Recursive Utility Under Uncertainty

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Abstract. This paper provides an axiomatization of recursive utility functions in an infinite horizon stochastic setting. In addition, some recently developed atemporal non-expected utility theories are integrated axiomatically into an intertemporal framework. The key axioms deal with intertemporal consistency and attitudes towards the temporal resolution of uncertainty.

1. Introduction

Consider intertemporal utility functions defined for consumption programs which extend over an infinite horizon. In deterministic models, the specification which dominates the capital theory literature is the intertemporally additive function with a constant rate of time preference, while generally in stochastic models the latter specification is adopted for the von Neumann-Morgenstern index and expected utility theory is assumed. For a framework of certainty, Koopmans (1960) has provided an axiomatic basis for a class of utility functions which weaken additivity over time. In this paper, we extend his axiomatic framework to admit uncertainty and a broad new class of intertemporal utility functions is characterized. These functions are called *recursive* and a key axiom is called recursivity, because of the recursive functional relation which defines intertemporal utility. Various subclasses of recursive utility are identified in Figure 1, which is clarified below.

Recursive utility functions need not conform with expected utility theory even in ranking restricted pairs of consumption programs, such as those in which all uncertainty is resolved immediately. Rather, the ranking of such "timeless gambles" may conform with one of several atemporal non-expected utility theories which have recently been proposed (e.g., Chew (1983) and (1989), Dekel (1986), Chew, Epstein and Segal (1990)).

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Indeed, the paper provides an axiomatic integration of the Chew and Dekel theory into an infinite horizon temporal framework.[1]

Two limiting features of the (expected) additive utility model which have been identified in the literature are: (i) the constancy of the rate of time preference (see Lucas and Stokey (1984) and Epstein and Hynes (1983) for recent discussions), and (ii) the fact that the two conceptually distinct aspects of preference, intertemporal substitutability and risk aversion, are intertwined in the within-period felicity function. Koopman's generalization of additivity endogenizes the rate of time preference and thus rectifies (i). The recursive utility functions developed here go further and also permit the disentangling of substitution from risk aversion. Moreover, this is accomplished in a framework where intertemporal consistency of choice prevails and dynamic programming techniques may be applied to solve optimization problems. (For an example of such an application see Epstein and Zin (1989).) The property of the expected additive utility model which is dropped in order to make the above separation possible is indifference to the way in which uncertainty resolves over time, in the sense of Kreps and Porteus (1978). We propose two weaker postulates regarding attitudes towards the timing of resolution, which are pivotal in the characterization of the subclasses of recursive utility considered below.

The stationarity provided by the infinite horizon framework plays an important simplifying role in our representation theorems (e.g., contrast with Chew and Epstein (1989)). But the infinite horizon and the need to distinguish between consumption programs which differ only in the way some common uncertainty resolves over time, requires that we adopt a domain for utility that has a complicated mathematical structure. The domain resembles the space of the infinite hierarchy of beliefs of players in Bayesian games (for example, see Mertens and Zamir (1985) and Myerson (1985)). In the literature on intertemporal utility theory, related domains have appeared in Kreps and Porteus (1978) in the finite horizon case and in Epstein and Zin (1989) for the case where consumption in the initial period is deterministic.

We proceed as follows: The domain is described in Section 2. The

[1] For a related analysis in a two-period model see Chew and Epstein (1989).

general class of recursive utility functions is defined and characterized in Section 3. A number of subclasses are explored in Section 4. Proofs are collected in an appendix.

2. Consumption

Adopt the following notation: For any metric space Y, $\mathrm{B}(Y)$ denotes the Borel σ-algebra and $M(Y)$ is the space of Borel probability measures on Y endowed with the weak convergence topology. The probability measure which assigns unit mass to $\{y\}$ is denoted $\delta[y]$. As a result of the map $y \to \delta[y]$, Y may be identified as a subspace of $M(Y)$. We write $Y' \subseteq Y$ if Y' is homeomorphic to a Borel subspace of Y. In that case $M(Y')$ can be identified with a subspace of $M(Y)$ via the map which takes $m' \in M(Y')$ into $m \in M(Y)$, $m(B) \equiv m'(B \cap Y')\ \forall B \in B(Y)$.

Consumption in each period t, $t = 0, 1, \ldots$, is restricted to lie in a compact metric space X.[2] Deterministic programs $(c_0, c_1, \ldots, c_t, \ldots)$ lie in Ω, the infinite Cartesian product of X. Under the product topology, Ω is also a compact metric space.

