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THE STRUCTURE OF PREFERENCES AND ATTITUDES TOWARDS THE TIMING OF THE RESOLUTION OF UNCERTAINTY*

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This paper is concerned with the phenomenon of preference for timing in the temporal resolution of uncertainty and its implications for the structure of utility functionals defined on multiperiod consumption programs. Several postulates concerning attitudes towards timing are stated using a new definition of timing premium for early resolution of uncertainty. The analysis provides an axiomatic basis for the specifications of expected utility and the more general weighted utility and implicit weighted utility functionals in temporal models.

1. INTRODUCTION

This paper is concerned with the phenomenon of preference for timing in the temporal resolution of uncertainty and its implications for the structure of utility functionals defined on uncertain consumption programs in a multiperiod framework. A series of postulates is considered, beginning with indifference to timing and then proceeding to various forms of nonindifference. Some of the latter postulates are formulated in terms of a new definition of timing premium for early resolution of uncertainty. In the former case expected utility functionals are implied (given other axioms) while nonexpected utility functionals are admissible when indifference to timing is weakened. In particular, a rationale is provided for the specification of implicit weighted and weighted utility functionals in multiperiod contexts. These functionals have been axiomatized in atemporal frameworks in Chew (1983, 1989) and Dekel (1986), where it is shown that they are capable of explaining Allais-type violations of expected utility (see also Machina 1983).

Our analysis is axiomatic. The first central axiom is the intertemporal consistency of preferences. The issue of consistency has been recognized as important in dynamic analysis since the seminal paper by Strotz (1956) and consistency is maintained in the vast majority of studies in economic dynamics. Expected utility specifications are intertemporally consistent but Johnsen and Donaldson (1985) show that consistency alone does not imply an expected utility ordering.

The other key axioms below concern attitudes towards the way in which uncertainty resolves over time. It is recognized (Kreps and Porteus 1979; Machina 1984) that this timing is generally significant when considering a preference ordering for random income streams induced from preference for consumption

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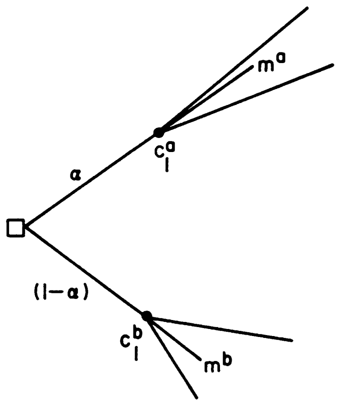

probability tree for $d \in D$,

$d = a\delta[c_1^a, m^a] + (1-a)\delta[c_1^b, m^b]$,

where $m^a, m^b \in M(X)$.

FIGURE 1

streams, since in such contexts earlier resolution can improve planning. But the desirability of earlier resolution is not at all clear at the primitive level of consumption streams. Indeed, indifference to the timing of resolution is widely maintained, e.g., in expected utility specifications where the expected value is computed with respect to the joint distribution of consumption levels, the latter distribution determines the ranking of the consumption program independently of the way in which uncertainty resolves over time.

Some of our axioms are closely related to those that have appeared in atemporal axiomatizations. For example, consistency and timing indifference are closely related to the well-known key axioms of expected utility theory—the independence axiom (IA) and the reduction of compound lotteries axiom (ROCLA). But they are nonetheless distinct sets of axioms. In this regard it is important to distinguish between the formal axioms IA and ROCLA and the informal verbal explanations which often accompany them. The former are atemporal but the latter frequently introduce sequential variants of the original single period choice setting (see Raiffa 1970, p. 82; and McClennen 1983). Thus mixtures of probability measures are often described as "two stage lotteries" where time is assumed to pass between stages. Even in such informal discussions, time is conceptual rather than real. In particular, monetary prizes realized at the final stage of the sequential lottery, rather than intertemporal consumption profiles, are the ultimate source of utility. The consistency and timing indifference axioms of this paper are the formal hypotheses which correspond to the Raiffa argument, extended to a real time framework where consumption-savings decisions can be modelled. Moreover, in our real time framework, nonindifference to timing is *plausible* and an analysis of related issues (e.g. the definition of premia for early resolution) can be undertaken.

Just as in the case of the atemporal literature, our interest in nonexpected utility orderings is motivated by the evidence against the empirical validity of expected utility theory which has accumulated in the behavioural experimentation literature originating with the Allais paradox. (See Machina 1982 for a discussion of this evidence.) But in a multiperiod setting there is an additional reason for considering more general utility theories—the expected utility specification is too rigid to permit the separation of intertemporal substitution from the degree of risk aversion. Such a separation is important, for example, in life-cycle based asset pricing models (Hansen and Singleton 1983) where some researchers have conjectured that it could improve the poor empirical performance of these models.[2]

Section 2 contains the body of the paper. Some concluding remarks are offered in Section 3. Proofs are collected in the Appendix.

2. PREFERENCE ORDERINGS

Our formal analysis is conducted in a two period model since that suffices to convey the crux of our argument. The extension to an arbitrary number of periods is described briefly.

