Asset Pricing with Stochastic Differential Utility

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Asset pricing theory is presented with representative-agent utility given by a stochastic differential formulation of recursive utility. Asset returns are characterized from general first-order conditions of the Hamilton-Bellman-Jacobi equation for optimal control. Homothetic representative-agent recursive utility functions are shown to imply that excess expected rates of return on securities are given by a linear combination of the continuous-time market-portfolio-based capital asset pricing model (CAPM) and the consumption-based CAPM. The Cox, Ingersoll, and Ross characterization of the term structure is examined with a recursive generalization, showing the response of the term structure to variations in risk aversion. Also, a new multicommodity factor-return model, as well as an extension of the "usual" discounted expected value formula for asset prices, is introduced.

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In this article we explore asset pricing theory, conventionally based on additive intertemporal von Neumann–Morgenstern utility, under a continuous-time version of recursive utility.

The conventional asset pricing model in financial economics, the consumption-based capital asset pricing model (CCAPM) of Lucas (1978) and Breeden (1979), assumes that agents' preferences have a time-additive von Neumann–Morgenstern representation. The model has been criticized for two (possibly related) reasons. First, it does not perform well empirically [see, e.g., Hansen and Singleton (1983); Mehra and Prescott (1985); Mankiw and Shapiro (1986); and Breeden, Gibbons, and Litzenberger (1989)]. Second, the above specification of utility confounds risk aversion and intertemporal substitutability within the instantaneous felicity function, while it would be clearly advantageous to the modeler to be able to disentangle these two conceptually different aspects of preference.

Motivated by these two drawbacks of the standard model, Epstein and Zin (1989) and Weil (1990) study recursive intertemporal utility functions in a discrete-time setting. These utility functions permit a degree of separation to be achieved between substitution and risk aversion [this separation having been used to advantage by Epstein (1988), Campbell (1990), and Kandel and Stambaugh (1990)], and also imply relations between asset returns and rates of consumption that match data more closely [see, e.g., Epstein and Zin (1991a, 1991b), Giovannini and Weil (1989), and Bufman and Leiderman (1990)]. Moreover, these objectives are achieved at a reasonable "cost" in terms of the required relaxation of the axioms of intertemporal expected utility theory. The principal relaxation is to allow nonindifference to the temporal resolution of consumption risk in the sense first discussed by Kreps and Porteus (1978). On introspective grounds, such nonindifference seems to us at least reasonable, if not compelling. [For relevant axiomatizations of recursive utility, see Kreps and Porteus (1978), Chew and Epstein (1991), and Skiadas (1991); for further discussion of the reasonableness of recursive utility, see Epstein (1992, sec. 3).]

In a precursor to this article [Duffie and Epstein (1992)], we formulate and analyze a continuous-time form of recursive utility, which we call stochastic differential utility. In this article, we demonstrate the tractability and usefulness of these utility functions by applying them in a representative-agent framework to derive a number of new asset pricing models. We take full advantage of the analytical power afforded by continuous time—our asset pricing results are sharper and simpler than those achieved in a discrete-time framework.

We derive two alternative extensions of Breeden's CCAPM. In the first, assuming that the representative agent's utility function is homothetic, we show that expected excess returns on assets are given by a linear two-factor model, in which the factors are aggregate consumption growth and the return on the market portfolio. That is, one has a linear combination of the CCAPM and the static CAPM of Sharpe (1964) and Lintner (1965). In a discrete-time setting, such a two-factor model has been derived only as an approximation under the assumption that consumption and asset returns are jointly lognormally distributed [Epstein and Zin (1991a)].

The above two-factor model is susceptible to the Roll (1977) critique of CAPM—the aggregate portfolio presumably includes non-traded assets, so returns to this portfolio are imperfectly observable. Therefore, we describe an alternative extension of the CCAPM that is immune to this criticism. We require that consumption at each instant has at least two components, and thereby derive (under specified assumptions) a multifactor model of expected excess returns that is distinguished from the CCAPM by the inclusion of a relative price factor. For example, if consumption consists of "ordinary" consumption and leisure, then the covariance of an asset's return with the real wage is one of the determinants of the systematic risk of the asset. For another example, if the agent is representative of a small open economy, and if the consumption vector consists of one traded good and one nontraded good, then covariation with the real exchange rate is one explanatory factor for the excess mean return of an asset.

In our final asset pricing model, we highlight the theoretical advantages of the flexibility provided by stochastic differential utility in disentangling risk aversion and intertemporal substitution. This is done by examining a generalization of the Cox, Ingersoll, and Ross (1985a) model of the term structure of interest rates in which one can characterize the response of the term structure to variations in risk aversion. We emphasize that such a characterization is not possible given the additive von Neumann–Morgenstern utility specification adopted by Cox, Ingersoll, and Ross—a change in the concavity of the instantaneous felicity function affects both risk aversion and the degree of intertemporal substitution and thus the interpretation of the corresponding comparative statics results is ambiguous.

We include one further asset pricing result of note—an extension of the "usual" expected discounted value formula for pricing a stream of dividends. This formula is used in our analysis of the term structure, but it is of broader interest.

To conclude this introduction, we relate our work to a body of literature that studies alternative generalized preference structures and the asset pricing implications that they deliver [see, e.g., Singleton (1990), Sundaresan (1989), Constantinides (1990), Hindy and Huang (1990), Detemple and Zapatero (1989), and Heaton (1991)]. These authors study utility functions that relax intertemporal additivity in an attempt to model, for example, habit formation or the durability of consumption goods. A consequence is that past consumption "matters" directly in affecting current preferences, while, for recursive utility, past consumption influences current choices only via its effect on current wealth.[1] While these alternative routes to generalizing intertemporal utility have considerable intuitive appeal, they do not deliver a separation between substitution and risk aversion [see Epstein (1992)], nor do they yield the new asset pricing models described below. It is a still unresolved empirical question whether recursive utility or one of the above alternatives, or perhaps a suitable composite, best explains and helps to organize the observed behavior of consumption and asset returns. There are of course other research directions that may also serve to better explain equilibrium asset prices, such as incomplete markets, transactions costs, and so on.

We proceed as follows: In Section 1, we provide a brief and user-oriented outline of stochastic differential utility. Asset pricing implications are described in Section 2.

1. Stochastic Differential Utility

1.1 Background

In this section, we describe an extension of the standard additive utility specification, in which the utility at time t for a consumption process c is defined by

$$V_t = E_t\left[\int_{s \geq t} e^{-\beta(s-t)} u(c_s)\, ds\right], \qquad t \geq 0, \tag{1}$$

where E_t denotes expectation given information available at time t. The more general utility functions, called *stochastic differential utility* (SDU), exhibit intertemporal consistency and admit Bellman's characterization of optimality. Much of the tractability of (1) is therefore preserved. The SDU model is a continuous-time analogue of the Epstein–Zin discrete-time utility, in which the utility is defined recursively by

$$V_t = W(c_t, m(\sim V_{t+1} \mid \mathcal{F}_t)), \tag{2}$$

where W is a function in two variables, $\sim V_{t+1} \mid \mathcal{F}_t$ denotes the distribution of the utility V_{t+1} at time $t+1$ conditional on the information

[1] Another difference between the two classes of utility functions is that, for the class in which the past matters, it is assumed that the von Neumann–Morgenstern model describes attitudes toward gambles in consumption, while the recursive utility class admits more general risk preferences. However, there exist recursive utility functions that conform with the von Neumann–Morgenstern axioms, for which past consumption does not influence current preferences, but which are not intertemporally additive (see Example 2 in Section 1.3).