The domain D of consumption programs is constructed inductively. Let $D_{-1} \equiv \Omega = X \times \Omega$ and then for each $t \geq 0$ define[3]

$$D_t \equiv M(X \times D_{t-1}). \tag{2.1}$$

By Parthasarathy (1967, p. 43), each D_t is a compact metric space. Elements of D_1 are probability measures on Ω and can be interpreted as consumption programs in which the gamble underlying the probability measure is played out, and all uncertainty is resolved, at $t = 0$. Similarly, elements of D_t can be interpreted as programs in which all uncertainty is resolved at or before time t. Since $X \times \Omega \subset M(X \times \Omega)$, it follows by induction that

$$D_t \subset D_{t+1}, \quad t = -1, 0, \ldots.$$

[2] If "compact metric" is everywhere replaced by "separable metric," the principal result (Theorem 2.1) remains valid. Compactness is, however, convenient for the subsequent utility analysis.

[3] The counterpart definition in the construction of belief spaces (see references in the introduction) has the form $D_0 = M(X)$ and $D_t = M(X \times D_{t-1})$ for $t \geq 1$.

The set $\bigcup_0^\infty D_t$ contains all consumption programs for which uncertainty persists only for finitely many periods. But we would like D to contain also programs in which uncertainty is resolved only asymptotically. For that purpose proceed as follows: Given any $d_1 \in D_1$, we can "collapse" the uncertainty in d_1 so that it is resolved completely by $t = 0$ rather than $t = 1$. Formally, define the map

$$f_0 : D_1 \to D_0 = M(\Omega), \quad f_0(d_1)(B) \equiv E_{d_1} T_B(\cdot\,,\cdot), \quad B \in \mathbb{B}(\Omega),$$

where $T_B : X \times M(\Omega) \to \mathbb{R}^1$,

$$T_B(c, \nu) \equiv \nu\{w \in \Omega : (c, w) \in B\}.$$

We inductively define

$$f_t : D_{t+1} \to D_t, \quad t \geq 1$$

by $f_t(d_{t+1})(B) \equiv d_{t+1}\{(c, d_t) \in X \times D_t : (c, f_{t-1}(d_t)) \in B\}$, where $d_{t+1} \in M(X \times D_t)$ and $B \in \mathbb{B}(X \times D_{t-1})$. Then d_{t+1} and $f_t(d_{t+1})$ induce the identical probability measure $f_0 f_1 \ldots f_t(d_{t+1})$ on Ω, but they differ in the temporal resolution of the common uncertainty. Of course, if d_{t+1} already lies in D_t, then $f_t(d_{t+1}) = d_{t+1}$; in fact, for each $t \geq 0$ and $d_{t+1} \in D_{t+1}$,

$$f_t(d_{t+1}) = d_{t+1} \Leftrightarrow d_{t+1} \in D_t. \tag{2.2}$$

We are now ready to define the space of consumption programs D. Let

$$D \equiv \{(d_0, d_1, \ldots) : d_t \in D_t \text{ and } d_t = f_t(d_{t+1})\ \forall t \geq 0\}. \tag{2.3}$$

The following intuition underlies this definition: Elements of D are intended to represent infinite probability trees in which each branch corresponds to an element in Ω. Picture such a tree d and for each t imagine "collapsing" everything beyond t in the sense that all uncertainty which in d resolves at t or later is now completely resolved at t. This transformation generates a new tree d_t. As t increases, d_t provides a better approximation to the initial tree d and the approximation error vanishes asymptotically. Thus the infinite sequence of such approximations $(d_1, \ldots, d_t, \ldots)$ accurately represents d.

Each D_t is a compact metric space and thus so is $X_0^\infty D_t$. Endow D with the relative topology it inherits as a subspace of $X_0^\infty D_t$. Then we can prove (by adapting the arguments in Epstein and Zin (1989)) the following result:

Theorem 2.1. *The space D is a compact metric space such that D is homeomorphic to $M(X \times D)$. Moreover, $\bigcup_0^\infty D_t$ is homeomorphic to a dense subset of D.*

As a result of the indicated homeomorphism each $d \in D$ can be identified with a probability measure on $X \times D$. This is a reflection of the fact that as one moves along an infinite probability tree, the "future" remaining conditional on information received at time 0 is also an infinite tree in D. Such "stationarity" of the domain is essential for the investigation of recursive utility functions (see the stationarity and recursivity axioms and the functional structure in Theorem 3.1).

