Probing Questions about Keys: Tonal Distributions through the DFT

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Abstract. Pitch-class distributions are central to much of the computational and psychological research on musical keys. This paper looks at pitch-class distributions through the DFT on pitch-class sets, drawing upon recent theory that has exploited this technique. Corpus-derived distributions consistently exhibit a prominence of three DFT components, f_5, f_3, and f_2, so that we might simplify tonal relationships by viewing them within two- or three-dimensional phase space utilizing just these components. More generally, this simplification, or filtering, of distributional information may be an essential feature of tonal hearing. The DFTs of probe-tone distributions reveal a subdominant bias imposed by the temporal aspect of the behavioral paradigm (as compared to corpus data). The phases of f_5, f_3, and f_2 also exhibit a special linear dependency in tonal music giving rise to the idea of a tonal index.

Keywords: tonality, key finding, DFT, phase space, probe tone

1 Introduction

Few studies in music psychology have stimulated as much interest and debate as Carol Krumhansl and Edward Kessler's 1982 article on tonal hierarchy ([12]). While it is important for establishing the probe-tone technique as a behavioral correlate of the sense of key—and the consequent focus on pitch-class distributions in research on the topic—central also to its impact, one suspects, was the visualization of key relationships by deriving a toroidal space from the probe-tone data. This two-dimensional toroidal geometry of key distances, derived by applying multi-dimensial scaling (MDS) algorithms to the correlations between pitch-class distributions, was not necessary to establishing the efficacy of the probe-tone method, nor was it a necessary component of the distributional model or subsquently developed key-finding algorithms (which use correlations between distributions directly, not filtered through the two-dimensional simplification of the MDS solution). Yet the validation of common habits of thought and language relating to musical keys, spatial metaphors of distance, direction, and region, and of widely used theoretical models (such as the Schoenberg-Weber chart of regions and the Tonnetz) kindled the imaginations of a wide range of subsequent researchers.

Spatial models raise a number of significant questions about the nature of musical keys, many of which have been examined in the music perception and cognition literature on the topic. This paper demonstrates that the discrete Fourier transform (DFT) on pcsets can clarify these questions and in some cases suggest novel solutions.

Krumhansl ([11], 99–106) noted that the spatial representation of keys in Krumhansl and Kessler 1982 could be reproduced, without recourse to MDS, by taking the third and fifth phase components of the Fourier analysis of the key profiles. For Krumhansl this theoretical reformulation of the space is primarily an expedient allowing for the plotting of various kinds of information (expert key assignments, distributional data in the music) in a fixed space. The practical problems can be overcome by clever use of computational techniques like self-organizing maps, as [13] and [16] have shown. But, as I will argue here, Krumhansl's simplification using the DFT is of considerable theoretical interest in its own right, especially in light of more recent applications of this same type of space ([3], [4], [28], [29]). In particular, basic mathematical properties of the DFT allow us to draw more far-reaching conclusions about this space and its significance to the nature of tonality.

Much of this research has produced different kind of pitch-class distributions that can be analyzed using the DFT on pitch-class vectors, as described by [15], [19], [4], [27], and [29]. The terminology used here is taken from [27]. The entries in the DFT vector are referred to as "components" and denoted $f_0, f_1, f_2, \ldots$. They are converted to polar coordinates with magnitude $|f_n|$ and phase ϕ_n, but with phases converted to a pitch-class scale and designated $Ph_n = 2\pi(\phi_n)/12$

2 Tonal Distributions

The large body of research that has grown out of Krumhansl's work has produced an abundant crop of tonal distributions. These come in two or three basic forms. Krumhansl and Kessler's ([12]) original distributions are probe-tone ratings from human subjects. Subsequent studies, such as [7], [6], and [22] applied the probe-tone technique in varying contexts, or other experimental tasks that produce comparable distributional data, such as the wrong-note detection technique used by [9]. Another method that has produced many distributions for key-finding algorithms (further discussed in the next section) is to derive distributions from the frequency of occurrence of scale degrees in a corpus. Finally, other distributions (e.g. in [23] and [21]) are created "by hand" to optimize the performance of key-finding algorithms.