Consumption c_t in either period $t = 1, 2$ is constrained to lie in X, a bounded interval in the nonnegative real line. The space of Borel probability measures on X, endowed with the weak convergence topology, is denoted $M(X)$. Let $D \equiv M(X \times M(X))$, the space of Borel probability measures on $X \times M(X)$, again endowed with the weak convergence topology. Elements of D are intertemporal consumption lotteries (or programs). They can be represented by probability trees as in Figure 1. (Note that for $i = a, b$, $\delta[c_1^i, m^i] \in D$ denotes the measure which assigns all mass to $\{(c_1^i, m_i)\}$.) In Figure 1, think of α and $(1 - \alpha)$ as describing the probability distribution of a random variable which is correlated with consumption levels in both periods and whose realization is observed at the start of period 1. The $t = 1$ consumption level is also observed at the start of period 1, after which the remaining future is described by a probability measure $m^i \in M(X)$ for second period consumption.

At $t = 1$, the utility function U^1: $D \rightarrow R$ represents the preference ordering on consumption programs. At the start of period 2, preferences on the remaining future, namely random period 2 consumption, are represented by U^2: $X \times M(X) \rightarrow R$. The utility functions U^1 and U^2 are related by some of the axioms below, but at this stage they may be specified independently of one another. The dependence of $U^2(c_1; m)$ on c_1 reflects the dependence of the $t = 2$ preference ordering on the consumption level c_1 realised in the previous period. But note that (in common with Johnsen and Donaldson 1985) it is assumed that preferences are independent of *unrealised* past alternatives.

[2] In the empirical literature, Hall (1985) and Zin (1986) have argued for the importance of such separation. They achieve it by adopting the preference specification of Selden (1979) and Selden and Stux (1978). Other non-expected utility specifications which permit the disentangling of ordinal and risk properties of preferences can be found in Kocherlakota (1986) and Chew and Epstein (1987). The inadequacy of multiperiod expected utility theory in this regard is demonstrated in the latter paper.

We now consider several axioms for these utility functions. The first is a common technical assumption.

Continuity (*C*). For each $c_1 \in X$, $U^2(c_1; \cdot)$ is continuous on $M(X)$.

The existence of such utility functions could be proven from more basic postulates on preference orderings (Debreu 1954).

The first key axiom is consistency. It parallels that adopted by Johnsen and Donaldson (1985) in their contingent commodity framework.

Consistency (*CS*). For all $c \in X$, $\alpha \in (0, 1)$ and m, m' and p in $M(X)$,

$$U^2(c; m) \geq U^2(c, m') \Leftrightarrow U^1(\alpha\delta[c, m] + (1 - \alpha)\delta[c, p]) \geq U^1(\alpha\delta[c, m'] + (1 - \alpha)\delta[c, p]). \tag{1}$$

The two consumption programs ranked on the right side of (1) are represented in Figure 2. Both involve certain consumption c at $t = 1$ and a "coin flip" at $t = 1$ which determines whether m or p (m' or p resp.) is the probability distribution corresponding to $t = 2$ consumption. If the equivalence in (1) were violated, then with probability $\alpha > 0$ the choice made at $t = 1$ would be regretted at $t = 2$.

Notice that (1) is not the independence axiom. For instance, the latter is a statement regarding one utility function, while (1) involves the utility functions in both periods. The precise link between IA and CS depends on which of the remaining alternative axioms is adopted.

Timing Indifference (*TI*). For all $\alpha \in [0, 1]$, $c \in X$ and m and m' in $M(X)$,

$$U^1(\alpha\delta[c, m] + (1 - \alpha)\delta[c, m']) = U^1(\delta[c, \alpha m + (1 - \alpha)m']). \tag{2}$$

The two consumption lotteries in (2) are portrayed in Figure 3. They share a common certain period 1 consumption level and a common probability distribution $\alpha m + (1 - \alpha)m'$ for period 2 consumption. But they differ in the timing of the resolution of uncertainty as defined in Kreps and Porteus (1978). In one lottery, a "coin flip" is performed at $t = 1$ to determine whether m or m' applies for $t = 2$, while in the other lottery the "coin flip" does not occur until $t = 2$.

As mentioned in the introduction, TI is widely maintained, if only implicitly; for instance, that is the case whenever $M(X \times X)$ rather than D is taken to be the choice space since the former identifies the two lotteries in (2). (A specific example is the specification in Selden and Stux 1978; see footnote 1.) But TI leads back to expected utility theory as described in the following theorem and the ensuing discussion.