We interpret the denseness of $\bigcup_0^\infty D_t$ in D as the statement that for programs in D all uncertainty is resolved asymptotically.

3. Recursive Utility

We begin with a (complete and transitive) preference ordering $\succsim$ on D. A number of axioms are proposed. They imply a utility function representation for $\succsim$ which has a recursive structure.

By Theorem 2.1 we can identify each element of D with a measure on $X \times D$. By $(c, d) \in D$ we mean the degenerate measure $\delta[c, d]$ in which initial consumption c is certain and the uncertain future is represented by d. Of course, deterministic programs $(c_0, c_1, \dots)$ may be identified as elements of D.

First we impose continuity.

Continuity. *For each $d \in D$, the sets $\{d' \in D : d' \succsim d\}$ and $\{d' \in D : d \succsim d'\}$ are closed.*

The next two axioms parallel Koopmans' Postulates 3b and 4 respectively.

Risk Separability. *For all $(c, c') \in X^2$ and $(d, d') \in D^2$, $(c, d) \succsim (c, d') \Leftrightarrow (c', d) \succsim (c', d')$.*

Stationarity. *For some $\bar{c} \in X$ and all $(d, d') \in D^2$, $(\bar{c}, d) \succsim (\bar{c}, d') \Leftrightarrow d \succsim d'$.*

Consider the comparison between two programs sharing a common deterministic period 0 consumption. Risk separability requires that the rankings of such programs and the uncertainty about the future which they embody, be independent of the level of initial consumption. If stationarity is also adopted, then the relative ranking of any pair of programs (c, d) and (c, d') is not only independent of c, but is also the same as would prevail if $\gtrsim$ were applied at $t = 1$ as though time began then. In this sense the passage of time and past consumption do not affect preferences.

The situation dealt with in the stationarity axiom is special in that there is no uncertainty resolved between $t = 0$ (when the choice between (c, d) and (c, d') is made) and $t = 1$ when the choice between d and d' is possibly reconsidered. Typically, some new information is forthcoming to an agent in the interval between two instants at which decisions are taken. Thus in a model with uncertainty the above axioms are not sufficient to imply the intertemporal consistency of preferences. The latter is guaranteed by the following important axiom:

Recursivity. *For all* $\alpha_i \in (0,1)$ *and* (c_i, d_i) *and* (c'_i, d'_i) *in* $X \times D$, $i = 1, \ldots, n$, *such that* $\sum \alpha_i = 1$, *if* $(c_i, d_i) \gtrsim (c'_i, d'_i)$ *for all* i, *then* $\sum \alpha_i \delta[c_i, d_i] \gtrsim \sum \alpha_i \delta[c'_i, d'_i]$. *Moreover, the latter preference is strict if* $(c_i, d_i) > (c'_i, d'_i)$ *for some* i.

The numbers $\alpha_1, \ldots, \alpha_n$ represent the probabilities corresponding to an experiment conducted at $t = 0$. If the i^{th} outcome of the experiment occurs, then (c_i, d_i) or (c'_i, d'_i) will follow depending upon the choice made at $t = 0$ between the probability mixtures $\sum \alpha_i \delta[c_i, d_i]$ and $\sum \alpha_i \delta[c'_i, d'_i]$. If the former leads to a preferred program ex post regardless of the outcome which is realized, then recursivity requires that the former lottery should be chosen ex ante. Otherwise, the ex ante choice would be regretted ex post with probability 1 and would be revised if possible.[4]

If $\gtrsim$ satisfies all of the above axioms, we refer to it as a *recursive preference ordering.* Any utility function which represents $\gtrsim$ is called a *recursive utility function.* We turn now to the functional structure of recursive utility.

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[4] Recursivity is essentially identical to the notion of consistency explored by Johnsen and Donaldson (1985). For an alternative view of consistency see Machina (1989).

Our representation theorem requires the following definition and notation. Let $\hat{M} \subset M(\mathbb{R}^1)$ and let $\mu : \hat{M} \to \mathbb{R}^1$. Say that μ is a *certainty equivalent* if (i) μ is increasing in the sense of first degree stochastic dominance, and (ii) $\mu(\delta[v]) = v$ for all $\delta[v]$ in $\hat{M}$. Secondly, for any $V : D \to \mathbb{R}^1$ and $d \in D$, denote by $m_V(d)$ that probability measure on $V(D)$ defined by

$$m_V(d)(B) \equiv d\{(c, d') \in X \times D : V(c, d') \in B\}, \quad B \in \mathrm{B}(V(D)). \quad (3.1)$$

We have identified d with a probability measure on $X \times D$ under the homeomorphism between D and $M(X \times D)$. Thus $m_V(d)$ describes the probability distribution of utility $V(c, d')$ induced by d.

