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Generalized *Tonnetze* and *Zeitnetz*, and the Topology of Music Concepts

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The music-theoretic idea of a *Tonnetz* can be generalized at different levels: as a network of chords relating by maximal intersection, a simplicial complex in which vertices represent notes and simplices represent chords, and as a triangulation of a manifold or other geometrical space. The geometrical construct is of particular interest, in that allows us to represent inherently topological aspects to important musical concepts. Two kinds of music-theoretical geometry have been proposed that can house *Tonnetze*: geometrical duals of voice-leading spaces, and Fourier phase spaces. Fourier phase spaces are particularly appropriate for *Tonnetze* in that their objects are pitch-class distributions (real-valued weightings of the twelve pitch classes) and proximity in these space relates to shared pitch-class content. They admit of a particularly general method of constructing a geometrical *Tonnetz* that allows for interval and chord duplications in a toroidal geometry. The present article examines how these duplications can relate to important musical concepts such as key or pitch-height, and details a method of removing such redundancies and the resulting changes to the homology the space. The method also transfers to the rhythmic domain, defining *Zeitnetze* for cyclic rhythms. A number of possible *Tonnetze* are illustrated: on triads, seventh chords, ninth-chords, scalar tetrachords, scales, etc., as well as *Zeitnetze* on a common types of cyclic rhythms or timelines. Their different topologies – whether orientable, bounded, manifold, etc. – reveal some of the topological character of musical concepts.

Keywords: *Tonnetz*, topology, geometry, Fourier transform, voice leading

2010 Mathematics Subject Classification:
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1. Definitions of *Tonnetze*

For the purpose of this article, I will define a *Tonnetz* very generally according to the maximum-overlap or common-tone principle, following the suggestion made in a well-known article by Richard Cohn (1997). However, it will be useful to think of *Tonnetze* in turn at different levels of abstraction, as networks of pitch-class sets, as topological objects, or as geometrical objects. The network construct is the most general, the geometrical one the most specific. In addition, while the definitions are set out initially with pitch-class sets in mind (subsets of the twelve-note scale understood as integers mod 12), they may also be applied to beat-class sets, looped rhythms understood as subsets of some periodically repeating set of evenly spaced time points. We will therefore construct *Tonnetze* in any possible *universe*, u, where this can refer to any equal division of an octave or equal division of a time-cycle. Beat-class sets bring other possible universes into play in a way that may lead to a wider range of applications than considering, e.g., other

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possible equal divisions of the octave. When generalized this way, the term "*Tonnetz*" is not entirely appropriate, so I will call this a "*Zeitnetz*." All of the following definitions may be modified to apply to *Zeitnetze* by replacing "pitch class" everywhere by "beat class." The two constructions are mathematically indistinguishable.

We begin from what is usually referred to as the "dual *Tonnetz*," in which nodes represent chords rather than tones, which I will call the *Tonnetz graph*.

Definition 1.1 A cardinality-n *Tonnetz graph* is a connected graph whose vertices correspond to pitch-class sets of cardinality n in some universe u, such that two vertices may share an edge only if their corresponding sets share $n-1$ elements in common, and the entire network maps onto itself under any transposition of universe u.

The last criterion, transposability, implicitly requires that, for any pitch-class set in the network, all of its transpositions must also be in the network.[1] However, it allows for the network to contain multiple distinct set classes. The other defining feature is that the connections reflect only the maximum-overlap relationships. The definition does allow for a possible maximum-overlap relationship to be excluded, because there may be geometric reasons for such an exclusion, as we will see. However, the connectivity condition may often require that all available connections must be included, in which case we can deduce a unique *Tonnetz* graph given just the set types (transposition-types) it contains. This also means that the choice of set types is not entirely arbitrary, because there must be a way to relate every type to every other through a series of maximum-overlap relationships.

Tonnetz graphs are common currency in much recent mathematical music theory, and therefore their theoretical value appears to be well recognized. What is often described as the dual of the standard triadic *Tonnetz*, or "chicken-wire torus" (Douthett and Steinbach), is a *Tonnetz* graph. The standard *Tonnetz* only involves one chord type and its inversion (major and minor triads) and this is often assumed to be an essential property for generalizing the *Tonnetz* – in that sense the above definition is unusually broad. Assuming only one set class, however, is too restrictive when we move beyond the case of three-note sets. The definition of *Tonnetz* graphs also encompasses a wider range of networks that have been proposed such as those of Douthett and Steinbach (1998), and Žabka (2014), all of which are *Tonnetz* graphs. Similarly, Tymoczko's 2011; 2012 "chord-based networks" are *Tonnetz* graphs. In much of this work (particularly in Tymoczko's), an additional constraint is put upon the networks, that adjacent chords must be close in a voice-leading sense. That is, the unique non-common elements must be a "step" apart in pitch-class space (one semitone, one or two semitones, or a generic scale-step, depending on the context). That restriction is lifted here, but we will find that under certain conditions it emerges naturally.

Historically a *Tonnetz* is a network of tones, where chords are represented by groups of connected pitch classes. By mapping the chords in a *Tonnetz* graph to simplices in a network of tones, we can define a *simplicial Tonnetz*.

Definition 1.2 A cardinality-n *simplicial Tonnetz* is a simplicial complex whose nodes are pitch-classes in some universe u, such that a *Tonnetz* graph is given by mapping n-simplices to nodes and connecting nodes that overlap in $(n-1)$-simplices, and such that every node and edge belong to some n-simplex participating in this mapping.

[1] An invertibility condition similar to the transposability condition might also be added to this definition, so that the *Tonnetz* graph must include all representatives of each $\mathrm{T}_n\mathrm{I}$-type set class. All the *Tonnetze* and *Zeitnetze* appearing as examples below would satisfy such a condition.

Generalizations of the *Tonnetz* as a simplicial complex have been studied by Bigo et al. (2013, 2015) and Cantazaro (2011), under somewhat more general definitions than the one proposed here, which fixes the cardinality of the maximal simplices and inherits transposability from the *Tonnetz* graph. It is clear that a *Tonnetz* graph can map to a simplicial *Tonnetz* in a canonical way, and vice versa, so the simplicial *Tonnetz* is essentially the same level of generalization as the *Tonnetz* graph. However, geometrical realizations of generalized simplicial *Tonnetze* are a more specific construction that has been less studied (examples include Tymoczko 2012 and Yust 2018a):

Definition 1.3 A cardinality-n *geometrical Tonnetz* is an embedding of a cardinality-n simplicial *Tonnetz* in some n-dimensional geometric space, such that the pitch classes are each associated with unique points in the space, all transpositions correspond to rigid transformations of the space, and the regions defined by the n-simplexes are disjoint and cover the space – i.e., they constitute a triangulation of the space.

The geometrical *Tonnetz* realizes the transposability condition for *Tonnetz* graphs geometrically, as rigid transformations. Since the number of simplices is finite, the space must be compact. The shift from the simplicial to geometrical *Tonnetz* is momentous where musical interpretation is concerned, because through a musical interpretation of the geometry we may associate specifically geometrical constructs such as paths and regions with musical concepts. Two significant ways of doing this have been proposed: through voice leading (Callender, Quinn, and Tymoczko 2008; Tymoczko 2012) and through harmonic quality (Quinn 2006; Amiot 2013, 2016; Yust 2015b) as defined by the discrete Fourier transform on pitch-class sets. With the latter type of geometry, a very general method of construction is defined in Yust 2018a that can embed any simplicial Tonnetz in a geometric space, specifically a *phase space*. To illustrate the main points of interest – the two ways of defining musical geometries, the methods of folding, and the music-conceptual significance of each of these – we begin by working carefully through the specialized cases with a high degree of redundancy, generated collections. These methods are then applied to two-interval generated collections, which include a number of examples of great musical interest (including the standard triadic *Tonnetz*).

2. Generated collections

2.1. *Dyadic Tonnetze*

At the extreme of intervallic redundancy are generated collections, sets generated by multiple iterations of a single interval. Generated collections are special in that they are accurately captured by a *Tonnetz* graph in the form of a plain cycle graph with a single set type. This is true of all dyadic *Tonnetze*, which are trivially generated. So, for example, a *Tonnetz* graph for ic1 dyads would look like:

$$\ldots — \{B,C\} — \{C,C\sharp\} — \{C\sharp,D\} — \{D,E\flat\} — \{E\flat,E\} — \ldots$$

Its associated simplicial *Tonnetz* is a one-dimensional cyclic space, whose edges correspond to the nodes of the *Tonnetz* graph, overlapping in individual vertices:

$$\ldots. — B — C — C\sharp — D — E\flat — \ldots$$

Similarly a *Tonnetz* graph of perfect fourths/fifths,

$$\ldots — \{C,G\} — \{G,D\} — \{D,A\} — \{A,E\} — \ldots$$

has an associated simplicial *Tonnetz*:

$$\ldots — \mathrm{C} — \mathrm{G} — \mathrm{D} — \mathrm{A} — \mathrm{E} — \ldots$$

Recall that at this level, the *Tonnetze* are understood as simplicial complexes – in this case made up of 1-simplices or edges. To turn them into geometrical *Tonnetze* is essentially an act of interpretation, assigning a geometrical interpretation to the circle, in this case the pitch-class circle and the circle of fifths.

One interpretation is offered by the application of the discrete Fourier transform (DFT) to pitch-class sets, which is discussed at length elsewhere (Quinn 2006; Amiot 2013, 2016; Yust 2015a,b, 2016, 2017a): these can be understood as *phase spaces*, where the position on the cycle represents the phase value of a certain Fourier component, denoted $\mathrm{Ph}_{k/u}$ for component k in universe u, or simply Ph_k where the universe is understood (e.g. for pitch-class sets, $u = 12$). It is not necessary to re-introduce all of the mathematics of the Fourier transform here: for present purposes, we can understand each Ph_k as an interval cycle in the sense of pitch-class set theory (Perle 1977; Straus 2016), where Ph_1 is the integers mod u and Ph_k is given by multiplying the pitch-class numbers by k mod u. For consistency with previous publications (reflecting the standard definition of the DFT) Ph_1 is treated throughout as the inverse of the pitch-class circle, so that ascent by 1 in Ph_1 corresponds to descent by semitone (or going back by one beat class). Other pitch-class sets, or multisets or weighted pitch-class sets, can then be located in the phase space by taking circular averages of their constituent pitch-classes. This gives the space a robust geometry: points in-between pitch classes have specific meaning. In the circle of fifths, or Ph_5, for example, the point halfway between C and G represents the C-G dyad.[2] To get a dense topology, we consider not only regular pitch-class sets, but also weighted ones, AKA pitch-class *distributions*. Weightings may have multiple musical applications, a common one being to represent the probability or frequency of appearance of a pitch class within a certain timespan (Huron 2006; Temperley 2007). The phase spaces provide a one-to-one complete list of ways of arranging the pitch classes that are true to the group-theoretic properties of the transposition group $\mathbf{Z}_{12}$ (or more generally, $\mathbf{Z}_u$) in the sense that the pitch classes map uniquely to points in the space, and transposing them corresponds to some rigid transformation of the space.

Another way to give geometrical meaning to these cycles is through Callender, Quinn, and Tymoczko's (2008) idea of voice-leading spaces. We might first note that the pitch-class circle is the basic one-dimensional voice-leading space, the space of log-frequency modulo the octave.[3] This is a different interpretation than Ph_1: a point half-way between C and C♯, e.g., represents a C quarter-sharp rather than the dyad {C,C♯}. However, this interpretation is not really native to the theoretical universe of the *Tonnetz*, where the simplices are usually taken to represent something like chords or collections, because the ic1 dyads in the one-dimensional voice-leading space represent segments of the log-frequency continuum, not dyadic collections. A more appropriate derivation from voice-leading geometries begins from the two-dimensional voice-leading space. By taking the geometric dual of this space, as proposed by Tymoczko (2012), we get points that correspond to individual pitch-classes. Here it is more appropriate to consider the second example above, the circle of fifths. Each point in the space represents a line in the dyadic voice-leading space, the line corresponding to all voice leadings that hold a particular

[2]This relationship, however, N.B., is not one-to-one. The point representing {C,G} in Ph_5 also represents {F,D}. The relationship only becomes one-to-one if we consider *all* possible phase spaces $0 \leq k \leq u/2$ and the magnitudes of each of these Fourier coefficients.

[3]Thoughout, "voice-leading spaces" will refer to what Callender, Quinn, and Tymoczko (2008) specifically call "OP-space." That is, it is voice-leading space folded to recognize octave and permutational equivalences.

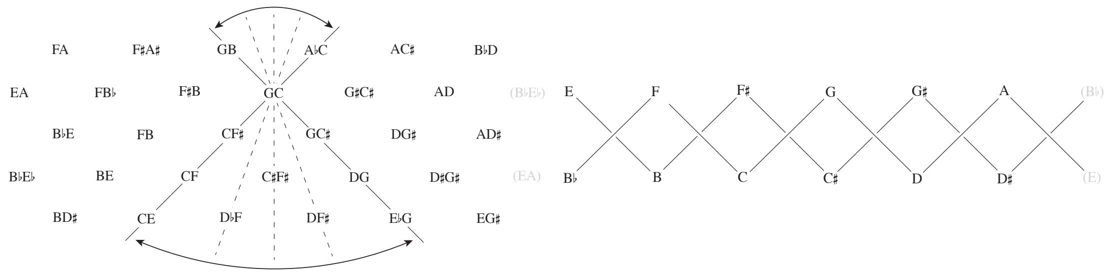

Figure 1. Rotation around a perfect-fifth dyad in voice-leading space, and the *Tonnetz* space as the geometrical dual of dyadic voice-leading space

pitch-class constant in one voice while moving the other voice. Motion between these represents a rotation of this line around a fixed dyad (some perfect-fifth dyad). This is illustrated on the left side of Figure 1. The point half-way between C and G on the circle of fifths then corresponds to a different line, one that gives a wedge voice leading (or *balanced* voice leading) around the perfect-fifth dyad.

The circle of fifths is not actually smooth according to this interpretation. The dual of the dyadic voice-leading space is shown on the right of Figure 1. Points in this space are lines in the voice-leading space (with pitch classes again indicating lines that hold a pitch-class constant), and motion to the left and right here corresponds to sliding the line along the axis of transposition. Perpendicular to this (up and down) are rotations of the line around the point where they cross the center of the space (the tritone axis). Diagonal lines combine both of these kinds of motion, equivalent to rotating around some other fixed point. These spaces are dual in the sense that lines in the *Tonnetz* space correspond to points in the voice-leading space just as lines in the voice-leading space correspond to points in the *Tonnetz* space. The circle of fifths is a zig-zag line that circles the *Tonnetz* space twice, changing direction each time it hits a pitch class. A similar zig-zag line could be defined for the ic1 intervals, but it would be an extremely inefficient way to move through the space, circling almost the entire space for each dyad, and crossing multiple other ic1 lines.[4] Following the line of transposition, on the other hand, would give a very efficient representation of the pitch-class circle, but does not actually reflect the idea of holding a pitch class constant as we go from one ic1 dyad to the next, which is essential to the idea of the *Tonnetz* as a network of common-tone relationships. Voice leading therefore prioritizes certain possible *Tonnetze* over others in a way that phase spaces do not.