THEOREM 1. *Let U^1 and U^2 satisfy continuity, consistency and timing indifference. Then for each $c_1 \in X$, $U^2(c_1; \cdot)$ is an expected utility functional on $M(X)$.*

The Theorem implies that period 2 preferences, but not necessarily period 1 preferences, conform to expected utility theory. But the multiperiod extension of

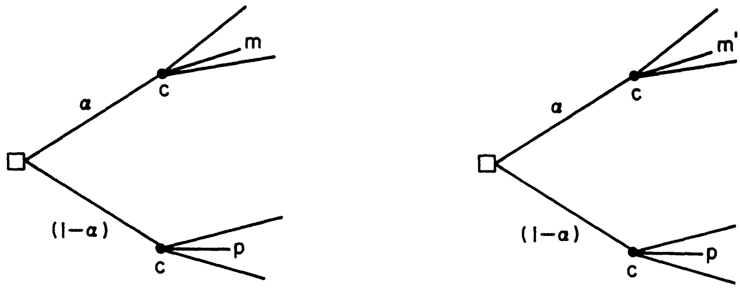

FIGURE 2

the Theorem, which we now describe, presents a more striking picture: Consumption lotteries which extend over $T > 2$ periods may be defined as in Kreps and Porteus (1978), henceforth KP (1978). The C, CS and TI axioms have natural extensions to this larger domain.[3] Then it is immediate that the preference orderings prevailing at each period t, $t = 2, \ldots, T$, must conform to expected utility theory. Note that if the model at $t = 1$ is a snapshot of an ongoing modelling process, then it follows from the $t = 0$ version of the Theorem that an expected utility ordering prevails at $t = 1$ also and hence in all periods. Finally, it follows from KP (1978, Corollary 3) that utility in any period (say $t = 1$) is the expected value of some utility index of remaining consumption, where the expected value is computed with respect to the joint probability distribution of consumption that is induced by the consumption lottery. In particular, there exist functions $A(\cdot)$ and $B(\cdot)$, the latter positive, such that

$$(3)\quad U^1(\alpha\delta[c, m]+(1-\alpha)\delta[c, m']) = A(c) + B(c)U^2(c;\, \alpha m + (1-\alpha)m')$$
$$= A(c) + B(c)[\alpha E_m u^2(c;\cdot) + (1-\alpha)E_{m'}u^2(c,\cdot)].$$

This leads to the standard intertemporal expected utility specification on the choice space $M(X^T)$.[4] The recursive structure in (3) reflects the CS axiom while the linearity of the right side with respect to U^2 reflects TI. (Contrast with (4) below.)

If, for the reasons mentioned in the introduction for example, one wishes to admit more general preference specifications, then the axioms in Theorem 1 must

[3] See Epstein (1986) for more details and also for an infinite horizon analysis.

[4] Kreps and Porteus (1978) provide an axiomatization for this intertemporal expected utility specification. But one of their axioms (Axiom 2.3 or 4.3) imposes the substitution axiom of expected utility theory on a subset of consumption lotteries, which unduly biases the analysis against non-expected utility specifications. In our analysis, this substitution axiom is a consequence of CS and TI.

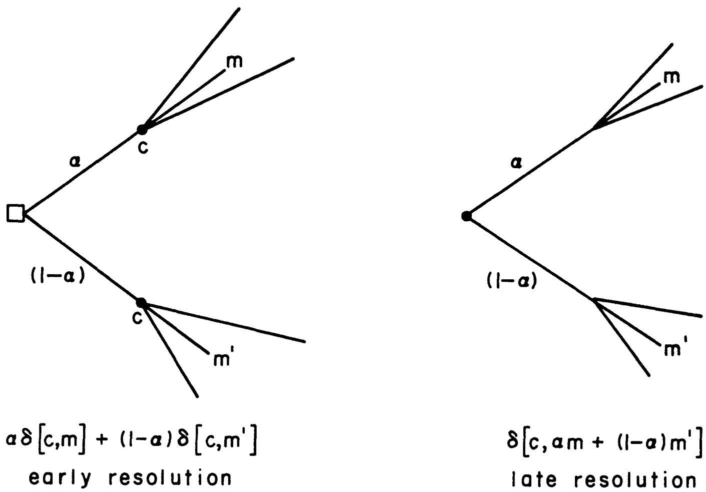

FIGURE 3

be weakened. One possibility is to weaken TI. It is, after all, perfectly "rational" for an individual to prefer either early or late resolution of uncertainty. To illustrate, consider the following example: A Ph.D. student is committed to spending a fully paid for, month long, vacation in France. His comprehensive exams have been graded but the results have not yet been publicized. If the individual wishes, he can learn the grade before embarking on his trip. Otherwise, he will discover it upon his return. Some people may prefer to know the test result before departing, others may prefer to live with the uncertainty, while still others may be indifferent.

A central feature of this example is that future consumption prospects are presumably influenced by the test result, i.e., prospects are brighter if a passing grade has been achieved than otherwise. But suppose instead that the student has actually completed his Ph.D. program and is awaiting job offers from two schools between which he is indifferent. An offer arrives before his departure but he is not certain from which school. It seems plausible that in these circumstances indifference to the time at which the facts are learned will be more common than in the case above. This example describes the intuition underlying the following weakening of TI:

Quasi-timing Indifference (*QTI*). The equality (2) holds for all $\alpha \in [0, 1]$, $c \in X$ and $m, m' \in M(X)$ such that $U^2(c; m) = U^2(c; m')$.

