(\omega^t) \leqslant (Tg)_t(\omega^t) + \beta\|f_{t+1} - g_{t+1}\|$. Thus $|(Tf)_t(\omega^t) - (Tg)_t(\omega^t)|/b^t \leqslant \beta b\|f_{t+1} - g_{t+1}\|/b^{t+1}$, and further $\|Tf - Tg\| \leqslant \beta b\|f - g\|$, proving that T is a contraction. *Q.E.D.*

PROPOSITION A.2 (Existence of Utility): *For each $c \in \mathscr{D}$ there exists a unique $V(c) \in \mathscr{D}$ such that* (2.4.1) *holds for all t and ω^t.*

PROOF: Define a map $T: \mathscr{D} \to \mathscr{D}$ by $\forall f \in \mathscr{D}$

$$(Tf)_t(\omega^t) = u\big(c_t(\omega^t)\big) + \beta \int f_{t+1}(\omega^t, \cdot)\, d\mathscr{P}(\omega_t, \cdot).$$

By the continuity of $\mathscr{P}$, $(Tf)_t$ is continuous. Next,

$$\sup_{\omega^t} (Tf)_t(\omega^t)/b^t \leqslant \sup_{\omega^t} u\big(c_t(\omega^t)\big)/b^t + \frac{\beta}{b^t} \sup_{\omega^t}\left\{\int f_{t+1}(\omega^t, \cdot)\, d\mathscr{P}(\omega_t, \cdot)\right\}$$
$$= \sup_{\omega^t} u\big(c_t(\omega^t)\big)/b^t + \beta b\|f_{t+1}\|/b^{t+1}.$$

Since u is increasing, concave, and $u(0) = 0$, we have $\infty > u(\|c\|) \geqslant u(c_t/b^t) \geqslant u(c_t)/b^t$ and

$$\|Tf\| = \sup_t \sup_{\omega^t} \frac{(Tf)_t(\omega^t)}{b^t} \leqslant u(\|c\|) + \beta b \sup_t \|f_{t+1}\|/b^{t+1} \leqslant u(\|c\|) + \beta b\|f\|.$$

Therefore, T is well-defined. Monotonicity and discounting for T are obvious. By Lemma A.1, T has a unique fixed point, which is the solution of (2.4.1). *Q.E.D.*

PROPOSITION A.3 (Approximation of Utility): *Fix $c \in \mathscr{D}$. For each T, define $\{V_t^T(c)\}_{t=1}^{\infty} \in \mathscr{D}$ by $V_t^T \equiv 0$ for $t > T$ and*

$$V_t^T(c; \omega^t) = u\big(c_t(\omega^t)\big) + \beta \int V_{t+1}^T(c; \omega^t, \cdot)\, d\mathscr{P}(\omega_t, \cdot)$$

for $0 \leqslant t \leqslant T$. Then $\lim_{T\to\infty} V_t^T(c; \omega^t) = V_t(c; \omega^t)$ for all t and ω^t.

PROOF: Verify that, for any $t \leqslant T$ and ω^t,

$$V_t^T(c;\omega^t) \leqslant V_t(c;\omega^t) \leqslant V_t^T(c;\omega^t) + \|V(c)\|(\beta b)^{T-t+1} b^t. \tag{A.1}$$

Q.E.D.

PROPOSITION A.4 (Continuity of Utility): *If u satisfies the growth condition, then $V_t(c;\omega^t)$ is continuous in (c, ω^t).*

PROOF: Under the growth condition,

$$\big|V_t(c;\omega^t)\big| \leqslant \frac{k_1}{1-\beta} + k_2 \sum_{j=1}^{\infty} \beta^j \|c_{t+j}\|.$$

Hence

$$\|V(c)\| \leqslant \frac{\beta k_1}{1-\beta} + \frac{\beta b k_2 \|c\|}{1 - \beta b}.$$

Thus it follows from (A.1) that

$$\text{(A.2)} \qquad V_t(c;\omega^t) - V_t^T(c;\omega^t) \leqslant \left(\frac{\beta k_1}{1-\beta} + \frac{\beta b k_2\|c\|}{1-\beta b}\right)(\beta b)^{T-t+1} b^t.$$

Let $c^n \to c$. For fixed t,

$$|V_t(c^n;\omega^t) - V_t(c;\omega^t)| \leqslant |V_t(c^n;\omega^t) - V_t^T(c^n;\omega^t)| + |V_t^T(c^n;\omega^t) - V_t^T(c;\omega^t)| + |V_t(c;\omega^t) - V_t^T(c;\omega^t)|.$$