Theorem 3.1. *$\succsim$ satisfies continuity, risk separability, stationarity and recursivity if and only if it can be represented numerically by a continuous function V which satisfies*

$$V(d) = \mu(m_V(d)) \quad (3.2a)$$

and

$$V(c, d) = W(c, V(d)) \quad (3.2b)$$

$\forall (c, d) \in X \times D$, where $m_V(d)$ is defined in (3.1), $\mu : \hat{M}(V(D)) \to \mathbb{R}^1$, $\hat{M}(V(D)) \equiv \{m_V(d) : d \in D\}$, μ is continuous and is a certainty equivalent, $W : X \times V(D) \to \mathbb{R}^1$ is continuous and $W(c, \cdot)$ is increasing on $V(D)$ for every $c \in X$.

The functional representation of V has two components—a certainty equivalent function μ and an *aggregator* W.[5] They define the following recursive relation, obtained by combining (3.2a) and (b), which V must satisfy:

$$V(c, d) = W(c, \mu(m_V(d))). \quad (3.3)$$

[5] Both μ and W are degenerate if $V(D)$ consists of a single point, which occurs if and only if all programs in D are indifferent according to $\succsim$. Such degeneracy could be ruled out by a sensitivity axiom similar to Postulate 2 of Koopmans (p. 291). Note also that W and μ change if V is replaced by an ordinally equivalent utility function. Thus there exist many (W, μ) pairs that represent the identical ordering in the sense of (3.2).

The recursive relation (3.3) is readily interpreted. Given a program (c, d), period 0 consumption c is nonrandom but the future, represented by d, is uncertain. Consequently, the value of intertemporal utility at $t = 1$ is uncertain from the perspective of the initial period. According to (3.3), $V(c, d)$ is computed in two stages: (i) the certainty equivalent of random future utility is computed, and then (ii) it is combined with current consumption via the aggregator W. If there is no uncertainty, μ drops out and (3.3) reduces to the recursive relation derived by Koopmans:

$$V(c_0, c_1, c_2, \ldots) = W(c_0, V(c_1, c_2, \ldots)).$$

One desirable feature of the structure of recursive utility functions, particularly of (3.3), is that dynamic programming techniques may be applied to solve optimization problems (see Epstein and Zin (1989)). The other important feature is that a degree of separation is achieved between intertemporal substitutability (encoded in W) and risk aversion (encoded in μ) in the sense which we now describe.

Definition. *If $\succsim^*$ and $\succsim$ are preference orderings on D, $\succsim^*$ is more risk averse than $\succsim$ if any gamble rejected by $\succsim$ is also rejected by $\succsim^*$, i.e., $\forall d \in D$ and $\forall C = (c_0, c_1, \ldots) \in \Omega$,*

$$C \succsim d \Rightarrow C \succsim^* d.$$

For the result regarding comparative risk aversion we need one additional assumption.

Postulate 1. *For every $d \in D$ $\exists C$ and C' (deterministic programs) in Ω, such that $C \succsim d \succsim C'$.*

Theorem 3.2. *Let $\succsim$ and $\succsim^*$ be two recursive preference orderings on D which also satisfy Postulate 1. Then $\succsim^*$ is more risk averse than $\succsim$ if and only if there exist corresponding utility functions V^* and V satisfying: (i) $V^* \mid \Omega = V \mid \Omega$; (ii) $W^* = W$ and $\mu^* \leq \mu$, where (W^*, μ^*) and (W, μ) satisfy the appropriate forms of (3.2a) and (b) for $\succsim^*$ and $\succsim$ respectively.*

Roughly speaking, given a recursive utility function V, we can increase risk aversion without changing the rankings of deterministic programs, by keeping the same aggregator and adopting a new certainty equivalent function μ^*; the latter should assign lower certainty equivalents to probability measures in its domain than does μ.

Besides Koopmans (1960), other papers which have studied aggregators include Lucas and Stokey (1984) and Boyd (1990). For examples of aggregator functions, see Koopmans, Diamond and Williamson (1964) and Epstein (1983); for the latter example see also (4.1) below. The next section describes a number of classes of certainty equivalent functions and also relates the general recursive utility specification to the standard expected additive utility form and another specification ((4.3) below) which has appeared in the literature.