The following is a weaker version of TI which makes use of the notion of a timing premium. For each $\alpha \in (0, 1)$ and (c, m), (c, m') in $X \times M(X)$, $U^1(\delta[c, m)] \geq U^1(\delta[c, m'])$, we define the *timing premium* correspondence for early resolution by:

$$\pi(\alpha, m, m', c) \equiv \frac{\beta/(1-\beta)}{\alpha/(1-\alpha)} - 1,$$

for any $\beta \in (0, 1)$ such that

$$U^1(\alpha\delta[c, m] + (1-\alpha)\delta[c, m']) = U^1(\delta[c, \beta m + (1-\beta)m']).$$

Note that π is not measured in units of consumption and may more accurately be termed a "timing probability premium" or a "timing odds ratio premium". (See the discussion at the end of this section.)

Existence of Timing Premium (*ETP*). For each $\alpha \in (0, 1)$, $c \in X$ and m, $m' \in M(X)$ such that $U^1(\delta[c, m]) \geq U^1(\delta[c, m'])$, $\pi(\alpha, m, m', c)$ is nonempty.

TI is the special case of ETP where $\pi \equiv 0$. ETP permits the timing of resolution to matter, but it requires that for any early coin flip there exist some other coin which yields an indifferent consumption lottery if it is flipped late. Note that (by Lemma A in the Appendix) π can be taken to be single-valued whenever it is nonempty. Also by Lemma A, if $U^2(c; m) > U^2(c; m')$, then early resolution is preferred iff $\beta > \alpha$ iff $\pi > 0$. Thus π can be interpreted as a premium for early resolution, where the premium is measured in probability units.

If TI is replaced by QTI or ETP, then expected utility theory is no longer implied. Rather, preferences will lie in a more general class which is characterized primarily by the following weakening of the independence axiom: If v: $M(X) \to R$, then v satisfies *betweenness* if for all $\alpha \in [0, 1]$ and m, m' in $M(X)$,

$$v(m) = v(m') \Rightarrow v(\alpha m + (1-\alpha)m') = v(m).$$[5]

Chew (1989) and Dekel (1986) describe the functional structure of such utilities. A utility functional v: $M(X) \to R$ is *implicit weighted* if $\exists\ \phi$: $M(X) \times Rng(v) \to R$ such that ϕ is linear in its first argument, and $\forall\ m \in M(X)$, $\phi(m, v(m)) = 0$. If we define $\phi(x, a) = \phi(\delta[x], a)$, then continuity implies that $v(m)$ is given by the solution "a" to

$$\int_X \phi(x, a)\, dm(x) = 0.$$

For the next theorem we require a weak monotonicity property.

Elementary Monotonicity (*EM*). For all c, a and b in X, $a < b$,

$$\alpha < \beta \Rightarrow U^2(c; (1-\alpha)\delta[a] + \alpha\delta[b]) < U^2(c; (1-\beta)\delta[a] + \beta\delta[b]).$$

This axiom is weaker than the requirement that $U^2(c; \cdot)$ be increasing in the sense of first degree stochastic dominance.

[5] This is a weaker form of betweenness than appears in Chew (1983) and Dekel (1986) who assume that the "better than" and "worse than" sets are both convex in the mixture sense. See Lemma A in the Appendix.

THEOREM 2. *Let* U^1 *and* U^2 *satisfy continuity, consistency and either*

(i) *elementary monotonicity and quasi-timing indifference, or*
(ii) ETP.

Then for each $c_1 \in X$, $U^2(c_1; \cdot)$ *is an implicit weighted functional on* $M(X)$.

As in the case of Theorem 1, a multiperiod extension is straightforward. That extension (formulated at $t = 0$) implies that the preference orderings prevailing at each $t = 1, \ldots, T$ can be represented by implicit weighted functionals.

The following is a stronger version of ETP.

Constant Timing Premium 1 (CTP1). For each $c \in X$ and $m, m' \in M(X)$, such that $U^1(\delta[c, m]) \geq U^1(\delta[c, m'])$, $\pi(\alpha, m, m', c)$ is nonempty and constant in α on (0, 1).

The constant timing premium axiom will lead to weighted utility functionals (e.g., Fishburn 1983; Chew 1983), where v: $M(X) \to R$ is a *weighted utility functional* if there exist real valued functions v and w on X such that $w > 0$ and

$$v(m) = \int w(x)v(x)\, dm(x) \Big/ \int w(x)\, dm(x) \qquad \forall\, m \in M(X).$$

The above corresponds to the special case of implicit-weighted utility with $\phi(m, a) = w(m)[v(m) - a]$, where $w(m) = \int w(x)\, dm(x)$. In the above, we adopt the convention that $v(x) = v(\delta[x])$ and $w(x) = w(\delta[x])$.

THEOREM 3. *Let* U^1 *and* U^2 *satisfy continuity, consistency and* CTP1. *Then for each* $c_1 \in X$, $U^2(c_1; \cdot)$ *is a weighted utility functional on* $M(X)$.

The multiperiod extension of the Theorem provides a rationale for the specification of weighted utility functions in temporal models.

Finally, we consider an alternative and intuitively plausible form of constancy for the timing premium, in which $\pi(\alpha, m, m'c)$ depends on m and m' only through the period 2 utility levels which they imply.