For all c^n such that $\|c^n - c\| < 1$, $|c_t^n(\omega^t)| \leqslant (\|c\|+1)b^T$ for $t \leqslant T$. It follows from the continuity of u that the second term converges to zero as $c^n \to c$ and that the convergence is uniform in ω^t. By (A.2), the first and third terms on the right side converge to zero as $T \to \infty$ uniformly in n and ω^t. Therefore, $V_t(\cdot;\omega^t)$ is continuous at c uniformly in ω^t. The desired joint continuity now follows from the continuity of $V_t(c;\omega^t)$ in ω^t. *Q.E.D.*

The remaining properties asserted for utility can be proven by standard arguments from the theory of recursive utility (see Lucas and Stokey (1984), Stokey and Lucas (1989), and Epstein and Zin (1989), for example). Footnote 8 clarifies the link with the recursive utility literature; note that W defined there is increasing and concave. For dynamic consistency, note that if $\mathscr{P}$ has full support, then (2.2.6) applies.

APPENDIX B: PROOF OF THEOREMS IN SECTION 3

For the convenience of the reader, we provide here statements of two results invoked in Section 3. The first is the version of Fan's Theorem employed in the derivation of the Euler inequalities (3.2.9). A stronger form is proven in Sion (1958, Theorems 4.2 and 4.2′).

FAN'S THEOREM: *Let X and Y be metrizable convex and compact subsets of some linear topological spaces, and f a continuous real-valued function on $X \times Y$ that satisfies* (i) $f(\cdot, y)$ *is concave on X for each y; and* (ii) $f(x,\cdot)$ *is convex on Y for each x. Then*

$$\max_x \min_y f(x,y) = \min_y \max_x f(x,y).$$

Second, the argument surrounding (3.3.2) relies on the following selection theorem that is slightly stronger than that which is explicitly stated in Michael (1956). This result is also needed below in the proof of Theorem 3.

LEMMA B.1 (Michael): *Suppose that X is paracompact, Y is a topological linear space, and Z is a convex closed subset of Y containing 0 that has a base $\{B_n\}$ for the neighborhoods of 0, consisting of symmetric and convex sets such that $B_{n+1} \subset \frac{1}{2}B_n$. Suppose that $\psi: X \to Z \subset Y$ is a lower semicontinuous convex-valued correspondence such that $\psi(X) + B_n \subset Z$. Suppose further that for each $y \in \psi(X)$, $y + B_n$ is open in Z. Then the correspondence $\bar{\psi}$ defined by $\bar{\psi}(x) = \overline{\psi(x)}$ admits a continuous selection.*

PROOF: See the proofs of Lemma 4.1 and Theorem 3.2″ in Michael (1956), or the proof of Theorem 9.G of Zeidler (1986). *Q.E.D.*

PROOF OF THEOREM 2: *Part (a)*: It remains to show only that if a price process q satisfies (3.2.4), or equivalently (3.2.2), then it is an equilibrium. Let (c,θ) be any (t,ω^t)-feasible plan for which $\theta_{t-1}(\omega^{t-1}) = 0$. It follows from (3.2.2) with $\Delta = \theta_\tau$ that there exist $\pi_\tau: \Omega^\tau \to \mathscr{M}(\Omega)$, $\tau \geqslant t$, such that $\pi_\tau(\omega^\tau) \in Q(\omega_\tau)$ for each ω^τ and

$$\text{(B.1)} \qquad \theta_\tau \cdot q_\tau u'(e_\tau) \geqslant \beta E_{\pi_\tau(\omega_\tau,\cdot)}\{u'(e_{\tau+1})\theta_\tau \cdot (q_{\tau+1} - d_{\tau+1})\}.$$

It follows from the budget constraints that

$$\text{(B.2)} \qquad e_\tau - c_\tau - \theta_\tau \cdot q_\tau = -(q_\tau + d_\tau)\cdot\theta_{\tau-1}.$$

By the concavity of u,

$$\text{(B.3)} \qquad u(e_\tau) \geqslant u(c_\tau) + u'(e_\tau)(e_\tau - c_\tau).$$