Figs. 1–3 plot distributional data from a variety of sources in three different Fourier phase spaces. (The locations of all major and minor triads and two diatonic scales are also given for reference.) Fig. 1 shows part of the Ph_3/Ph_5 space used by Krumhansl ([11]) and which is also the basis of Amiot ([3], [4]) and my ([28]) continuous Tonnetz. Despite a great variety of techniques represented by corpus-derived distributions—using entire pieces with or without accounting for modulations, using just the initial or final measures, using melodies only or polyphonic textures, counting pitch-classes in different ways—all are bunched very closely together, near the tonic triad but typically with a slightly higher Ph_5, possibly reflecting a bias towards the dominant. Temperley's ("CBMS") and Sapp's hand-made distributions are close enough to these, but lie on the fringes of the pack.

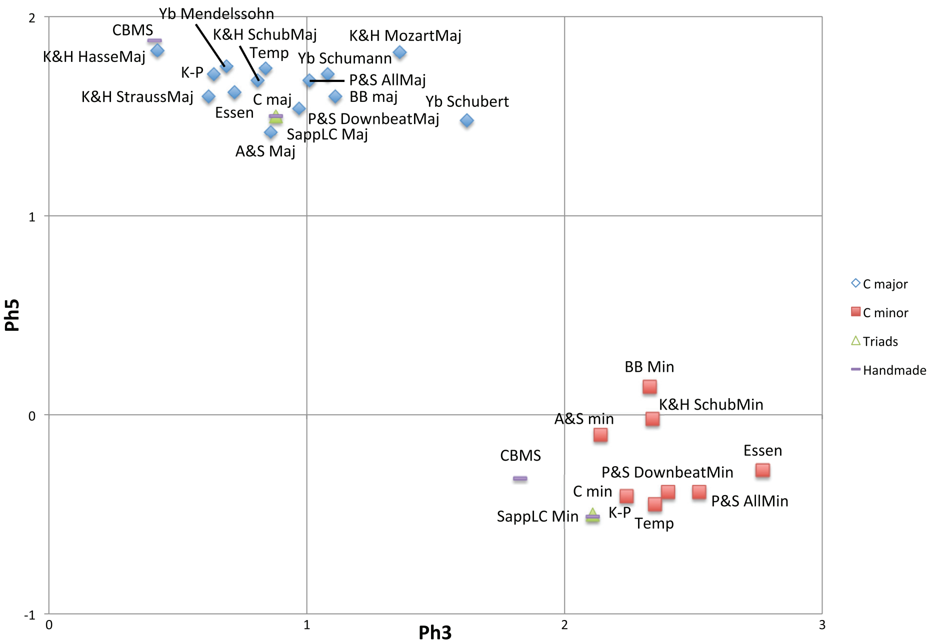

Fig. 1. $Ph_{3/5}$ plot of corpus distributions from a variety of sources (Yb: [26], K&H: [10], K-P, Essen, Temp: Kotska-Payne, Essen, and Temperley corpora from [24] , BB: Bellman-Budge from [21], P&S: [18], A&S: [2])

Fig. 2 plots the same data with Ph_2 replacing Ph_3. Major-key data spreads out a little more in the Ph_2 dimension, but on the whole we can reach the same conclusions. The interchangeability of Ph_2 and Ph_3 relates to a basic property of tonality, the tonal index, explored further below. Other components do not provide the same kind of essential tonal information, as the $Ph_{1/4}$ plot in Fig. 3 illustrates. Major-key data are particular unfocused in the Ph_4 dimension and the minor-key data in the Ph_1 dimension. Even where a certain amount of consistency might be found, such as the minor-key profiles in the Ph_4 dimension, it is closer to unrelated triads like B major, B minor, and D minor.

The probe-tone profiles are more variable, but reliably close to the corresponding corpus data. Fig. 4 gives a variety of major-key probe-tone data reflecting a variety of experimental paradigms. Cuddy and Badertscher ("C&B") include major-triad and major scale contexts on three levels of musical back-