4. Attitudes Towards Temporal Resolution

We will consider a number of subclasses of recursive utility corresponding to different hypotheses about attitudes towards the way in which uncertainty resolved over time, in the sense first formalized by Kreps and Porteus (1978). It is convenient to define the projection π from D onto D_0,

$$\pi(d) = \pi(d_0, \dots, d_t, \dots) \equiv d_0. \tag{4.1}$$

Interpret $\pi(d) \in M(\Omega)$ as representing the "marginal" distribution of consumption inherent in the program d, where information about the temporal resolution of uncertainty has been eliminated.

The first and most common hypothesis is that temporal resolution is a matter of indifference.

Timing Indifference. *For all* $\alpha \in [0,1]$, $c \in X$ *and* $(d, d') \in D^2$,

$$\alpha\delta[c,d] + (1-\alpha)\delta[c,d'] \sim \delta[c, \alpha d + (1-\alpha)d'].$$

The two programs in the statement of the axiom involve the same uncertainty about the future in the sense that they have identical images under the projection π. But in $\delta[c, \alpha d + (1-\alpha)d']$ one learns only at $t = 1$, after the realization of the random variable corresponding to the $(\alpha, 1-\alpha)$ probability distribution, whether it is d or d' that is to be faced. In the other lottery, that information is forthcoming at $t = 0$. In the absence of an ability to exploit earlier information to improve planning, and planning advantages are not an issue in this section, one may wish to hypothesize timing indifference.

The timing indifference axiom has strong implications for recursive preferences.

Theorem 4.1. *The preference ordering $\succsim$ is recursive and satisfies timing indifference if and only if it can be represented by a continuous utility function V for which the aggregator and certainty equivalent functions of Theorem 3.1 take the form:*

$$W(c, v) = u(c) + B(c)v, \tag{4.1}$$

for some $u : X \to \mathbb{R}^1$ and $B : X \to \mathbb{R}^1_{++}$, and

$$\mu(m) = E(m), \tag{4.2}$$

the expected value of m, for $m \in \hat{M}(V(D))$.

The aggregator (4.1) is the discrete-time version of the function proposed by Uzawa (1968). In conjunction with (4.2) it implies the expected utility model

$$V(d) = E_{\pi(d)} \sum_{0}^{\infty} u(c_t)B(c_0) \cdots B(c_{t-1}), \tag{4.3}$$

where $E_{\pi(d)}$ is the expected value with respect to the measure $\pi(d) \in M(\Omega)$. (An alternative axiomization of (4.3), under the maintained hypothesis of expected utility theory, may be found in Epstein (1983).) The specification (4.3) is the most general recursive utility function consistent with expected utility theory. But it does not permit comparative risk aversion in the sense of Definition 3.1 and Theorem 3.2, since if $W^* = W$ (= (4.1) without loss of generality), then $\mu^* = \mu$ (= (4.2)).

For completeness we also describe an axiomatic basis for the standard additive model which consists of (4.1)–(4.3) with B constant. It is straightforward to show that the following additional axiom, similar to Postulate (3′a) of Koopmans (p. 307), will suffice.

Future Independence. *For all $(c_0, c_1, c_0', c_1') \in X^4$ and deterministic programs C and C' in Ω, $(c_0, c_1, C) \succsim (c_0', c_1', C) \Leftrightarrow (c_0, c_1, C') \succsim (c_0', c_1', C')$.*

This axiom deals only with deterministic programs and requires that trade-offs between consumptions in periods 0 and 1 be independent of consumption levels in later periods.

In view of the unfortunate consequence of timing indifference noted above, we are led to reconsider that axiom. We would argue that it is perfectly "rational" for an individual to care about the temporal resolution of uncertainty. For example, early resolution might be preferred by a "nervous" or "edgy" person who does not like living with uncertainty. On the other hand, an affinity for surprises or the pleasure derived from hope, for, or anticipation of, favorable events which have some chance of occurring in the future, could lead to a preference to defer the resolution of uncertainty.[6] Thus we turn to a number of axioms which admit limited forms of nonindifference to timing. In all cases, the implied utility functions are sufficiently flexible that comparative risk aversion analysis in the sense of Definition 3.1 becomes possible. The subclasses of recursive utility obtained in this way and their interrelationships are indicated in Figure 1.