Tymoczko (2012) does not actually consider the *Tonnetz* of Figure 1 as a candidate for a dyadic voice-leading space, because its two-semitone voice leadings are not minimal. Note that when the line segment corresponding to a fifth in the dual space (the right side of Figure 1) rotates from one fifth to another, it passes through the tritone. For example, between {F,C} and {C,G} is {F♯,C}. We can also see this in the voice-leading space itself. According to the logic of voice leading, then, Figure 1 presents an incomplete, one might even say inaccurate, picture. Voice leading from one fifth to another with a common tone *necessarily* passes through an intermediate tritone, and the voice leading {F,C}→{B,F♯} is the same size, or smaller, than {F,C}→{C,G}, not six times as large, as it is in our *Tonnetz*. For this reason, Tymoczko only considers the possibility of *Tonnetze*

[4]A common-tone voice leading between ic1 dyads that is equally efficient to the one on ic5 dyads is in fact possible (moving one note by two semitones at each stage) but it involves a voice crossing. Stated with respect to the dual space, we can say that edges can only rotate through a vertical axis (corresponding to the wedge voice leading) not the horizontal axis (corresponding to transposition). Rotations of the latter type correspond to a voice crossing).

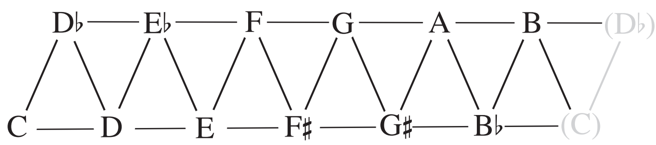

Figure 2. Simplicial *Tonnetz* of chromatic trichords

that include the most even chord types (such as tritones) before including the next-most even and so forth. This highlights a crucial difference with Fourier phase spaces: In Fourier spaces, the tritone is not necessarily an intermediary between the two fifths; this would be true in Ph_2-space, which reflects, in a certain sense, the logic of dyadic voice leading. But it is not true in Ph_5. The intermediary between {F,C} and {C,G} in *all* phase spaces, including Ph_5, is the shared pitch class C.

Whether one includes tritones or not, the voice-leading interpretation gives geometrical meaning to a *Tonnetz*, but does not technically satisfy the definition of a geometrical *Tonnetz* given in section 1, because the regions corresponding to the perfect-fifth 2-simplexes are not disjoint: tritone-related perfect fifths cross at the wedge voice leading that relates them. If the size of the universe were odd (hence sharing no factors with 2), this would not occur and voice leading would give a good *Tonnetz* geometry on maximally even dyads (for instance, fifths in a generic-diatonic, $u = 7$, universe).

Because of the relative simplicity of phase spaces as a geometrical realization of *Tonnetze* (with pitch classes and higher-cardinality chords in the same space rather than dual spaces) and because the relationship of phase spaces to interval cycles gives this interpretation a valuable level of generality, we will use phase spaces for the remainder of this section and return to voice leading in sections 4 and 5.

2.2. *Generated trichordal Tonnetze*

As we increase the cardinality of the generated collection, we can continue to use the same form of *Tonnetz* graph. For instance, the chromatic trichord has a graph:

$$\ldots — \{\text{C,C}\sharp\text{,D}\} — \{\text{C}\sharp\text{,D,E}\flat\} — \{\text{D,E}\flat\text{,E}\} — \{\text{E}\flat\text{,E,F}\} — \ldots$$

And a chromatic tetrachord:

$$\ldots — \{\text{C,C}\sharp\text{,D,E}\flat\} — \{\text{C}\sharp\text{,D,E}\flat\text{,E}\} — \{\text{D,E}\flat\text{,E,F}\} — \{\text{E}\flat\text{,E,F,F}\sharp\} — \ldots$$

And so forth. A simplicial complex of chromatic trichords might look like Figure 2. Despite a simple cyclic *Tonnetz* graph, the simplicial *Tonnetz* requires a second dimension, bounded by the ic2s, the intervals unique to a given chromatic trichord, because a 2-simplex (triangle) requires a space of a minimum of two dimensions. To embed this simplicial complex in a geometry raises the question of what this second dimension might represent musically.

One way to address this question is a procedure I suggest in Yust 2018a: begin by defining a *Tonnetz* for the chromatic trichord in a two-dimensional phase space that has duplications of ic1, and then eliminate the duplications by defining a folding, resulting in a space like Figure 2 that inherits geometric meaning in the bounded dimension from the original phase space. A trichordal Tonnetz can be embedded in a two-dimensional phase space Ph_{k_1,k_2} (the direct product $\text{Ph}_{k_1} \times \text{Ph}_{k2}$) given almost any choice of k_1 and

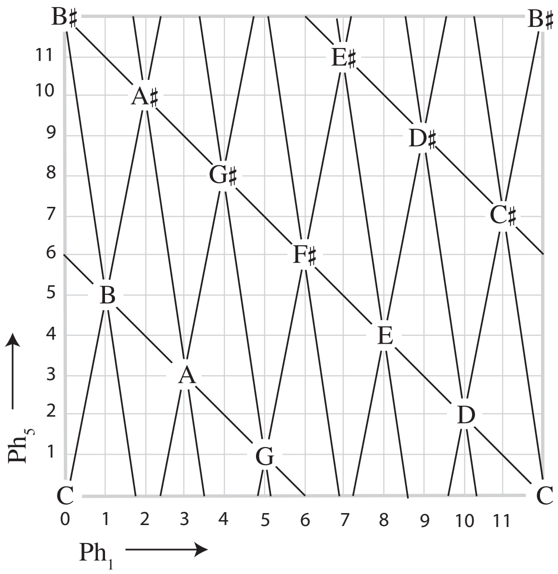

Figure 3. Geometrical *Tonnetz* of spelled chromatic trichords in $\mathrm{Ph}_{1,5}$-space

k_2,[5] but certain choices are better than others. Here, it will be useful for Ph_{k_1} to order the pitch classes by the generating interval, for reasons that will become clear shortly, so we choose $k_1 = 1$. For the other dimension, any value of k_2 will do, but since the function of this dimension is to split apart the two whole-tone collections, a good choice will be one that does so most effectively. The ideal choice from this perspective would be Ph_6, but Ph_6 (or $\mathrm{Ph}_{k/u}$ with $k = u/2$) is degenerate in a 12-tone universe: it can logically take only two values (0 and 6), so it is not really possible to ascribe any meaning to a Ph_6 path (except by "oversampling" to a larger universe). The next best choice is Ph_5, which represents the average circle-of-fifths position of a collection. Figure 3 shows a $\mathrm{Ph}_{1,5}$ space triangulated by chromatic trichords. The simplicial Tonnetz embedded in this geometry has twice as many 2-simplexes than the one in Figure 2, two for each chromatic trichord, and it is topologically toroidal, with two cyclic dimensions rather than just one.

A simplicial Tonnetz *triangulates* a phase space, partitioning it into regions, such that each region is associated with a trichord (or, more generally $(n+1)$-note chord) from the set of trichords belonging to the given *Tonnetz*. These regions correspond to the all of the points belonging to the given trichord, in the sense that each possible weighting of the notes in the trichord corresponds to some point within the given region.[6] For a generated trichord, such as the chromatic trichords of Figure 3, there are actually two such regions, because there is an orientational ambiguity in one dimension. For instance, in $\mathrm{Ph}_{1,5}$ space, the note F is an equal distance above or below the dyad {E,F♯}. If the trichord {E,F,F♯} is weighted more towards E than F♯, then it will fall in a region below the {E,F♯} dyad. If it is weighted more towards F♯, then it will fall in a region above. Alternately, we could imagine adding other notes to provide the context that distinguishes the two versions of the trichord. In the presence of, say, {C♯,D♯,G♯}, the chromatic trichord is {E,E♯,F♯}. In

[5] The exception is a choice such that k_1, k_2, and u are not mutually prime; see Yust 2015b. Ideally we should also apply some constraints on the choice of k_1 and k_2 relating to the compactness of regions, but determining exactly how this should be done would be somewhat involved and therefore beyond our present scope.

[6] Our drawings of the regions, with straight lines as boundaries, however, are not completely accurate and should be seen as an idealization. In fact, if the interval defining a given boundary has a larger magnitude in one dimension than the other, this imparts a sinusoidal curve to the boundary. In practical situations, the amount of curvature is typically slight and therefore can be safely disregarded.

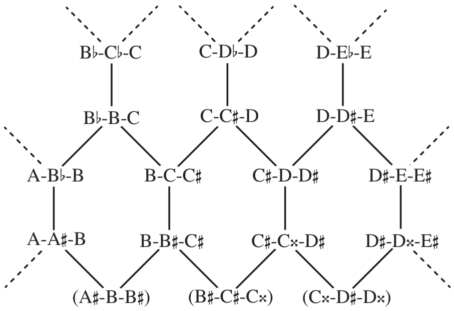

Figure 4. *Tonnetz* graph of spelled chromatic trichords

the context of {G,A,D} it is {E,F,F♯}.

Values of Ph_5, average circle-of-fifths positions, are of significant musical interest: Amiot (2017a) shows that the size of its associated Fourier coefficient, f_5, is the best measure of the *diatonicity* of pitch-class sets,[7] and it is overwhelmingly the most prominent harmonic quality of pitch-class distributions that occur across tonal music (Yust 2017b, 2019). For this reason, Ph_5 is the principal criterion for distributional key-finding algorithms and models of key perception (Cuddy and Badertscher 1987; Krumhansl 1990; Yust 2017b). Given this relationship to keys, Ph_5 paths associated with ic1 distinguish between diatonic semitones – the shorter way around the Ph_5 cycle and the one that occurs within a diatonic collection – and chromatic semitones, which go the longer way around the Ph_5 cycle and span beyond the limits of a single diatonic collection.[8]

In the *Tonnetz* of Figure 3, each chromatic trichord is made up of both kinds of semitone, one that corresponds to -5 in Ph_5 and one that corresponds to $+7$ in Ph_5, which sum to the regular whole tone: $(-1,-5)+(-1,7) = (-2,2)$. Because there are two ways to order the chromatic and diatonic semitones, there are two kinds of chromatic trichord. Therefore the *Tonnetz* graph corresponding to Figure 3 is not the same as the one corresponding to the simplicial complex in Figure 2; the *Tonnetz* graph for this space is the one shown in Figure 4, with 24 vertices and cyclic in two dimensions. Note that the sets at the bottom of the figure in parentheses are equivalent to the ones at the top: the identities of the pitch classes are still enharmonically identified. The distinctions are made only with respect to the intervals within the pitch-class sets. B♭-C♭-C is therefore distinct from B♭-B-C but equivalent to A♯-B-B♯. The graph has two kinds of cycles, one that goes through all of the pitch classes following the horizontal zig-zags (a cycle of twelve elements), and one that alternates swapping the semitone types (vertical edges) with left and right moves on the horizontal cycles (a cycle of four elements).

In the toroidal geometry, the two kinds of chromatic trichord occupy different regions of the space, with distinct boundaries (overlapping in the shared whole-tone interval)

[7]Dmitri Tymoczko (personal communication, Oct. 2018) has challenged this point with the argument that diatonicity is not an intuitive description of f_5 in other universes (say, for quarter-tones, $u = 24$). The obvious response to this is that f_5 has only been equated with diatonicity in contexts where 12-tET is taken for granted – in other words, "diatonicity" or Quinn's (2006) "diatonicness" has only ever referred to $f_{5/12}$, not $f_{5/u}$ for all u. Nonetheless, Tymoczko's challenge raises an interesting point, which is that certain concepts related to harmonic qualities should generalize away from a particular discretization of the octave. This is true of diatonicity: we have an intersubjectively robust intuitive sense of diatonicity in, say, 31-tone equal temperament. The appropriate generalization to capture this for $u > 12$ is $|f_{7/u}||f_{12/u}|/|f_0|^2$. That is, diatonicity is determined, generally, by the combined size of two components, f_7 and f_{12} (normalized by the total power, f_0^2). The special status of these particular components ultimately comes, presumably, from the fact that they give the best approximations to the acoustic perfect fifth, $\log_2(3/2)$, so an alternate, more basic, definition might be given by the correlation with the spectrum of the perfect fifth with f_0 (and values of k exceeding some threshold) excluded.

[8]For more on this distinction see Yust (2015b, 2018b).

but the same vertices. The same vertices can bound totally different regions of space because Ph_5 is cyclic. The differences between two trichords with the same pitch classes is contingent upon the meaning of Ph_5: to the extent that this is relevant, the distinction is relevant. This would be true given a tonal context: for instance in a context of A major, the chromatic trichord {E,E♯,F♯} has a different meaning (a move to the sharp side, tonicization of the relative minor) than {E,F,F♯} (a move to the flat side, mode mixture). For composers of the nineteenth century, orientations along the circle of fifths can have crucial musical meaning (see, for example, analyses of Schubert's tonal plans in Yust 2015b, 2018b, Forthcoming). In such contexts, spelling distinctions are used to indicate orientations around the circle of fifths, which is how they are commonly, though not exclusively, used in tonal music.[9] These Ph_5 distinctions might not be meaningful, in other contexts, such as in Webern's music. In that case we want a different kind of geometry, one that embeds the simplicial Tonnetz of Figure 2. We can derive such a geometry by folding a toroidal one like $\mathrm{Ph}_{1,5}$-space.

2.3. *Mathematical definition of phase-space foldings*

In the two-dimensional, trichordal, case, defining a folding for a toroidal Tonnetz is easy to do given the appropriate conditions: a geometric *Tonnetz* in which triangles representing the same trichord overlap in one interval, and where the generating (duplicated) intervals project onto one axis without overlapping. The later condition can be met by the selection of the appropriate Ph_k for the given generator. In particular, for generator g/u (g and u coprime), the appropriate k is one that satisfies the equation $kg =_{\mathrm{mod}(u)} \pm 1$. Through this relationship a phase space is uniquely associated with a specific interval – it will be the essential dimension for *Tonnetze* on sets of any cardinality generated by that interval.[10] For ic1-generated trichords, $\mathrm{Ph}_k = \mathrm{Ph}_1$, and for the ic5-generated trichords it is Ph_5. Or, for another example, the triad is a generated trichord in a mod-7 universe (Figure 5) and its generating interval ($g = 2$) projects neatly onto a Ph_3 axis ($2 \times 3 =_{\mathrm{mod}7} -1$). Many important rhythms are also generated: The *tresillo* is a three-note rhythm generated by interval 3 in a minimal embedding universe of 8 (usually a dotted quarter in 4/4), and as a *Zeitnetz* in $\mathrm{Ph}_{1,3}$ space (Figure 6) it has duplications. The duplicated rhythms are differently oriented in time, since Ph_1 is the dimension representing simple temporal proximity. This could be understood as two ways of grouping the same rhythm, 3-2-3 or 2-3-3, as illustrated on the right side of Figure 6. Such a distinction may be musically important in some contexts, in which case we would want to retain this toroidal version of the *Zeitnetz*. If it is not, however, we may want to fold this geometry to produce a simple cyclic *Zeitnetz*.

Let us refer to the axis that takes a projection of the generator cycle a *principal dimension* and the other as a *disambiguating dimension*, whose function is to separate

[9] Spelling itself is by definition a notational phenomenon, and as such it is accurately described by a system of voice leading on seven-note collections, or generalized key signatures (Tymoczko 2005). Although Ph_5 mimics the central axis of this voice-leading space when restricted to relatively even seven-note collections (Yust 2016), the two are derived from a very different set of assumptions and should not be conflated (Tymoczko and Yust 2019). However, it is routine in music to treat enharmonic notation is often an imperfect tool for expressing an underlying musical reality, often relating to key or harmonic function (though by no means limited to that). To the extent that one believes key perception relates to pitch-class distributions – a theory that remains debatable, but is supported at this point by a large amount and variety of empirical evidence (Krumhansl and Cuddy 2010) – tonal composers' use of spelling often reflects an underlying reality expressed by Ph_5 relationships.