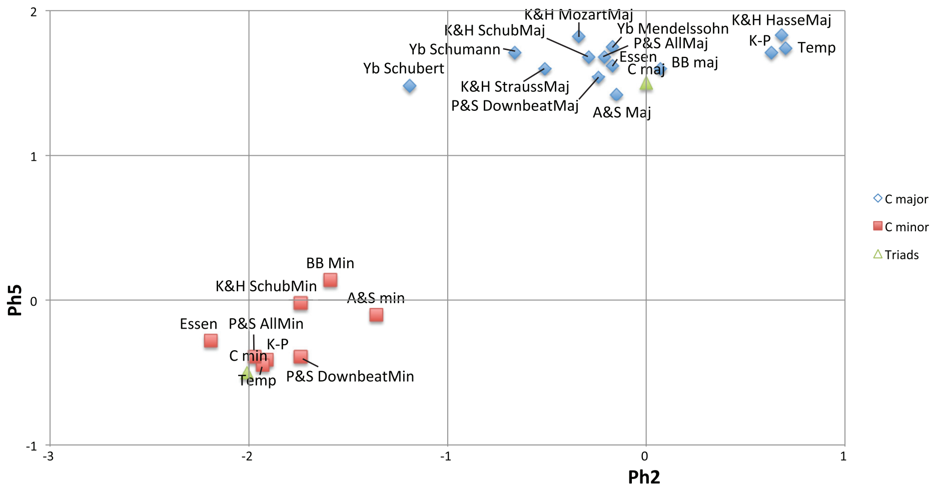

Fig. 2. $Ph_{2/5}$ plot of corpus distributions.

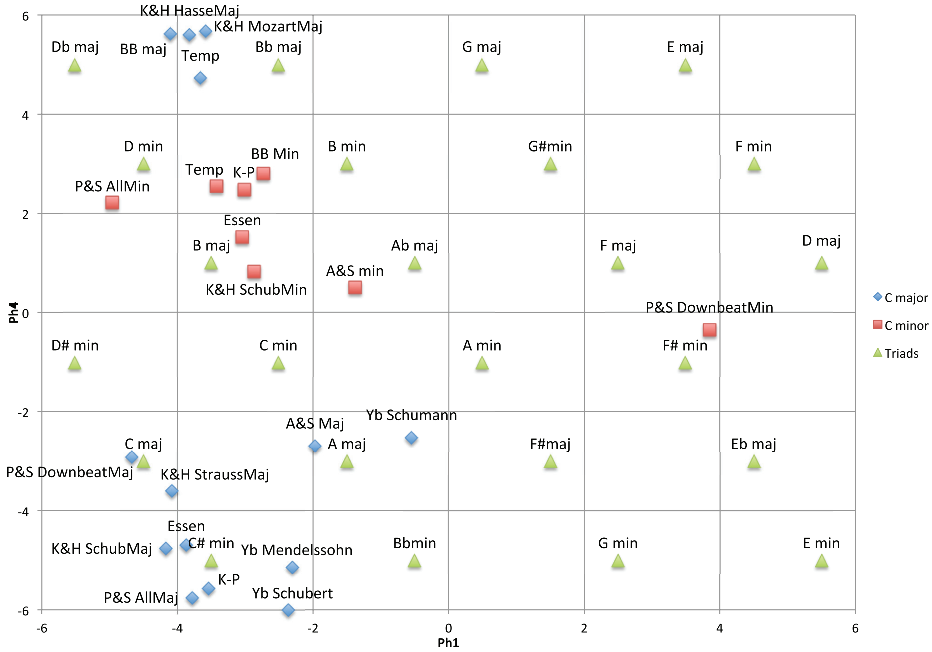

Fig. 3. $Ph_{1/4}$ plot of corpus distributions.