Suppose that the uncertainty to be resolved is whether d or d' will be faced in the future and suppose further that d and d' are indifferent to one another. Then the psychic costs or benefits of early resolution are less apparent and timing indifference is plausible. More precisely, consider the following axiom:

Quasi-Timing Indifference. *For all* $\alpha \in [0,1]$, $c \in X$ *and* $(d,d') \in D^2$, *if* $d \sim d'$ *then* $\alpha\delta[c,d] + (1-\alpha)\delta[c,d'] \sim \delta[c, \alpha d + (1-\alpha)d']$.

Next suppose that $d > d'$ and that

$$\alpha\delta[c,d] + (1-\alpha)\delta[c,d'] \sim \delta[c, \beta d + (1-\beta)d']$$

[6] It is difficult to find direct evidence on attitudes towards timing, since people's observed choices invariably reflect both the planning advantages of early resolution and the psychic costs or benefits upon which we focus here. But revealed preference for late resolution, in spite of the planning cost that entails, would constitute an a fortiori case for psychic preference for late resolution. An example would be a preference for not resolving early uncertainty about date of death. Some experimental evidence regarding attitudes towards timing is provided in Cook (1989).

for $\beta > \alpha$. Then the fact that indifference holds only when the preferred prospect d is given a large weight β in the late resolution case, reflects a preference for early resolution. In fact, the difference between β and α, or alternatively the normalized expression $[\beta(1-\alpha)/\alpha(1-\beta)]-1$, can be interpreted as a probability premium which would be demanded if late resolution were substituted for early resolution. It is intuitive that the premium should be unaffected if d and d' are replaced by respectively indifferent programs. Thus consider the following axiom:

Constant Timing Premium. *For all* $(\alpha, \beta) \in [0,1]^2$, $c \in X$ *and* $(d, d', e, e') \in D^4$ *with* $d \sim e$ *and* $d' \sim e'$, *if*

$$\alpha\delta[c,d] + (1-\alpha)\delta[c,d'] \sim \delta[c, \beta d + (1-\beta)d'],$$

then

$$\alpha\delta[c,e] + (1-\alpha)\delta[c,e'] \sim \delta[c, \beta e + (1-\beta)e'].$$

The above axioms have no implications for the aggregator function of a recursive ordering, but they do restrict the certainty equivalent function μ, albeit not as much as does (4.2). We now describe the implied functional forms for μ, all of which have been studied in the atemporal literature on non-expected utility theories.

Let μ be defined on $M(S)$ where $S \subset \mathbb{R}^1$. Say that μ is a *betweenness* function if $\exists$ continuous $\phi : S \times \mathbb{R}ng(\mu) \to \mathbb{R}^1$ such that $\phi(s,s) \equiv 0$, ϕ is increasing in its first argument and $\forall m \in M(S)$, $\mu(m)$ is the unique solution to

$$\int_S \phi(s, \mu(m))dm(s) = 0. \tag{4.4}$$

(See Dekel (1986) and Chew (1989).) The special case where $\phi(s, \cdot)$ is linear in $\mu(\cdot)$ leads to an explicit expression for μ of the following form:

$$\mu(m) = u^{-1}\left[\int u(s)w(s)dm(s) \Big/ \int w(s)dm(s)\right], \tag{4.5}$$

for suitable u and w. Such μ's are called *weighted utility* functions (Chew (1983)). If w is constant, the latter specializes to the *expected utility* uncertainty equivalent

$$\mu(m) = u^{-1}\left(\int u(s)dm(s)\right). \tag{4.6}$$

Our final result relates these functional forms to the timing axioms (see also Figure 1).[7]

