Chapter 1

Geometric Generalizations of the Tonnetz and their Relation to Fourier Phases Spaces

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Some recent work on generalized Tonnetze has examined the topologies resulting from Richard Cohn's common-tone based formulation, while Tymoczko has reformulated the Tonnetz as a network of voice-leading relationships and investigated the resulting geometries. This paper adopts the original common-tone based formulation and takes a geometrical approach, showing that Tonnetze can always be realized in toroidal spaces, and that the resulting spaces always correspond to one of the possible Fourier phase spaces. We can therefore use the DFT to optimize the given Tonnetz to the space (or vice-versa). I interpret two-dimensional Tonnetze as triangulations of the 2-torus into regions associated with the representatives of a single trichord type. The natural generalization to three dimensions is therefore a triangulation of the 3-torus. This means that a three-dimensional Tonnetze is, in the general case, a network of *three* tetrachord-types related by shared trichordal subsets. Other Tonnetze that have been proposed with bounded or otherwise non-toroidal topologies, including Tymoczko's voice-leading Tonnetze, can be understood as the embedding of the toroidal Tonnetze in other spaces, or as foldings of toroidal Tonnetze with duplicated interval types.

1. Formulations of the Tonnetz

The Tonnetz originated in nineteenth-century German harmonic theory as a network of pitch classes connected by consonant intervals. Initially presented as a grid of perfect fifth and major third intervals, Arthur von Oettingen and Hugo Riemann used it to show acoustical relationships between tones, but in Ottokar Hotinský's reformulation, it became a triangular network of tones connected by perfect fifths, major thirds, and minor thirds. Riemann adopted this form of the network as a map of tonal relationships for analysis of chromatic harmony.[1,2] The important feature of this triangular network is that each triangle contains all the pitch classes of a major or minor triad, and triads sharing two pitch classes are adjacent across a common edge.

In an influential article, Richard Cohn[3] suggests that the Tonnetz can be generalized by changing the intervals of the network, so that the triangles correspond to different possible trichord types. He also points out a feature unique to the triadic Tonnetz, not shared by other possible Tonnetze within the 12-tone system: for any two adjacent triads, the unique tones that distinguish them are always separated by a small distance, one or two semitones.

As a result of this long history, the Tonnetz has acquired, as Dmitri Tymoczko points out,[4] at least three theoretical meanings: A map of acoustical relationships between tones, a map of common-tone relationships between triads, and as a map of minimal voice-leading relationships between triads. Cohn's generalization makes little sense according to the acoustical understanding of the Tonnetz. It also prioritizes common-tone relationships over stepwise voice leading, which Tymoczko takes as the essential property of a Tonnetz. Nonetheless, Cohn's common-tone based formulation is of significant musical interest, and Tymoczko's voice-leading Tonnetze may in fact be understood as special cases of it. Considering the topic from a geometrical perspective, rather than a purely topological approach—as reflected in a number of recent mathematical papers on generalized Tonnetze[5–7]—helps to enrich their potential music-theoretic applications. In this sense the the work below shares a kinship with Tymoczko's recent work, also fundamentally grounded in geometric considerations, but the geometries derived here have a different mathematical basis, the DFT.

For the Tonnetz as a network, the triangles are mathematically significant as its *maximal cliques* (maximal sets of vertices all of which are adjacent). The corresponding geometrical concept is the *simplex.* Thus, as Louis Bigo and others have observed,[6] the Tonnetz may be understood as a simplicial complex, and generalized along these lines. From this perspective, Hotinský's "completion" of the network was necessary and inevitable, because Oettingen's original grid contains no cliques that could be identified with 2-simplexes (triangles). The Tonnetz is more than just a simplicial complex, however, because it also fills an entire space with 2-simplexes, making it a *triangulation* of that space. This will have important ramifications for three-dimensional Tonnetze.

2. Definition and Construction of 2-Dimensional Tonnetze

We can define a Tonnetz geometry with some basic stipulations:

- There are a finite number of pitch classes corresponding to some $\mathbf{Z}_u$
- Each pitch class is assigned a unique point in the space
- *Transposability*: All translations of $\mathbf{Z}_u$ (musical transpositions) correspond to some rigid geometric transformation of the space that maps the pitch-classes appropriately.

The transposability condition is crucial; it demands that the cyclic structure of the pitch-class universe be reflected in the geometry as rigid transformations. It also implies that intervals of $\mathbf{Z}_u$ of the same size are represented by the same distances.

A straightforward construction that maps musical transpositions to translations in a cyclic space can immediately generate a family of spaces satisfying the above stipulations for any $\mathbf{Z}_u$. The procedure is as follows:

(1) Choose a number of dimensions
(2) For each dimension, n, choose a mapping $1 \mapsto 2\pi x_n/u$ where x_n is an integer between 0 and $n-1$, such that all of the x_n are mutually coprime to u.
(3) Place the pitch classes by iterations of interval $1 = (x_1, x_2, ...)$.

In step (3), and throughout, a constant multiplier of $2\pi/u$ is assumed. It is clear that this procedure will produce a geometric embedding of the pitch classes that satisfies the transposability condition, where translations of the space by $(kx_1, kx_2, ...)$ correspond to transposition by k. The coprime condition ensures that each pitch class has a unique position in the space.

Given a space satisfying the above conditions, we can define a generalized Tonnetz as the *triangulation* of the space whose vertices correspond precisely to the pitch classes, and where the dimension of the simplexes matches the dimension of the space. The following construction for the 2-dimensional case will be generalized to 3 and n dimensions below.