Define V_t^T as in Proposition A.3. We have the following lengthy but elementary chain of inequalities:

$$
\begin{aligned}
&V_t^T(c;\omega^t) - V_t(e;\omega^t)\\
&\quad = u(c_t) + \beta\int V_{t+1}^T(c;\omega^t,\cdot)\,d\mathcal{P}(\omega_t,\cdot) - u(e_t) - \beta\int V_{t+1}(e;\omega^t,\cdot)\,d\mathcal{P}(\omega_t,\cdot)\\
&\quad \leqslant u'(e_t)(c_t - e_t) + \beta\int V_{t+1}^T(c;\omega^t,\cdot)\,d\mathcal{P}(\omega_t,\cdot) - \beta\int V_{t+1}(e;\omega^t,\cdot)\,d\mathcal{P}(\omega_t,\cdot)\\
&\quad = -u'(e_t)\theta_t\cdot q_t + \beta\int V_{t+1}^T(c;\omega^t,\cdot)\,d\mathcal{P}(\omega_t,\cdot) - \beta\int V_{t+1}(e;\omega^t,\cdot)\,d\mathcal{P}(\omega_t,\cdot)\\
&\quad \leqslant -u'(e_t)\theta_t\cdot q_t + \beta E_{\pi_t(\omega^t,\cdot)}\{V_{t+1}^T(c;\omega^t,\cdot)\} - \beta E_{\pi_t(\omega^t,\cdot)}\{V_{t+1}(e;\omega^t,\cdot)\}\\
&\quad \leqslant \beta E_{\pi_t(\omega^t,\cdot)}\{-u'(e_{t+1})\theta_t\cdot(q_{t+1}+d_{t+1})\} + \beta E_{\pi_t(\omega^t,\cdot)}\{V_{t+1}^T(c;\omega^t,\cdot)\}\\
&\qquad\qquad - \beta E_{\pi_t(\omega^t,\cdot)}\{V_{t+1}(e;\omega^t,\cdot)\}\\
&\quad = \beta E_{\pi_t(\omega^t,\cdot)}\Big\{-u'(e_{t+1})\theta_t\cdot(q_{t+1}+d_{t+1}) + u(c_{t+1}) + \beta\int V_{t+2}^T(c;\omega^{t+1},\cdot)\\
&\qquad\qquad \times d\mathcal{P}(\omega_{t+1},\cdot) - u(e_{t+1}) - \beta\int V_{t+2}(e;\omega^{t+1},\cdot)\,d\mathcal{P}(\omega_{t+1},\cdot)\Big\}\\
&\quad = \beta E_{\pi_t(\omega^t,\cdot)}\Big\{u'(e_{t+1})(e_{t+1} - c_{t+1} - \theta_{t+2}\cdot q_{t+1}) + u(c_{t+1}) + \beta\int V_{t+2}^T(c;\omega^{t+1},\cdot)\\
&\qquad\qquad \times d\mathcal{P}(\omega_{t+1},\cdot) - u(e_{t+1}) - \beta\int V_{t+2}(e;\omega^{t+1},\cdot)\,d\mathcal{P}(\omega_{t+1},\cdot)\Big\}\\
&\quad \leqslant \beta E_{\pi_t(\omega^t,\cdot)}\Big\{-u'(e_{t+1})\theta_{t+2}\cdot q_{t+1} + \beta\int V_{t+2}^T(c;\omega^{t+1},\cdot)\,d\mathcal{P}(\omega_{t+1},\cdot)\\
&\qquad\qquad - \beta\int V_{t+2}(e;\omega^{t+1},\cdot)\,d\mathcal{P}(\omega_{t+1},\cdot)\Big\}\\
&\quad \leqslant \beta E_{\pi_t(\omega^t,\cdot)}\Big\{\beta E_{\pi_{t+1}(\omega^{t+1},\cdot)}\Big\{-u'(e_{t+2})\theta_{t+2}\cdot(q_{t+2} - d_{t+2})\\
&\qquad\qquad + u(c_{t+2}) + \beta\int V_{t+3}^T(c;\omega^{t+2},\cdot)\,d\mathcal{P}(\omega_{t+2},\cdot) - u(e_{t+2})\\
&\qquad\qquad - \beta\int V_{t+3}(e;\omega^{t+2},\cdot)\,d\mathcal{P}(\omega_{t+1},\cdot)\Big\}\Big\}\\
&\qquad \vdots\\
&\quad \leqslant \beta E_{\pi_t(\omega^t,\cdot)}\Big\{\cdots\ \beta E_{\pi_{t+T}(\omega^{t+T},\cdot)}\Big\{-u'(e_{t+T+1})\theta_{t+T}\cdot(q_{t+T+1} + d_{t+T+1})\\
&\qquad\qquad + u(c_{t+T+1}) - u(e_{t+T+1})\\
&\qquad\qquad - \beta\int V_{t+T+2}(e;\omega^{t+T+1},\cdot)\,d\mathcal{P}(\omega_{t+T+1},\cdot)\Big\}\cdots\Big\}\\
&\quad \leqslant \beta E_{\pi_t(\omega^t,\cdot)}\Big\{\cdots\ \beta E_{\pi_{t+T}(\omega^{t+T},\cdot)}\Big\{-u'(e_{t+T+1})\theta_{t+T}\cdot q_{t+T+1}\\
&\qquad\qquad - \beta\int V_{t+T+2}(e;\omega^{t+T+1},\cdot)\,d\mathcal{P}(\omega_{t+T+1},\cdot)\Big\}\cdots\Big\}\\
&\quad \leqslant \beta E_{\pi_t(\omega^t,\cdot)}\{\cdots\ \beta E_{\pi_{t+T}(\omega^{t+T},\cdot)}\{u'(e_{t+T+1})\bar{K}\cdot q_{t+T+1}\}\cdots\}\\
&\quad \leqslant b^t(\beta b)^{T-t+1}\|u'(e^*)\bar{K}\cdot q\| \to 0
\end{aligned}
$$