[10] The relationship of intervals to Fourier components (i.e., the index of a phase space, Ph_k) has been explored by Quinn (2006) and Amiot (2007). Both show that a one-to-one relationship does not hold for interval *content* per se, but does for intervals as generators.

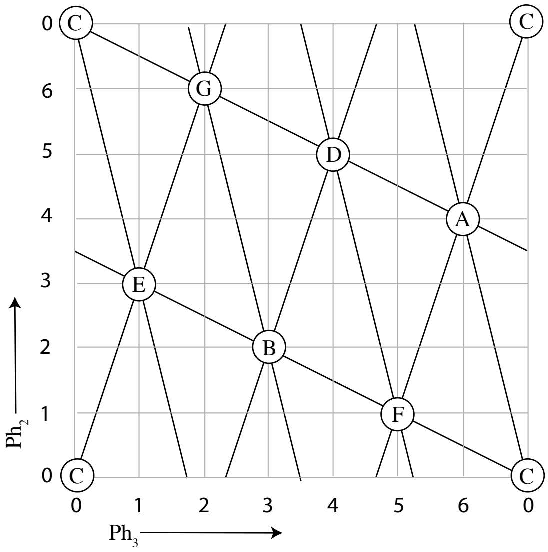

Figure 5. Geometrical *Tonnetz* of generic triads in $\mathrm{Ph}_{3,2/7}$-space

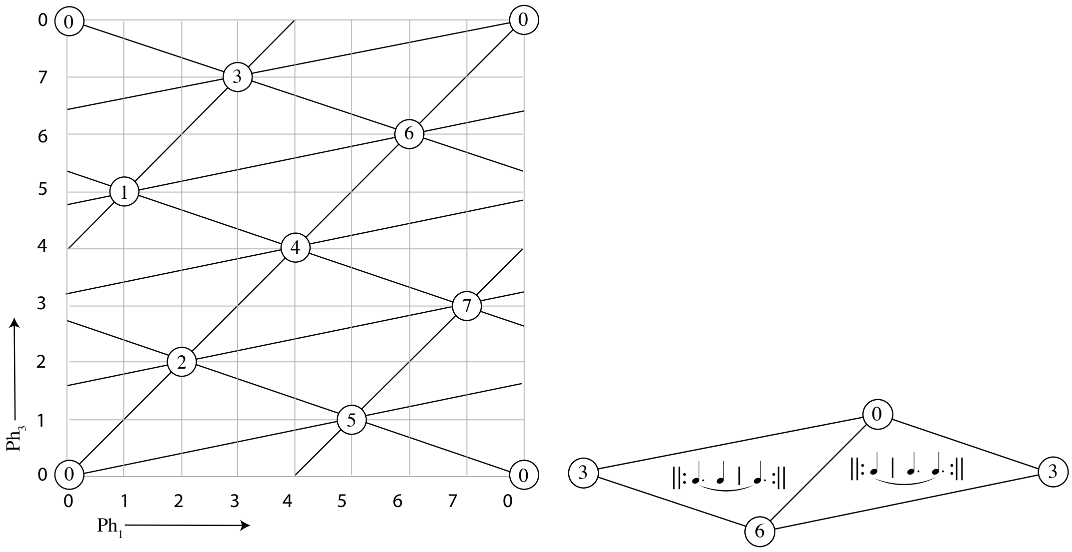

Figure 6. Geometrical *Tonnetz* of tresillo rhythms in $\mathrm{Ph}_{1,3/8}$-space

two forms of the generating interval.

Given a two-dimensional generated *Tonnetz* with axes x and y, let $f(x, y) = 0$ be an equation for the intervallic axis for the non-generating interval, which I will call a *boundary function.* If u is even (as in Figure 3 and Figure 6) then half of the pitch classes will satisfy $f(x, y) =_{\mathrm{mod}(u)} 0$ and the other half will satisfy $f(x, y) =_{\mathrm{mod}(u)} u/2$. If u is odd, as in Figure 5, all pitch-classes will satisfy $f(x, y) =_{\mathrm{mod}(u)} 0$. For the (012) *Tonnetz*, we have $f(\mathrm{Ph}_1, \mathrm{Ph}_5) = \mathrm{Ph}_1 + \mathrm{Ph}_5$. For the tresillo *Tonnetz* we have, similarly, $f(\mathrm{Ph}_1, \mathrm{Ph}_3) = \mathrm{Ph}_1 - \mathrm{Ph}_3$. For the generic triad *Tonnetz* we have $f(\mathrm{Ph}_3, \mathrm{Ph}_2) = \mathrm{Ph}_3 + 2\mathrm{Ph}_2$. Note that the boundary functions necessarily pass through the origin and therefore have no constants.

In the case where u is even, we can reparameterize by retaining the principal dimension, x, and replacing the disambiguating dimension with $\delta = u/2 - |f(x, y)_{\mathrm{mod}(u)} - u/2|$,

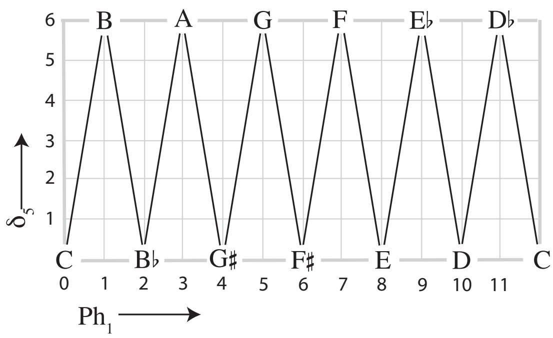

Figure 7. Folded chromatic trichord *Tonnetz*

which gives the distance from the line given by the boundary function. The value of z is maximum where $f(x, y) = u/2$, and zero on the boundary function. The new space parameterized by x and δ equates the two forms of the generating interval. It has only one kind of cycle, the x cycle; in the δ dimension there are "mirror" boundaries, meaning that the image of a straight line (such as the one for the generating interval) "reflects" off the boundaries. It is useful also to add a subscript k to δ that recalls the original dimension $y = \mathrm{Ph}_k$. Figure 7 shows a folded space for the (012) *Tonnetz* in Figure 3, where $\delta_5 = 6 - |(\mathrm{Ph}_1 + \mathrm{Ph}_5)_{\mathrm{mod}12} - 6|$. The ic1 intervals the image of a single straight line.

This satisfies our original goal: a well-defined geometry that embeds the simpler form of the generated trichordal *Tonnetz*, without duplications. However, the purpose of defining such a geometry, rather than simply using the simplicial form of the *Tonnetz*, is that it is potentially richer in meaning. The meaning of the cyclic dimension of the generating interval remains intact from the original phase space to the folded one, but the meaning of the newly defined δ parameter, and to what extent it inherits some meaning from the original phase space, demands further examination. The musical interpretation of the original space derives from the positions it assigns to other pitch-class sets and distributions, and the same is true of the folded space. Therefore understanding the meaning of the δ parameters involves consideration of how other kinds of pitch-class sets occupy the space.

An an illustrative example, let us consider two versions of an (027)-*Tonnetz*. The optimum space for this *Tonnetz* (excluding the use of Ph_6) is the same as for chromatic trichords, $\mathrm{Ph}_{1,5}$. However, we might also be interested in thinking of (027)s as an element of $\mathrm{Ph}_{3,5}$ space. Krumhansl (1990) proposed this as a basic space of harmonies and keys in tonal music, and her space has been applied widely in music theory and music cognition research. I relate it to a number of issues in the analysis of tonal music in Yust 2015b. The Ph_3 parameter corresponds to a nearest even division of the octave into three parts, which can be understood as the three positions (root-third-fifth) of a triad, agnostic as to which position corresponds to the root. When coupled with Ph_5 and applied to primarily high-diatonicity distributions (as in tonal music), Ph_3 distinguishes dominant-side and subdominant-side harmonies with respect to a key (note that Ph_5 can then determine which triadic position corresponds to the root).[11] I will therefore refer to this as the triadic space.

Figure 8 shows each of these (027) *Tonnetze*. In the $\mathrm{Ph}_{1,5}$ space, the two kinds of ic5 interval distinguish fourths from fifths. The boundary space (whole-tone axis) is defined

[11] See also Yust 2019.

by $\mathrm{Ph}_1 + \mathrm{Ph}_5 = 0$. In the $\mathrm{Ph}_{3,5}$ space, the difference is one of triadic status. The shorter Ph_3 distance between notes a fourth apart corresponds to thinking of them as differing by one triadic position, as they normally would as the fifth and root of a major or minor triad. Taken the long way around, we understand the *fifth* to span one triadic position and the fourth to span two. For instance, in a context centered on a D major triad, the note C would be understood as a downward displacement of the root, and G an upward displacement of the third, so the fifth C-G spans one triadic position (root to third) while the fourth G-C spans two (third to root). The large Ph_3 distance of this interval reflects the relative unusualness of this situation. The boundary space is the whole-tone axis made by the sum of these two different kinds of fourth: $\mathrm{Ph}_3 + 3\mathrm{Ph}_5 = 0$.

The lower panels of Figure 8 show the result of folding each space to eliminate duplications of the (027) trichords. Considering just the locations of pitch classes and the *Tonnetz* itself, the two spaces look identical. However, if we consider the locations of various pitch-class sets or distributions, we see substantial differences, beginning with the (027)s themselves. In the δ_1 space, each (027) is located at the midpoint of its (02) dyad, since the (02) dyad determines the Ph_1 balance of the set. In the δ_3 space, on the contrary, the (027) is understood as a kind of triad, with both fourths representing potential triadic intervals, so it is located at the single pitch class shared by both fourths. Figure 8 shows a series of pitch-class sets plotted in both the $\mathrm{Ph}_{1,5}$ and $\mathrm{Ph}_{3,5}$ spaces and the corresponding folded spaces, beginning from DEA and gradually adding, one at a time, pitch classes from the (027) a major third above (F♯, G♯, C♯) or below (B♭, C, F). The changes of Ph_5 are the same in both spaces (monotonically sharpward or flatward). But in δ_1 the sets remain anchored to the even whole-tone scale. In δ_3 they cross the space to the opposite side, reflecting the triadic significance of the added notes (which fill in a diatonic stack of thirds).

If u is odd, then we may define $\delta = u - |f(x,y)_{\mathrm{mod}(2u)} - u|$. By taking $f(x,y)$ mod $2u$ rather than mod u we divide the line in half at the midpoint of its cycle from (0, 0) to (0, 0). The first half of the boundary line takes on a minimum value $z = 0$ and the second half the maximum $z = u$. For the generic triad *Tonnetz* in Figure 5, $\delta = 7 - |(\mathrm{Ph}_3 + 2\mathrm{Ph}_2)_{\mathrm{mod}14} - 7|$, resulting in the space in Figure 9. The line from $(0,0)$ to $(0,u)$ is identified with (u,u) to $(u,0)$, resulting in a Möbius strip (a non-orientable space).

2.4. *Higher-order foldings*

As we increase the cardinality of generated collections, we can define higher-dimensional *Tonnetze* in toroidal spaces by further differentiating forms of the generating interval. Then a similar method of folding may be applied to turn cyclic dimensions into bounded ones, until a single cyclic chain of intersecting n-simplexes remains, reflecting the basic *Tonnetz* graph. After the space is fully folded, it is homotopy equivalent to the circle that acts as its principal dimension.

For example, a *Tonnetz* of chromatic tetrachords may be defined in $\mathrm{Ph}_{1,4,5}$ space by differentiating one ic1 interval in the Ph_4 dimension and another in the Ph_5 dimension. The *Tonnetz* contains six copies of each (0123) tetrachord based on the possible permutations of the three forms of ic1, which correspond to tangent-space vectors $(-1,-4,-5)$, $(-1,-4,7)$, and $(-1,8,-5)$. The sum of these is the minor third interval, $(-3,0,-3)$. By taking the cross product of one of the ic1s with the ic3, we get the equation of a plane for one of the (013)s shared by "enharmonically equivalent" tetrachords. There are three of these (one for each ic1 type): $(-1,-4,-5) \times (-3,0,-3) = (12,12,-12)$,

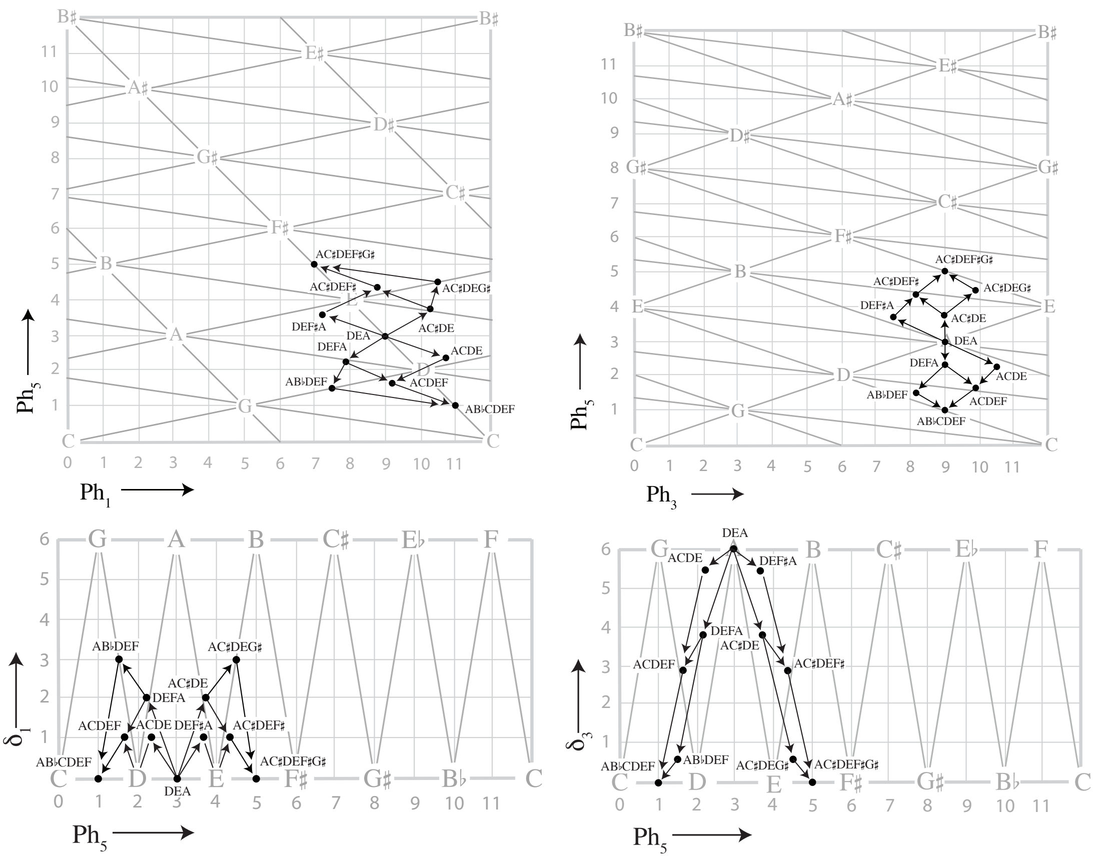

Figure 8. An (027)-*Tonnetz* in two spaces, and foldings showing the different resulting positions of major and minor triads

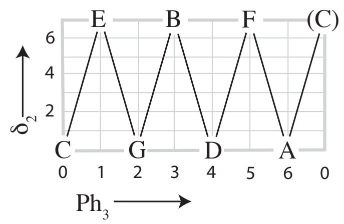

Figure 9. A folded generic-triad *Tonnetz*

$(-1,-4,7)\times(-3,0,-3) = (12,-24,-12)$, and $(-1,8,-5)\times(-3,0,-3) = (24,-12,-24)$. Reducing the products by a factor of 12, we get the equations of three boundary planes: $\mathrm{Ph}_1 + \mathrm{Ph}_4 - \mathrm{Ph}_5 = 0$, $\mathrm{Ph}_1 - 2\mathrm{Ph}_4 - \mathrm{Ph}_5 = 0$, and $2\mathrm{Ph}_1 - \mathrm{Ph}_4 - 2\mathrm{Ph}_5 = 0$. By design only two of the three planes are linearly independent; the third is a sum of the other two: $(1,1,-1) + (1,-2,-1) = (2,-1,-2)$.