(1) Define a two-dimensional space for the given $\mathbf{Z}_u$ as specified above.
(2) Choose a line segment that connects pitch-class 0 to any other pitch class without passing through another pitch class.
(3) Translate this line segment to all pitch classes.
(4) Choose another line segment from pitch-class 0 to any other that is not parallel to the first line segment and does not cross it or any of its

translates. Translate this to all pitch classes. The result is a skewed square lattice.

(5) Choose a diagonal of one of the parallelograms and translate it to each pitch class.

A simple example is given in Fig. 1 for $\mathbf{Z}_6$, represented as a whole-tone universe. The pitch classes are set by choosing $1 \mapsto (1, 2)$. We make a lattice by first connecting the major thirds with line segments $(2, -2)$ from each pitch class, then the major seconds with (1, 2) line segments. There are two possible diagonals to bisect this lattice, one of which (connecting the tritones) creates a (026)-Tonnetz. The other possibility would give an (024)-Tonnetz by creating a distinct set of major-second edges. This possibility of interval duplications is inherent to the formalism and will be further explored below.

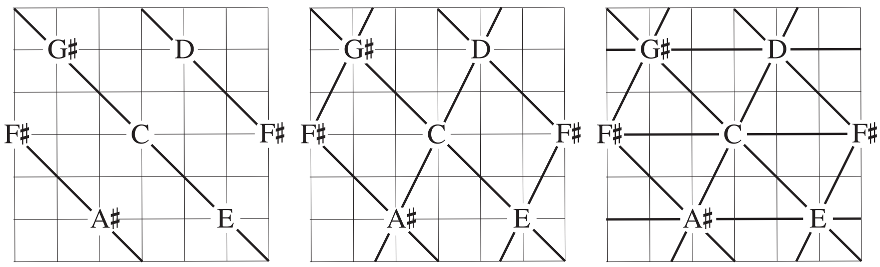

Fig. 1. Construction of a Tonnetz in a whole-tone universe.

Other Tonnetz formulations that have been proposed relate to this toroidal construction, although they may display other topological features. For instance, Tymoczko's triadic voice-leading Tonnetz (Fig. 2) is formulated in 3-note chord space, which is a bounded, non-orientable triangular prism whose ends are glued with a twist.[8] The geometry satisfies the transposability condition through translation along the central axis of the space and screw rotation around the axis. The dimensionality of the space can be reduced, however, by taking just the toroidal surface surrounding this axis where all the pitch classes lie, making it equivalent to one of the toroidal spaces derivable from the construction proposed above. The purpose of Tymoczko's extension of this space is to show the augmented triads as triangles in the added dimension cutting across the torus, essential to his minimal voice leading requirement.

Other kinds of spaces can be derived in certain cases by folding toroidal

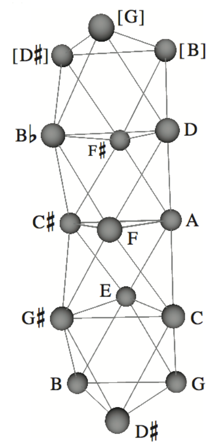

Fig. 2. Tymoczko's Triadic Tonnetz in Voice-Leading Space.[4]

ones, resulting in non-orientable, bounded, and spherical topologies. Examples are described below.

3. Geometric Tonnetze and Fourier Phase Spaces

Emmanuel Amiot[9] first noted a connection between Fourier phase spaces and the Tonnetz. Phase spaces are defined by applying the DFT to the characteristic function of a pitch-class set, then taking the phase values of certain coefficients as coordinates. Because phases are cyclic, the resulting spaces are toroidal. Amiot observed that using phases of the third and fifth coefficients yields an arrangement of major and minor triads mimicking the dual of the standard Tonnetz. I investigated this further[10] observing that the Tonnetz was a triangulation on the pitch classes, and that such a triangulation could be defined for any trichord in any 2-dimensional phase space. Fig. 3 shows the standard Tonnetz in a $\mathrm{Ph}_{3,5}$-space (where "$\mathrm{Ph}_{x,y}$" refers to a two-dimensional phase space on phases of coefficients x and y.[11]) This space is optimal for this Tonnetz, but a well-formed Tonnetz for any trichord is possible in this space provided that $\mathbf{Z}_{12}$ is its minimal embedding universe. For example, we can draw an (013)-Tonnetz, although the $\mathrm{Ph}_{3,5}$-space is clearly not optimal for making compact (013) regions.

This naturally raises the question of whether the reasoning can be reversed, deriving the Fourier phase spaces from some elementary principles about the geometric embedding of a Tonnetz. The above construction

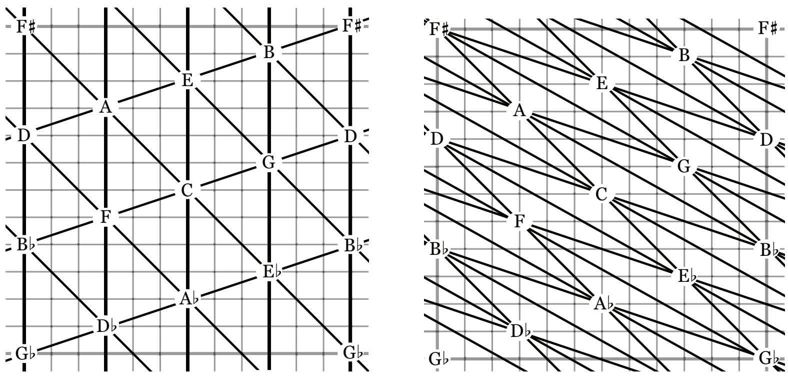

Fig. 3. The standard Tonnetz and (013)-Tonnetz in $\mathrm{Ph}_{3,5}$-space.

shows that this is indeed the case, provided we limit the transposability condition to translations, enforcing a simple toroidal topology. The one-dimensional cyclic spaces defined by the condition $1 \in \mathbf{Z}_u \mapsto 2\pi x_n/u$ are precisely the possible one-dimensional phase spaces for the DFT of a pitch-class set in universe u, and an n-dimensional phase space is a direct product of n one-dimensional spaces.