where $\bar{K} \in \mathbb{R}_+^n$ has all components equal to $\inf_{i,\tau,\omega^\tau} \theta_{i,\tau}(\omega^\tau)$. The first inequality follows from (B.3); the second equality follows from (B.2) and $\theta_{t-1} = 0$; the second inequality follows from $\pi_t(\omega^t) \in Q(\omega_t)$; the third inequality follows from (B.1); the third and fourth equalities follow from (2.4.1) and (B.2); the fourth inequality follows from (B.3); the fifth inequality follows from (B.1) and $\pi_{t+1}(\omega^{t+1}) \in Q(\omega_{t+1})$; the seventh inequality follows from (B.2) and (B.3); the eighth inequality follows from the short selling constraint $\theta \geqslant -\bar{K}$ and the nonnegativity of utility; and the last inequality follows from the fact that the process $\{u'(e_t)\bar{K} \cdot q_t\}$ is in $\mathscr{D}$. Thus $V_t(c;\omega^t) - V_t(e;\omega^t) \leqslant 0$, which implies that q is an equilibrium.

Part (*b*): We need to show only that Q admits a continuous selection. Then the claim of (b) follows from (a) and the arguments in the text surrounding (3.3.2).

Let $C^*(\Omega)$ be the dual of $C(\Omega)$ endowed with the weak* topology. Then $\mathscr{M}(\Omega)$ is a compact subset of $C^*(\Omega)$. Since $C(\Omega)$ is separable, there exists a countable family $\{f_n\}$ that is a dense subset of the closed unit ball of $C(\Omega)$. Let Z be the closed ball with radius 4 in $C^*(\Omega)$, i.e.,

$$Z \equiv \{m \in C^*(\Omega) : \|m\| \leqslant 4\},$$

where the norm is the usual norm on the dual space. Define a metric on Z by

$$d(P,Q) = \sum_n \frac{1}{9^n}\left|\int f_n \, dP - \int f_n \, dQ\right|.$$

This metric induces the weak* topology on Z. In particular, it induces the weak convergence topology on $\mathscr{M}(\Omega)$, which is a subset of Z. Under this metric, Z is a convex and compact metric space. Define

$$B \equiv \left\{m \in Z : \sum_n \frac{1}{9^n}\left|\int f_n \, dm\right| < 1\right\}, \qquad B_n = \frac{1}{2^n}B,$$

and apply Lemma B.1.

Part (*c*): Follows from the dynamic consistency of the utility process under the assumption of full support for $\mathscr{P}$. *Q.E.D.*

PROOF OF THEOREM 4: See text.

PROOF OF THEOREM 3: *Part* (*b*): (*i*) *Proof of* (3.3.3) *and* (3.3.4): Define contraction mappings $\bar{T}_i : \mathscr{D} \to \mathscr{D}$ and $\underline{T}_i : \mathscr{D} \to \mathscr{D}$ by, for each $f \in \mathscr{D}$,

$$(\bar{T}_i f)_t(\omega^t) = \beta \max_{m \in Q(\omega_t)} E_m\{f_{t+1} + u'(e_{t+1})\, d_{i,t+1}\},$$

$$(\underline{T}_i f)_t(\omega^t) = \beta \min_{m \in Q(\omega_t)} E_m\{f_{t+1} + u'(e_{t+1})\, d_{i,t+1}\}.$$

Denote their unique fixed points by $\bar{f}_i$ and $\underline{f}_i$ and define $\bar{q}_{i,t}$ and $\underline{q}_{i,t}$ by $\bar{q}_{i,t}(\omega^t) = \bar{f}_{i,t}(\omega^t)/u'(e_t(\omega^t))$ and $\underline{q}_{i,t}(\omega^t) = \underline{f}_{i,t}(\omega^t)/u'(e_t(\omega^t))$. By construction, $\{\bar{q}_{i,t}\}$ and $\{\underline{q}_{i,t}\} \in \mathscr{D}$ and satisfy (3.3.3).