$\mathcal{F}_t$ available at time t, and m is a certainty-equivalent functional. For example, a version of the Kreps–Porteus (1978) model can be obtained by taking the special case of the expected-utility certainty-equivalent m defined by $m(\sim X) = h^{-1}(E[h(X)])$, where h is a von Neumann–Morgenstern utility function.

The outline of stochastic differential utility provided here is designed to provide adequate background for understanding the asset pricing applications. For further details and for proofs of some of the assertions made below, the reader is referred to our companion article [Duffie and Epstein (1992)].

1.2 The definition

We begin with a typical setup for continuous-time asset pricing: a standard Brownian motion B in $\mathbb{R}^d$ on a given probability space, and information given by the standard filtration $\{\mathcal{F}_t\}$ of B. That is, $\mathcal{F}_t$ is the σ-algebra generated by $\{B_s: 0 \leq s \leq t\}$ and augmented [see Karatzas and Shreve (1988) or Chung and Williams (1991) for technical definitions that we do not provide here]. The time horizon $T \in (0, \infty]$ is finite unless otherwise indicated. Consumption processes are chosen from the space D of square-integrable[2] progressively measurable processes valued in $C = \mathbb{R}^l_+$, for some number l of commodities. (We take $l = 1$ except where indicated.)

The stochastic differential utility $U: D \to \mathbb{R}$ is defined as follows by two primitive functions, $f: C \times \mathbb{R} \to \mathbb{R}$ and $A: \mathbb{R} \to \mathbb{R}$. When well defined, the utility process V for a given consumption process c is the unique Ito process V with $V_T = 0$ having a stochastic differential representation of the form

$$dV_t = [-f(c_t, V_t) - \tfrac{1}{2}A(V_t)\|\sigma_v(t)\|^2]\, dt + \sigma_v(t)\, dB_t,$$

where σ_v is an $\mathbb{R}^d$-valued square-integrable progressively measurable process, and where the dependence of V on c is suppressed in the notation. We think of V_t as the continuation utility for c at time t, conditional on current information, and $A(V_t)$ as a variance multiplier, applying a penalty (or reward) as a multiple of the utility "volatility" $\|\sigma_v(t)\|^2$. If, for each consumption process c, there is a well-defined utility process V, the stochastic differential utility function U is defined by $U(c) = V_0$, the initial utility. The pair (f, A) generating V is called an aggregator. Roughly speaking, the connection between (f, A) and the pair (W, m) employed in the discrete-time formulation (2) of recursive utility is that f is a differential counterpart of W, while A is a measure of the local risk aversion of m.

[2] The square-integrability requirement on a process c is that $E(\int_0^T \|c_t\|^2\, dt) < \infty$, or when $T = +\infty$, the requirement is that $E(\int_0^\infty e^{\nu t}\|c_t\|^2\, dt) < \infty$, where ν is a constant characterized in Appendix C of Duffie and Epstein (1992). (This appendix is co-authored with Costis Skiadas.)

Since $V_T = 0$ and $\int \sigma_v(t)\ dB_t$ is a martingale, we can equally well write

$$V_t = E_t\left(\int_t^T \left[f(c_s,\ V_s) + \frac{1}{2}A(V_s)\|\sigma_V(s)\|^2 \right] ds\right), \tag{3}$$

where E_t denotes expectation given $\mathcal{F}_t$. In special situations, we may vary from the convention taken here of having zero terminal utility V_T.

On occasion (Section 2.4), we will consider the infinite horizon case. For each consumption process c, the infinite horizon utility process V_t is defined as the pointwise limit, as $T \to \infty$, of V_t^T, where we let V^T denote the solution of (3). That is,

$$V_t \equiv \lim_{T\to\infty} V_t^T. \tag{3'}$$

In Appendix C of Duffie and Epstein, co-authored with Costis Skiadas, it is shown that the utility process V_t defined in this way is the unique solution V, satisfying a suitable transversality condition, to

$$V_t = E_t\left(\int_t^T \left[f(c_s,\ V_s) + \frac{1}{2}A(V_s)\|\sigma_V(s)\|^2 \right] ds + V_T\right), \qquad T \geq t.$$

The special case (1) of additive utility is obtained by letting $A = 0$ and $f(c, v) = u(c) - \beta v$, as can be checked with an application of Ito's Lemma. Though (3) seems considerably more complicated than (1), a principal thrust of our earlier article and a major objective here is to show that (3) is nevertheless "as tractable" as (1). This is accomplished by establishing a number of desirable properties for the defining relation (3).

First, under specified assumptions on the aggregator, there exists a unique Ito process V satisfying (3). Thus, the stochastic differential utility function U is well defined by (3), though an explicit closed-form representation for U generally does not exist. Second, under natural conditions on the aggregator, U has a range of natural properties. For example, U is monotonic and risk averse if $A(\cdot) \leq 0$ and if f is jointly concave and increasing in consumption. A third desirable property is the existence of a number of canonical parametric functional forms, in addition to the standard specification (1), two of which are described below. Fourth, Bellman's characterization of optimality can be applied in such a way that state variables reflecting past consumption are unnecessary. The fact that past consumption does not matter, in the sense that the continuation utility V_t is independent of consumption prior to t, is a consequence of the forward-looking nature of (3). The Bellman equation is described in Section 2.

A final attractive feature of stochastic differential utility, and one that distinguishes it from the additive expected utility form (1), is the flexibility to partially disentangle intertemporal substitution from risk aversion. It is apparent from (3) that, for deterministic consumption processes, the variance multiplier A is irrelevant since it multiplies a zero variance. Thus "certainty preferences," including the willingness to substitute consumption across time, are determined by f alone. We conclude that only risk attitudes are affected by a change in A (holding f fixed). In particular, let

$$A^*(\cdot) \leq A(\cdot), \tag{4}$$

and suppose that U and U^* are the intertemporal utility functions corresponding to (f, A) and (f, A^*), respectively. Then U^* is *more risk averse than* U in the sense that any consumption process c rejected by U in favor of some deterministic process $\bar{c}$ would also be rejected by U^*. That is,

$$U(c) \leq U(\bar{c}) \Rightarrow U^*(c) \leq U^*(\bar{c}). \tag{5}$$

We emphasize that this establishes the significance of the variance multiplier for comparative rather than absolute risk aversion and that this significance arises only for a given f. [That is, the comparison above is between (f, A) and (f, A^*) with f common.] In particular, if A^* is everywhere negative, then it is correct to say that U^* corresponding to (f, A^*) is more risk averse than U corresponding to $(f, 0)$, but it is not necessarily true that U^* is risk averse [in the sense of preferring any c to the mean consumption process for which time t consumption equals $E(c_t)$].

1.3 Examples

We offer the following examples, which are also found in Duffie and Epstein (1992).