Constant Timing Premium 2 (CTP2). For each $c \in X$, $\alpha \in (0, 1)$ and $m, m', p, p' \in M(X)$, $\pi(\alpha, m, m', c)$ is nonempty; in addition, if $U^2(c; m) = U^2(c; p)$ and $U^2(c; m') = U^2(c; p')$ then $\pi(\alpha, m, m', c) = \pi(\alpha, p, p', c)$.

THEOREM 4. *Let* U^1 *and* U^2 *satisfy continuity, consistency and* CTP2. *Then for each* $c_1 \in X$, $U^2(c_1; \cdot)$ *is an expected utility functional on* $M(X)$.

As in the case of Theorem 1, an expected utility functional is implied. But there is a significant difference between the implied functional structures which becomes evident when the multiperiod extensions of the theorems are considered. The extension of Theorem 4 implies that utility functionals in all periods conform with expected utility theory. Thus the hypotheses of KP (1978, Theorem 1) are satisfied and we can conclude, for example, that there exists a function Φ, with suitable domain and increasing in its second argument, such that

$$U^1(\delta[c, m]) = \Phi(c, U^2(c; m)) = \Phi(c, E_m u^2(c, \cdot)), \tag{4}$$

and

$$U^1(\alpha\delta[c, m] + (1-\alpha)\delta[c, m']) = \alpha\Phi(c, E_m u^2(c, \cdot)) + (1-\alpha)\Phi(c, E_{m'}, u^2(c, \cdot)).$$

(See also KP 1979, Proposition 1.) The structure in (3) is the special case where $\Phi(c, \cdot)$ is linear. The more general recursive structure in (4) admits nonindifference to timing but necessarily conforms with CTP2. KP (1979) have termed the preference ordering corresponding to (4) as *temporal* von Neumann-Morgenstern to distinguish it from the von Neumann-Morgenstern structure corresponding to (3).[6]

KP identify the curvature of $\Phi(c, \cdot)$ as the determinant of attitudes towards timing, with convexity (concavity) corresponding to a preference for early (late) resolution. This identification is consistent with our timing premium approach to measuring attitudes towards timing. Thus, for example, if the convexity of $\Phi(c, \cdot)$ is increased by means of an increasing and convex transformation, then $\pi(\alpha, m, m', c)$ rises for all $\alpha \in (0, 1)$, $c \in X$ and $m, m' \in X$ such that $U^2(c; m) > U^2(c; m')$. Indeed π can be interpreted as a probability premium measure of the curvature of $\Phi(c, \cdot)$ as in Pratt (1964, p. 126). If z and h are defined by means of

$$U^2(c; m) = z + (1-\beta)h, \qquad U^2(c; m') = z - \beta h,$$

then

$$\pi(\alpha, m, m', c) = \frac{h}{2}\left[\frac{-\Phi_{22}}{\Phi_2}(c, z)\right] + o(h).$$

3. CONCLUDING REMARKS

We have explored some axioms underlying the specification of intertemporal utility functionals in a stochastic setting. First, we identified consistency and timing indifference as the basis for expected utility theory. One of them must be weakened if non-expected utility specifications are to be admissible. In this paper, the weakening of timing indifference has been explored. An alternative route, of course, is to weaken consistency which is the approach taken in Chew and Epstein (1987).

The choice of which route to follow in modelling consumption-savings behaviour could be enlightened if some empirical evidence could be brought to bear on the validity of timing indifference. There is a need to determine whether widespread and systematic violations of timing indifference can be uncovered in

[6] Note that CTP2 and our remaining axioms provide an alternative basis for temporal von Neumann-Morgenstern preference. Kreps and Porteus (1978) simply *assume* that each U^t is an expected utility functional.

various experimental or market settings. We have suggested several specific forms of timing nonindifference which should be explored empirically.

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APPENDIX

PROOF OF THEOREM 1. Show that $U^2(c; \cdot)$ satisfies IA on $M(X)$, i.e.,

$$U^2(c; m) = U^2(c; m') \Rightarrow U^2(c; \alpha m + (1-\alpha)p) = U^2(c; \alpha m' + (1-\alpha)p).$$

Consider the following four consumption lotteries:

$$d^1 \equiv \alpha\delta[c, m] + (1-\alpha)\delta[c, p], \qquad d^2 \equiv \alpha\delta[c, m'] + (1-\alpha)\delta[c, p],$$

$$d^3 \equiv \delta[c, \alpha m + (1-\alpha)p], \qquad d^4 \equiv \delta[c, \alpha m' + (1-\alpha)p].$$

Then

$$\begin{aligned} &U^1(d^1) = U^1(d^3) && \text{by TI} \\ &U^1(d^1) = U^1(d^2) && \text{by CS} \\ &U^1(d^2) = U^1(d^4) && \text{by TI} \\ \Rightarrow\ &U^1(d^3) = U^1(d^4) && \\ \Rightarrow\ &U^2(c; \alpha m + (1-\alpha)p) = U^2(c; \alpha m' + (1-\alpha)p) && \text{by CS.} \end{aligned}$$

Finally, apply the continuity of $U^2(c; \cdot)$. Q.E.D.