The folded space for this *Tonnetz* will retain Ph_1 as the cyclic dimension. The other dimensions $(\mathrm{Ph}_4/\mathrm{Ph}_5)$ are replaced by barycentric coordinates λ_1, λ_2, and $\lambda_3 = 1 - \lambda_1 - \lambda_2$, defined for each cross-section of the space $\mathrm{Ph}_1 = x$ by the points $(x, 0, x)$, $(x, 4, x+4)$, and $(x, 8, x+8)$ where the three boundary planes intersect. These points define a triangular lattice of the cross-section with six triangles, as shown on the right side of Figure 10, and the barycentric coordinates map all of these triangles onto one

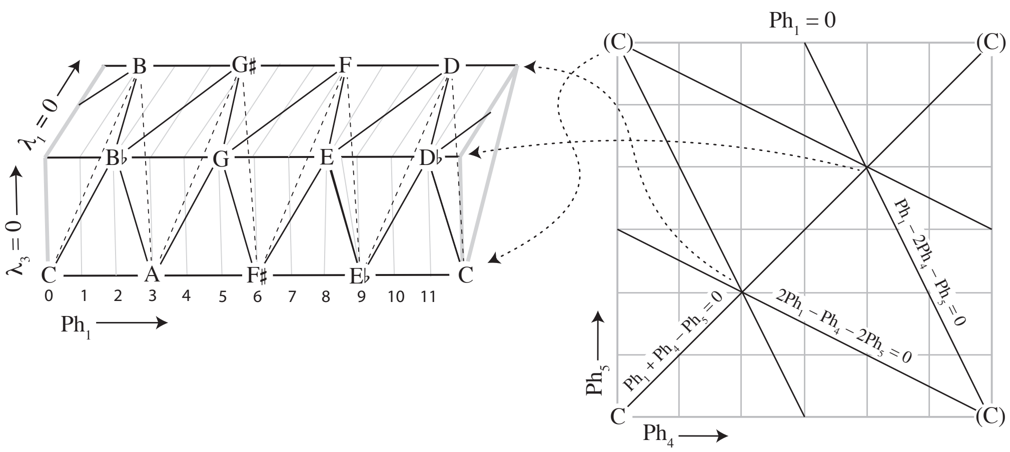

Figure 10. Folded (0123)-*Tonnetz* and a cross-section of the toroidal space it is derived from.

another.[12] The resulting space is illustrated on the left side of Figure 10.

For comparison, consider a *Tonnetz* of generic seventh chords generated by 2 mod 7 (a generic third). Starting with a $\mathrm{Ph}_{1,2,3}$ space, we can define three forms of the generating interval, $(-2,3,1)$, $(-2,-4,1)$, and $(5,3,1)$. Taking cross products with the sum $(1,2,3)$ (which represents the interval of the seventh), we have boundary planes $\mathrm{Ph}_1+\mathrm{Ph}_2-\mathrm{Ph}_3$, $-2\mathrm{Ph}_1+\mathrm{Ph}_2$, and $\mathrm{Ph}_1-2\mathrm{Ph}_2+\mathrm{Ph}_3$. The principle dimension is Ph_3, so for cross-section $\mathrm{Ph}_3=x$ barycentric coordinates can be defined on $(\frac{x}{3},\frac{2x}{3},x)$, $(\frac{x+7}{3},\frac{2x+14}{3},x)$, and $(\frac{x+14}{3},\frac{2x+7}{3},x)$. The result is the *Tonnetz* shown in Figure 11. This geometry is similar to the one for the (0123) *Tonnetz*, but different in one important respect: because the universe is not divisible by three, there is a twist where the two ends of the triangular prism reconnect.[13] Furthermore, because the chord is so large relative to the universe ($n>u/2$), every note in the universe is connected to every other in this *Tonnetz*. Therefore, if it were treated merely as a graph, we could find a complete subgraph for every possible set in the universe, including every tetrachord. However, geometrically only the seventh-chord tetrahedra define a simplicial decomposition. A different generating interval, such as a fifth, will make a complete cycle in the Ph_3 dimension of the space (e.g., C-G-D-A-(C)), so it will not actually outline a region, even though all of its notes are connected in one way or another. Thus, defining Ph_3 as the principal dimension is essential to making this the space for a generic seventh-chord *Tonnetz* (as opposed to, say, an (0134), diatonic stack of fourths, *Tonnetz*).

The use of barycentric coordinates can be generalized to simplices of any dimensionality, so this method of folding is generalizable to any one-interval generated collection. The entire process is:

(1) Define an n-dimensional phase space that includes one dimension with non-overlapping projections of the generating interval (i.e., Ph_k such that $|gk| =_{\mathrm{mod}(u)} 1$).
(2) Define n unique tangent-space vectors that correspond to the generating interval, and are linearly independent

[12] There are three barycentric coordinates given by projections onto the lines from each vertex of the triangle to the midpoint of the opposite edge, and their values are the proportion of the distance from the opposite edge. The three coordinates are constrained to sum to a constant, so only two of the three are independent.

[13] Compare to the trichord space of Callender, Quinn, and Tymoczko (2008) and Tymoczko (2011), which has the same topology. This space is derived differently, from voice-leading considerations, but is similar to the spaces discussed here in that it can be understood as a quotient of a direct product of three Ph_1 spaces.

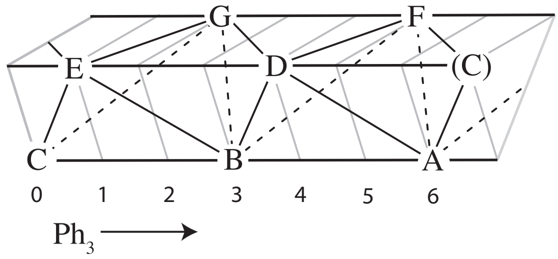

Figure 11. Folded generic seventh-chord *Tonnetz*

(3) Take the cross product of each of these with the remainder interval. This defines n hyperplanes containing the intersections of two versions of the set.
(4) For each cross section, $\mathrm{Ph}_k = x$, define barycentric coordinates using the n points where these hyperplanes intersect.
(5) The resulting space defined by Ph_k and the barycentric coordinates contains the desired non-redundant *Tonnetz* geometry.

This general scheme can be understood to capture the $n = 2$ case also, where the boundary hyperplanes are one-dimensional. These are split into one or two edges where the boundary function crosses each cross section, and the δs give distances from the endpoints of the edge, which can be understood as a trivial instance of barycentric coordinates.

2.5. *The heteromorphic diminished seventh chord*

The diminished seventh chord is a special kind of generated collection, a perfectly even collection where the remainder interval is equal to the generating interval. As such it represents a logical extreme of the kind of collections explored in the previous section: a *Tonnetz* of diminished seventh chords, considered just as basic *Tonnetz* graph or simplicial *Tonnetz*, is trivial, containing only one chord. As a musical object, though, a diminished seventh chord is potentially rich in multiple meanings, something that made it a an essential resource to eighteenth- and nineteenth-century composers. If the process of embedding a simplicial *Tonnetz* in a robust geometry is one of investing it with musical meaning, then even a chord that gives rise to a trivial *Tonnetz* might have a non-trivial network of meanings when embedded in different spaces.

The minimal embedding universe of a diminished seventh chord is $u = 4$ – i.e. the chord itself. Accordingly, its only available phase space is $\mathrm{Ph}_{1/4}$, with $\mathrm{Ph}_{2/4}$ being degenerate. Since three dimensions are required to make a four-note chord into a simplex, the procedure outlined in the previous section using phase spaces requires beginning from the direct product of three copies of $\mathrm{Ph}_{1/4}$, which implies associating different meanings with each of these copies. Since we are interested in the diminished seventh chord as a component of tonal harmony, it makes sense to embed it in the non-minimal $u = 12$, in which case $\mathrm{Ph}_{1/4}$ can be interpreted as $\mathrm{Ph}_{1/12}$, $\mathrm{Ph}_{3/12}$, $\mathrm{Ph}_{5/12}$, all of which order the notes of the diminished seventh in the same way. We already have developed interpretations for each of these in the context of tonal harmony: $\mathrm{Ph}_{1/12}$ is a position on the pitch-class circle, $\mathrm{Ph}_{3/12}$ is a triadic location, and $\mathrm{Ph}_{5/12}$ is position on the circle of fifths. I will use $\mathrm{Ph}_{11/12}$ in place of $\mathrm{Ph}_{1/12}$, however, so that interval values have their conventional interpretation (with 1 as an ascending semitone). The notes of the diminished seventh

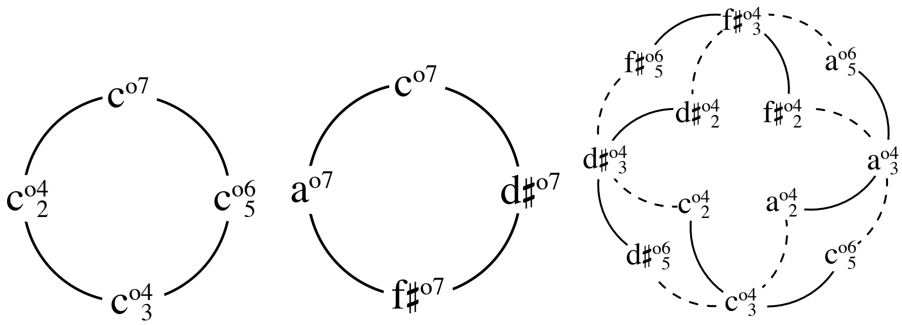

Figure 12. Graphs for folded (0369)-*Tonnetze*. On the right, dashed lines connect chords with the same bass, and solid lines chords with the same nominal root.

are equally spaced in all of these dimensions.

In each of these phase spaces we can realize the generating interval of the diminished seventh either as an interval of length 3 (or -3) or -9 (resp. 9). In Ph_{11}, the distinction is between an ascending 3-semitone interval and a descending 9-semitone interval. This dimension can therefore represent differences of voicing or figured bass. The complete diminished seventh consists of three intervals of size 3 and one of size -9. The ordering of these will determine the position of the chord: 7, $\frac{6}{5}$, $\frac{4}{3}$, or $\frac{4}{2}$. For instance, if we begin from C and fix the spelling of notes as C-E♭-G♭-B𝄫, then ordering the Ph_{11} interval values 3-3-3-(-9) produces a $\mathrm{c}^{\circ 7}$, 3-3-(-9)-3 produces a $\mathrm{c}^{\circ}{}^{6}_{5}$, and so on. The left panel of Figure 12 shows a *Tonnetz* graph for these different positions of the $\mathrm{c}^{\circ 7}$.

Spelling can be used to distinguish between the two possible Ph_5 orientations of the 3-semitone interval, as a minor third (-3) or augmented second (9). This changes the figured bass along with the nominal root of the chord, so, with the Ph_{11} interval ordering fixed at 3-3-3-(-9) (from pitch-class 0 to 9), the Ph_5 ordering (-3)-(-3)-(-3)-9 gives $\mathrm{c}^{\circ 7}$, (-3)-(-3)-9-(-3) gives $\mathrm{a}^{\circ}{}^{6}_{5}$, ($-3$)-9-($-3$)-($-3$) gives $\mathrm{f}\sharp^{\circ}{}^{4}_{3}$, and 9-($-3$)-($-3$)-($-3$) gives $\mathrm{d}\sharp^{\circ}{}^{4}_{2}$. A corresponding *Tonnetz* graph is shown in the middle of Figure 12.

We can combine these two kinds of intervallic distinctions to make a *Tonnetz* of different possible nominal roots and positions, by defining the intervals of the seventh chord in $\mathrm{Ph}_{11}\times\mathrm{Ph}_5$ as $(3,9)$, $(-9,-3)$, and two copies of $(3,-3)$. For instance, $(3,9)$-$(-9,-3)$-$(3,-3)$-$(3,-3)$ is a $\mathrm{d}\sharp^{\circ}{}^{6}_{5}$. By permutation of these intervals, twelve possible forms of the same diminished seventh are possible, and they make the *Tonnetz* graph shown on the right of Figure 12. This *Tonnetz* excludes the root position chords, because they have a different set of intervals: $(3,-3)$-$(3,-3)$-$(3,-3)$-$(-9,9)$. The root-position diminished seventh is an essentially one-dimensional structure because it has only two kinds of intervals.

The diminished seventh chord is perhaps the earliest instance where harmonic theory very clearly separated interpretations of the same set pitch classes as distinct harmonic objects. Bach (1949, p. 438) (orig. 1753), for instance, makes it clear that the idea of a diminished seventh as a common entity with distinct enharmonic interpretations was well established by the mid-eighteenth century. If we consider that the pitch-class distributions of tonal music very consistently maintain a strong diatonicity (high $|f_5|$), diminished sevenths, with their pitch classes equally spaced on the circle of fifths, were the earliest method that composers used to weaken f_5 and thereby create an ambiguity of diatonic position.

The distinctions made in Figure 12 might not be entirely sufficient to reflect the possible functions of a given diminished seventh, though. In particular, the notion of "nominal root" determined solely by Ph_5 orientation is somewhat unsatisfactory. For typical harmonies (triads and seventh chords), the concept of a root is associated with triadic distinctions, Ph_3 orientations, as well as Ph_5 orientations. We have noted above that Ph_3 orientations divide the pitch-classes up between three triadic positions (root, third, fifth, but without specifying which position corresponds to the root). A seventh chord, since it has four notes, must have two notes assigned to the same triadic position. With most seventh chords (dominant, minor, half-diminished) there is no ambiguity about the triadic orientation: the root and seventh are a step apart and belong to a single triadic position. Either the seventh is a displacement of the root or, in certain contexts, the root might be considered a displacement of the seventh (in which case the third might be called the "true root"). In a diminished seventh chord, since it is equally spaced, any two notes might be equally likely to be assigned to the same triadic position. Furthermore, there is not necessarily any reason why the Ph_5 orientation of notes (as sharpward or flatward, determining the spelling of the chord and therefore assignment of nominal root) should have to correlate in any particular way to their Ph_3 orientations – that is, the spelled root and seventh need not belong to the same triadic position, as they normally would in a diatonic seventh chord.