Example 1 (Standard Additive Utility). The standard additive expected utility function (1), with the utility process

$$\bar{V}_t = E_t\left[\int_{s \leq t} u(c_s)\, e^{-\beta(s-t)}\, ds\right],$$

corresponds to the aggregator $(\bar{f}, \bar{A})$, where

$$\bar{f}(c, v) = u(c) - \beta v, \qquad \bar{A} = 0. \tag{6}$$

If $C = \mathbb{R}_+$ and u has the usual properties, then we can also define the aggregator

$$f(c, v) = \beta \frac{u(c) - u(v)}{u'(v)}, \qquad A(v) = \frac{u''(v)}{u'(v)}. \tag{6'}$$

Ito's Lemma applied to $u(V_t)$ shows that the corresponding utility process V satisfies

$$V_t = u^{-1}(\beta \bar{V}_t) = u^{-1}\left(E_t\left[\beta \int_{s \geq t} u(c_s) e^{-\beta(s-t)}\, ds \right] \right).$$

In particular, the utility functions U defined by (f, A) and $\bar{U}$ defined by $(\bar{f}, \bar{A})$ are ordinally equivalent and thus represent the same preference ordering of consumption processes. Therefore, in a sense clarified further below, the aggregators $(\bar{f}, \bar{A})$ and (f, A) are "equivalent."

Example 2 (Uzawa Utility). Let the aggregator (f, A) be defined by

$$f(c, v) = u(c) - \beta(c)v, \qquad A(v) = 0, \tag{7}$$

which extends (6) by allowing β to vary with c. Then an application of Ito's Lemma yields the stochastic differential utility U:

$$U(c) = E_0\left[\int_0^\infty u(c_s) \exp\left[-\int_0^s \beta(c_\tau)\, d_\tau \right] ds \right]. \tag{8}$$

This functional form was proposed by Uzawa (1968) in the context of certainty, and was subsequently studied in a setting of uncertainty by Epstein (1983) and applied to asset pricing issues by Bergman (1985). Restricted to deterministic consumption processes, U violates the strong intertemporal separability of (1) if $\beta(\cdot)$ is not constant. In particular, the marginal rate of substitution [see (35) below and the ensuing discussion] between consumption at two instants s and t, with $s < t$, is independent of consumption at times preceding s, but depends on consumption at all other times. Since the variance multiplier A is fixed in (7), however, there is no scope for changing risk attitudes without at the same time affecting certainty preferences or the willingness to substitute across time. The next example provides such flexibility.

Example 3 (Kreps–Porteus Utility). Let $C = \mathbb{R}_+$, $0 \neq \rho \leq 1$, $0 \leq \beta$, $0 \neq \alpha \leq 1$, and define, for $v > 0$,

$$f(c, v) = \frac{\beta}{\rho} \frac{c^\rho - v^\rho}{v^{\rho-1}}, \qquad A(v) = \frac{\alpha - 1}{v}. \tag{9}$$

Although a closed-form expression for the corresponding utility function is not available, this example has some nice features. First, the function f coincides with (6′) if $u(c) \equiv c^\rho/\rho$. Thus, for deterministic consumption processes, the utility function is of the additive CES form, with elasticity of intertemporal substitution $(1 - \rho)^{-1}$. If $\alpha = \rho$, then the standard additive and Kreps–Porteus utilities coincide even for random processes since the aggregators defined by (6′) and (9) are identical. This is not so, however, if $\alpha \neq \rho$. It can be shown (by applying Ito's Lemma and taking limits as the length of a time interval goes to zero) that the corresponding stochastic differential equation for the utility process V is the continuous-time limit of the homogeneous CES specification examined in discrete time by Epstein and Zin (1989). It is shown there that risk aversion of the intertemporal ordering increases as α falls. A corresponding result for our continuous-time framework is implied by the discussion surrounding (4) above, since $A(\cdot)$ falls as α does. In a well-defined sense, therefore, ρ and α can be interpreted as intertemporal substitution and risk aversion parameters, respectively.

1.4 Ordinally equivalent utility processes

Only the ordinal properties of a utility function are of interest and, as illustrated in Example 1 above, there may exist different aggregators generating ordinally equivalent utility functions. Moreover, for analytical convenience we may wish to employ a particular representation of intertemporal utility and the corresponding aggregator. In order to explore this, we consider a change of variables in the form of a twice continuously differentiable $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ that is strictly increasing with $\varphi(0) = 0$. Two utility functions U and $\bar{U}$ are ordinally equivalent if there is a change of variables φ such that $\bar{U} = \varphi \circ U$. Two aggregators (f, A) and $(\bar{f}, \bar{A})$ are then defined to be *ordinally equivalent* if they generate ordinally equivalent utility functions.

With the aid of Ito's Lemma, it can be seen that two aggregators (f, A) and $(\bar{f}, \bar{A})$ generating utility functions are ordinally equivalent if there is a change of variables φ with

$$\begin{aligned} f(c, z) &= \frac{\bar{f}[c, \varphi(z)]}{\varphi'(z)}, \qquad (c, z) \in C \times \mathbb{R}, \\ A(x) &= \varphi'(x)\bar{A}[\varphi(x)] + \frac{\varphi''(x)}{\varphi'(x)}. \end{aligned} \tag{10}$$

For example, beginning with (f, A), an often simplifying change of variables is some such φ that eliminates the variance multiplier A from the formulation. That is, consider the possibility of choosing φ so that the new variance multiplier $\bar{A}$ defined by (10) is zero. It is enough that φ satisfies the differential equation $\varphi''(x) = A(x)\varphi'(x)$. Solutions are defined by

$$\varphi(v) = C_2 + C_1 \int_{v_0}^{v} \exp\left[\int_{v_0}^{u} A(x)\, dx\right] du, \tag{11}$$

where v_0 is arbitrary and C_2 and C_1 are constants, with $C_1 > 0$, chosen so that $\varphi(0) = 0$.

If $\bar{V} = \varphi \circ V$ is integrable, it follows that V satisfies (3) if and only if

$$\bar{V}_t = E_t\left[\int_t^T \bar{f}(c_s, \bar{V}_s)\, ds\right], \qquad t \in [0, T]. \tag{12}$$

By the above construction, any aggregator (f, A) has an ordinally equivalent aggregator $((\bar{f}, \bar{A})$ whose variance multiplier $\bar{A}$ is zero. We refer to $(\bar{f}, \bar{A})$, or $\bar{f}$ itself, as the normalized version of (f, A).

Since normalizing an aggregator yields the substantial simplification of (3) represented by (12), this is advantageous for proving the existence of stochastic differential utility [see, for existence results, Duffie and Epstein (1992) as well as Duffie and Lions (1990)]. In this article, the primary reason for being interested in the normalized version of an aggregator is that it leads to a simpler form for the Bellman equation.

We emphasize that the reduction to a normalized aggregator $(\bar{f}, 0)$ does not mean that intertemporal utility is risk neutral (even though $\bar{A} = 0$) or that we have lost the ability to disentangle substitution from risk aversion. Such disentangling is not easily expressed in terms of $\bar{f}$, however, since both components of the original aggregator (f, A) are involved in $\bar{f}$, and the exercise of "keeping f fixed while changing A" is not readily expressed in terms of $\bar{f}$ alone. Thus, we suggest the following two-step procedure to the modeler:

1. Specify an unnormalized aggregator (f, A) such as (9); each component admits an unambiguous interpretation as in the Kreps–Porteus example.