Given $q \in \mathscr{E}$, let $\{\xi_t\}$ be as in (3.3.7) and (3.3.8). Denote by $\bar{\mathscr{D}}$ the set of processes satisfying the requirements in the definition of $\mathscr{D}$ with the possible exception of continuity. Define contraction mappings T_i: $\bar{\mathscr{D}} \to \bar{\mathscr{D}}$ by

$$(T_i f)_t(\omega^t) = \beta E_{\xi_t(\omega^t,\cdot)}\{f_{t+1} + u'(e_{t+1})\, d_{i,t+1}\}.$$

Its unique fixed point is $f_i \in \bar{\mathscr{D}}$. By (3.3.8) and the uniqueness of the fixed point, $u'(e_t)q_{i,t} = f_{i,t}$ on Ω^t. Now (3.3.4) follows from the monotonicity of the three maps $\bar{T}_i$, $\underline{T}_i$, and T_i and the observation that $(\underline{T}_i f)_t \leqslant (T_i f)_t \leqslant (\bar{T}_i f)_t$. *Q.E.D.*

For the next step, we need the following lemma concerning the existence of ε-optimal continuous policies. Bertsekas and Shreve (1978, Section 8.2) contains a parallel result for measurable policies.

LEMMA B.2: *Suppose that X is paracompact, Y is a topological linear space, and Z is a convex closed subset of Y containing* 0 *that has a base $\{B_n\}$ for the neighborhoods of* 0, *consisting of symmetric and convex sets such that $B_{n+1} \subset \frac{1}{2}B_n$. Suppose $\Gamma : X \to Z \subset Y$ is a continuous, compact and* *convex-valued correspondence such that* $\psi(X)+B_n\subset Z$. *Suppose further that for each* $y\in\Gamma(X)$, $y+B_n$ *is open in* Z. *Let* F: $X\times Z\to\mathbb{R}$ *be continuous. Define* f, g: $X\to\mathbb{R}$ *by*

$$f(x)=\min_{y\in\Gamma(x)}F(x,y);\qquad and \qquad g(x)=\max_{y\in\Gamma(x)}F(x,y).$$

(a) *If* $F(x,y)$ *is convex in* y, *then for any* $\varepsilon>0$ *there exists a continuous function* h: $X\to Y$ *such that* $h(x)\in\Gamma(x)$ *and* $F(x,h(x))\leqslant f(x)+\varepsilon$ *for all* $x\in X$.

(b) *If* $F(x,y)$ *is concave in* y, *then for any* $\varepsilon>0$ *there exists a continuous function* h: $X\to Y$ *such that* $h(x)\in\Gamma(x)$ *and* $F(x,h(x))\geqslant g(x)-\varepsilon$ *for all* $x\in X$.

PROOF: We prove (a). Fix $\varepsilon>0$. Define a correspondence ψ: $X\to Z\subset Y$ by

$$\psi(x)=\{y\in\Gamma(x):F(x,y)<f(x)+\varepsilon\}.$$

By the convexity of $F(x,y)$ in y, $\psi(x)$ is convex. Suppose $y\in\psi(x_0)$. Then $F(x_0,y)<f(x_0)+\varepsilon$. By the continuity of $f(x)+\varepsilon$ (via the Maximum Theorem) and F, there exists a neighborhood $N(x_0)$ of x_0 such that $\forall x\in N(x_0)$, $F(x,y)<f(x)+\varepsilon$, which implies that $\forall x\in N(x_0)$, $y\in\psi(x)$, which in turn implies that for any open set V, the set $\{x:\psi(x)\cap V\neq\varnothing\}$ is open. Therefore ψ is lower semicontinuous. By Lemma B.1, ψ admits a continuous selection, say h. Since $\psi(x)\subset\{y\in\Gamma(x):F(x,y)\leqslant f(x)+\varepsilon\}$, we have $F(x,h(x))\leqslant f(x)+\varepsilon$ for all $x\in X$. *Q.E.D.*

LEMMA B.3: *If* $q_i\in\mathscr{D}$ *and satisfies* (3.3.8) *for some* $\{\xi_t\}$, *then*

$$u'(e_t)q_t\leqslant b^t\frac{u(\|e\|)\beta b}{1-\beta b}.$$

PROOF: Apply (3.3.8) and the concavity of u. *Q.E.D.*

(ii) *Proof of* (3.3.5): We show the existence of q^1. The existence of q^2 can be shown similarly. In the following, the superscript 1 is suppressed and, without essential loss of generality, we set $t=0$.