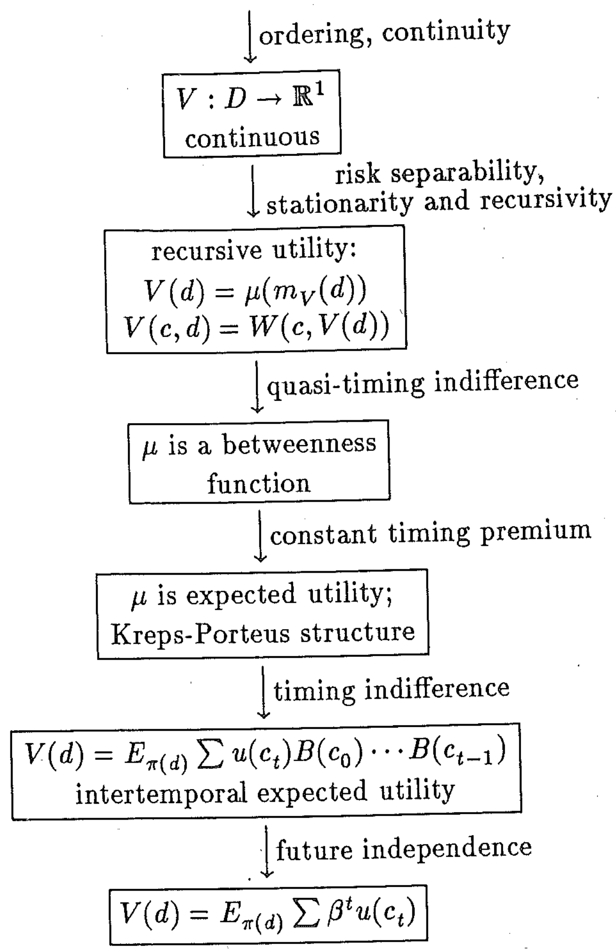

Figure 1

Theorem 4.2. *The preference ordering* $\succsim$ *is recursive and satisfies respectively*

[7] A timing axiom characterization of the weighted utility certainty equivalent is not included here. See Chew and Epstein (1989) for a characterization in a two-period model.

(i) quasi-timing indifference,
(ii) quasi-timing indifference and constant timing premium,

if and only if it can be represented by an aggregator W and certainty equivalent μ as in Theorem 3.1, where in addition μ lies in the following functional form classes respectively:

(i′) betweenness,
(ii′) expected utility.

The last case corresponds to the functional structure first explored by Kreps and Porteus (1978). Note that even though μ is expected utility based, the corresponding intertemporal utility function V does not satisfy the independence axiom on D and does not conform with (4.3).

Both of the intertemporal utility theories described in the theorem are sufficiently flexible to model comparative risk aversion. For example, in the expected utility case, if in the appropriate forms of (4.6) u^* is more concave than u, then $\mu^* \leq \mu$. For the betweenness case, let μ and μ^* be represented by ϕ and ϕ^* respectively as in (4.4). Suppose that ϕ^* is a concave transform of ϕ in their first arguments, i.e.,

$$\phi^*(s,z) \equiv h(\phi(s,z),z),$$

for (s,z) in the domain of ϕ, where h is defined on an appropriate domain and is an increasing and concave function of its first argument there. (The function h must also satisfy $h(0,z) \equiv 0$ in order that $\phi(s,s) \equiv 0$ and $\phi^*(s,s) \equiv 0$.) Then, by Jensen's inequality applied to (4.4), $\mu^* \leq \mu$.

It would clearly be of interest to exploit the flexibility of the intertemporal utilities described in the Theorem, and indeed of the general recursive utilities in Theorem 3.1, to reexamine standard issues in capital theory, such as asset pricing or optimal stochastic growth. Some results in the asset pricing context may be found in Epstein (1988) and Epstein and Zin (1989).

Appendix

Proof of Theorem 3.1. The necessity of the axioms is clear. We prove their sufficiency.

By Debreu (1954), $\succsim$ can be represented numerically by a continuous real-valued function V defined on D. Define W on $X \times V(D)$ by

$$W(c, v) \equiv V(c, d) \quad \text{for any} \quad d \in V^{-1}(v). \tag{A1}$$

Risk separability and stationarity imply that W is well-defined and that $W(c, \cdot)$ is increasing on $V(D)$ for each $c \in X$. Since D is compact and V is continuous, we can show that W is continuous on $X \times V(D)$. Evidently, (3.2b) is satisfied.

Let ψ be the map which takes $d \in D$ into $m_V(d) \in M(V(D))$. If f is any continuous (and therefore also bounded) real-valued function on $V(D)$ then $f(V(\cdot))$ is continuous (and bounded) on D. It follows that ψ is a continuous map.

Define μ on $\hat{M}(V(D))$ by

$$\mu(m) \equiv V(d), \quad \text{for any} \quad d \in D \quad \text{such that } m = m_V(d).$$

We need to show that μ is well-defined, i.e., that

$$m_V(d) = m_V(d') \Rightarrow V(d) = V(d'). \tag{A2}$$

The functions ψ and V are continuous and the subset of $M(X \times D)$ consisting of measures having finite support is dense in $M(X \times D)$. Therefore, it suffices to prove (A2) for the case where d and d', identified as elements of $M(X \times D)$, each has finite support. In that case, if $m_V(d) = m_V(d')$, then $\exists \alpha_i, c_i, d_i, c_i'$ and d_i', $i = 1, \ldots, n$, such that $V(c_i, d_i) = V(c_i', d_i') \; \forall i$, $d = \sum \alpha_i \delta[c_i, d_i]$ and $d' = \sum \alpha_i \delta[c_i', d_i']$. Thus recursivity implies that $V(d) = V(d')$.