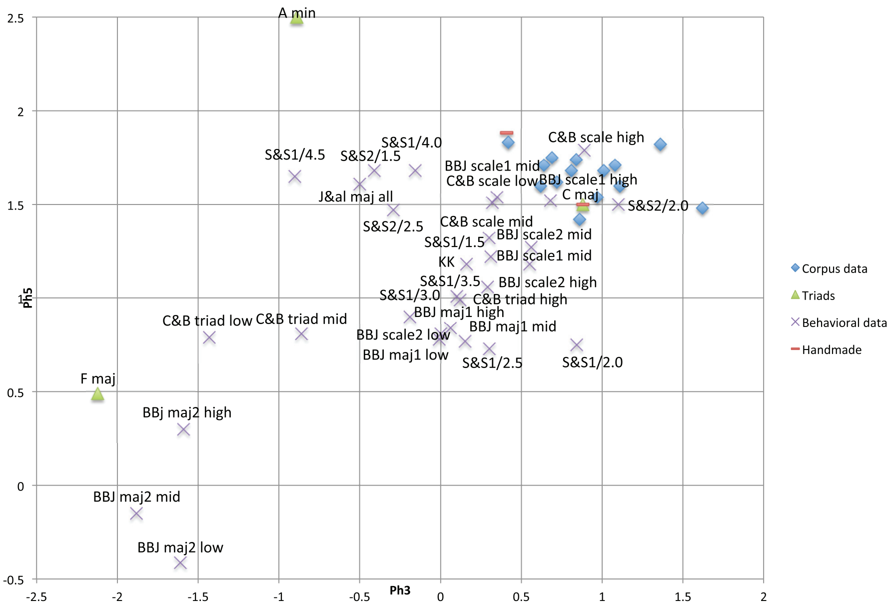

Fig. 4. $Ph_{3/5}$ plot of corpus distributions and probe tone distributions from a variety of sources (KK: [12], C&B: [7], BBJ: [6], S&S: [22], J&al: [9]).

ground. Brown, Butler, and Jones ("BBJ") replicate these ("triad1," "scale1") and also test contexts that reorder the tones of each ("triad2," "scale2"). Smith and Schmuckler randomly generate contexts using Krumhansl and Kessler's profiles weighting tones by duration ("S&S1") or frequency ("S&S2") at varying levels. Janata et al. ("J&al") use a very different method of wrong-note detection. Despite such differences in experimental paradigm, these data very consistently deviate from the corpus data on the subdominant side. The difference may result from the temporal aspect of the probe-tone task: listeners evaluate, not a note merely *in* the given context, but *after* it, and motion to the left in Ph_3 (descending thirds or fifths) is much more typical of tonal music than to the right, particularly at endings and moments of resolution. Particularly striking is Brown, Butler, and Jones's reordering of the arpeggiated triad, which appears to consistently imply F major more strongly than C major.

To examine the matter more closely, let us focus on a single, fairly rich, body of corpus data collected by Prince and Schmuckler ([18]). Tables 1 and 2 show the DFTs for their data collapsed over metric position but divided by composer.[1] These data are average tonal profiles for each composer, with all pieces transposed to C major or C minor, but with no accounting for modulations. The data for Bach, Mozart, Beethoven, and Chopin represent relatively large sam-

[1] Jon Prince generously shared this raw data through personal correspondence.

ples (between 20,000 and 120,000 quarter notes for each data point) while those for Schubert, Liszt, Brahms, and Scriabin are smaller (1700–16,000). Despite the wide range of harmonic styles represented, one very clear conclusion can be drawn from the DFT magnitudes: With one exception, f_5 is always very large, followed by f_3, then f_2. This agrees with results from [8] whose ic5 category may be roughly equated with $|f_5|$ through Quinn's [19] "intervallic half-truth."[2] The last three components are negligibly small, in most cases less than 1% of the total "amplitude" of the distribution. The one exception is Liszt-major, which differs from all the other distributions in that f_3 and f_5 are equally prominent. While this may point to something special in Lizst's harmonic style, we should not make too much of this distribution, since it represents only three pieces (Grande étude de Paganini 4, Liebesträume 3, and Transcendental Etude 5).[3] One other discernable stylistic difference is the greater emphasis on diatonicity (f_5) in Bach versus all later composers. This is particularly pronounced in the minor mode, where later composers typically put more weight on f_2 and f_3 at the expense of f_5.

Table 1. DFTs of corpus data from Prince and Schmuckler, [18], for major keys. Squared magnitudes are multiplied by 10^4.

Composer	$\lvert f_1\rvert^2$	$\lvert f_2\rvert^2$	$\lvert f_3\rvert^2$	$\lvert f_4\rvert^2$	$\lvert f_5\rvert^2$	$\lvert f_6\rvert^2$	Ph_1	Ph_2	Ph_3	Ph_4	Ph_5	Ph_6
Bach	2	79	150	10	2095	2	9.89	0.96	0.38	3.74	1.96	6
Mozart	9	243	310	1	2158	4	9.34	11.91	0.96	8.04	1.66	0
Beethoven	4	182	287	7	1427	0	8.31	11.53	1.34	6.43	1.53	6
Schubert	7	127	337	18	1931	0	8.25	0.18	1.06	7.70	1.65	0
Chopin	10	186	357	11	1638	1	7.43	11.72	1.03	8.22	1.26	0
Brahms	2	49	224	5	1009	2	7.74	0.27	0.68	9.37	1.59	0
Liszt	5	85	402	30	394	43	6.21	0.12	0.68	9.37	1.59	0
Scriabin	11	117	352	47	2154	2	8.93	1.09	0.89	7.12	1.93	6