Choose T such that

$$2\frac{(\beta b)^T u(\|e\|)\beta b}{u'(\|e_t\|)(1-\beta b)}<\beta\varepsilon.$$

By Lemma B.2 (with $X=\Omega^t$, Z as in the proof of part (b) of Theorem 2, $\Gamma\equiv Q$, $F(\omega^t,m)\equiv E_m\{u'(e_{t+1})(d_{i,t+1}+q_{i,t+1})\}$ and noting that the right side of the last expression is a continuous function of (ω^t,m)), there exists, for each t, $\pi_t:\Omega^t\to\mathscr{M}(\Omega)$ continuous such that $\pi_t(\omega^t)\in Q(\omega_t)$ and

$$\beta E_{\pi_t(\omega^t,\cdot)}\{u'(e_{t+1})(d_{i,t+1}+\underline{q}_{i,t+1})\}\leqslant u'(e_t(\omega^t))\underline{q}_{i,t}(\omega^t)+u'(\|e^*\|)(1-\beta)^2\varepsilon.$$

By the proof of Theorem 2(b), there is a unique equilibrium price process q in $\mathscr{E}$ associated with $\{\pi_t\}$ as in (3.3.8) with ξ_t replaced by π_t. Now we show that $q_{i,0}$ satisfies the appropriate form of (3.3.5). For this purpose, define $q_i^T\in\mathscr{D}$ by $q_{i,t}^T\equiv 0$ for $t>T+1$, $q_{i,T+1}^T=\underline{q}_{i,T+1}$ and

$$q_{i,t}^T=\beta E_{\pi_t}\left\{\frac{u'(e_{t+1})}{u'(e_t)}\left(d_{i,t+1}+q_{i,t+1}^T\right)\right\}\qquad\text{for } t\leqslant T.$$

Then we claim for $t\leqslant T$,

$$\text{(B.4)}\qquad u'(e_t)q_{i,t}^T\leqslant u'(e_t)\underline{q}_{i,t}+u'(\|e^*\|)(1-\beta)^2(\varepsilon+\beta\varepsilon+\cdots+\beta^{T-t}\varepsilon).$$

This is true when $t=T$, since

$$u'(e_T)q_{i,T}^T=\beta E_{\pi_T}\{u'(e_{T+1})(d_{i,T+1}+\underline{q}_{i,T+1})\}\leqslant u'(e_t)\underline{q}_{i,T}+u'(\|e^*\|)\varepsilon(1-\beta)^2.$$

Assume that (B.4) is true for some $t+1 \leqslant T$. Then

$$\begin{aligned} u'(e_t)q_{i,t}^T &= \beta E_{\pi_t}\{u'(e_{t+1})(d_{i,t+1}+q_{i,t+1}^T)\} \\ &\leqslant \beta E_{\pi_t}\{u'(e_{t+1})d_{i,t+1}+u'(e_{t+1})\underline{q}_{i,t+1} \\ &\qquad + u'(\|e^*\|)(1-\beta)^2(\varepsilon+\beta\varepsilon+\cdots+\beta^{T-t-1}\varepsilon)\} \\ &\leqslant u'(e_t)\underline{q}_{i,t}+u'(\|e^*\|)(1-\beta)^2(\varepsilon+\beta\varepsilon+\cdots+\beta^{T-t}\varepsilon). \end{aligned}$$

Thus (B.4) is established.

Setting $t=1$ and letting $T\to\infty$ on the right side of (B.4), we obtain

$$q_{i,1}^T \leqslant \underline{q}_{i,1}+\varepsilon(1-\beta).$$

Now by Lemma B.2 and straightforward calculation,

$$0 \leqslant q_{i,1}-q_{i,1}^T \leqslant 2\frac{(\beta b)^T u(\|e\|)}{u'(\|e_0\|)}\frac{\beta b}{1-\beta b}.$$

Then by our choice of T, $q_{i,1} \leqslant q_{i,1}^T+\beta\varepsilon \leqslant \underline{q}_{i,1}+\varepsilon$. *Q.E.D.*