From the eighteenth century on, the diminished seventh has been conventionally understood as a displacement of a dominant seventh, or a dominant minor ninth with a missing root. This implies that the nominal fifth (as the seventh of the implied dominant) and nominal seventh (the minor ninth, or displacement of the root) belong to a single triadic position. This orientation in $Ph_3 \times Ph_5$ space is shown on the left of Figure 13, for a d$\sharp^{\circ 7}$ chord, whose displaced root would be B. This conventional interpretation can be generalized as a particular ordering of $Ph_3 \times Ph_5$ intervals: $(3, 9)$-$(3, -3)$-$(3, -3)$-$(-9, -3)$. Here the notes interpreted as triadically stable, D♯-F♯ are also the furthest sharp. If we fix the spelling (Ph_5 orientation) but shift the Ph_3 orientation in the tonic direction (to the left), we get a different interval ordering, $(3, 9)$-$(3, -3)$-$(-9, -3)$-$(3, -3)$, shown on the right. The furthest flat note (nominal seventh), C, is now triadically stable, while F♯ and A now share a triadic position and may be treated as displacements of G. The D♯, though triadically stable, is remote in Ph_5 and therefore can resolve to E to give a C major triad.[14] In other words, this $Ph_3 \times Ph_5$ orientation corresponds to the usual common-tone interpretation of the diminished seventh, and shows why this distinct interpretation cannot be captured by spelling alone. Logically we can then extrapolate to a third possible interval ordering, $(3, 9)$-$(-9, -3)$-$(3, -3)$-$(3, -3)$, in which the notes A and C are triadically stable and the sharp-side notes, D♯-F♯, are displacements of a hypothetical fifth, E. This inversion of the usual dominant-functioning interpretation might be understood as a minor version of the common-tone diminished seventh (a displacement of A minor).

The diminished seventh, as a 3-simplex, can be embedded in $\mathcal{T}^3$, thus potentially admitting of all three of these kinds of distinctions simultaneously. Specifically, we can make a *Tonnetz* on the diminished seventh {C,E,F♯,A} whose intervals in $Ph_{11} \times Ph_3 \times Ph_5$ are $(3, 3, 9)$, $(3, -9, -3)$, $(-9, 3, -3)$ and $(3, 3, -3)$. The twenty-four possible permutations of these intervals partition the $Ph_{11,3,5}$ space into 24 tetrahedral regions, all with *the same four vertices*, differing only in their orientation in the space. Even though they are, in a sense, all versions of the same chord, they make a non-trivial *Tonnetz*, with certain interpretations overlapping in shared tetrachords and others being more distant.[15]

[14] Theoretically this reasoning could be inverted so that the chord resolves to G♯ minor with A and C resolving down by semitone. This is less compelling because of the general tonal tendency to associate descent in Ph_5 (decending fifths, ascending leading tones) with resolution.

[15] Note that the total is 24 (rather than $4^3 = 64$) because certain intervals are excluded, namely those that are long in two dimensions at once. We already noted that the diminished seventh interval has this property (it is

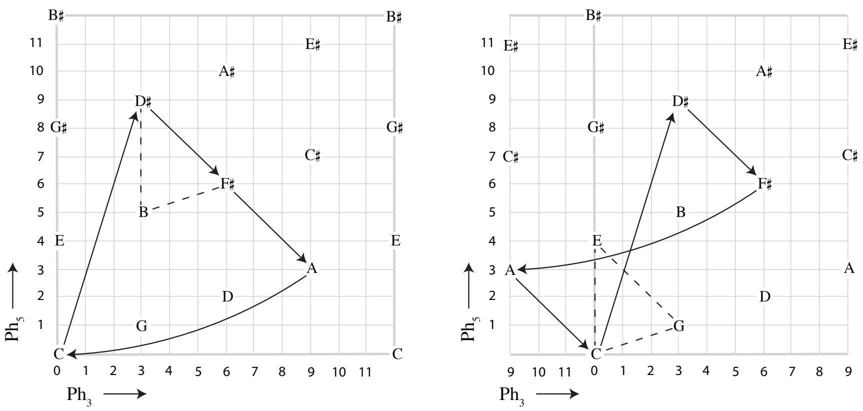

Figure 13. Different triadic orientations of the same diminished seventh chord, as a dominant (displacement of B major in an E minor context) or tonic-functioning, or common-tone, diminished seventh (displacement of C major).

We may then simplify this space through various types of folding, which remove the intervallic distinctions (and thereby some of the types of diminished seventh) and the cyclic homology of one of the dimensions. If we forget the triadic (Ph_3) distinctions we get a *Tonnetz* of 12 regions for the graph on the right side of Figure 12. If we forget the distinctions relating to direction around the pitch-class circle (Ph_{11}), 3 semitones in one direction versus 9 in the other, and we have a *Tonnetz* of 12 triadic/diatonic forms of a diminished seventh (three functions by four spellings), whose graph is isomorphic to the one for the 12 distinctions of spelling and position. In this latter space (cyclic in Ph_3 and Ph_5 and folded in Ph_{11}) the diminished seventh chord lives in a universe of interval classes, where it is not possible to orient along the pitch-class circle. However, spellings and triadic distinctions remain in effect, so sharp-flat orientation and functional orientation are possible. Also, recall from above that even after a dimension is folded, it still effects the underlying topology, so that proximity in Ph_{11} continues to effect proximity of points within the regions (insofar as these points represent pitch-class distributions), even as the global sense of orientation in Ph_{11} has been lost.

We can also go one step further, folding two dimensions of the large diminished-seventh *Tonnetz*, to get the simple cycles of interpretations on the left of Figure 12. This is precisely analogous to the folded chromatic tetrachord and diatonic seventh-chord *Tonnetze* of the previous section, differing only in the cardinality of the resulting cyclic *Tonnetz* graph (with 4 elements instead of 12 or 7). However, the diminished seventh chord is different from these in that its remainder interval is equivalent to its generating interval, which makes it possible to have a *Tonnetz* in which *all three dimensions are folded*. This fully simplified diminished seventh *Tonnetz*, with no interval duplications, is in a bounded tetrahedral space with the pitch classes of the diminished seventh as vertices, all of the intervals between them (minor thirds and tritones) as edges, and the four diminished triads as faces bounding the entire space. This can be derived directly from the full $\mathrm{Ph}_{11,3,5}$ space by defining barycentric coordinates based on the distance of each point from the nearest plane for each diminished triad, mapping the 24 tetrahedral regions defined by the diminished seventh onto one another. Pitch classes not in the *Tonnetz* fall on the face opposite the pitch class two semitones away. So, for instance, pitch classes D

long in both Ph_1 and Ph_5). Similarly the long Ph_3 interval (the minor third that does not span triadic positions) is never the augmented second (indeed, it would hard to interpret the tonal meaning such an interval), and also is never allowed to be the major sixth interval.

and F♯ are mapped onto one another and fall on the (E♭, F♯, A) face (opposite C). The resulting set of pitch classes on a common face therefore, interestingly, forms an (01478), which is special as the only example of a simple pitch-class set in $u = 12$ that is perfectly balanced ($|f_1| = |f_5| = 0$) but not transpositionally symmetrical (Amiot 2017b; Milne, Bulger, and Herff 2017).

3. Two-interval generated *Tonnetz*

Thus far we have only considered *Tonnete* on one-interval-generated collections, but the principles of embedding a *Tonnetz* in a toroidal phase space and folding dimensions to equate duplicated intervals can be applied more widely. Here we consider another class that includes most of the other examples of theoretical interest, collections generated by two distinct interval types. The primary difference is that instead of a *Tonnetz* on a single transposition-type, we will now combine multiple transposition-types in a single *Tonnetz* or *Zeitnetz*, including, but not limited to, inversionally related transposition-types (e.g., major and minor triads).

The simplest case of two-interval generated *Tonnetze* are the trichordal *Tonnetze*, such as the standard triadic *Tonnetz*. Any trichord that is not one-interval generated can be generated by two distinct intervals a-b, and will have a remainder $(-a-b)_{mod(u)}$. Its inversion will then be given by taking these in the opposite order, b-a. For instance, the standard triadic *Tonnetz* results from setting $a = 4$ and $b = 3$, giving major triads, 4-3-5, and minor triads, 3-4-5, where $5 = (-a-b)$ is the remainder. Note, however, that this method of generating the standard *Tonnetz* is not unique. We could also use, e.g., $a = 5$ and $b = 4$, giving major triads in "second inversion," 5-4-3, and minor triads in "first inversion," 4-5-3.

Trichordal *Tonnetze* generated by two intervals are not foldable in the way described in the previous section, because all of their intervals are unique.[16] Larger two-interval generated *Tonnetze*, however, are. One example is a seventh-chord *Tonnetz* described by Douthett (1997) and Yust (2018a). This extends the standard triadic *Tonnetz* with another minor third. To get the transposition types for this *Tonnetz*, we list all unique permutations of the intervals 3, 3, and 4, and append the remainder $12 - a - b - c = 2$. The result is three types of seventh chords: dominant sevenths, 4-3-3-2; minor sevenths, 3-4-3-2; and half-diminished sevenths, 3-3-4-2. Note that inversionally related transposition types will always be included. Nonetheless we still count them separately, since some transposition types will have inversional symmetries (like the minor seventh). The folding for this seventh-chord *Tonnetz* equates the two minor thirds, so that there are no duplications of the minor seventh chords. The resulting topology may be understood as a fattened 2-dimensional torus – that is, it has two cyclic dimensions and one bounded dimension. Each boundary contains a whole-tone collection with a (026) trichordal *Tonnetz*. These (026) trichords are connected to points on the opposite boundary to complete dominant or half-diminished seventh chords. Minor seventh-chord simplexes then also result from these connections, filling in the space between the 3-3-4-2 and 4-3-3-2 tetrahedra to give a complete simplicial decomposition of the space.

We can construct other two-interval generated tetrachordal *Tonnetze* similarly, and they will have one bounded dimension (the disambiguating dimension) and two cyclic dimensions (principal dimensions), which results in either the same topology as the seventh-chord *Tonnetz* (an orientable fattened 2-dimensional torus) or a similar non-

[16] However, this is not the only possible method of folding. See the section on spherical *Tonnetze* below.

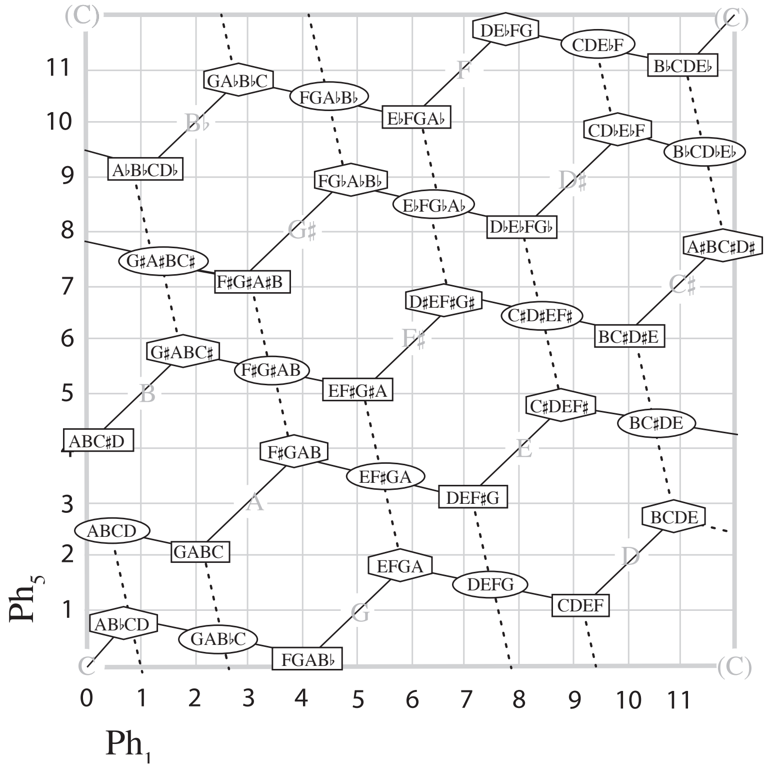

Figure 14. *Tonnetz* of scalar tetrachords, 1-2-2-7, 2-1-2-7, and 2-2-1-7, projected onto $Ph_{1,5}$-space. Dashed lines show "voice-leading"-type transformations that hold the remainder interval (the bounding fourth) constant. Solid lines show scalar extensions that remove the lowest note and add a note a fifth above, or vice versa.

orientable topology. An example of the latter type, also mentioned in Yust 2018a, is a space of scalar tetrachords generated by 1, 2, and 2 – that is, tetrachords in the older sense. The tetrachords are the "phrygian" 1-2-2-7, "dorian" 2-1-2-7, and "ionian" 2-2-1-7. The remainder is coprime to 12, so the second method of folding needs to be used in this case, resulting in a non-orientable space with a single boundary. This boundary houses a trichordal (015)-*Tonnetz*. Connecting the (015)s across the bounded dimension to the note that fills in the major third makes phrygian and ionian tetrachords, and the dorian tetrachords fill in the space between these.

Since two-interval generated *Tonnetze* always have two cyclic dimensions remaining after folding, they can be effectively visualized by projection onto a two-dimensional torus. For the scalar-tetrachord *Tonnetz*, for example, the best choice of cyclic dimensions (which can be determined by maximizing Fourier coefficients) are Ph_1 and Ph_5. In Figure 14 the tetrahedra are projected into $Ph_{1,5}$-space by taking the Fourier phase values of their tetrachords (i.e. taking circular averages of their pitch classes in each dimension). The $Ph_1 \times Ph_5$ space may be compared to Noll's (2018) *degree space* and corresponding concept of width-height duality, or height versus tone character (Clampitt and Noll 2011). Scalewise tetrachords embody this duality in the sense that they are equally compact in Ph_1 and Ph_5.

Although I have advocated throughout for the importance of an underlying geometry, including dimensions that may be rendered homologically trivial through folding, throughout this section and the next two, we will operate at a somewhat more general level, remaining agnostic about the dimension that has been folded and working instead with the *Tonnetz* graph and the topologically significant dimensions of the phase-space geometry (the ones that define the cyclic components of the homology group). In the (0135)/(0235) *Tonnetz*, the topologically significant dimensions are Ph_1 and Ph_5. Projection into this two-dimensional space eliminates the unspecified third dimension.

Two-interval generated *Tonnetze*, with 3 and 4 (in $\mathbf{Z}_{12}$) as the two intervals, formalize the common musical notion of tertian harmony. Such constructions generalize the standard *Tonnetz* to higher cardinality by considering the generating thirds as an essential feature, and increasing the cardinality, which corresponds to dimensionality in the topological construction. The Douthett seventh-chord *Tonnetz* is one example. A more exotic set of seventh chords would result from using two major thirds, as opposed two minor thirds (augmented-major sevenths, 4-4-3-1; major sevenths, 4-3-4-1; and minor-major sevenths, 3-4-4-1). We can also extend further to cardinality 5, or ninth chords. As the cardinality increases, we may also increase the number of foldings, which means the number of cyclic dimensions remains constant at two. All such *Tonnetze* will house a standard triadic *Tonnetz*, whose original form can be recovered by projecting it onto the cyclic dimensions.