The relation to phase spaces may be a key to successful music-analytic application of non-triadic Tonnetze. Thus far, such applications have been limited; examples include an (013)-Tonnetz analysis of a Bach fugue subject by David Lewin,[12,13] Van den Toorn and McGinness's octatonic-diatonic (025) Tonnetz,[14] Stephen Brown's ic1/ic5 Tonnetz,[15] and Siciliano's[16] Cohn-cycle analyses which could be readily plotted in (026) and (014) Tonnetze. These analyses typically require a high degree of saturation of a musical passage with a single trichord type, and must leave by the wayside aspects of the harmony not accounted for by that trichord. Yet recent research has indicated a wide range of analytical applications of Fourier phase spaces.[11,17–19] The inherent limitation of the Tonnetz as a network to single trichord-type may be loosened up by the embedding in phase spaces, which situate it in a conceptually richer music-theoretic context. In addition, Amiot's[18] application of the DFT to beat-class sets introduces the possibility of rhythmic Tonnetze.

4. Intervallic Duplications and Foldings

One important feature of toroidal spaces is that there are an infinite number of conceivable line segments connecting any two given points. This means that a musical interval may be identified with a vector in a phase space but not necessarily a specific realization of that vector as a path. I have previously[10] proposed an expanded concept of musical interval that allows for such distinctions, and a corresponding concept of *intervallic axes* in phase spaces. The possibility of multiple realizations of a given interval has important consequences for generalized Tonnetze.

Consider the (026)-Tonnetz of Fig.1. Each tritone occurs twice as an interval in the network and the distinction is significant. The tritone F♯–C on the left, for instance, is filled in differently (by G♯ or A♯) than the one oriented as C–F♯ on the right (which is filled in by D or E). Fig. 4 provides a similar example in the rhythmic domain, where numbers refer to beats of a 4/4 measure and superscripts to eighth-note offsets. The chosen phase space represents temporal proximity on the x-axis and the possible syncopations of the half-note pulse on the y-axis. The Tonnetz is made up of ♩.♪𝅗𝅥 and ♪♩.𝅗𝅥 rhythms shifted to different places in the measure. A given half-note pulse is represented in two forms differing in which half of the measure is filled.

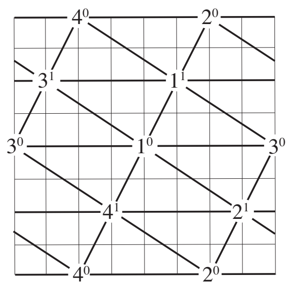

Fig. 4. A rhythmic Tonnetz.

The necessity of duplicated intervals in toroidal Tonnetz explains why Cantazaro[7] finds non-toroidal topologies in his computations, because his procedure does not recognize the possibility of duplicated intervals. In many situations, the distinction between forms of the same interval may be musically significant, and hence it is preferable to retain the toroidal space. Yet, if the distinctions are not significant, it is possible to fold the space over a given intervallic axis, creating a bounded space. For example, the triadic $\mathbf{Z}_7$-Tonnetz in Fig. 5 has two copies of each third, and, because the triad is symmetrical in this universe, there are also two copies of each triad. Therefore, it is possible to shear and fold the space over the fifths axis such that the two forms of third map onto each other and the equivalent triads. The result is the triadic Möbius strip discussed by Muzzulini[20] and Mazzola.[21] The same kind of folding can be performed in any instance where a symmetry exists in the Tonnetz, and the result is a bounded space that will be orientable in an even-cardinality universe and non-orientable in an odd-cardinality universe. Muzzulini's folding simplifies the triadic Tonnetz, but the toroidal version might also be of theoretical value. One might, for instance, treat the upwards oriented third as major and the downward oriented one as minor, so that a distinction between major (upwards pointing) and minor (downwards pointing) triads can be reflected in the Tonnetz despite the fact that chromatic variants of a given generic note-name are not distinguished in the space.

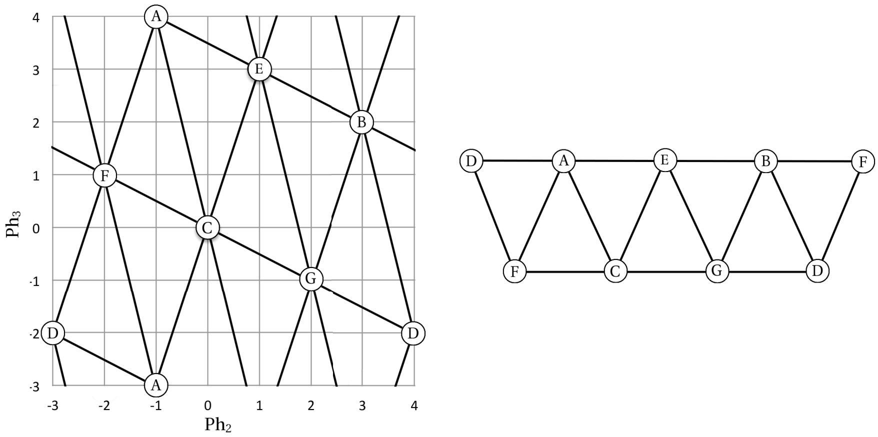

Fig. 5. A triadic Tonnetz in $\mathbf{Z}_7$ and a folding of it into a triadic Möbius strip.