(iii) *Proof of* (3.3.6): "Only if" follows from (3.3.4). For the converse, assume (3.3.6). By choosing ε sufficiently small in (3.3.5) and noting the proof of the latter, it follows that there exist two equilibria q^0 and q^1, with $q_i^0 \neq q_i^1$ and corresponding (in the sense of (3.3.2)) sequences of continuous functions $\{\pi_t^0\}$ and $\{\pi_t^1\}$ from Ω^t to $\mathscr{M}(\Omega)$ with $\pi_t^i(\omega^t)\in Q(\omega^t)$ for $i=0,1$ and all $\omega^t\in\Omega^t$. For each $\alpha\in[0,1]$, define $\pi_t^\alpha=\alpha\pi_t^0-(1-\alpha)\pi_t^1$. By the proof of Theorem 2(b), there exists a unique $q(\alpha)\in\mathscr{E}$ such that

$$q_{i,t}(\alpha,\omega^t)=\beta E_{\pi_t^\alpha(\omega^t,\cdot)}\left\{\frac{u'(e_{t+1})}{u'(e_t)}(q_{i,t+1}(\alpha)+d_{i,t+1})\right\}.$$

If it can be shown that for each i and t, the map $\alpha\mapsto q_{i,t}(\alpha)\in C(\Omega^t)$ is continuous, then the proposition is proven because $q_{i,t}^0\neq q_{i,t}^1$ implies that $q_{i,t}^0(\omega^t)\neq q_{i,t}^1(\omega^t)$ for some ω^t. Then $q_{i,t}(\alpha,\omega^t)$ as a continuous function of α assumes at least two distinct values and hence must assume a continuum of distinct values.

It remains to show that $q_{i,t}(\alpha)$ is continuous in α. Let $\varepsilon>0$. By Lemma B.3,

$$\text{(B.5)}\qquad \begin{aligned} \|q_{i,t}(\alpha)-q_{i,t}(\alpha_0)\| &\leqslant \|q_{i,t}(\alpha)-q_{i,t}^T(\alpha)\|+\|q_{i,t}^T(\alpha)-q_{i,t}^T(\alpha_0)\| \\ &\quad +\|q_{i,t}^T(\alpha_0)-q_{i,t}(\alpha_0)\| \\ &\leqslant 2\frac{b^t(\beta b)^{T-t+1}u(\|e\|)\beta b}{u'(\|e_t\|)(1-\beta b)}+\|q_{i,t}^T(\alpha)-q_{i,t}^T(\alpha_0)\|, \end{aligned}$$

where $q_i^T(\alpha)\in\mathscr{D}$ is defined by $q_{i,t}^T(\alpha)\equiv 0$ for $t>T$ and

$$q_{i,t}^T(\alpha)=\beta E_{\pi_t^\alpha}\left\{\frac{u'(e_{t+1})}{u'(e_t)}(q_{i,t+1}^T(\alpha)+d_{i,t+1})\right\}\quad \text{for } t\leqslant T.$$

The continuity of $q_{i,t}^T(\alpha)$ as a function from $[0,1]$ to $C(\Omega^t)$ follows by straightforward induction. This implies that the second term of (B.5) can be made less than $\varepsilon/2$ by choosing $|\alpha-\alpha_0|$ arbitrarily small. Finally, choose T such that the first term of (B.5) is less than $\varepsilon/2$. *Q.E.D.*

Part (a): Let $q^n\in\mathscr{E}$ and $q^n\to q\in\mathscr{D}^n$. Then, by the Maximum Theorem, (3.2.2) is satisfied for q. Therefore, $q\in\mathscr{E}$ and $\mathscr{E}$ is closed.

Define $P\mathscr{E}\subset\mathscr{E}$ to consist of those equilibria q for which (3.3.8) is satisfied by some sequence $\{\xi_t\}$ as in (3.3.7), except that "measurability" is strengthened to "continuity." As in the proof of (3.3.6), $P\mathscr{E}$ can be shown to be path-connected and hence also connected. Secondly, $P\mathscr{E}$ is dense in $\mathscr{E}$. (The argument is similar to the proof of (3.3.5); in particular, ε-optimal continuous policies are used.

A detailed proof is available from the authors upon request.) We conclude (Dugundji (1966, Theorem 1.6, p. 109)) that $\mathscr{E}$ is connected. *Q.E.D.*

APPENDIX C: Aggregation in a Heterogenous Agent Economy

We provide the details to support the example in Section 3.4 dealing with heterogeneous agents.