It is clear that betweeness is implied by the following:

Strong Betweenness (*SB*). For every $c \in X$, $m, m' \in M(X)$,

$$U^2(c; m) < U^2(c; m') \Rightarrow U^2(c; \alpha m + (1-\alpha)m') \in (U^2(c; m), U^2(c; m')) \qquad \text{for all } \alpha \in (0, 1).$$

The above is equivalent to:

Mixture-Monotonicity (*MM*). For every $c \in X$, $m, m' \in M(X)$,

$$U^2(c; m) < U^2(c; m') \Rightarrow U^2(c; \alpha m + (1-\alpha)m') < U^2(c; \beta m + (1-\beta)m'), \qquad \text{if } 0 \le \beta < \alpha \le 1.$$

LEMMA A. *Axioms C, ETP and CS imply SB for* $U^2(c; \cdot)$.

PROOF. Suppose to the contrary that $U^2(c; \alpha^0 m + (1-\alpha^0)m') \le U^2(c; m)$, for some $\alpha^0 \in (0, 1)$. (The other case is similar.) It follows from Axiom C that $\exists\ \mu \in (0, 1]$ such that $U^2(c; m) = U^2(c; \mu\alpha^0 m + (1-\mu\alpha^0)m')$. Since the set $\{\mu \in (0, 1]: U^2(c; m) = U^2(c; \mu\alpha^0 m + (1-\mu\alpha^0)m')\}$ is closed and nonempty, it contains a least element μ^0. Let $\beta^0 = \mu^0\alpha^0$. Then $U^2(c; m) = U^2(c; \beta^0 m + (1-\beta^0)m')$. It is a consequence of the beginning of the proof of Theorem 3 (see note in bracket) that

CS and ETP imply the existence of $\gamma^0 \in (0, 1)$ such that

$$U^2(c; \beta^0 m + (1-\beta^0)m') = U^2(c; \gamma^0[\beta^0 m + (1-\beta^0)m'] + (1-\gamma^0)m)$$
$$= U^2(c; \gamma^0\mu^0\alpha^0 m + (1-\gamma^0\mu^0\alpha^0)m').$$

This gives rise to a contradiction since $\gamma^0\mu^0 < \mu^0$. Q.E.D.

PROOF OF THEOREM 2. Repeat the argument in the proof of Theorem 1 but fix $p = m$. Then QTI can be used in place of TI to establish that $U^2(c_1; \cdot)$ satisfies betweenness. Alternatively, ETP can be used according to Lemma A to establish that $U^2(c; \cdot)$ satisfies strong betweenness. The implicit weighted functional structure follows from Dekel (1986, Appendix A). The separate proof presented here, which is adapted from Chew (1989), is shorter and more elementary. By the preceding argument and Lemma A, each of hypotheses (i) or (ii) implies that $U^2(c; \cdot)$ satisfies EM and betweenness, which properties are sufficient for the following argument.

Denote by $\gtrsim$ the ordering represented by $U^2(c; \cdot)$. Let $[A, B] = X$. Construct v: $M([A, B]) \to [0, 1]$ via: $\forall\, m \in M([A, B])$, $m \sim S_{v(m)}$, where $S_p = p\delta_B + (1 - p)\delta_A$. Construct λ: $M([A, B]) \times [0, 1] \to R$ below.

$$\text{For } a \in [0, v(m)], \qquad S_a \sim \lambda(m, a)m + (1 - \lambda(m, a))\delta_A.$$
$$\text{For } a \in (v(m), 1], \qquad S_a \sim \lambda(m, a)m + (1 - \lambda(m, a))\delta_B.$$

The existence of λ follows from continuity and EM, the latter of which implies that

$$\delta_A \lesssim S_a \lesssim m \quad \text{for} \quad a \in [0, v(m)],$$

and

$$m \lesssim S_a \lesssim \delta_B \quad \text{for} \quad a \in (v(m), 1].$$

Define ϕ: $m([A, B]) \times [0, 1] \to R$ via:

$$\phi(m, a) = \begin{cases} \left[\dfrac{1 - \lambda(m, a)}{\lambda(m, a)}\right] a, & a \in [0, v(m)] \\[2ex] -\left[\dfrac{1 - \lambda(m, a)}{\lambda(m, a)}\right](1 - a), & a \in (v(m), 1]. \end{cases}$$

Clearly, $\phi(m, v(m)) = 0$. It suffices to show that ϕ is linear in its first argument. Let $0 < \beta < 1$ and consider the unit tetrahedron $Bm'mA$ where $m\ m'$. (See Figure 4.) Any point $Q(q_1, q_2, q_3, q_4)$ in the tetrahedron represents a probability mixture $q_1 A + q_2 m + q_3 m' + q_4 B$ where q_1, q_2, q_3 and q_4 denote the volumes of the sub-tetrahedrons $Qmm'B$, $Qm'BA$, $QBAm$ *and* $QAmm'$ respectively.