The $\mathbf{Z}_4$ Tonnetz presents a unique case in which a spherical topology is possible through folding. The full Tonnetz is shown in Fig. 6 as a Tonnetz on (036) subsets of a diminished seventh chord. Due to the symmetry of the trichord, a folding is possible over the tritone axis. Because the resulting boundaries are tritone axes, they also contain duplications that can be identified. This folding reduces the edge count by two without changing the number of vertices or triangles. Hence, the Euler characteristic of the space changes from 0 (the value for all toroidal Tonnetze, which in the two-dimensional case have u vertices, $3u$ edges, and $2u$ triangles) to 2. The spherical Tonnetz satisfies the transposability conditions through rotations of the sphere, a cyclic subgroup of the tetrahedral symmetry group.

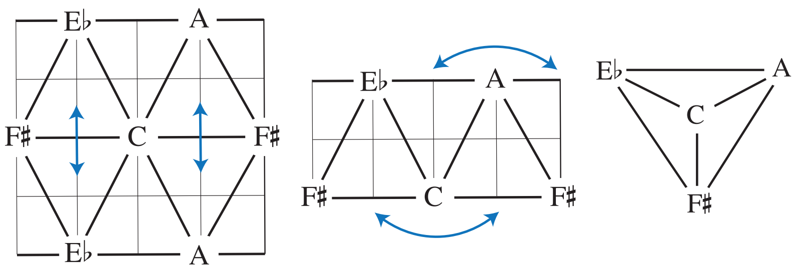

Fig. 6. Deriving a spherical Tonnetz through folding of the $\mathbf{Z}_4$ Tonnetz

5. Optimization

The observation that any Tonnetz can be drawn in any phase space for the given universe motivates the search for a method of optimizing the space to the Tonnetz. The obvious criterion for optimization is to seek the most compact spaces, avoiding stringy regions like those in the (013)-Tonnetz of Fig. 3. However, this could potentially be operationalized in different ways. One method would be to minimize the perimeter of the regions, which amounts to minimizing the number of times each axis cycles the space. However, this measure turns out to be somewhat coarse. The connection to the DFT offers another more sensitive method, which is to take the magnitude of the DFT components for the given trichord (in the two-dimensional case). These indicate how compact the trichord is in each dimension, so maximizing them optimizes the fit.

The existence of duplicated intervals in the set class, however, complicates the optimization process, because the space must allow for two different line segments representing the same interval, such that neither is excessively long. This is made possible by choosing one dimension that *minimizes* the DFT value of this interval. A complete optimization strategy then is first to ensure that there is a minimizing dimension for each duplicated interval, then to choose the remaining dimensions so as to maximize the DFT components of the set class.

Table 1 shows the results for each trichord type of $\mathbf{Z}_{12}$. The wrapping number gives the total number of times all of the axes cycle the space in either dimension. In two cases, (014) and (037), the DFT magnitudes distinguish between two possibilities with the same wrapping number. The association of trichords with optimum spaces (excluding degenerate spaces $\mathrm{Ph}_{1,1}$ and $\mathrm{Ph}_{5,5}$) is as close to one-to-one as possible given that there are more possible spaces than trichords. $\mathrm{Ph}_{1,2}$ and $\mathrm{Ph}_{2,5}$ appear as alternate spaces for the same trichord, (016), and $\mathrm{Ph}_{1,5}$ and $\mathrm{Ph}_{3,4}$ appear only as second-best spaces for two possible trichords.

Table 1. Optimum Tonnetz spaces for each trichord of $\mathbf{Z}_{12}$.

Trichord	Best space	DFT mag's	Wrapping	Other space	DFT mag's	Wrapping
(012)	$\mathrm{Ph}_{1,6}$	2.73, 0.73	16			
(013)	$\mathrm{Ph}_{1,4}$	2.39, 1.73	14	$\mathrm{Ph}_{1,5}$	2.39, 1.51	16
(014)	$\mathrm{Ph}_{1,3}$	1.93, 2.24	14	$\mathrm{Ph}_{3,4}$	2.24, 1.73	14
(015)	$\mathrm{Ph}_{2,3}$	2, 2.24	14			
(016)	$\mathrm{Ph}_{1,2}$	1, 2.65	16			
	$\mathrm{Ph}_{2,3}$	2.65, 1	16			
	$\mathrm{Ph}_{2,5}$	2.65, 1	16			
(025)	$\mathrm{Ph}_{4,5}$	1.73, 2.39	14	$\mathrm{Ph}_{1,5}$	1.51, 2.39	16
(027)	$\mathrm{Ph}_{5,6}$	2.73, 0.73	16			
(037)	$\mathrm{Ph}_{3,5}$	2.24, 1.93	14	$\mathrm{Ph}_{3,4}$	2.24, 1.73	14

Fig. 7 shows an (012) Tonnetz in $\mathrm{Ph}_{1,6}$ space. The (012)s are not very compact in the Ph_6 dimension, but this is optimal because it allows for two distinct ic1 intervals splitting the Ph_6 cycle in half. The upward-pointing and downward-pointing (012)s are enharmonically the same, but are distinguished by spelling, taking advantage of the redundancy.