Since D is compact and ψ is continuous, we can show that μ is continuous. Recursivity implies that μ is increasing in the sense of first degree stochastic dominance. The definition of μ immediately implies that $\mu(\delta[v]) = v \; \forall v \in V(D)$, and so μ is a certainty equivalent. Finally, it is clear that (3.2a) is satisfied. Q.E.D.

Proof of Theorem 3.2. Suppose $\succsim^*$ is more risk averse. Then $\succsim^*$ and $\succsim$ must rank deterministic programs identically. Thus the representations V and V^*, provided by Theorem 3.1, can be chosen so that

$$V \mid \Omega = V^* \mid \Omega. \tag{A3}$$

By Postulate 1, $V(D) = V^*(D)$. By the construction of the aggregator (see (A2)), $W = W^*$. In light of (A3), $\succsim^*$ being more risk averse amounts to $V^* \leq V$ on D. Thus $\mu^* \leq \mu$ follows from the appropriate forms of (3.2a).

Consider the converse. We need only show that

$$V^* \leq V \quad \text{on } D. \tag{A4}$$

Prove the inequality first on D_0 (recall (2.1)). Any $d \in D_0$ can be viewed as a measure on Ω. Since $V \mid \Omega = V^* \mid \Omega$, $m_V(d) = m_{V^*}(d)$. Therefore, we can use (3.2a) and $\mu^* \leq \mu$ to show that $V^*(d) \leq V(d)$. Proceed inductively, (exploiting also (3.2b)), to prove that $V^* \leq V$ on $\bigcup_0^\infty D_t$. Since the latter is dense in D, (A4) follows. Q.E.D.

Proof of Theorem 4.1. Let V represent $\succsim$ on $M(X \times D)$ as provided by Debreu (1954). We show first that V satisfies the independence axiom, i.e., $\forall \alpha \in [0,1]$,

$$V(d) = V(d') \Rightarrow V(\alpha d + (1-\alpha)e) = V(\alpha d' + (1-\alpha)e).$$

Consider the following four programs, where $c \in X$ is arbitrary:

$$d^1 = \alpha\delta[c,d] + (1-\alpha)\delta[c,e], \quad d^2 = \alpha\delta[c,d'] + (1-\alpha)\delta[c,e],$$
$$d^3 = \delta[c, \alpha d + (1-\alpha)e], \qquad d^4 = \delta[c, \alpha d' + (1-\alpha)e].$$

Then

$$\begin{aligned} V(d^1) &= V(d^3) && \text{by timing indifference,} \\ V(d^1) &= V(d^2) && \text{by recursivity,} \\ V(d^2) &= V(d^4) && \text{by timing indifference,} \\ \Rightarrow V(d^3) &= V(d^4) \Rightarrow && \text{by (3.2b)} \end{aligned}$$

$V(\alpha d + (1-\alpha)e) = V(\alpha d' + (1-\alpha)e)$, as desired.

We conclude that, after redefining V if necessary,

$$V(d) = E_d U(\cdot) \quad \forall d \in M(X \times D), \tag{A5}$$

where $U : X \times D \to \mathbb{R}^1$ is a von Neumann-Morgenstern utility index. Since $U(c,d') = V(\delta[c,d'])$, we see that (4.2) is a restatement of (3.2a).

To show that W defined in (A1) satisfies (4.1), i.e., is linear in its second argument, apply (3.2b), the mixture linearity of V represented by (A5) and timing indifference as follows:

$$\begin{aligned}\alpha W(c, V(d)) + (1-\alpha)W(c, V(d')) &= \alpha V(c,d) + (1-\alpha)V(c,e) \\ &= V(\alpha\delta[c,d] + (1-\alpha)\delta[c,d']) \\ &= V(c, \alpha d + (1-\alpha)d') \\ &= W(c, \alpha V(d) + (1-\alpha)V(d')).\end{aligned}$$

Q.E.D.

Proof of Theorem 4.2. The sufficiency of the functional forms is readily verified. For necessity, adapt the proof of Theorem 4.1. For example, show that (i) implies that V satisfies the betweenness axiom of Chew (1989) and Dekel (1986) and then invoke their representation results. In the case of (ii), show that V satisfies the independence axiom. More details for the two-period context may be found in Chew and Epstein (1989). Q.E.D.

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