The two ninth-chord *Tonnetze* of greatest musical interest are those generated by 3, 3, 3, and 4, and by 3, 3, 4, and 4. The former is a *minor ninth-chord Tonnetz* with remainder 11 and the latter is a *major ninth-chord Tonnetz* with remainder 10. The foldings are somewhat different in that the first identifies three intervals (the minor thirds) and the second identifies two pairs of intervals. Therefore the minor-ninths *Tonnetz* is somewhat simpler, including only four chord types. Figure 15 shows its projection in $\mathrm{Ph}_{4,5}$-space. The edges of the *Tonnetz* graph are also shown, with dotted lines for voice-leading–type relationships, meaning that they hold the position of the ninth interval constant while moving one other note by semitone. Solid lines show the extensions of the chain of thirds, where the initial note in the chain is removed and another note added at the end, or vice versa. The major-ninths *Tonnetz*, with six chord types, has a optimum projection in $\mathrm{Ph}_{3,5}$ space, as shown in Figure 16. Even this projection is not perfect, though, because two chord types fall in the same location, the dominant ninth (4-3-3-4-10) and the minor-major ninth (3-4-4-3-10 – i.e. a minor triad with a major seventh and major ninth). To show the *Tonnetz* graph, then, doubled dotted lines indicate two distinct connections to the co-located dominant and minor-major ninths, while the solid lines to and from these show individual connections (from left to right, augmented ninth → dominant ninth → half-diminished ninth, or half-diminished ninth → minor-major ninth → augmented ninth).

The minor-ninths *Tonnetz* has an interesting relationship to the diminished-seventh *Tonnetze* discussed in section 2.5, because diminished sevenths are one of the shared subsets (between dominant-minor and diminished-minor ninth chords). The minor ninths *Tonnetze* at different levels of folding therefore all have diminished-seventh sub-*Tonnetze*. In the fully ramified 5-toroidal version of the *Tonnetz*, the sub-Tonnetz is the full 4-toroidal diminished seventh *Tonnetz*. When folded down to two cyclic dimensions, as in Figure 15, we see that the diminished-seventh *Tonnetz* retains one cyclic dimension, here shown as Ph_5 distinctions, so that the same diminished seventh chord corresponds to four vertically aligned edges in the network. The historical origins of diminished-seventh enharmonicism are therefore reflected in this *Tonnetz*: originally conceived as a dominant minor ninth chord with an elided root, or dominant seventh with a displaced root, it is required to retain a distinct enharmonic identity, as reflected in the dimensionality of the folded minor-ninths *Tonnetz*.

We can make an interesting *Zeitnetz* that is very similar to the major-9ths *Tonnetz*, generated by 3 and 4, but with $u = 16$, giving a remainder of 2 rather than 10. This *Zeitnetz* includes a number of interesting timelines that are used in some Latin American

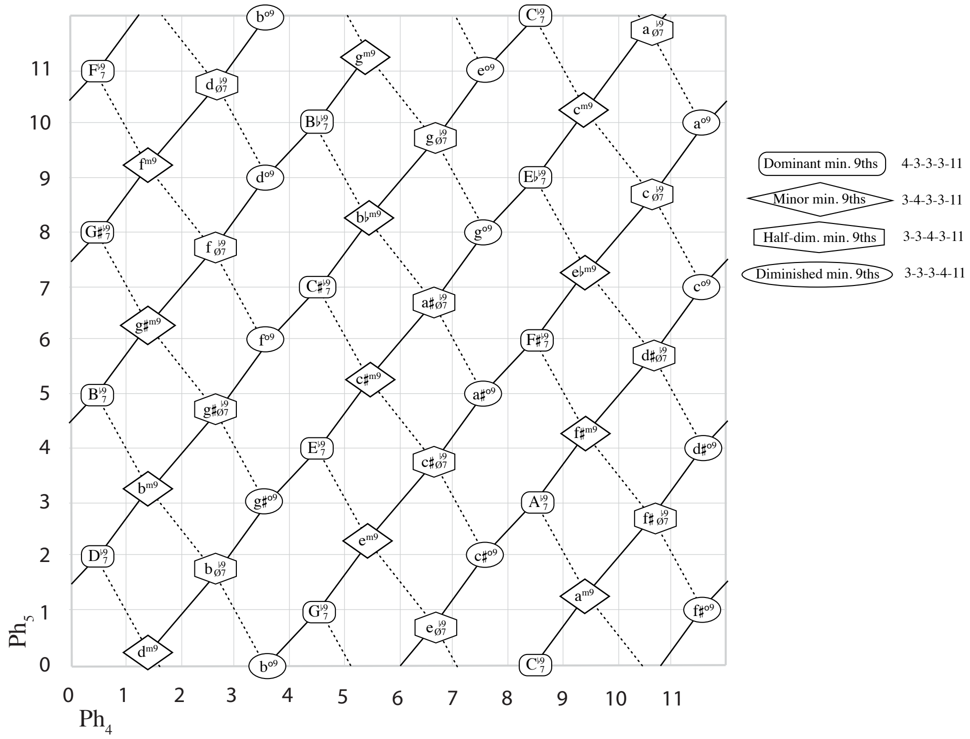

Figure 15. *Tonnetz* of minor ninth chords projected onto $\mathrm{Ph}_{4,5}$-space. Dashed lines show "voice-leading"-type connections, which keep the remainder interval in place. Solid lines show extensions of the chain of thirds.

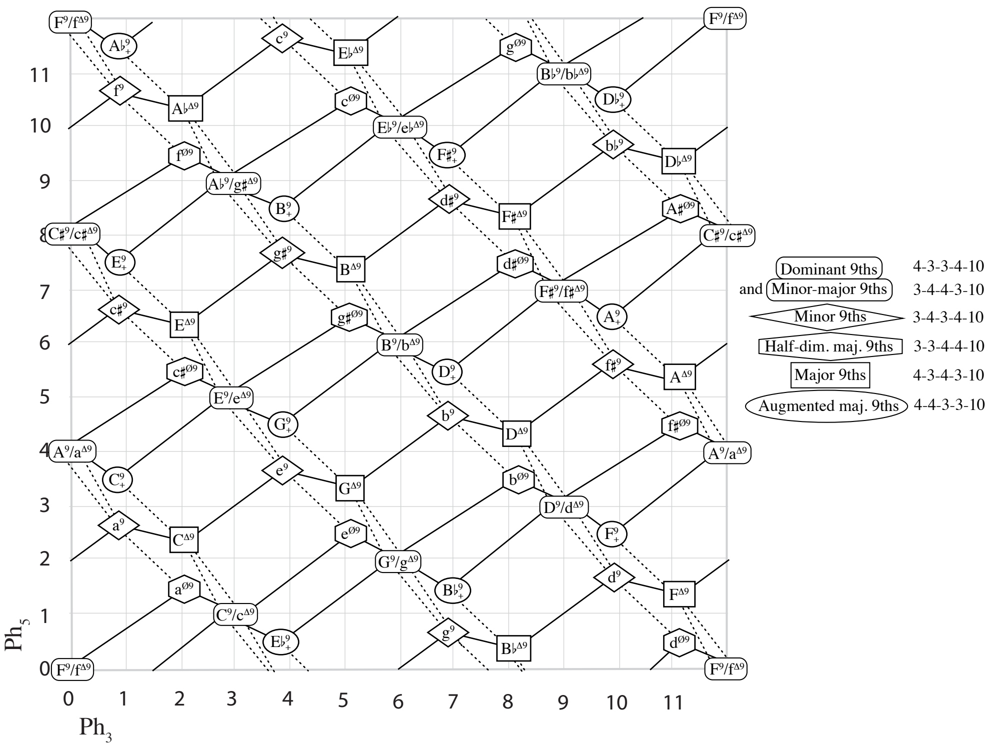

Figure 16. *Tonnetz* of major ninth chords projected onto $\mathrm{Ph}_{4,5}$-space. Dominant ninths and minor-major ninths are represented by the same points in the space, and double lines show independent connections to each of these. Dashed lines show "voice-leading"-type connections, which keep the remainder interval in place. Solid lines show extensions of the chain of thirds.

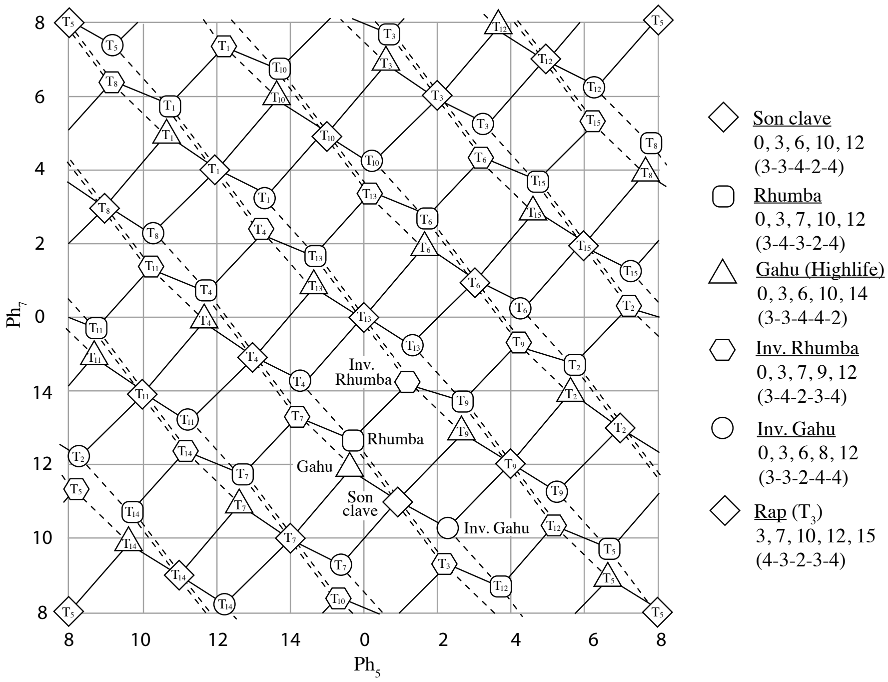

Figure 17. *Zeitnetz* generated by 3, 3, 4, and 4 in $u = 16$ projected in $\mathrm{Ph}_{5,7/16}$-space. Son clave and the rap timeline occupy the same positions in the space, and double lines show independent connections to each of these. Dashed lines show "voice-leading"-type connections, which keep the remainder interval in place.

and West African musical genres, the best known being the Son clave of Afro-Cuban music (or Kpanlogo timeline of Ghana – see Agawu 2003, 157–61): 3-3-4-2-4 (a rotation of 4-3-3-4-2). This *Tonnetz* also includes other timelines discussed by Toussaint (2013), the Rhumba timeline (3-4-3-2-4, a rotation of 4-3-4-3-2) and the Gahu or Highlife timeline (3-3-4-4-2), as well as the inversions (i.e. retrogrades) of these, and one that he refers to as the "rap timeline" (4-3-2-3-4, a rotation of 3-4-4-3-2). The best projection of this *Zeitnetz* is in $\mathrm{Ph}_{5,7/16}$-space, which much like the major-ninths *Tonnetz*, superposes two rhythms ("rap" and Son clave), as shown in Figure 17. The rap timeline is therefore given in T_3, the rotation that is superposed on the standard rotation of the Son clave. This phase space represents the rhythms as approximations to maximally even 5-in-16 (interval pattern 3-3-3-3-4) rhythms and subsets of maximally even 7-in-16 (interval pattern 3-2-2-3-2-2-2) rhythms.

4. Uniform *Tonnetze*

Another type of two-interval generated *Zeitnetz* can be constructed on what Osborn (2014) calls *Euclidean rhythms*, following Gómez-Martín, Taslakian, and Toussaint (2009) and Toussaint (2013), which include many rhythms common in popular music. These are defined as the beat-class sets that can be generated by permuting the adjacency intervals of a maximally even rhythm.[17] Although maximally even sets are generated when n and u are coprime (Clough and Douthett 1991), if the generator is not an adjacency interval, then distinct collections can result from permuting these.

[17]Toussaint's definition is different than Osborn's, effectively only including the maximally even patterns (which, it turns out, can be generated though a version of the Euclidean algorithm). Osborn's definition is somewhat preferable since a new term is not needed for maximally even patterns.

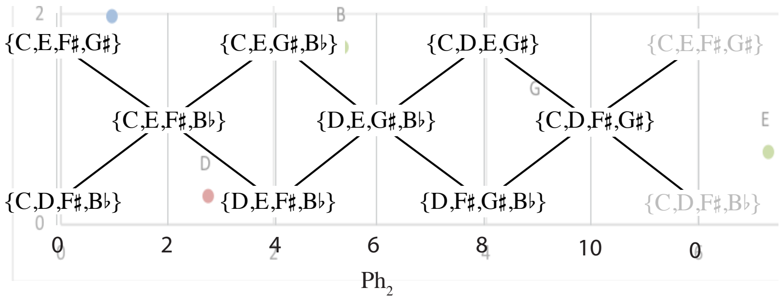

Figure 18. Graph for a uniform *Tonnetz* generated by 2, 2, 4, 4.

For present purposes, what is significant about Euclidean rhythms is that they can be understood as two-interval generated patterns where the *remainder interval is equal to one of the generating intervals*. Because the intervals need not relate to the Euclidean algorithm in the manner described by Toussaint, I will call these *uniform Tonnetze*. In uniform *Tonnetze* as compared to other two-interval generated *Tonnetze*, an additional folding is possible, so that it can be reduced to one cyclic dimension.

The simplest non-trivial example of such a *Tonnetz* is generated by 2 and 1 mod 6, which could be realized as quarter notes and eighth notes in 3/4, or as steps and thirds in a whole-tone scale.[18] The latter realization gives a *Tonnetz* consisting of (0248) and (0268) tetrachords in 12-tET. This is folded according to the process described above, resulting in a space with only one cyclic dimension remaining, as it would be for a *Tonnetz* on generated tetrachords. The boundaries of the space are two augmented-triad Möbius strips. Figure 18 gives its graph, projected in $\mathrm{Ph}_{2/6}$ but with the two set types separated. Note the lack of connection between sets that share a common augmented triad. The augmented triads occur on a boundary where two of the major-third intervals (2 whole-tone steps) are equated, but one, the sum of the two whole-tone intervals of the set, remains distinct. Therefore there are actually three geometrically distinct forms of each augmented triad.

A similar *Tonnetz* generated by the intervals of the maximally even pentatonic scale, 2, 2, 2, 3, 3, contains two chord types, pentatonic scales and dominant major ninth chords, as shown in Figure 19. The choice of generating intervals differing by a minimal amount (1) guarantees, for a uniform *Tonnetz*, that one of the chord types will be maximally even. Assuming that the cardinality n is coprime to u, the maximally even collection is generated and the generating interval will project onto the axis Ph_n or $\mathrm{Ph}_{(u-n)}$ without overlapping. In other words, to use a folded phase space for this kind of *Tonnetz*, the dimension that we retain as a cyclic dimension should be the one indexed by the cardinality of the collection. This is Ph_5 for the pentatonic *Tonnetz*. The form of the graph is determined by the number of intervals, two of one type and three of the other (as opposed to two of both types, as in Figure 18). For instance, a *Tonnetz* generated by 2, 2, 1, 1, 1 (pentatonic subsets and ninth-chord subsets in a mod-7 universe) would look like Figure 19, only cycling after seven of each chord type instead of twelve. Similarly, a *Zeitnetz* generated by 2, 2, 2, 1, 1, would have a central cycle of eight rotations of a *cinquillo* rhythm (2-1-2-1-2) with detours through rotations of 2-2-2-1-1.