6. Three-Dimensional Tonnetze

The idea of a Tonnetz as a triangulation gives new perspective on a persistent question of how to generalize the Tonnetz to tetrachords. Edward Gollin[22] proposed a three-dimensional network of tetrahedra as a seventh-chord Tonnetz. However, Gollin (and others) have also assumed that a Tonnetz must be specific to a single set class, which means that the (0258) tetrachords of Gollin's Tonnetz sometimes share a complete triangular face with another tetrachord, but most of the time link to one another in, at best, a shared edge, leaving a lot of empty space between tetrahedra. Gen-

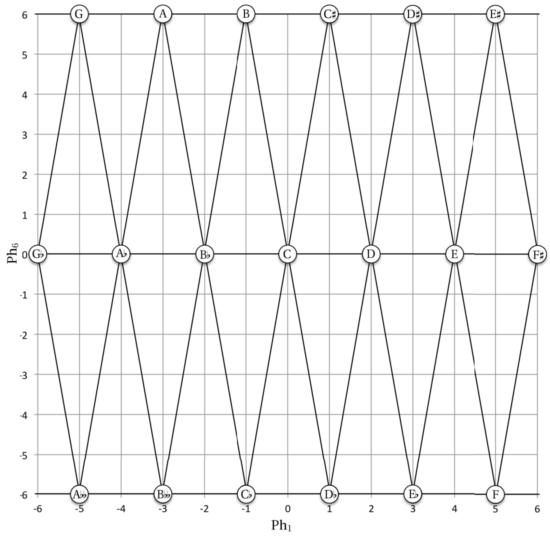

Fig. 7. An enharmonic (012) Tonnetz.

eralizing the idea of a complete triangulation of a toroidal space instead motivates discarding the requirement of a single tetrachord type.

The construction of a three-dimensional Tonnetz follows roughly the same procedure as the construction of the two-dimensional Tonnetz:

(1) Choose a three-dimensional phase space for $\mathbf{Z}_u$.
(2) Choose any line segment from pitch class 0 to some pitch class as an edge that does not pass through any other pitch class. Translate it to each pitch class in the space.
(3) Choose another line segment from 0 to some pitch class that is not parallel to and does not cross any of the previous line segments, and translate it to each pitch class.
(4) Repeat the previous step. The result will be a skewed regular cubic lattice partitioning the space into u regions (parallelepipeds).
(5) Choose a parallelepiped incident upon pitch-class 0. For each of its three faces, add a diagonal from 0, and translate these to each other pitch class in the space.
(6) Add diagonal for the entire parallelepiped from 0, and translate this similarly.

The entire process results in a total of seven distinct sets of intervallic axes defining the three-dimensional Tonnetz. This means that for $u = 12$, at least one interval type must be duplicated, since there are only six interval classes. Fig. 8 illustrates the process schematically through a particular choice of intervals. The cubic lattice of step (4) defines three planes and their intersections with three intervallic axes. These three intervallic axes (which correspond to interval classes) plus the choice of orientation for each (which correspond to intervals proper) determine the rest of the process, and hence the entire Tonnetz. Choosing the same interval classes with different orientations would amount to orienting the parallelepiped in Fig. 8 from a different corner, and would result in different Tonnetze. However, reversing orientations of *all* of the intervals (orienting from the F♯) results in the same Tonnetz.

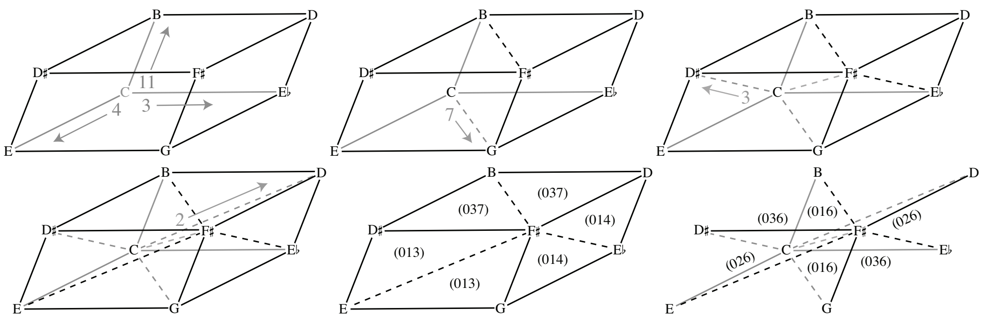

Fig. 8. Diagram (not to scale) of the construction of a three-dimensional Tonnetz.

Fig. 8 shows steps 5–6 of the process in three stages. The first interval (7) defines a new plane that cuts the parallelepipeds into two triangular prisms. It also completes a triangulation of the plane made by intervals 3 and 4 into a Tonnetz, which happens to be a triadic Tonnetz in this case. The next stage cuts (0147) tetrachords (BCD♯F♯ and CE♭F♯G) out of the corner of each triangular prism and also triangulates the plane defined by intervals 4 and 11. The new axis is a distinct axis associated with interval class 3—here understood as an augmented second, since it makes an (014) Tonnetz. Note that the D♯ in the upper left corner must be the same point in the space as the E♭ in the lower right corner, connected to C twice from different directions. The two planes that have been added at this stage intersect in the middle of the parallelepiped, in the tritone axis going through C–F♯. This tritone axis, corresponding to the interval added in step 6 of the process, also triangulates the two new planes. Finally, in the third stage of step 5 one more plane is added that cuts the each of the remaining pyramid-shaped regions each into two tetrahedra and also triangulates the plane made by intervals 3 and 11. This plane also intersects the others created in step 5 in the central tritone. The new tetrahedra correspond to (0137)—CEF♯G, BCDF♯—and (0236)—CD♯EF♯, CDE♭F♯—tetrachords.