2. In order to solve intertemporal optimization problems via the Bellman equation, apply the change of variables (11) to obtain the normalized version $(\bar{f}, 0)$, keeping in mind the interpretation of the components of $\bar{f}$ provided by Step 1. For example, the normalized aggregator $\bar{f}$ corresponding to (9) is given by

$$\bar{f}(c, v) = \frac{\beta}{\rho}\frac{c^{\rho} - (\alpha v)^{\rho/\alpha}}{(\alpha v)^{(\rho/\alpha - 1)}},$$

where the parameters ρ and α have been previously interpreted.

1.5 Homotheticity

We will make use of the following definition in the asset pricing discussion. A utility function U is homothetic if, for any consumption processes c and c' and any scalar $\lambda > 0$,

$$U(\lambda c') \geq U(\lambda c) \Leftrightarrow U(c') \geq U(c).$$

The SDU function generated by an aggregator (f', A') is shown in Duffie and Epstein (1992) to be homothetic if (and in a natural sense only if) there is an ordinally equivalent aggregator (f, A) satisfying (i) f is homogeneous of degree 1, and (ii) the variance multiplier A is linearly homogeneous of degree -1 in that, for some k, $A(v) = k/v$ for all $v > 0$. In fact, the nonordinal utility function U generated by such an (f, A) is homogeneous of degree 1—that is, $U(\lambda c) = \lambda U(c)$ for all c and $\lambda > 0$. An example is the Kreps–Porteus utility generated by the aggregator (9).

2. Asset Pricing Results

Following the approach of Merton (1973) and Breeden (1979), as well as Cox, Ingersoll, and Ross (1985b), we can explore the implications for asset returns of the Bellman equation determining optimal consumption and portfolio choice. Svensson (1989) has done independent work on the portfolio choice problem with a class of recursive utility functions. [The more abstract approach of Duffie and Skiadas (1990) mentioned below can be used to obtain the same results in a non-Markovian setting and with weaker differentiability assumptions.]

We take as given an n-dimensional Markov state process X satisfying the stochastic differential equation

$$dX_t = b(X_t, t)\,dt + a(X_t, t)\,dB_t, \qquad X_0 \in \mathbb{R}^n, \tag{13}$$

where b: $\mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ and a: $\mathbb{R}^n \times [0, T] \to \mathbb{R}^{n \times d}$. (It is enough for the existence and strong uniqueness of X that a and b are Lipschitz.) We follow the line of exposition in Duffie (1988, exercise 25.12), and assume that the value function for optimal utility is a $C^{2,1}$ function J: $\mathbb{R}^n \times \mathbb{R} \times [0, T] \to \mathbb{R}$, with $J(x, w, t)$ denoting the maximum utility in state x with wealth w at time t. In state x at time t, let $\lambda(x, t) \in \mathbb{R}^N$ denote the vector of expected rates of return of the N given risky assets in excess of the riskless instantaneous return $r(x, t)$, and let $\sigma(x, t)$ denote the $N \times d$ matrix of diffusion coefficients of the risky asset prices, normalized by the asset prices, so that $\sigma(x, t)\sigma(x, t)^{\top}$ is the "instantaneous covariance" matrix for asset returns. The combined state process $\{(X_t, W_t)\}$ is defined by the underlying Markov "shock" process X and by the wealth process W satisfying

$$dW_t = [W_t z_t^{\top}\lambda(X_t, t) + W_t r(X_t, t) - c_t]\,dt + W_t z_t^{\top}\sigma(X_t, t)\,dB_t, \tag{14}$$

where $\{(c_t, z_t)\}$ is the consumption-portfolio control process, with $z_t \in \mathbb{R}^N$ representing the fractions of wealth W_t invested in the N risky assets at time t.

For a normalized aggregator f, Duffie and Epstein show that the Bellman equation for optimal control is

$$\sup_{(c,z)\in C\times\mathbb{R}^N} \mathcal{D}^{(c,z)}J(x, w, t) + f[c, J(x, w, t)] = 0, \tag{15}$$

where

$$\mathcal{D}^{(c,z)}J(x, w, t) = J_t + J_x b + J_w(wz^\top\lambda + wr - c) + \tfrac{1}{2}\mathrm{tr}(\Sigma),$$

with

$$\Sigma = \begin{pmatrix} a \\ wz^\top\sigma \end{pmatrix}^\top \begin{pmatrix} J_{xx} & J_{xw} \\ J_{wx} & J_{ww} \end{pmatrix} \begin{pmatrix} a \\ wz^\top\sigma \end{pmatrix}.$$

This reduces to the familiar Bellman equation if f is given by (6), corresponding to the additive expected utility model of preferences. The simplicity of (15) is notable, especially in light of the complexity of (3) and even (12).

2.1 A two-factor CAPM

We will continue to adopt, without explicit mention, differentiability assumptions suggested by our line of attack. The first-order condition for the Bellman equation for optimal interior c is $f_c = J_w$. Assuming that the optimal consumption policy is given by a smooth function C of states [i.e., $c_t = C(X_t, W_t, t)$], we can differentiate $J_w = f_c$ with respect to w and obtain $J_{ww} = f_{cc}C_w + f_{cv}J_w$. Likewise, differentiating with respect to x leaves

$$J_{wx} = f_{cc}C_x + f_{cv}J_x. \tag{16}$$

The first-order condition with respect to the portfolio vector z is

$$J_w w\lambda + J_{ww}\sigma\sigma^\top z w^2 + \sigma a^\top J_{xw} w = 0. \tag{17}$$

Simplifying, the vector λ of excess expected rates of return satisfies

$$-\lambda = \frac{f_{cc}}{J_w}\sigma\sigma_c^\top + f_{cv}\sigma\sigma^\top z w + \frac{f_{cv}}{J_w}\sigma a^\top J_x^\top, \tag{18}$$

where

$$\sigma_c = C_x a + C_w w z^\top \sigma \tag{19}$$

is the diffusion function for consumption. Assuming a single representative agent, $\Sigma_{RC} \equiv \sigma\sigma_c^\top$ is the vector of "instantaneous covariances" of asset returns with aggregate consumption increments. Likewise, since market clearing implies that $z_t W_t$ is the vector of total market values of the securities, we can view $\Sigma_{RM} \equiv \sigma\sigma^\top z_t W_t$ as the vector of instantaneous covariances of asset returns with increments in the value of the market portfolio. For the third term on the right-hand side of (18), however, it is difficult to give an observable interpretation without further assumptions.

Suppose, for example, that C_x and J_x are co-linear. This is true, for instance, under homothetic preferences, for which J is homogeneous with respect to wealth—that is, of the form $J(x, w, t) = j(x, t)w^\gamma$ for some j and γ.[3] By (16), therefore,

$$f_{cc}C_x = (\gamma w^{-1} - f_{cv})J_x. \tag{20}$$

In this case, using (19), the third term of (18) is a linear combination of Σ_{RC} and Σ_{RM}, and we are left with a restriction on asset returns of the form

$$\lambda = k_1\,\Sigma_{RC} + k_2\,\Sigma_{RM}, \tag{21}$$

for scalar-valued functions k_1 and k_2 on $\mathbb{R}^n \times [0, T]$. In other words, we have a linear two-factor model for excess expected returns that is a linear combination of the consumption-based CAPM of Breeden (1979) and the market-portfolio-based CAPM of Sharpe (1964), Lintner (1965), Merton (1973), and Chamberlain (1988).