The generally low magnitudes of f_1, f_4, and f_6 explain another feature of the distributions: the lack of consistency in phases for these components. Phase values should become more volatile as the magnitudes approach zero where the phase becomes undefined. However, it is logically possible to expect variability in phase values for the well-represented components (f_2, f_3, and f_5). Such variation

[2] There is also agreement on the minor-key data which shows smaller f_5s and correspondingly fewer ic5-category designations. The DFT data is less equivocal on the secondary features of tonality, however, which clearly relate to f_3 and f_2. This surfaces in Honingh and Bod's results in the form of ic3- or ic4-category pcsets, but it is hard to draw as clear-cut a conclusion from this aspect of their results.

[3] With the assistance of Matthew Chiu, I have recently assembled a larger data set of distributions that confirms these conclusions, including the pronounced low diatonicity of Liszt's music, especially in the minor mode where it reaches a level approximately equal to that of f_3. The tendency can also be seen in Wagner and Scriabin, but not quite as strongly as in Liszt.

Table 2. DFTs of corpus data from Prince and Schmuckler, [18], for minor keys. Squared magnitudes are multiplied by 10^4.

Composer	$\|f_1\|^2$	$\|f_2\|^2$	$\|f_3\|^2$	$\|f_4\|^2$	$\|f_5\|^2$	$\|f_6\|^2$	Ph_1	Ph_2	Ph_3	Ph_4	Ph_5	Ph_6
Bach	8	69	208	49	1489	3	8.72	10.80	2.29	2.60	11.60	6
Mozart	17	195	656	10	1515	6	8.97	10.53	2.24	0.15	11.91	6
Beethoven	2	239	457	5	1150	16	6.33	10.18	2.51	1.73	11.70	6
Schubert	9	238	540	14	1815	1	9.20	10.47	2.62	11.76	11.93	0
Chopin	0	149	336	6	1002	6	6.57	10.36	2.39	2.21	11.88	6
Brahms	9	194	390	2	847	19	8.23	9.89	1.87	2.14	11.68	6
Liszt	0	254	651	1	1179	14	4.83	10.75	1.75	1.30	0.40	6
Scriabin	5	237	399	48	1524	19	10.10	10.88	1.25	0.61	11.91	0

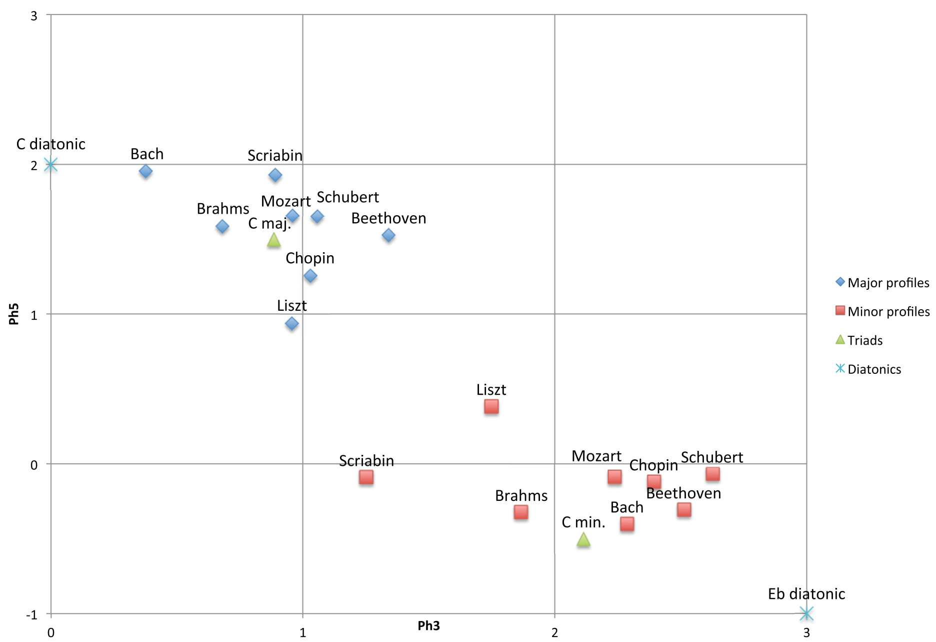

Fig. 5. $Ph_{3/5}$ plot of corpus distributions from Prince and Schmuckler, [18].

could reflect real differences between composers, who represent a wide range of styles. On the whole, however, we do not see much variability in these phases, and the data, as in Fig. 1, tends to cluster close to the Ph_2, Ph_3, and Ph_5 values for the tonic triads, as can be seen in Fig. 5. The only stylistic differences evident here are the greater tendency of Bach's distributions toward the diatonic scales, especially in major, and the opposite tendency of Liszt's distributions toward the parallel keys.