The entire process thus defines three tetrachord types (each appearing in two inversions) to make a complete simplicial partition. It also creates, in total, six planes, each of which houses its own two-dimensional Tonnetz. The choice of intervals in Fig. 8 results in (037), (013), (014), (016), (026), and (036) Tonnetze. The number of planes could also be determined combinatorially by noting that each of the three tetrahedron types has four distinct faces, and each face is shared by two tetrahedra. Note also that the construction satisfies the requirement of a Euler characteristic of zero: u vertices, $7u$ edges, $12u$ triangles, and $6u$ tetrahedra: $\chi = u - 7u + 12u - 6u = 0$. The odd number of edges is perhaps at first surprising, until we note that the process creates two types of edges. The intervallic axes added in steps 4 and 6 belong to all tetrachord types, whereas those added in step 5 belong only to two out of the three (or four out of the six, if we distinguish the rotationally related tetrahedra corresponding to inversionally related tetrachords). Thus, four of the interval classes—minor third, major third, semitone, and tritone in the example—occur at the intersection of six tetrachords (all types) where three—augmented second, perfect fifth, and major second—occur at the intersection of four (two types). This also checks out combinatorially ($6 \times 6 = 4 \times 6 + 3 \times 4$).

The example given in Fig. 8 duplicates a single interval class, ic3. The duplication may be theoretically justified by a spelling distinction—the ic3s that occur in (037)s and (013)s are understood as minor thirds, whereas those occurring in (014)s are augmented seconds. The (036)s have both types, making them asymmetrical. The (0137) tetrachords may therefore be understood as key-defining diatonic subsets, the (0147)s as triad + leading-tone subsets of harmonic major/minor scales, and the (0236)s as scale segments containing an augmented second. One can thus imagine potential analytical uses and theoretical interest for this Tonnetz dealing with harmonic major and minor scales.

For a given space and universe, the number of possible Tonnetze is technically infinite, since the initial intervallic axes may be freely chosen and in a toroidal space any two points may be connected by an infinite number of lines. These may be reduced to a finite number by equating Tonnetze that result in the same set of tetrachords, but the number is still large and contains many degenerate examples. It is useful to classify the possibilities according to the number and kind of interval duplications required for each, since an excess of interval duplications results in impracticable examples. Table 2 lists seven such classes (A–G) for $u = 12$. Class A, with the fewest possible duplicated intervals, includes the example just discussed plus five others. In most cases the duplicated interval is ic3, but duplications of ic1 and ic5 are also possible. Class A Tonnetze are all based on one of the all-interval tetrachords, (0137) or (0146). Classes B, C, and D duplicate two intervals and omit one. They are subdivided on the grounds of what kinds of intervals are duplicated, whether they are what we might call primary (those belonging to all tetrachord types) or secondary (those belonging to just two types). In Classes B and C, both duplicated intervals occur once as a primary and once as a secondary interval (they differ only in how the intervals are distributed among trichords). In Class D, one duplicated interval occurs twice as a primary interval while the other occurs twice as a secondary interval. This group, characterized by duplicated tetrachords, is especially large. Class E and G Tonnetze both have a duplicated secondary interval in addition to a tripled interval, while class F and H have three different duplicated intervals. Classes E and G differ in that the duplicated interval is secondary in class E and primary in class G, while class H has two duplicated primary intervals and class F only one. The list includes every Tonnetz with no 0 intervals and with $\mathbf{Z_{12}}$ as its minimal universe.

Table 2 also gives one or two optimal spaces for each Tonnetz, using a modification of the procedure described above for the two-dimensional case. When there are three variants of an interval (classes E and G), two dimensions must be included that minimize the DFT of that interval (maximize its spread in that dimension).

A musical interpretation of any of these possible Tonnetze these may begin by identifying a musical distinction that can justify and take advantage of the interval duplications. For example, an (0235)-(0135) Tonnetz (class D) has duplications of ic2 and ic3 and two forms of (013) and (025) trichords depending on which form of ic2 and ic3 they contain. The different possible arrangements of major seconds also lead to two possible forms of (0135) and make the inversions of (0235) distinguishable. This is all true also of the traditional just intonation scales: they have two forms of major second, a larger "Pythagorean" whole step defined by the frequency ratio 9/8, and a smaller "just" whole step of ratio 10/9. When combined with a semitone, these make two kinds of minor third, Pythagorean (32/27), and just (6/5). Using abbreviations P and J for the two forms of whole tone

Table 2. List of three-dimensional Tonnetze.