In the previous section, we distinguished two kinds of links in two-interval generated *Tonnetze*, those of voice-leading type, where the remainder interval is fixed, and others that extend the chain of generating intervals in one direction or the other. In the

[18] The trivial examples would be those in which a one-interval generated collection is treated as a two-interval generated collection by treating the remainder as a generating interval

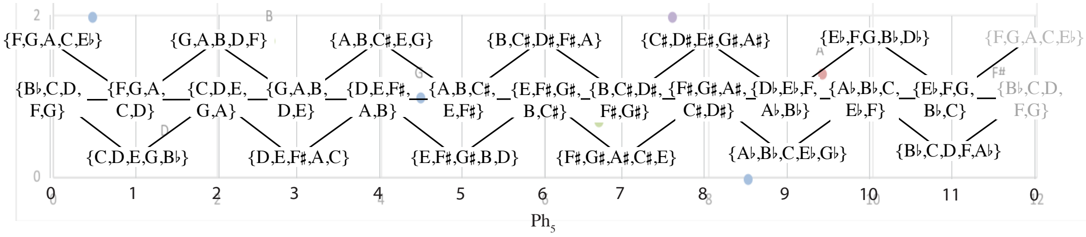

Figure 19. Graph for a uniform *Tonnetz* of pentatonic scales and dominant ninths generated by 2, 2, 2, 3, 3.

uniform *Tonnetze*, this distinction breaks down, because the remainder interval is not distinguishable from a generating interval. Therefore, in cases like these, where the generating intervals differ by 1, the *Tonnetz* connections given by simple permutations are precisely the set of of single-semitone voice leadings. Networks like this that include all and only the single-semitone voice leadings are discussed by Tymoczko (2011). If the pentatonic sets in Figure 19 are replaced by their complements, the result is a network of diatonic and acoustic scales, an important voice-leading network for Tymoczko.

This coincidence should be surprising, because the two kinds of network are derived from very different theoretical premises. But note that they are equivalent only as chord-based networks, not as geometrical *Tonnetze*. Section 2 above explained how Tymoczko (2012) creates a geometrical *Tonnetz* out of chord-based voice-leading networks by defining a dual space. For the pentatonic example, we can take as our objects hyperplanes in the five-dimensional voice-leading space. The possible rotations and translations of these hyperplanes define a dual five-dimensional space, where certain points represent pitch classes in the sense of representing a hyperplane in voice-leading space that holds a certain pitch class constant. This is precisely analogous to the two-dimensional (dyadic) case. The important point is that in all cases, the dual space has a cyclic dimension for translation that corresponds to the cyclic transpositional dimension of the voice-leading space, or the *directed voice-leading sum* (Cohn 1998). The *Tonnetz* graph or the simplicial *Tonnetz* can therefore be geometrically embedded in voice-leading space with no essential change to Figure 19 except for changing the meaning of the x-axis from Ph_5 to transposition of a five-note collection. It turns out that, for any size of collection n, these two quantities, Ph_n and the projection onto the central axis of the n-dimensional voice-leading space, are close approximations of one another as long as we restrict our attention to relatively even chord types. This is informally presented in Yust 2015b and addressed in greater mathematical detail in Tymoczko and Yust 2019. Therefore, these two kinds of geometry will embed the same simplicial *Tonnetze*, with the same topology – the significant differences between them only emerge if we consider what kind of musical objects occupy the space between the pitch classes.

Another interesting example of a uniform *Zeitnetz* can be built from the maximally even 6-in-16 beat-class set 3-3-2-3-3-2. This rhythm may be understood as a repeated tresillo (3-3-2). Permutation of these intervals gives distinct beat-class sets 3-3-3-2-3-2 or 3-3-3-3-2-2, or some rotation of these. Biamonte (2014) calls the latter, a common rhythm in rock, ragtime, and other popular music genres, the “double tresillo” (see Cohn 2016). The resulting *Tonnetz* can be folded into one cyclic dimension, giving the graph shown in Figure 20.

Higher-dimensional uniform *Tonnetz* of scale types are also possible. The 7-in-12 maximally even scale is the diatonic, 1-2-2-1-2-2-2, and permutation gives the acoustic scale or melodic minor, 1-2-1-2-2-2-2, and the “whole-tone plus one” scale 1-1-2-2-2-2-2. The *Tonnetz* graph in Figure 21 is similar to that for the pentatonic *Tonnetz*, but with more

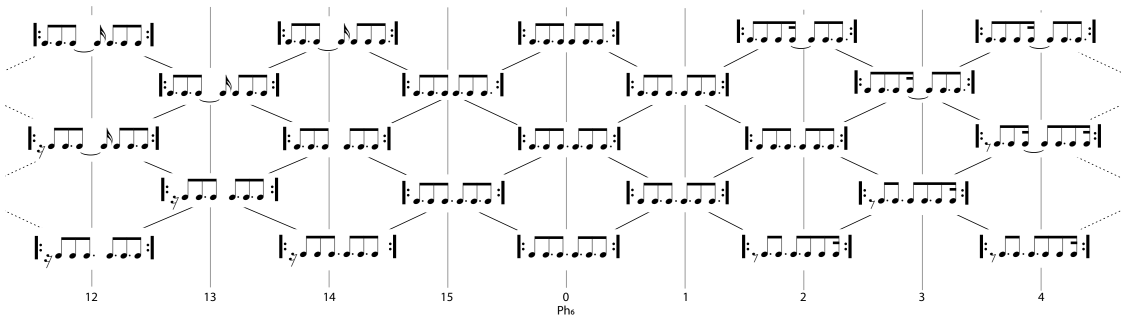

Figure 20. Portion of a graph for a uniform *Zeitnetz* generated by 3, 3, 3, 3, 2, 2. In the middle row are the maximally even repeated tresillos (3-3-2-3-3-2), the top and bottom rows have rotations of the double tresillo (3-3-3-3-2-2), and between these are the other pattern, 3-3-3-2-3-2. The graph cycles after going through the eight distinct rotations of the repeated tresillo, of which five are shown.

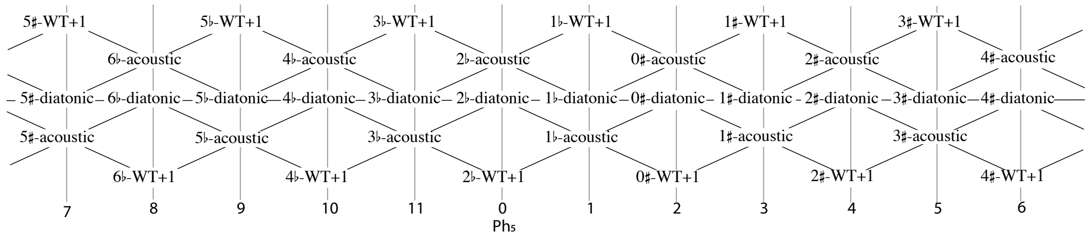

Figure 21. The graph of a uniform *Tonnetz* of scales generated by 2, 2, 2, 2, 2, 1, 1. The scales are labeled using Hook (2011)'s system for spelled heptachords, which counts the number of sharps and flats needed to spell the collection with seven distinct letter names. For instance {G♯, A, B♭, C, D, E, F♯} is the 1♯ whole-tone-plus-one scale, because it is spelled with two sharps and one flat, and the difference then is one sharp.

scale types. The cyclic dimension is Ph_5 and a spine of connected diatonic collections tours the cycle with short detours to acoustic collections and long detours to whole-tone-plus-one collections. (Note, however, that, as with the other graphs in this section, only the horizontal positions labeled with the cyclic axis reflect the *Tonnetz* geometry. The vertical positions are chosen for convenience to separate collections that project onto the same Ph_5 location.)

An interesting feature of maximally even patterns like the diatonic, however, is that they can be defined as two-interval generated patterns through multiple distinct pairs of generators. For example, we could treat the thirds as generators of the diatonic, constructing a scalar *Tonnetz* with a tertian logic. As Meredith (2011) observes, this captures what are typically taught as the standard tonal scales, including the harmonic minors and majors (4-4-3-4-3-3-3) as well as the melodic minor/acoustic (4-4-3-3-4-3-3), but not the whole-tone-plus-one. In addition, this *Tonnetz* includes a more exotic type of collection including a diminished second (i.e., enharmonic variant of a single pitch class), 4-4-4-3-3-3-3, e.g., C-D♯-E♭-F♯-G-A-B.

5. Spherical *Tonnetze* and other topologies

Each of the n-dimensional *Tonnetz* geometries constructed includes a collection of hyperplanes with $(n-1)$-dimensional *Tonnetze* that make up the boundaries of the regions of the n-dimensional *Tonnetz*, corresponding to the subsets of the set classes involved. For instance, regions of the (0258)/(0358) *Tonnetz* are bounded by planes with (025), (026), (036), and (037) *Tonnetze*. In the n-torus (unfolded) *Tonnetz*, these sub-*Tonnetze* are

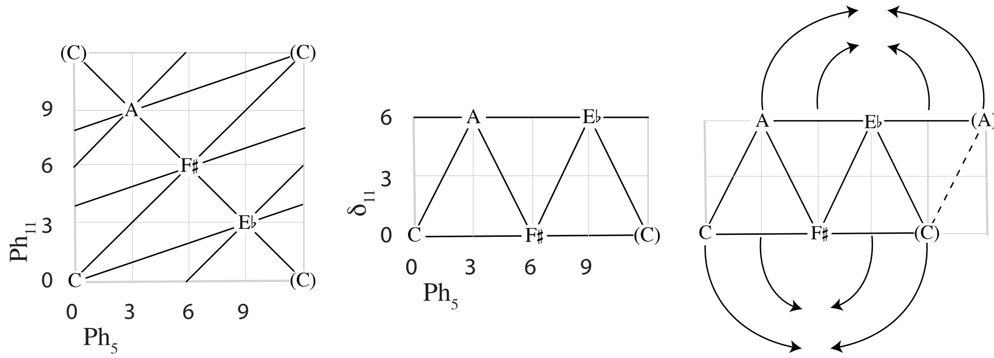

Figure 22. (036)-*Tonnetze* at different levels of folding (toroidal, cyclic, and spherical)

$(n-1)$-toruses, and they retain this topology when a single dimension of the n-torus is folded.[19] With higher-order foldings, the homologies of the sub-*Tonnetze* are simplified in the same way as the parent *Tonnetz*. For instance, the uniform pentatonic *Tonnetz* of Figure 19 is a network of 4-simplexes (pentatopes) folded to have only one cyclic dimension (represented by Ph_5). The sub-*Tonnetze* are tetrahedral, with (0246) and (0258) *Tonnetze* making the boundary of the space, (0257) making the boundary between the two pentachord types, and (0257) as the boundary within the central cycle of pentatonic scales. Each of these tetrahedral *Tonnetz* are also cyclic in one dimension.

What happens, then, in the extreme case discussed in section 2.5, of a totally bounded *Tonnetz* on a perfectly even collection? Figure 22 shows the process undergone by an (036) sub-*Tonnetz* at different levels of folding of the parent diminished seventh *Tonnetz*. It begins as a 2-toroidal *Tonnetz* and remains so with one dimension folded (i.e. when the diminished sevenths contain three different types of minor thirds based on distinctions in two dimensions, such as $\mathrm{Ph}_{1,5}$ or $\mathrm{Ph}_{3,5}$). With two foldings, when the diminished seventh *Tonnetz* becomes a cyclic network of four tetrahedral regions, the diminished triad *Tonnetz* is also reduced to a simple cyclic network of triangles bounded by tritone axes, as shown in the middle panel of Figure 22. In this (036) *Tonnetz* there are no longer distinct types of minor third, but there are two kinds of tritone. For instance, if Ph_5 is the cyclic dimension, the sum of two minor thirds is a diminished fifth, but the remainder of two minor thirds is an augmented fourth In other words, all of the (036)s share two common tones, but those that share a common ic3 are adjacent in the *Tonnetz*, whereas those that share a tritone are not. When the parent diminished seventh *Tonnetz* is reduced to its simplest form, as a single tetrahedral space with no duplications, these duplicated tritones are also equated, as shown on the right side of Figure 22, resulting in an (036) *Tonnetz* that has, as the surface of a tetrahedron, a spherical topology. This means that it is a compact surface with a trivial fundamental group. In other words, this (036) *Tonnetz* is the logical extreme of removing the homological elements of a toroidal *Tonnetz*.

A spherical surface can be illustrated in stereographic projection, as on the left of Figure 23, which places one pole in the center, the opposite pole in a circle around the

[19]Although they may be reduced or mapped onto one another. For instance, in the 3-torus version of the seventh-chord *Tonnetz*, there are two distinct planes with (025) *Tonnetze* and two with (037) *Tonnetz*. These are mapped onto one another in the folded version. Each (036) *Tonnetz* has eight triangles as a plane in the 3-torus, but only four in the fattened 2-torus.

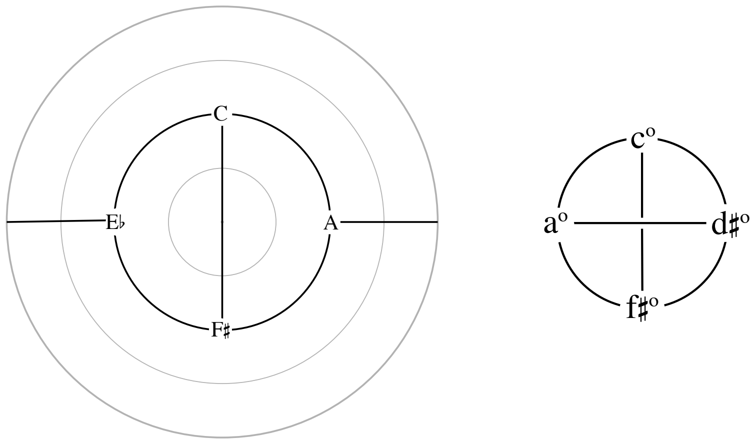

Figure 23. The tetrahedral *Tonnetz* in stereographic projection, and its *Tonnetz* graph

perimeter (i.e. this circle actually corresponds to a single point), and shows longitude lines as radii. We can see from this how transpositions by minor third and tritone map onto rigid symmetries of the sphere. The graph of the spherical (036) *Tonnetz*, on the right of Figure 23, is different than the graph of the cyclic and toroidal (036) *Tonnetze*. While all duplications of (036) trichords are already removed from the cyclic version (the one in the middle of Figure 22), the *Tonnetz* graph adds additional edges linking the tritone-related (036)s. The result, then, is a complete graph on four vertices (the graph-theoretic analog of a 3-simplex).

The construction of the tetrahedral *Tonnetz* can be generalized to higher-dimensional simplices, creating a family of n-dimensional hyperspherical *Tonnetze*. However, as a trichordal, 2-dimensional, *Tonnetz*, the construction is unique to the (036) set class (or, equivalently, a ♩ ♩ 𝅗𝅥 rhythm), because it is the only generated trichord with the remainder interval of a tritone, which can be equated with its inversion. Or, we might observe that the particular symmetries of the sphere used to satisfy the transposability criterion are order-4 and order-2, so the only available intervals are $8^{\text{ve}}/4$ and $8^{\text{ve}}/2$. However, there is one other way to construct a spherical trichordal *Tonnetz* with a different symmetry group.

One difference between the spherical topology and all the other *Tonnetze* we have constructed by folding n-dimensional phase spaces is its Euler characteristic, χ, which is an important topological invariant. This is defined as $\chi := V - E + F$, where V is the number of vertices in the simplicial complex, E the number of edges, and F the number of faces. The Euler characteristic is a topological property shared by the simplicial *Tonnetz* and the geometric *Tonnetz*. For all torii, $\chi = 0$. The folding process defined in section 2 preserves χ, because it removes an equal number of edges and faces. But the one illustrated in Figure 22 removes two edges without removing any faces. This simplicial complex therefore has the Euler characteristic of a sphere, which is $\chi = 2$. To remove an edge by folding without changing the vertices, we need a cycle with exactly two vertices on it – i.e. a tritone axis. To get $\chi = 2$ we need exactly two of these, which occurs only in the (036) or equivalent *Tonnetz*.