This striking result suggests a signal processing analogy to explain tonality as a kind of band-pass filter for pitch-class information. The tonal filter supresses certain frequencies (f_1, f_4, and f_6) while amplifying others (f_2, f_3, and f_5). As a definition of tonality, this has the advantage that it can be treated either as a property of music or as a way of hearing or interpreting music. That is, to the extent that music is tonal, it will tend to feature harmonic content that emphasizes f_2, f_3, and f_5, and tonal interpretations of music are those that filter out f_1, f_4, and f_6, possibly with disregard for a prominent status for one of those components. For instance, octatonic music (such as certain pieces by Messiaen) will have a prominent f_4, but a tonal interpretation of octatonic music will suppress this feature in order to amplify f_2, f_3, and f_5, which may be controlled by choice of subsets or emphasized notes within the given octatonic context. This means that a three-dimensional phase space, $Ph_{2/3/5}$, may be a sufficient and more stable tonal state space than the original 12-dimensional space of pitch-class distributions, since each key occupies a distinct region of $Ph_{2/3/5}$-space. However, we have also found that a two-dimensional toroidal space appears to be sufficient for distinguishing keys. This reflects an additional constraint that seems built into tonal syntax, a linear dependence between Ph_2, Ph_3, and Ph_5. This linear constraint, $Ph_2 + Ph_3 - Ph_5 \approx 0$, gives rise to a "tonality index" that will be further discussed below. Given such a linear constraint, the three-dimensional space of tonality may be projected onto any of its two-dimensional subspaces with (ideally) no essential information loss.

3 Key Finding

Many studies have approached the question of key from the standpoint of artificial intelligence, by developing and testing key-finding algorithms. Distributional approaches emerge overwhelmingly as state-of-the-art from a survey of the key-finding literature since Krumhansl and Schmuckler [11] developed the first distributional algorithm. A number of similar algorithms have been proposed, with the major points of distinction being the use of different ground truth distributions for each key and differences in how distributions are calculated for each piece.

While most key-finding algorithms use correlation between profiles to determine a best key, Albrecht and Shanahan ([2]) show good results for an algorithm that uses Euclidean distances. The Euclidean distance of two distribution is simply $\sqrt{\sum(x_i - y_i)^2}$ over the twelve pitch-classes with the distributions normalized such that $\sum(x_i) = \sum(y_i) = 1$. The DFT helps illuminate the differences between these approaches.

Example 6 compares the two methods. For the distributions typical of tonal music, like the Bach example, they match very closely, and both reflect circle-of-fifths distances. (The preferred key according to both methods, G major, however, is incorrect for this E minor chorale.) The second example is a distribution from a tonally ambiguous eight-measure theme that suggests both A minor and E minor. The tonal ambiguity is reflected by the similar scores for these two keys, but still the two methods, Euclidean and correlational, give very similar results.

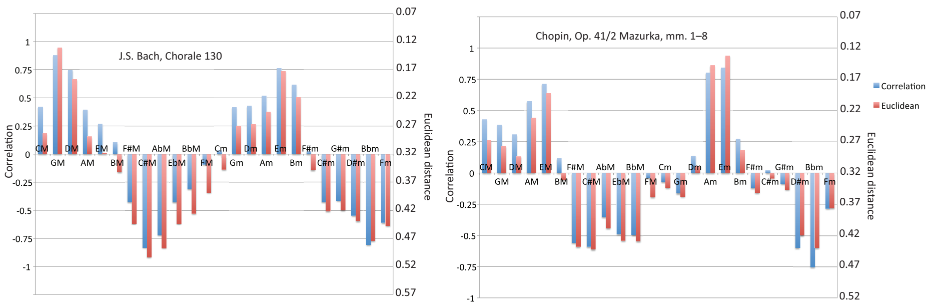

Fig. 6. Correlations and Euclidean distances of Albrecht and Shanahan's, [2], key profile and distributions from two tonal pieces.