Class	Properties	Tetrachords	Duplications	Omits	Opt. Space
A	One symmetric trichord	(0125) (0126) (0146)	ic1		$\mathrm{Ph}_{1,2,6}$
		(0237) (0157) (0137)	ic5		$\mathrm{Ph}_{2,5,6}$
		(0136) (0236) (0146)	ic3		$\mathrm{Ph}_{1,2,4}$
		(0147) (0236) (0137)	ic3		$\mathrm{Ph}_{2,3,4}$
		(0147) (0258) (0146)	ic3		$\mathrm{Ph}_{2,3,4}$
		(0136) (0258) (0137)	ic3		$\mathrm{Ph}_{2,4,5}$
B	Two symmetric trichords	(0124) (0125) (0135)	ic1, ic2	ic6	$\mathrm{Ph}_{1,3,6}$
		(0135) (0237) (0247)	ic5, ic2	ic6	$\mathrm{Ph}_{3,5,6}$
C	One duplicated trichord	(0126) (0127) (0157)	ic1, ic5, (016)	ic3	$\mathrm{Ph}_{1,2,6}/\mathrm{Ph}_{2,5,6}$
D	Duplicated tetrachord	(0134) (0236)×2	ic1, ic3, (013), (014)	ic5	$\mathrm{Ph}_{1,4,6}$
		(0145) (0125)×2	ic1, ic4, (014), (015)	ic6	$\mathrm{Ph}_{1,3,5}$
		(0156) (0126)×2	ic1, ic5, (015), (016)	ic3	$\mathrm{Ph}_{1,2,6}$
		(0156) (0157)×2	ic1, ic5, (015), (016)	ic3	$\mathrm{Ph}_{2,5,6}$
		(0235) (0135)×2	ic2, ic3, (013), (025)	ic6	$\mathrm{Ph}_{1,3,5}$
		(0235) (0136)×2	ic2, ic3, (013), (025)	ic4	$\mathrm{Ph}_{1,2,4}/\mathrm{Ph}_{2,4,5}$
		(0347) (0147)×2	ic3, ic4, (014), (037)	ic2	$\mathrm{Ph}_{2,3,4}$
		(0158) (0237)×2	ic4, ic5, (015), (037)	ic6	$\mathrm{Ph}_{1,3,5}$
		(0358) (0258)×2	ic3, ic5, (025), (037)	ic1	$\mathrm{Ph}_{4,5,6}$
E	Duplicated tetrachord	(0145) (0148)×2	ic4 (×3), ic1, (014), (015)	ic2, ic6	$\mathrm{Ph}_{1,3,5}$
	with augmented triad	(0347) (0148)×2	ic4 (×3), ic3, (014), (037)	ic2, ic6	$\mathrm{Ph}_{2,3,4}$
		(0158) (0148)×2	ic4 (×3), ic5, (015), (037)	ic2, ic6	$\mathrm{Ph}_{1,3,5}$
F	Duplicated tetrachord	(0134) (0124)×2	ic1, ic3, ic2, (013), (014)	ic5, ic6	$\mathrm{Ph}_{1,3,6}$
	with symmetric trichords	(0358) (0247)×2	ic5, ic3, ic2, (025), (037)	ic1, ic6	$\mathrm{Ph}_{3,5,6}$
G	Duplicated symmetric	(0123) (0124)×2	ic1 (×3), ic2, (012), (013)	ic5, ic6	$\mathrm{Ph}_{1,3,6}$
	trichord	(0257) (0247)×2	ic5 (×3), ic2, (025), (027)	ic1, ic6	$\mathrm{Ph}_{3,5,6}$
H	Quadrupled trichord	(0167) (0127)×2	ic1, ic5, ic6, (016)×4	ic3, ic4	$\mathrm{Ph}_{1,2,5}$
I	Tripled tetrachord	(0123)×3	ic1×3, ic2×3	ics4,5,6	$\mathrm{Ph}_{1,3,4}$
		(0257)×3	ic5×3, ic2×3	ics1,4,6	$\mathrm{Ph}_{3,4,5}$

and *s* for a semitone, the tetrachords are, in consecutive intervals, P*s*J, J*s*P, *s*JP, *s*PJ, PJ*s*, and JP*s*. The traditional just major and minor scale patterns, PJ*s*PJP*s* and P*s*JP*s*PJ, contain all of these possible strings plus one other type that defines the bad fourth, P*s*P. Just scales may therefore be represented as paths in this Tonnetz that produce the sequence of overlapping tetrachords between the bad fourth and augmented fourth.

Duplications may also be used to fold the spaces. The Tonnetze of classes D–G, for instance, can be folded to equate their duplicated tetrachords. Such a folding of the (0258)-(0358) Tonnetz produces a seventh-chord Tonnetz described by Jack Douthett in an unpublished letter written in 1997.[23] The folding is over the whole-tone planes that house (026) trichords, turning these into boundaries. One way to imagine this space is to draw two (026) Tonnetze, as in Fig. 1, parallel to one another in a fattened

2-torus, one using the even whole-tone scale and one using the odd. Then connect all pitch classes from one plane to the other related by a consonant interval (ic3 or ic5, four connections for each pitch class) so as to link each pitch class to two adjacent (026)s on the opposite boundary, making overlapping dominant and half-diminished sevenths. For example, D on the even boundary connects to {G, B, F} and {B, F, A} on the odd one. According to Table 2, the optimal space for the (0358)-(0258) Tonnetz is $\text{Ph}_{4,5,6}$. The Ph_6 dimension is included for disambiguating the duplicated ic3s and ic5s. This dimension is therefore the one whose cycles are removed by the folding, so that it becomes simply a means of separating the two whole-tone planes between the boundaries of the fattened 2-torus. Therefore, this optimal realization of Douthett's Tonnetz can be accurately pictured through the projection onto $\text{Ph}_{4,5}$ space in Fig. 9. Each point shows the position of one tetrahedron, with dominant and half-diminished sevenths alternately pointing up and down, depending on their whole-tone affinity, and minor seventh chords connecting a major second on one plane to an obliquely directed major third on the other. Lines connect chords that share a face. The minor sevenths are fully connected, while the (0258)s each have one face on a boundary. Douthett's schematic diagram (Fig. 10) illustrates this.