For $a \in [v(m), v(\beta m' + (1 - \beta)m)]$, consider the indifference plane given by $(a, 0, 0, 1 - a)$, $(1 - \lambda, 0, \lambda, 0)$, $(0, \beta\lambda_\beta, (1 - \beta)\, \lambda_\beta, 1 - \lambda_\beta)$ and $(0, \lambda', 0, 1 - \lambda')$. (Note that Betweenness of U^2 guarantees that indifference surfaces on simplexes are

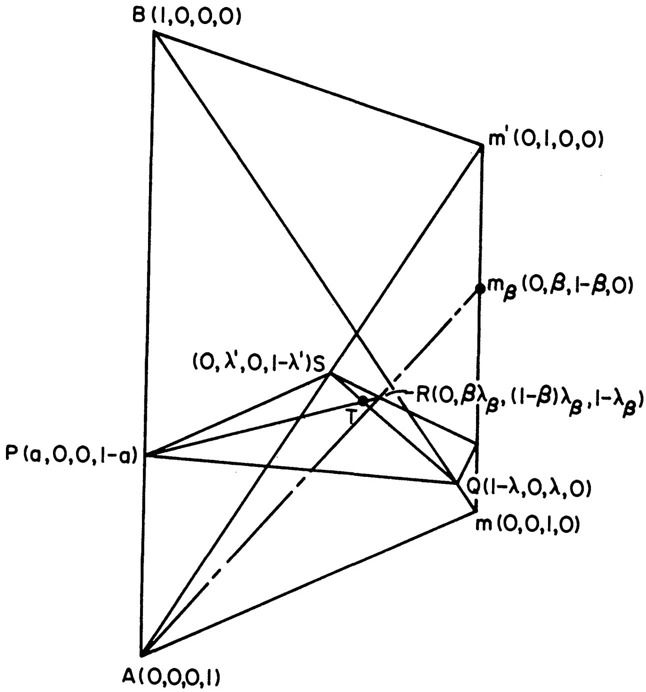

FIGURE 4

planes.) We can solve for $T = (t_1, t_2, t_3, t_4)$ given by:

$$T = \mu R + (1 - \mu)P = \sigma S + (1 - \sigma)Q.$$

It follows that:

$$t_1 = (1 - \mu)a = (1 - \sigma)(1 - \lambda) \qquad t_2 = \mu\beta\lambda_\beta = \sigma\lambda'$$

$$t_3 = \mu(1 - \beta)\lambda_\beta = (1 - \sigma)\lambda \qquad t_4 = \mu(1 - \lambda_\beta + (1 - \mu)(1 - a) = \sigma(1 - \lambda')$$

where $\lambda = \lambda(m, a)$, $\lambda' = \lambda(m', a)$, $\lambda_\beta = \lambda(\beta m' + (1 - \beta)m, a)$.

Dividing t_2 by t_3, we obtain:

$$\sigma = \beta\lambda/(\beta\lambda + (1 - \beta)\lambda').$$

Multiplying by λ'/β yields:

$$\mu\lambda_B = \lambda\lambda'/(\beta\lambda + (1 - \beta)\lambda').$$

From t_1, we have:

$$(1 - \mu)a = \frac{(1 - \beta)\lambda'(1 - \lambda)}{\beta\lambda + (1 - \beta)\lambda'}.$$

Finally, from t_4, we obtain:

$$\mu(1-\lambda_\beta) = \sigma(1-\lambda') + (1-\sigma)(1-\lambda) - (1-\mu)$$
$$= \frac{\beta\lambda(1-\lambda') + (1-\beta)(1-\lambda)\lambda'}{\beta\lambda + (1-\beta)\lambda'} - (1-\mu).$$

Observe that

$$\phi(\beta m' + (1-\beta)m, a) = \left(\frac{1-\lambda_\beta}{\lambda_\beta}\right)a$$
$$= \left\{\frac{\beta\lambda(1-\lambda') + (1-\beta)(1-\lambda)\lambda' - (1-\mu)[\beta\lambda + (1-\beta)\lambda']}{\lambda\lambda'}\right\}a$$
$$= \left[\beta\,\frac{(1-\lambda')}{\lambda'} + (1-\beta)\left(\frac{1-\lambda}{\lambda}\right)\right]a - \frac{(-\beta)\lambda'(1-\lambda)}{\lambda\lambda'}$$
$$= \beta\left(\frac{1-\lambda'}{\lambda'}\right)a - (1-\beta)\left(\frac{1-\lambda}{\lambda}\right)(1-a)$$
$$= \beta\phi(m', a) + (1-\beta)\phi(m, a).$$

For $a \in (0, v(m))$, we can solve for:

$$(0, \beta\lambda_\beta, (1-\beta)\lambda_\beta, 1-\lambda_\beta) = \sigma(0, \lambda', 0, 1-\lambda') + (1-\sigma)(0, 0, \lambda, 1-\lambda).$$

It follows that: $\beta\lambda_\beta = \sigma\lambda'$ and $(1-\beta)\lambda_\beta = (1-\sigma)\lambda$ so that

$$\sigma = \beta\lambda/[\beta\lambda + (1-\beta)\lambda'].$$

Observe that:

$$\phi(\beta m' + (1-\beta)m, a) = \left[\frac{1-\lambda\beta}{\lambda\beta}\right]a$$
$$= \left[\frac{\beta\lambda + (1-\beta)\lambda' - \lambda\lambda'}{\lambda\lambda'}\right]a$$
$$= \left[\beta\left[\frac{1-\lambda'}{\lambda'}\right] + (1-\beta)\left[\frac{1-\lambda}{\lambda}\right]\right]a$$
$$= \beta\phi(m', a) + (1-\beta)\phi(m, a).$$

The corresponding demonstration for a in the other intervals is similar. Q.E.D.