First define the subclass of our model of utility that corresponds to Schmeidler (1989) where beliefs are represented by a capacity. Say that the probability kernel correspondence $\mathscr{P}$ is *capacity-based* if for each $\omega \in \Omega$: (i) the mapping $A \to \mathscr{P}(\omega, A)$, from $\mathscr{B}(\Omega)$ into $[0,1]$, defines a convex capacity; and (ii) $\mathscr{P}(\omega) = \{m \in M(\Omega): m(A) \geqslant \mathscr{P}(\omega, A), \forall A \in \mathscr{B}(\Omega)\}$. In that case we have the following convenient Choquet integration formula for any $f \in C_+(\Omega)$:

$$\int f \, d\mathscr{P}(\omega) = \int_0^\infty \mathscr{P}(\omega, \{f \geqslant t\}) \, dt.$$

Moreover, and this is critical for what follows, for any two such functions f and g:

$$\text{(C.1)} \qquad \int (f+g) \, d\mathscr{P}(\omega) \geqslant \int f \, d\mathscr{P}(\omega) + \int g \, d\mathscr{P}(\omega)$$

and equality prevails if f and g are *comonotone*, that is, if

$$\text{(C.2)} \qquad [f(\omega') - f(\omega)][g(\omega') - g(\omega)] \geqslant 0, \qquad \forall \omega', \omega \in \Omega.$$

Note that the ε-contamination and belief function kernel examples are capacity-based. For these and other examples in a static setting, see Wasserman and Kadane (1990). Suppose further that $\mathscr{P}$ has full support and that $\mathscr{P}(\omega)$ is constant in ω.

Let β, $\{u^h\}$, $\{\alpha^h\}$, and e be as in the text. For each t and ω^t assign consumption to agent h given by the solution to (3.4.11) with $x = e^*(\omega_t)$. Denote the consumption processes defined in this way by $\bar{c}^h$ and the associated utility processes by $\bar{V}^h$. Then

$$\bar{V}_t^h(\omega^t) = u^h\big(x^{h*}(e^*(\omega_t))\big) + \text{constant},$$

where $x^{h*}(x)$, $h = 1, \ldots, H$, is the solution to (3.4.11). Since the latter functions are all nondecreasing, we see that

$(*)$ for each t, the functions $\{\bar{V}_t^h : h = 1, \ldots, H\}$ are pairwise comonotone.

That is, given the allocation $\{\bar{c}^h\}$, agents agree (weakly) in their induced rankings of states. This occurs because the i.i.d. assumption restricts the dependence of beliefs on the current state so that it does not offset the comonotonicity of current felicities $u^h(x^{h*}(e^*(\cdot)))$.

We now show that $\{\bar{c}^h\}$ solves (3.4.10) uniquely and in particular is Pareto optimal: For any other feasible utility processes $\{V_t^h\}_{t=0}^\infty$ for $h = 1, \ldots, H$, (2.4.1), (C.1), and (3.4.11) imply

$$\sum \alpha_h V_t^h(\omega^t) = \sum \alpha_h u^h\big(c_t^h(\omega^t)\big) + \beta \sum \alpha_h \int V_{t+1}^h(\omega^t, \cdot) \, d\mathscr{P}(\omega_t, \cdot)$$

$$\leqslant \sum \alpha_h u^h\big(\bar{c}_t^h(\omega^t)\big) + \beta \int \sum \alpha_h V_{t+1}^h(\omega^t, \cdot) \, d\mathscr{P}(\omega_t, \cdot)$$

whereas

$$\text{(C.3)} \qquad \sum \alpha_h \bar{V}_t^h(\omega^t) = \sum \alpha_h u^h\big(\bar{c}_t^h(\omega^t)\big) + \beta \int \sum \alpha_h \bar{V}_{t+1}^h(\omega^t, \cdot) \, d\mathscr{P}(\omega_t, \cdot).$$

By the contraction mapping arguments in Appendix A, it follows that

$$\text{(C.4)} \qquad \sum \alpha_h V_t^h(\omega^t) \leqslant \sum \alpha_h \bar{V}_t^h(\omega^t).$$

The full support assumption for $\mathscr{P}$ guarantees that $\{\bar{c}^h\}$ is the unique solution to (3.4.10).

In terms of the candidate representative agent's intertemporal utility U^{α} defined by (3.4.10), it follows from (C.3) and (C.4) that

$$U_t^{\alpha}(e;\omega^t)=u^{\alpha}(e^*(\omega_t))+\beta\int U_{t+1}^{\alpha}(e;\omega^t,\cdot)\,d\mathscr{P}(\omega_t,\cdot).$$

Moreover, a corresponding equality holds also if e is replaced by an arbitrary $c\in\mathscr{D}$, since the preceeding arguments extend to arbitrary endowment processes e. Therefore, U^{α} is generated by β, $\mathscr{P}$, and u^{α}, completing the arguments sketched in the text.

Finally, note that the assumption of i.i.d. beliefs was used above only to guarantee $(*)$. Indeed, the latter condition, assumed to hold not only for the given e but for all endowment processes in an open neighborhood of e in the norm topology of $\mathscr{D}$, suffices for our aggregation result.

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