After folding, the only duplications remaining in this Tonnetz are of the tritones on the boundaries. Cutting across the space are (036) Tonnetze with boundaries, making 2-dimensional Tonnetz-bands like the one in the middle panel of Fig. 6. Identifying the duplicated tritones in these (036) Tonnetze make them spherical, as also shown in Fig. 6. This identification eliminates the last of the interval duplications. It also turns the boundaries into a new set of tetrachords: note in Fig. 1 that the (026) Tonnetz actually connects every note of the whole-tone collection to every other. The distinction between tritones is the only reason that tricords like {F♯, G♯, C} and {C, D, F♯} do not combine into a single tetrachord. Identifying the tritones folds each boundary into a set of three (0268) tetrahedra. The resulting space is the one described by Tymoczko[4] as the voice-leading Tonnetz for four-note chords, topologically the direct product of a sphere and a circle. As with the augmented triad in Tymoczko's voice-leading Tonnetz for three-note chords (Fig. 2), the (0369) tetrachords have a special status in this space. Although they are fully connected in the network created by the triangulation, they are not properly elements of the triangulation as tetrachords. For Tymoczko they are essential intermediaries between dominant and half-diminished sevenths sharing three notes, so that the voice

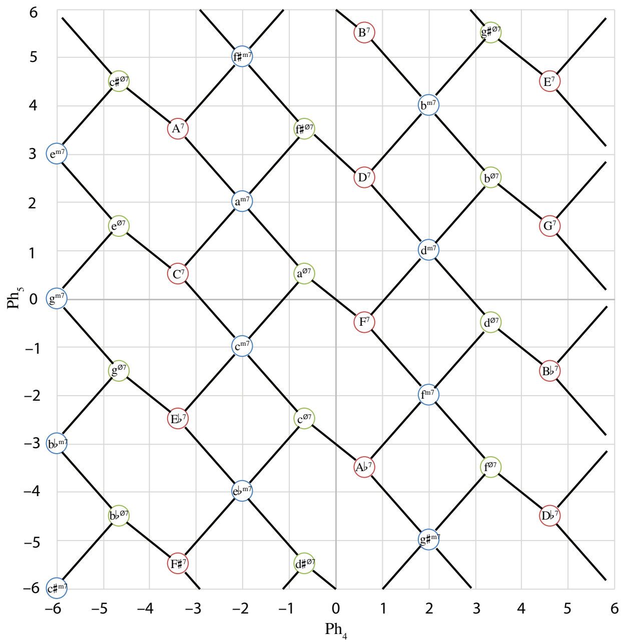

Fig. 9. The dual of Douthett's seventh-chord Tonnetz projected onto $\mathrm{Ph}_{4,5}$-space.

leading is entirely semitonal. But in the triangulation their status is equivalent to that of the two-dimensional planes that house trichordal Tonnetze made up of similar 2-faces of the triangulation, not of the tetrahedral (3-simplex) elements of the triangulation. The situation is analogous to that of Tymoczko's two-dimensional triadic Tonnetz, where the augmented triad, which he counts as an essential triangle (2-simplex), is, in the toroidal triangulation, an intervallic axis that happens to be limited to three elements. This distinction is essentially homological: the elements of the simplicial decomposition can be contracted to points in the space. Other maximal cliques, included in Tymoczko's networks but not in the simplicial decomposition, cannot.

7. The n-Dimensional Generalization

The construction defined for two and three dimensions above can be generalized to n dimensions:

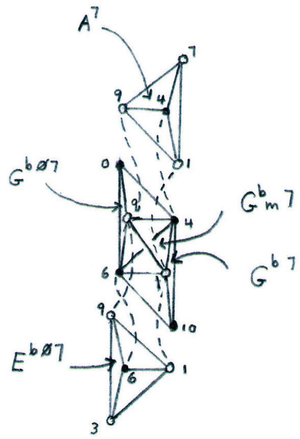

Fig. 10. Douthett's drawing[23] of the construction of a seventh-chord Tonnetz.

(1) Choose n distinct, linearly independent line segments, $a, b, c, \ldots$, extending from pitch-class 0 to some other pitch class, which do not pass through any other pitch-classes.
(2) Translate these u times (to each pitch class), making a skewed regular hypercubic lattice partitioning the space into u regions.
(3) Add the $\binom{n}{2}$ line segments obtained by taking vector sums of any two of $a, b, c, \ldots$ and extending these from pitch-class 0. These will triangulate each 2-face of the hypercube. Translate these to each pitch class to triangulate all of the planes of the hypercubic lattice.
(4) Continue this process for each m, $1 < m \leq n$, taking vector sums of m of $a, b, c, \ldots$, adding these to from each pitch-class to complete a simplicial decomposition of each m-face of the hypercubic lattice.

Note that the pitch classes used in the process need not be unique—one of the line segments may even connect a pitch class to itself. Therefore degenerate examples such as Tonnetze on a one-pitch-class universe are possible. The number of edges defined in this process is

$$e = \sum_{i=1}^{n} \binom{n}{i} u = (2^{n-1} - 1)u \qquad (1)$$

where $\binom{n}{i}$ gives the number of i-faces of the hypercube incident upon a given vertex.

For example, in the three dimensional case we can define a skewed cubic lattice with three intervals, a, b, c, triangulate the faces with intervals $a+b, b+c, a+c$, then complete the triangulation of the 3-faces, the skewed cubes themselves, with $a+b+c$. The tetrahedra correspond to all the permutations of a, b, and c. To get the augmented-second Tonnetz described above, let $a \to 3$ (a points from 0 to 3), $b \to 4$ (etc.), and $c \to 11$. Then $a+b \to 7$, $b+c \to 3$, $a+c \to 2$, and $a+b+c \to 6$. The tetrahedra may be constructed by stringing together a, b, and c in any order. In our example, abc and cba correspond to tetrachord type (0147), acb and bca to (0236), and bac and cab to (0137). We get the same Tonnetz by either negating all of a, b, and c, or exchanging any of a, b, or c with $(-a-b-c)$.

For $n = 4$, we get a Tonnetz of $24u$ 4-simplexes representing pentachords (12 types occurring in all transpositions and inversions). The number of intervallic axes is 15, so the number of duplications for $u = 12$ would be necessarily quite large.

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