It can be shown, as one would expect, that the functions k_1 and k_2 are invariant under the ordinal transformation (10). It is convenient to adopt the linearly homogeneous utility function U to represent preferences corresponding to an aggregator (f, A) such that f is linearly homogeneous and $A(v) = (\alpha - 1)/v$ for some $\alpha < 1$ (see Section 1.5). In that case, one can derive that

$$k_1 = \frac{\gamma\alpha}{c(1-\gamma)}, \qquad k_2 = \frac{1-\gamma-\alpha}{w(1-\gamma)}, \tag{22}$$

where $\gamma = -cf_{cc}/f_c$. We have seen [via (4) and (5)] that α can be interpreted as a comparative risk-aversion parameter. On the other hand, Epstein (1987) shows that γ provides an inverse measure of the degree to which consumption is substitutable across time in a deterministic setting. [For the Kreps–Porteus utility example (Example 3), $\gamma = 1 - \rho$, the reciprocal of the constant elasticity of intertemporal substitution.] Thus, both intertemporal substitution and risk aversion determine the "factor premiums" k_1 and k_2 in (21).

The assumptions underlying our two-factor model—the existence of a representative agent and homothetic utility—are special. Under homotheticity, demand aggregation in the strong sense of Gorman

[3] We are given the homothetic utility function U and aggregator $(f, 0)$. We apply Section 1.5 repeatedly to argue as follows: First, there exists an aggregator (f', A'), ordinally equivalent to $(f, 0)$, such that the implied utility function U' is homogeneous of degree 1 and such that $A'(v) = k/v$. If $k = 0$, then $A' = 0$, and by (10) and (12), $U' = KU$ for some constant K. If $k \neq 0$, then $\bar{U} = (U')^k/k$ has aggregator $(\bar{f}, 0)$, implying (as above) that $U = K\bar{U}$. In either case, U is homogeneous of some degree γ—that is, $U(\lambda c) = \lambda^\gamma U(c)$, for all $\lambda > 0$ and all c. The desired homogeneity property of J now follows from the fact that the consumption process c is optimal given initial conditions (x, w, t) if and only if the rescaled process $w^{-1}c$ is optimal given $(x, 1, t)$.

(1953) can be invoked to derive a representative agent, but only by assuming that individuals are identical in terms of preferences and information. In contrast, the single-factor CCAPM is valid even with heterogeneity across individuals; neither does it require homotheticity, though that is frequently assumed in empirical implementations. On the other hand, the two-factor model performs better empirically, as illustrated by the references cited in the introduction, and also by Mankiw and Shapiro (1986) and Bollerslev, Engle, and Wooldridge (1988).

Bergman (1985) derives a form of (18) corresponding to the Uzawa special case of recursive utility (see Example 2 of Section 1.3). He points out that, with nonseparability across time via endogenous discounting, Breeden's consumption-beta model does not apply. We find the Uzawa utility function to be better suited to a multicommodity setting as we now explain.

2.2 A multicommodity-factor model of returns

The preceding two-factor model assumes not only the homotheticity of utility, but also that all income is investment income. For example, labor income and exogenously endowed consumption were excluded in order to guarantee homogeneity (with respect to wealth) of the value function, which homogeneity was critical in deriving (21). One can view exogenous income streams as the dividends of some shadow asset, in which case (21) is still valid if the market portfolio is expanded to include the new asset. However, if the latter is not traded, then the return to the market portfolio is not readily observable or computable from available data.

Here, we describe an alternative set of assumptions, allowing one to replace the shadow prices J_x in (18) with observables. The resulting model of asset prices can accommodate the features noted above. In particular, a labor-supply decision could be part of the decision problem of the representative agent, and only returns to traded assets enter into the pricing formula.

Reconsider the basic consumption-portfolio choice model, but let the consumption rate at time t be in the form of a vector $(c_t, q_t) \in \mathbb{R}_+ \times \mathbb{R}^l_+$, $l \geq 1$. The first good is the numeraire; the remaining relative prices are defined by some smooth p: $\mathbb{R}^n \times [0, T] \to \mathbb{R}^l_+$. One of the components of q_t could represent leisure and the corresponding component of $p(X_t, t)$ would be a real wage rate. The wealth process W given by (14) is reformulated as

$$dW_t = [r(X_t, t)W_t + W_t z_t^\top \lambda(X_t, t) - c_t - p(X_t, t)^\top q_t + y(X_t, t)]\,dt + W_t z_t^\top \sigma(X_t, t)\,dB_t, \tag{23}$$

where y is an exogenous income process, which could include the value of the agent's skill and time endowments.

Suppose, as before, that the optimal consumption choice q_t is given in feedback form as $Q(X_t, W_t, t)$, for some smooth function Q. With $\Sigma_{RQ} \equiv \sigma\sigma_Q^{\top}$, where $\sigma_Q \equiv Q_x a + Q_w w z^{\top}\sigma$, the first-order conditions on the portfolio choice z as well as market clearing imply that

$$-\lambda = \Sigma_{RC}\frac{f_{cc}}{f_c} + \Sigma_{RQ}\frac{f_{cq}^{\top}}{f_c} + f_{cv}\left(\Sigma_{RM} + \frac{\sigma a^{\top} J_x^{\top}}{J_w}\right). \tag{24}$$

Consumption choices at each instant must satisfy

$$f_c(c_t, q_t, J) = J_w \tag{25}$$

and

$$f_q(c_t, q_t, J) = p^{\top} J_w. \tag{26}$$

Under an additional assumption, which we now describe, we can use (25) and (26) to replace the term in (24) involving $\sigma a^{\top} J_x^{\top}/J_w$ by a more readily interpretable expression. Evidently, for each commodity $i \in \{1,\ldots,l\}$,

$$\frac{f_{q_i}}{f_c}(c_t, q_t, J) = p_i, \tag{27}$$

which suggests that f_{q_i}/f_c is the marginal rate of substitution between c_t and q_{it}. (This equivalence can be established rigorously.) In the standard specification [replacing c by (c, q) in (25)], this marginal rate of substitution is independent of J since (c_t, q_t) is weakly separable from consumption at all times $t' \neq t$. In contrast, here we assume the opposite, in the form of the following assumption: There exists at least one $j \in \{1,\ldots,l\}$ such that, for each (c, q), the function

$$\Psi(\cdot) \equiv \frac{f_{q_j}}{f_c}(c, q, \cdot)$$

has a differentiable inverse Ψ^{-1}. In this case, we refer to c and q_j as being *invertibly nonseparable* (from the future). This is true, for example, if

$$\frac{\partial}{\partial v}\left[\frac{f_{q_j}}{f_c}(c, q, v)\right]$$

is everywhere strictly positive, in which case an increase in future prospects (as represented by "slightly larger utility V_t") shifts preferences toward consumption of commodity j and away from the numeraire commodity, given fixed levels of the other commodities.