Both methods may be better understood through basic Fourier theorems. Euclidean distances remain Euclidean distances after the DFT, measured in a direct product of complex planes and scaled by $1/\sqrt{12}$, by the unitarity principle (i.e., orthogonality). Furthermore, the convolution theorem says that correlations become dot products after the DFT: $\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$. When the magnitudes of DFT components match—e.g. when comparing two tonal distributions with largef_2, f_3, and f_5—both measures will reflect the phase differences of the prominent components, and therefore they will tend to agree, the only difference being that correlation will be even more strongly biased towards the components that are large in both distributions (and hence will favor f_5 more strongly when comparing tonal distributions). Therefore, a simple explanation of how distributional key finding works is that the scale is selected by Ph_5 and the mode by Ph_3 or Ph_2. The same results could therefore be derived from proximity in $Ph_{3/5}$-space.

When distributions emphasize different periodicities, particularly where a DFT component is large in one distribution and close to zero in the other, the two methods respond differently. Correlation will simply supress such components (since the influence of a component is weighted by a product of magnitudes). The Euclidean measure will include a constant value that is uninfluenced by changes of phase (i.e., transposition). Therefore the range of Euclidean distances will contract more noticeably when comparing non-tonal distributions to the 24 key profiles, but the difference should not usually affect the choice of best key.

Example 7 shows two instances where the two methods do choose different keys, the first eight measures of Schubert's song "Dass sie hier gewesen" and the subject of the F♯ minor fugue from Bach's WTC I. Both distributions suppress f_2 and f_3 in favor of some non-tonal component, f_4 in Schubert's case (because of a heavily emphasized vii$^{\circ 7}$/ii chord) and f_1 in Bach's (because the subject is very chromatic and restricted in range). As a result, all components except for f_5 are effectively canceled out for both the correlational and Euclidean criteria. In this situation, correlation is biased towards major keys, because the major-key distribution has a slightly higher $|f_5|$. Euclidean distance chooses the key that is closest in Ph_5, which happens to be minor in both instances, whereas correlation chooses the closest major key. As a result, correlation selects the correct key for Schubert (C major as opposed to G minor) but Euclidean distance selects the correct key for Bach (F♯ minor rather than E major). The bias of correlation towards the major mode is a likely explanation of the Albrecht and Shanahan's finding that an algorithm using Euclidean distances performs considerably better than others on minor-mode pieces, but somewhat worse on the major mode.

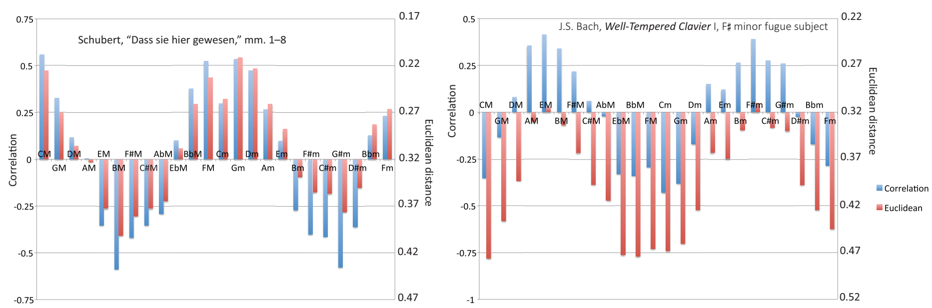

Fig. 7. Correlations and Euclidean distances of Albrecht and Shanahan's, [2], key profile and distributions from two tonal pieces.

Studies on human subjects have been much more attentive to the influence of temporal ordering on perceptions of key than the key-finding literature. Since distributions collapse the temporal dimension, they implicitly assume that the temporal order of pitch-classes does not influence the sense of key, even though experimental studies such as [5], [6], and [17] have amply demonstrated the importance of temporal order to key inferences. Approaches to key finding that deal with modulation by using windowed analysis, such as Temperley's ([23], [24]) and Sapp's ([21]), may partially address this concern. But these only allow for the sense of key to change over time; they do not propose means by which the temporal ordering of pitch classes may influence the sense of key beyond assuming that more recently occuring pitches will have a stronger influence. More promising is Quinn's [20] approach of treating progressions as the basic elements of tonality rather than chords (built upon by White [25]). The tonal filter may provide a way of "fuzzying" the concept of chord, with progressions as characteristic kinds of motions in $Ph_{2/3/5}$ space.