Proof of Theorem 3. Let $U^2(c; m) = U^2(c; m')$ and consider the following:

$$d^1 \equiv \alpha\delta[c, m] + (1-\alpha)\delta[c, p], \qquad d^2 \equiv \alpha\delta[c, m'] + (1-\alpha)\delta[c, p],$$
$$d^3 \equiv \delta[c, \beta m + (1-\beta)p], \qquad d^4 \equiv \delta[c, \gamma m' + (1-\gamma)p].$$

$\mathrm{CS} \Rightarrow U^1(d^1) = U^1(d^2)$.

$\mathrm{CTP} \Rightarrow U^1(d^1) = U^1(d^3)$ for $\beta = \dfrac{\alpha\tau_1}{\alpha\tau_1 + (1-\alpha)}$ and some constant τ_1.

$$\text{CTP} \Rightarrow U^1(d^2) = U^1(d^4) \quad \text{for} \quad \gamma = \frac{\alpha\tau_2}{\alpha\tau_2 + (1-\alpha)} \quad \text{and some constant } \tau_2.$$

Therefore, $U^1(d^3) = U^1(d^4)$ whenever

$$\text{(A1)} \qquad \frac{\gamma/(1-\gamma)}{\beta/(1-\beta)} = \frac{\tau_2}{\tau_1}.$$

By CS, $U^2(c;\ \beta m + (1-\beta)p) = U^2(c;\ \gamma m' + (1-\gamma)p)$ whenever (A1) holds. (Note that if we assume ETP in place of CTP, then we can conclude that for every $\beta \in (0,\ 1)$, there is a $\gamma \in (0,\ 1)$ such that $U^2(c;\ \beta m + (1-\beta)p) = U^2(c;\ \gamma m' + (1-\gamma)p)$.) If we let $p = m$, it follows from Lemma A that $U^2(c;\ \cdot)$ satisfies strong betweenness or mixture-monotonicity.

As in the proof of Theorem 2, $U^2(c;\ \cdot)$ is ordinally equivalent to $v(\cdot)$, where

$$\phi(m,\ v(m)) = \int_X \phi(x,\ v(m))\ dm(x) = 0.$$

Define $w(m,\ a) = \phi(m,\ a)/[v(m) - a]$. Then

$$\begin{aligned} v(\beta m + (1-\beta)p) = \hat{a} &= \frac{\beta w(m,\ \hat{a})v(m) + (1-\beta)w(p,\ \hat{a})v(p)}{\beta w(m,\ \hat{a}) + (1-\beta)w(p,\ \hat{a})} \\ &= \frac{\gamma w(m',\ \hat{a})v(m') + (1-\gamma)w(p,\ \hat{a})v(p)}{\gamma w(m',\ \hat{a}) + (1-\gamma)w(p,\ \hat{a})} \\ &= v(\gamma m' + (1-\gamma)p). \end{aligned}$$

The condition (A1) implies that:

$$\frac{\gamma/(1-\gamma)}{\beta/(1-\beta)} = \frac{w(m,\ \hat{a})}{w(m',\ \hat{a})} = \text{constant},$$

for $\hat{a} \in [v(p),\ v(m)]$ (or $[v(m),\ v(p)]$ if $v(p) > v(m)$). Since the choice of p is arbitrary, it follows that $w(m,\ a)$ is of the form $w(m)f(a)$, which is equivalent to weighted utility where $\phi(m,\ a) = w(m)[v(m) - a]$. Q.E.D.

Proof of Theorem 4. Adopt the notation in the statement of CTP2 and define

$$d^1 = \alpha\delta[c,\ m] + (1-\alpha)\delta[c,\ m'], \qquad d^2 = \alpha\delta[c,\ p] + (1-\alpha)\delta[c,\ p'],$$

$$d^3 = \delta[c,\ \beta m + (1-\beta)m'], \qquad d^4 = \delta[c,\ \beta p + (1-p)\beta].$$

By CS, $U^1(d^1) = U^1(d^2)$. By ETP (which is implied by CTP2), $U^1(d^1) = U^1(d^3)$ for some $\beta \in (0,\ 1)$. Then $\text{CTP2} \Rightarrow U^1(d^2) = U^1(d^4)$ for the same β. Thus, $U^1(d^3) = U^1(d^4)$ which implies, by CS, $U^2(c;\ \beta m + (1-\beta)m') = U^2(c;\ \beta p + (1-\beta)p')$.

Q.E.D.

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