Since invertible nonseparability is an assumption regarding f that is invariant to the transformation (10), it can be interpreted as a restriction on certainty preferences, that is, on the preference ordering of deterministic consumption processes. In particular, it is compatible with a utility function that conforms with von Neumann–Morgenstern theory in the ranking of stochastic consumption processes, though it is not compatible with the standard additive form (1). For example, invertible nonseparability is satisfied by the Uzawa utility function if the aggregator (7) satisfies, after an obvious change in notation and for all (c, q),

$$u_{q_j}\beta_c \neq u_c\beta_{q_j}.$$

Moreover, invertible nonseparability is satisfied also by the generated utility function U even if the variance multiplier A in (7) is not restricted to be zero. In that case, U generally violates the von Neumann–Morgenstern axioms (in the same way that the Kreps–Porteus function does), and a closed-form expression for U is not available. On the other hand, expanding the functional form by allowing the specification of A to vary is advantageous, since this provides the flexibility needed to disentangle substitution and risk aversion in the sense of (4) and (5).

Given an invertibly nonseparable pair (c, q_j), we can differentiate (with respect to x) the first-order condition

$$f_{q_j}(C, Q, J) = p_j f_c(C, Q, J), \tag{28}$$

and obtain

$$(p_j f_{cv} - f_{q_j v}) J_x = (f_{q_j c} - p_j f_{cc}) C_x + (f_{q_j q} - p_j f_{cq}) Q_x - f_c p_{jx}, \tag{29}$$

where p_{jx} denotes the (row) vector of partial derivatives of p_j with respect to x. A corresponding equation is obtained for J_w if (28) is differentiated with respect to w:

$$(p_j f_{cv} - f_{q_j v}) J_w = (f_{q_j c} - p_j f_{cc}) C_w + (f_{q_j q} - p_j f_{cq}) Q_w - f_c p_{jw}. \tag{30}$$

Substituting (29) and (30) into (24), simplifying, and letting $\Sigma_{Rp_j} \equiv \sigma a^\top p_{jx}{}^\top$, we can rewrite the asset return restriction (24) as

$$\begin{aligned} -\lambda = {} & \Sigma_{RC} f_c^{-1}[f_{cc} + f_{cv}(f_{q_j c} - p_j f_{cc})(p_j f_{cv} - f_{q_j v})^{-1}] \\ & + \Sigma_{RQ} f_c^{-1}[f_{cq}^\top + f_{cv}(f_{q_j q} - p_j f_{cq})^\top (p_j f_{cv} - f_{q_j v})^{-1}] \\ & - \Sigma_{Rp_j} f_{cv}(p_j f_{cv} - f_{q_j v})^{-1}. \end{aligned} \tag{31}$$

Covariances with all consumption levels *and* with the price of commodity number j are therefore important in determining mean excess returns. The intuition underlying (31) is apparent from its derivation—the relative price factor appears because p_j serves as a proxy for unobservable future utility.

With the additive expected utility model (25), we have

$$-\lambda = (\Sigma_{RC} f_{cc} + \Sigma_{RQ} f_{cq}^{\mathsf{T}})/f_c.$$

[This can be compared with the results of Breeden (1979, sect. 7), in which price factors enter but only if an asset return's covariance with each of the consumption quantities is replaced by the single covariance with total real expenditure.] In a cross-sectional regression of mean excess returns of the various securities, the appearance of the covariance term Σ_{Rp_j} (of returns with the price increments of commodity j) as an additional explanatory variable can therefore be interpreted as evidence against the weak separability of (c, q) and for the alternative of invertible nonseparability of (c, q_j). On the other hand, given the presence of Σ_{Rp_j}, the addition of Σ_{Rp_k}, $k \neq j$, to the regression should not lead to greater explanatory power if our model is valid.

Finally, suppose that, for any commodity $k \neq j$,

$$f_{cq_k} = f_{q_j q_k} = 0.$$

Then we have the three-factor model

$$-\lambda = k_1\, \Sigma_{RC} + k_2\, \Sigma_{Rq_j} + k_3\, \Sigma_{Rp_j}, \tag{32}$$

and Σ_{Rq_k} does not enter if $k \neq j$. This theoretical rationale for focusing on a subset of consumption betas in an empirical analysis may be useful if consumption data are limited.

The assumption of invertible nonseparability originated with Epstein and Zin (1990), although the factor structure of (31) was not evident in their discrete-time formulation. There is no empirical evidence yet available regarding (31).

2.3 An asset pricing formula

In this subsection, we present a map from the space of stochastic processes for security dividends, in units of the numeraire consumption commodity, to the space of stochastic processes for the corresponding security price. Merely the absence of arbitrage and some mild technical assumptions [as reviewed in Duffie (1992, chap. 6)] imply the existence of a positive "state-pricing process" π, defined by the property that, for any security promising a dividend rate process δ, the stochastic price process S for the market value of the security satisfies

$$S_t = \frac{1}{\pi_t} E_t\left(\int_t^T \pi_s \delta_s \, ds\right), \qquad t \in [0, T]. \tag{33}$$

(We suppose that both δ and π are square integrable, and maintain our finite time horizon T.) More generally, a security promising a cumulative dividend process D of finite variation has a price process S given by

$$S_t = \frac{1}{\pi_t} E_t\left(\int_t^T \pi_s \, dD_s\right). \tag{34}$$

For example, the price of a pure discount bond paying 1 unit of consumption at time s is, at any time $t < s$, equal to $E_t(\pi_s/\pi_t)$.

In the additively separable case (1), the state-price process π is identified within a representative-agent framework by $\pi_t = e^{-\beta t} u'(c_t)$ [assuming that u is differentiable and that $u'(c_t)$ is square-integrable]. This is well understood from the discrete-time work of Rubinstein (1976) and Lucas (1978); the continuous-time result is derived by Huang (1987) and Duffie and Zame (1989).

Here, we will informally identify a state-price process π for a single-agent economy with our stochastic differential utility formulation. Under technical regularity conditions and in terms of the normalized aggregator f, a state-pricing process π is defined by

$$\pi_t = \exp\left[\int_0^t f_v(c_s, V_s) \, ds\right] f_c(c_t, V_t), \tag{35}$$

which coincides with the additively separable case if $f(c, v) = u(c) - \beta v$. The interpretation of π_t/π_s from (35) in terms of intertemporal substitution of income between dates t and s is rather clear. We will derive (35) from an analysis of the Bellman equation, under assumed regularity of the value function. This representation (35) of the state-price process π has, subsequent to this approach, been derived directly from the infinite-dimensional first-order conditions for optimal consumption and portfolio choice by Duffie and Skiadas (1990). This is true even without Brownian information and leads to an alternative derivation of all our asset pricing results, without relying on the Bellman equation and its associated smoothness assumptions and Markovian information structure.

In order to derive (35) from the Bellman equation, we first derive a preliminary result of independent interest. The Bellman equation (15) implies that

$$\begin{aligned} f_v J_w &= -\frac{\partial}{\partial w} \mathcal{D} J \\ &= -\mathcal{D} J_w - J_w(z^\top \lambda + r) - w J_{ww} z^\top \sigma \sigma^\top z - z^\top \sigma a^\top J_{xw}. \end{aligned} \tag{36}$$

From (17),

$$-z^{\top}\lambda J_w = z^{\top}\sigma\sigma^{\top} z J_{ww} w + z^{\top}\sigma a^{\top} J_{xw}. \tag{37}$$

Letting π be defined by (35), we can use Ito's Lemma to write $d\pi_t = \mu_\pi(t)\,dt + \sigma_\pi(t)\,dB_t$ for appropriate μ_π and σ_π. In fact, substituting (37) into (36) and using the fact that $J_w = f_c$,

$$\mu_\pi(t) = -\pi_t r(X_t, t). \tag{38}$$

That is, the real short-term interest rate $r(X_t, t)$ is equal to the exponential rate of decline of the discounted marginal utility for wealth, extending the characterization of short-term interest rates by Cox, Ingersoll, and Ross (1985b, p. 373).