However, there is one other way to change χ with V fixed, and that is to add faces without adding edges. Consider the (024) whole-tone *Tonnetz* in Figure 24. After being folded to a cyclic trichordal *Tonnetz*, it has a major-third boundary, which has exactly three vertices and three edges. That is, understood as a *graph*, there is a 3-clique (triangle)

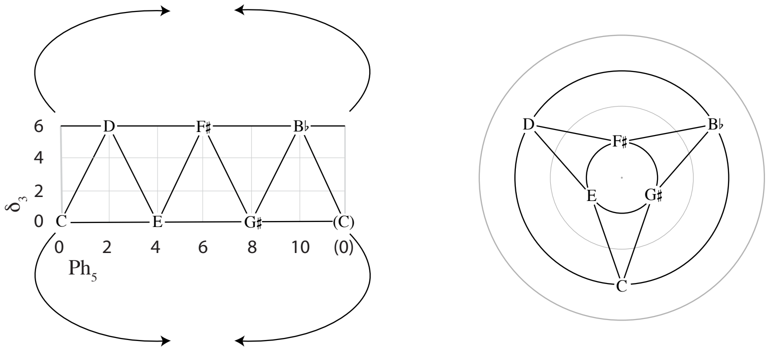

Figure 24. Derivation of the octahedral *Tonnetz* from a cyclic *Tonnetz*

corresponding to each augmented triad. Geometrically, however, the three edges are colinear, so they do not actually define a region in the space. If such a region did exist, as Figure 24 shows, the *Tonnetz* geometry would be spherical.[20] Nothing in the phase space corresponds to this region, though – that is, we can see that such a geometry, with an octahedral symmetry group, could exist in principle, but we do not yet have a musical interpretation of it.

We find a related phenomenon, however, which does have musical significance, if we return to voice leading as the basis of our geometry, specifically trichordal voice-leading *Tonnetz* where u is divisible by 3, such as the standard triadic *Tonnetz*. Cohn (1997) initially linked the standard triadic *Tonnetz* to the concept of voice leading while formalizing it as a network of common-tone relationships, yet Tymoczko (2012) ultimately demonstrated the full repercussions of the voice leading conception of the *Tonnetz* using voice-leading geometries. Recall from section 2 that geometrical *Tonnetze* can be constructed in dual voice-leading spaces. In the trichordal case, points in the dual space correspond to planes in the voice-leading space, while points in the voice-leading space correspond to planes in the dual space. The dual space therefore has a point for each pitch-class (corresponding to the voice-leading plane that holds that pitch-class constant), and each set of three pitch-classes in the dual space defines a plane, which corresponds to a point in the voice-leading space belonging to the given trichord. Motions along that dual-space plane correspond to rotations about the point for that trichord in voice-leading space. We therefore have a dual-space plane for each major and minor triad, and can define triangular regions on each of those planes bounded by the trichord intervals.[21] Rotations of these planes around shared dyads ("flips") then correspond to edges in the chord-based network (a *Tonnetz* graph) in voice-leading space. Therefore, piecing together all of these regions we have a two-dimensional subspace of the three-dimensional dual voice-leading

[20] Reasoning from χ can thus show that these are the only two examples of two-dimensional spherical *Tonnetze*. We could also observe that the existence of a spherical *Tonnetz* satisfying the transposability condition implies the existence of a Platonic solid with u vertices and whose symmetry group has a cyclic subgroup of order u. By process of elimination we are left then with only these two, tetrahedral and octahedral. For example, there is no cubic *Tonnetz* because the order-8 subgroups of the symmetry group of a cube are not cyclic.

[21] This can be done by specifying that the balanced voice leading plane occurs within the region, which is then defined as a convex hull around that point. Note that the balanced voice leading is the same plane for three major-third related triads, so these regions all cross one another in the space. For instance, the C major region and E major region cross in a line that corresponds to balanced voice leadings that hold E constant, C major and A♭ major cross in a line of balanced voice leadings holding C constant, and all three planes intersect in a point that corresponds to balanced voice leadings between trichords of sum class 11 ($= 0+4+7 = 11+4+8 = 0+5+8 \bmod 12$). The illustration in Tymoczko 2012 redraws the *Tonnetz* to eliminate these visually confusing intersections, which is helpful for visualizing its structure, but somewhat obscures the underlying geometry.

space corresponding to the traditional *Tonnetz*. Technically this subspace is not a surface because it intersects itself in multiple places, and for the same reason it does not properly satisfy the definition of a *Tonnetz* geometry given in section 1. This situation is analogous to that of the dyadic voice-leading *Tonnetz* in section 2.1.

This voice-leading version of the traditional *Tonnetz* gives a distinct geometrical meaning to the same *Tonnetz* graph and simplicial *Tonnetz* as the phase-space version described in section 3. The equivalent *Tonnetz* graphs result from the same general correspondence between Ph_n and the n-note voice-leading space that we found in uniform *Tonnetze* for maximally even collections in section 4: for relatively even chords, Ph_n approximates a projection onto the central axis of the n-note voice-leading space. As a result, the same basic *Tonnetz* graph can be geometrically realized in two distinct ways.

However, because major and minor triads are *not* maximally even, the voice-leading space we have constructed for the traditional *Tonnetz* is not a satisfactory, or at least an incomplete, representation of triadic voice leading. As Tymoczko (2011, 2012) points out, the traditional *Tonnetz* inaccurately represents two-semitone voice leadings like {CEG} → {CFA♭} or {CED} → {C♯EG♯} as relatively large as compared to the two-semitone "relative" voice leading, e.g. {CEG} → {CEA}. This is because a voice leading like {CEG} → {CEA} passes through an augmented triad {CEG♯} which has not been included in the network. Including these as a necessary intermediary between all two-semitone voice leadings between major and minor triads gives a network of single-semitone voice leadings that accurately represents distances in the voice-leading space. Adding augmented triads to the network is similar to doing so in the octahedral *Tonnetz* of Figure 24 above: each augmented triad adds a face to the space without adding any new vertices or edges. Therefore, the Euler characteristic of the standard triadic *Tonnetz*, $\chi = 0$, increases by 4 when considered as a proper voice-leading *Tonnetz*, even without accounting for the intersections of the major- and minor-triad regions. According to the classification theorem of 2-manifolds, there is no way to consider such a simplicial complex as belonging to a surface (a connected compact 2-manifold), because the maximum Euler characteristic of a surface is that of a sphere, $\chi = 2$.

Tymoczko's *Tonnetz* therefore raises an interesting question about topology and musical meaning: what is the significance of a *Tonnetz* being embeddable in a manifold? A manifold is defined as a topology that is locally Euclidean – specifically, in a 2-manifold, every point has a region that is homeomorphic to the Euclidean plane. The triadic voice-leading *Tonnetz* fails this criterion in multiple places: at the major third intervals, which border three triadic regions, and on the intersections within triadic regions that share a common pitch-class sum. The locally Euclidean feature might be likened to reading a map: while it may be impossible to assign consistent orientations globally (for instance, on the surface of the earth), if we restrict our attention to a local region, we can imagine that the dimensions are fixed and extend infinitely. In the standard *Tonnetz*, this imagined Euclidean space is what Harrison (2002) called the "unconformed" *Tonnetz*. This is clearly how Riemann thought of the *Tonnetz* (Gollin 2011). As it was originally conceived, as a map of just tuning relationships, it is in fact a noncompact Euclidean space, but Riemann applied it in contexts where no such literal tuning differences were present. Rather the Euclidean space is imagined, which is possible to do on the toroidal *Tonnetz* as long as we maintain that only a limited extent of the space is "visible" from a given point at one time. Clearly analysts following Riemann have found this locally Euclidean aspect of the *Tonnetz* to be of great value, particularly when talking about tonal harmony from a perceptual standpoint. When Rings (2011), for instance, talks about Riemann's "intentional paths" he explicitly models the perception of tonal harmony as something locally Euclidean. In fact, the qualities of brightness-darkness and dominantness-subdominantness Rings discusses track closely with the dimensions of the $\mathrm{Ph}_{3,5}$ space that can embed the standard *Tonnetz* and is posited as a perceptual space for tonal harmony by Krumhansl (1990). Cohn (2012, 64–67) explicitly advocates for the standard *Tonnetz* on the grounds of its locally Euclidean properties, even while maintaining that he is primarily interested in voice leading and that the *Tonnetz* does not accurately model this, for exactly the reasons that Tymoczko points out.

What Tymoczko shows us, then, is that when using the *Tonnetz* in this way, we cannot also imagine that it is a map of voice-leading relationships, at least not in the strict sense of voice leading modeled by voice-leading geometries. We can think of it as embedded within a two-dimensional phase space like $\mathrm{Ph}_{3,5}$ with the structure of a manifold, but it is important then to recognize chords as instances of pitch-class distributions, a different sort of musical object than the chords of voice-leading spaces. Indeed, this does seem to be native to, e.g., Cohn's thinking, when he says,

> There are advantages to maintaining individual tones as primary objects, rather than prepackaging them into triads. The *Tonnetz* can track continuities among individual pitch-classes, yielding analytical information that would otherwise be difficult to recover, and provides locations for other pitch-class combinations that arise in nineteenth-century music. These include consonant dyads and dissonant seventh chords that would otherwise need to be referred to a consonant triad via expansion or contraction, forcing interpretations that might be underdetermined or even arbitrary. (Cohn 2012, 67)

Cohn, that is, views triads as existing in a space that mixes chords of different cardinality, which is precisely what a space of pitch-class distributions does. One can make limited claims about voice leading in such a space, grounded in the mathematical association of Ph_n with the central axis of n-note chord space that holds for relatively even chords, but it is important to recognize that this does not amount to a robust theory of voice leading in Tymoczko's sense, and that if we want to talk about voice leading in this way, we cannot have a *Tonnetz* with the structure of a manifold.

Analogous points can be made about voice leading between seventh chords. As mentioned in section 3 above, Douthett (1997) constructs a *Tonnetz* of dominant, half-diminished, and minor seventh chords, generated by 3, 3, 4 (remainder 2). When folded to equate the minor thirds, the boundaries of this *Tonnetz* are two whole-tone planes containing the (026) subsets of the dominant and half-diminished chords. At this stage, the seventh-chord *Tonnetz* has a toroidal topology, with a bounded third dimension. However, it still has duplications of the tritones, which can be identified to create a spherical sub-*Tonnetz*, as illustrated in Figure 22 above.[22] When the tritones are identified, the folded phase space now has fully connected (0268) tetrachords ("French sixth chords"), and to continue to think of it as a 3-manifold, we would have to understand these (0268) simplices as regions (adding six faces to compensate for the six edges), which they are not in the original phase space. As a folded phase space, this seventh chord *Tonnetz* now only has one non-trivial homological element, a cyclic homology associated with Ph_4, oblique to the spherical (036)-*Tonnetze*, which is also the phase-space dimension that mimics a voice-leading axis for four-note chords. This folded phase space also has fully connected

[22] A three-dimensional toroidal *Tonnetz* was also proposed by Gollin (1998), but, lacking the minor seventh chords, it is impossible to think of Gollin's *Tonnetz* as actually filling any space, including a three-dimensional toroidal one. Tymoczko (2012) points out that Gollin did not really explicitly realize the geometry of his *Tonnetz*, and therefore he might have been thinking of something like a voice-leading *Tonnetz*, in which case the tritone duplications in his illustration would be superfluous. If on the other hand, he had something like Douthett's *Tonnetz* in mind, as he suggests in referring to the geometry as toroidal, the tritone duplications are necessary, and we could give a retroactive theoretical justification for them along the lines suggested in Yust 2018a, using phase spaces. Nevertheless, Tymocko's critique illustrates the importance of establishing clear theoretical foundations when proposing something like a geometry of chord relations.

diminished seventh chords. They do not define regions, however, just as the augmented triads of the toroidal triadic *Tonnetz* are fully connected but do not define regions. If were to treat them as such, we would obtain a *Tonnetz* graph equivalent to Tymoczko's (2011) seventh-chord voice-leading lattice. As in other cases, this voice-leading lattice can be realied geometrically in the dual voice-leading space. The seventh-chord regions of this voice-leading *Tonnetz* intersect where they share pitch-class sums, analogous to the 2- and 3-dimensional cases, and have singularities at (036) faces, which belong to three regions (dominant, half-diminished, and diminished). The diminished seventh chords are essential to accurately reflecting voice-leading distances because they are intermediaries in progressions like G^7-$b^{\varnothing 7}$, analogous to triadic relatives like C-a.

6. Topology of musical concepts

Tonnetz graphs, as defined at the outset of this article, are easy to produce and admit of many possibilities, and a variety of kinds of *Tonnetze* and *Zeitnetze* have been illustrated here. Using phase spaces, all such networks can be embedded in musical geometries, and we have found such geometries to relate to a number of significant existing musical concepts. Due to the special role of simplicial complexes in the study of topology, we have been able to readily apply this method to revealing some of the topology of musical concepts, such as the cyclic homology of diatonicity and harmonic function. More complex topologies can create musical heteromorphs, distinct meanings assigned to identical pitch-class intervals, but our geometries can also forget these differences, altering the global homology of the space but without loosing the local topology, the common-tone based concept of proximity and the local directionality of the parent phase spaces.

A notable example of a particularly heteromorphic musical object is the fully diminished seventh chord. Through variability in is orientation in pitch space, the circle of fifths, and functional space, it can potentially support a surprisingly large *Tonnetz* based entirely on changes of interpretation of a single chord. Such a *Tonnetz* might be useful in rationalizing the many ways diminished sevenths can be reinterpreted in nineteenth century music, not only enharmonic changes but also changes of function, such as common-tone function versus its standard dominant function, and more distant types of reinterpretation can be distinguished from less distant ones. The historical importance of the diminished seventh chord in giving rise to richly ramified heteromorphic play in chromatic harmony may be explained, in fact, by the role of $\mathrm{Ph}_{3,5}$-space as a basic syntactical space of keys and harmonic functions. As a perfect representative of the f_4 dimension of Fourier space, the diminished seventh is completely oblique to the $\mathrm{Ph}_{3,5}$ plane, and therefore has a unique ability to shape-shift within tonality.

Phase spaces are not the only way to impart geometrical meaning to the abstract simplicial networks of generalied *Tonnetz*, but they are particularly well suited to this task because, as a space of tonal distributions, proximity in Fourier space derives directly from common-tone content. Because of this, they offer a particularly general way of geometrically embedding possible *Tonnetze*. The other prominent way of imparting musically meaningful geometry to a *Tonnetz* is through voice-leading geometry. While this is also in principle somewhat generalizable, the desire for network distances to reliably reflect voice-leading distances gives rise to a more limited set of possibilities. Where the two give distinct musical meaning and topology to the same networks, we have a particularly interesting point of contact between theories based on very different basic principles.

It is worth reflecting here that a 12-dimensional vector space on pitch-class distributions can be defined directly, with each dimension corresponding to the weighting of one pitch-class. Because the Fourier transform is actually a change of basis, however, this space is actually the same as the larger Fourier space. What the Fourier transform allows us to do is to project this larger space into one of fewer dimensions, such as our phase spaces, which forget, in particular, differences between pitch-class sets that relate specifically to transposition type. Phases spaces are therefore not in a one-to-one relation with pitch-class distributions, but we can define canonical paths between points (such as simple interpolation, or "pitch-class cross-fade") that work in all phase spaces.

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