4 The Tonal Index

We have observed that typical distributions in tonal music feature three prominent components, f_2, f_3, and f_5, but also that a two-dimensional space using any selection from Ph_2, Ph_3, and Ph_5, is sufficient to represent the tonal implications of a particular distribution. The reason is that typical tonal distributions seem to be constrained to keep the quantity $Ph_2 + Ph_3 - Ph_5$, the *tonal index*, close to zero. Fig. 8 provides an example of how the tonal index tends to stay very consistently close to zero in the windowed analysis of a tonal piece.

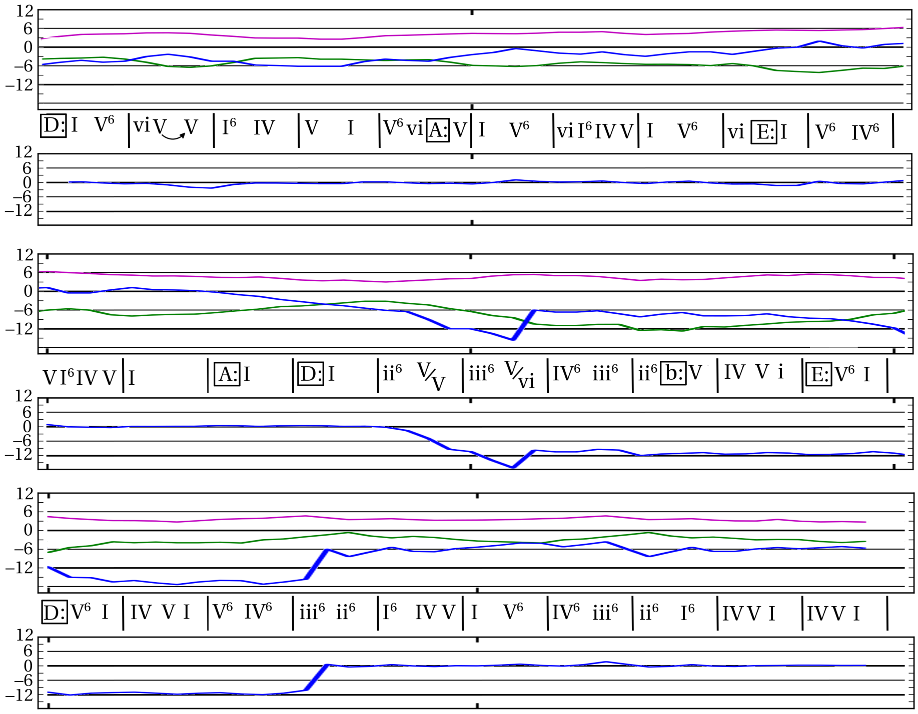

Fig. 8. Ph_2 (Green), Ph_3 (Blue), and Ph_5 (Pink) and the tonal index (blue, lower graphs) in a windowed analysis of Corelli's Violin Sonata Op. 5/1 mvt. 2, aligned with a harmonic summary of the score.

The tonal index is equal to zero for certain basic, mode-neutral, pitch-class sets: unisons, perfect fifths, and diatonic scales. This is related to the mathematical fact that, for generated collections, an index of this type can only take two values, 0 or 6.[4] For major and minor triads, it is small, ± 0.62. The non-composer-specific distributions in Figs. 1–3 range from -0.20 to -1.04 for major and 0.60 to 1.05 for minor averaging -0.67 and 0.84.[5] The Prince/Schmuckler data of Fig. 5 gives averages of -0.51 and 0.41 and some evidence of historical trends. In major, the index for composers up to Brahms ranges just from -0.79 to -0.42 averaging -0.64, very close to the major triad value. The late tonal styles of Liszt and Scriabin give values much higher and closer to zero, 0.14 and 0.05. In the minor mode, Bach stands out somewhat with an index of 0.49, late-eighteenth/early-nineteenth century composers range from 0.85 to 1.16, and Brahms seems to group with Liszt and Scriabin with indexes again close to zero: 0.08, 0.11, 0.23. This suggests that the late tonal style may be characterized by the attenuation of this aspect of the major-minor distinction.

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[4] This is a consequence of Amiot's [4] Proposition 4.3, which also appears in [27] but missing a $\pm$, crucial for the recognition 6 (or more generally π) as a possible value. This can also be extended to other inversionally symmetrical collections using Proposition 6.8 from [4]. Thanks to Emmanuel Amiot for these observations.

[5] The averaging is done in the complex plane on normalized values.

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