Consider a security with consumption dividend rate process $\{\delta(X_t, t)\}$ and price process S relative to the consumption numeraire. By definition of λ and σ,

$$dS_t = (S_t[r(X_t, t) + \lambda^i(X_t, t)] - \delta(X_t, t))\,dt + S_t\sigma^i(X_t, t)\,dB_t,$$

where λ^i and σ^i are the appropriate components of λ and σ, respectively. Let $Y_t = S_t\pi_t$, $t \in [0, T]$. From (38), (17), Ito's Lemma, and simple algebraic manipulation using the fact that $f_c = J_w$, we have $dY_t = \mu_Y(t)\,dt + \sigma_Y(t)\,dB_t$, where

$$\mu_Y(t) = -\pi_t\delta(X_t, t). \tag{39}$$

Since $S_T = 0$, we know that $Y_T = 0$, and therefore (provided σ_Y is square integrable), we have $Y_t = E_t[\int_t^T - \mu_Y(s)\,ds]$, or equivalently,

$$S_t = \frac{1}{\pi_t} E_t\left[\int_t^T \pi_s\delta(X_s, s)\,ds\right],$$

which matches (33). Since the security is arbitrary, we have confirmed that π is indeed the state-price process, at least for this Markov state-space setting and under our smoothness and integrability assumptions. [Duffie and Skiadas (1990) show more details.]

2.4 Term structure: CIR extension

We now study the effect of varying risk aversion, holding intertemporal substitution fixed, on the term structure of interest rates, extending the spirit of the model by Cox, Ingersoll, and Ross (1985a, sect. 3) (hereafter CIR).

State variable process. Let B be a standard Brownian motion in $\mathbb{R}^2$. Consider a "shock" process X defined by

$$dX_t = (aX_t - b)\,dt + \sqrt{X_t}\,\eta\,dB_t,$$

where $a < 0$ and $b < 0$ (mean reversion), while $\eta \in \mathbb{R}^2$ has $k \equiv \|\eta\| > 0$. An endowment process c is defined as the solution to the stochastic differential equation

$$dc_t = c_t(hX_t - \beta)\,dt + c_t\sqrt{X_t}\,\nu\,dB_t,$$

where $c_0 > 0$, while $h > 0$ is a constant and $\nu \in \mathbb{R}^2$ has $\epsilon \equiv \|\nu\| > 0$ and $h > \epsilon^2$. The correlation coefficient $\psi \equiv \nu\cdot\eta/(k\epsilon)$ is arbitrary.

Utility. Consider first the limiting form of the Kreps–Porteus utility model (9) corresponding to $\rho = 0$. As $\rho \to 0$, the aggregator in (9) approaches the aggregator (f, A) with

$$f(c, v) = \beta v(\log c - \log v), \qquad A(v) = (\alpha - 1)/v. \tag{9$'$}$$

The normalization $V_T = 0$ (see Section 1.2) can cause technical difficulties for this specification. In order to avoid them, we adopt the alternative normalization $V_T = \xi$, where ξ is a positive constant whose precise value will be of no consequence below. Equivalently, the utility process $V_t - \xi$ satisfies the zero terminal condition and [by an appropriate form of (10)] is generated by the aggregator f given by

$$f(c, v) = \beta(\xi + v)[\log(c) - \log(\xi + v)], \qquad A(v) = \frac{\alpha - 1}{\xi + v}.$$

This modification of (9′) is appealing for several reasons. First, it follows from (4) and (5) that risk aversion declines with α, as in the "unmodified" Kreps–Porteus model (9′). Second, we show below that, in the limit as $\alpha \to 0$, this yields the standard log-utility specification used by CIR. Finally, since the results to follow are valid for all $\xi > 0$, they are in a sense more robust than those that could be derived using the aggregator (9′).

If we apply the transformation φ defined by

$$\varphi(v) = \frac{(\xi + v)^\alpha - \xi^\alpha}{\alpha},$$

as described in (10), we obtain the normalized aggregator

$$\bar{f}(c, v) = \beta(\xi^\alpha + \alpha v)[\log c - \alpha^{-1}\log(\xi^\alpha + \alpha v)], \qquad \beta > 0, \quad \alpha < 1.$$

We note that, as $\alpha \to 0$, we get $\bar{f}(c, v)$ converging to $\beta[\log(c/\xi) - v]$, which yields

$$V_0 = \beta E_0\left[\int_0^T e^{-\beta t}\log\left(\frac{c_t}{\xi}\right)dt\right],$$

which is ordinally equivalent to the standard log-utility specification used by CIR. We will be limiting ourselves to values of α near zero and thus to small perturbations of the CIR specification.

An application of Ito's Lemma shows that, if utility process V for the given endowment process is of the form $V_t = J(t, X_t, c_t)$, then the Bellman equation (15) implies that J solves the semilinear parabolic partial differential equation:

$$\begin{aligned} J_t + J_x(ax - b) + cJ_c(bx - \beta) \\ + \tfrac{1}{2}x[J_{xx}k^2 + J_{cc}c^2\epsilon^2 + 2J_{cx}\psi k\epsilon] + \bar{f}(c, J) &= 0, \\ J(T, x, c) &\equiv 0, \end{aligned} \tag{40}$$

suppressing arguments of functions from the notation wherever convenient. Likewise, a solution J to (40) determines the utility process V, since V is uniquely characterized by (15). For more details on PDE solutions of stochastic differential utility, see Duffie and Lions (1990).

Our trial solution for J is given by

$$\alpha^{-1}\log(\xi^\alpha + \alpha J(t, c, x)) = q_t \log c + m_t x + n_t,$$

where q, m, and n are differentiable functions on the time interval into the real line. (Note that the left-hand side has limit $J + \log \xi$ as $\alpha \to 0$.) This candidate solution for J solves our PDE if and only if

$$q_t = 1 - e^{-\beta(T-t)},$$

$$\dot{m}_t + (a - \beta + \alpha q_t \psi k\epsilon) m_t + \tfrac{1}{2}\alpha m_t^2 k^2 + q_t b + \tfrac{1}{2}q_t(\alpha q_t - 1)\epsilon^2 = 0,$$

$$m_T = 0,$$

$$\dot{n}_t - bm_t - \beta q_t - \beta n_t = 0, \qquad n_T = \log \xi. \tag{41}$$

Short rate process. Let $\pi_t = \exp(\int_0^t \bar{f}_v(c_s, V_s)\, ds)\bar{f}_c(c_t, V_t)$ denote the state-price process, defined as in Section 2.3, with $V_t = J(t, X_t, c_t)$. The short-term riskless rate r is defined by $r_t = -\mu_\pi(t)/\pi_t$, where μ_π denotes the drift process associated with π, as explained in Section 2.3. If we do the calculations, we obtain

$$r_t = [b + (\alpha q_t - 1)\epsilon^2 + \alpha m_t \psi k\epsilon]X_t \equiv \Omega_t X_t, \qquad t \in [0, T]. \tag{42}$$

For α sufficiently close to zero, we note that $\Omega_t > 0$, and hence $r_t > 0$. In addition, $E[r_s \mid r_t] = \Omega_s E[X_s \mid X_t]$ and $\mathrm{var}[r_s \mid r_t] = \Omega_s^2 \mathrm{var}[X_s \mid X_t]$, with the obvious notation for conditional variance given X_t. We can show by elementary means that Ω_s increases with α if the instantaneous correlation between the state process X and the endowment process c is not excessively negative, in the sense that

$$\psi > 2a\epsilon/k(2b - \epsilon^2). \tag{43}$$

It follows that *both* the conditional mean and the conditional variance of the interest rate process *decline* with increased risk aversion, provided (43) is satisfied, which is certainly the case with $\psi = 0$. The opposite qualitative effects are also possible under conditions on the parameters. For example, in the infinite horizon version of the model (described below), greater risk aversion *increases* both the conditional mean and variance of the interest rate if $\psi < (a - \beta)\epsilon k^{-1}/(b - \epsilon^2/2)$.

Bond prices. Consider an equilibrium with a representative consumer as above. We will price discount bonds of varying maturities in this economy and thus determine the term structure.

We are interested in calculating the price $p_{t,s}$ at time t of a pure discount real bond maturing at s. By the definition of the state-price process,

$$p_{t,s} = \frac{1}{\pi_t} E_t(\pi_s) = \frac{E_t(\bar{f}_c(c_s, V_s)\exp[\int_t^s \bar{f}_v(c_\tau, V_\tau)\,d\tau])}{\bar{f}_c(c_t, V_t)}.$$

Since $\{(X_t, c_t)\}$ is a Markov process, we can immediately write $p_{t,s} = \hat{P}(t, s, X_t, c_t)$ for some measurable function $\hat{P}$. In fact, however, we show that $\hat{P}$ does not depend on c_t, given X_t. To see this, first note that the stochastic differential expression for c also implies that, for any times τ and $t \leq \tau$,

$$c_\tau = c_t \exp\left(\int_t^\tau \left[bX_s - \beta - \frac{\epsilon^2 X_s}{2}\right] ds + \int_t^\tau \sqrt{X_s}\nu \, dB_s\right).$$

Substituting

$$\bar{f}_c(c, v) = \beta(\xi^\alpha + \alpha v)/c$$

and

$$\bar{f}_v(c, v) = \beta\alpha[\log(c) - \alpha^{-1}\log(\xi^\alpha + \alpha v)] - \beta$$

and our expression for $V_s = J(s, c_s, X_s)$ into the above expression for $p_{t,s}$ then implies that $p_{t,s} = E_t(Z_X Z_c)$, where Z_X depends on X but does not involve terms in c, and where

$$Z_c = \exp\left[\log(c_t)\left(\alpha(q_s - q_t) + \beta\alpha\int_t^s e^{-\beta(T-u)}\,du\right)\right].$$

A calculation shows that $Z_c = 1$, implying that $p_{t,s}$ is the expectation of a measurable function of the distribution of $\{X_\tau\colon t \leq \tau \leq s\}$ conditional on X_t, and can therefore (since X is Markov) be written in the form $p_{t,s} = F(t, s, X_t)$ for some measurable F. In fact, we can use the stochastic differential equation for X and the boundary condition

$F(t, t, x) = 1$ to obtain a PDE for F, as in CIR. We will write this PDE instead for the function P defined by

$$P(t, s, r_t) \equiv F(t, s, \Omega_t^{-1} r_t).$$

We have

$$\begin{aligned} \tfrac{1}{2}\sigma_r^2(t) r P_{rr} + K(\theta_t - r) P_r + P_t - \lambda_t r P_r - rP = 0, \\ P(s, s, r) = 1, \end{aligned} \tag{44}$$

where, from (42),

$$dr_t = \left[K(\theta_t - r_t) + r_t \frac{\dot{\Omega}_t}{\Omega_t}\right] dt + \sqrt{\Omega_t r_t \eta}\, dB_t,$$

and

$$\begin{aligned} &\theta_t \equiv \Omega_t b/a, \qquad k = -a, \qquad \sigma_r^2(t) = \Omega_t \eta \cdot \eta, \\ &\lambda_t \equiv (1 - \alpha q_t)\epsilon k \psi - \alpha m_t k^2 - \dot{\Omega}_t/\Omega_t. \end{aligned} \tag{45}$$

When $\alpha = 0$ and when Ω_t and m_t are both independent of time, (44) and (45) have the forms derived by CIR. To obtain the CIR forms of $\alpha \neq 0$ and to facilitate further analysis, we consider an infinite horizon model. We will show later that, in the infinite horizon model, $q_t \equiv 1$ and $m_t \equiv m^*$, where m^* is the smaller positive root of

$$\tfrac{1}{2}\alpha m^2 k^2 + (a - \beta + \alpha \psi k \epsilon) m + h + \tfrac{1}{2}(\alpha - 1)\epsilon^2 = 0.$$

We make these substitutions and also substitute $\Omega(\alpha) \equiv [h + (\alpha - 1)\epsilon^2 + \alpha m^* \psi k \epsilon]$ into (44) and (45) in place of Ω_t. The interest rate process and the PDE for bond prices are exactly of the CIR form [their (17) and (22)]. Thus, their bond price formula (23) also applies, with θ, K, σ_r^2, and λ defined as above in terms of primitive parameters.

Finally, we consider the effect of risk aversion on the yield to maturity $R(t, s, r_t)$. By the CIR equation (28), the "long term" rate is given by

$$R(t, \infty, r_t) = \frac{2K\theta}{\gamma + K + \lambda}, \qquad \gamma^2 \equiv (K + \lambda)^2 + 2\sigma_r^2. \tag{46}$$

We can show that $R(t, \infty, r)$ *decreases with risk aversion locally near* $\alpha = 0$, provided that the instantaneous correlation between the state process and the endowment process is not excessively negative, in the sense that ψ satisfies (43) and

$$\psi > -\frac{K(2h - \epsilon^2)}{2\epsilon(\beta - \alpha)}. \tag{47}$$

In order to investigate short-term yields we note that $R(s, s, r_t) = r_t$ and focus on the partial derivative $R_s(t, s, r_t)\,|_{s=t}$. We compute

$$R_s(t, s, r_t)\,|_{s=t} = \tfrac{1}{2}[K\theta - (K + \lambda) r_t].$$

[See Vasicek (1977, p. 184).] It follows that *increasing risk aversion induces short-term yields to fall,* locally near the expected-additive-log-utility model, if the exogenously specified instantaneous correlation between consumption and the "shock" state is not too negative, in the sense of (43) and (47).

It remains only to justify our claims regarding the solution of the infinite horizon model. The only nontrivial claim is that regarding m^*. Infinite horizon utility is defined as the pointwise limit of the finite horizon utilities as in (3′). Thus, we need to show the following: Denote by $m(\cdot, T)$ the solution to the ODE (41), depending on the horizon T. We must show that $m(\cdot, T) \to m^*$ as $T \to \infty$. By elementary means we can show that, for all α sufficiently near 0,

(i) $m(\cdot, T) \geq 0$,
(ii) $m(t, \cdot)$ is increasing,
(iii) $m(\cdot, T) < m^*$, for all $T > 0$.

Finally, $m(0, T) \to m^*$ as $T \to \infty$. (Details can be provided upon request.) The desired conclusion follows.

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