A Space for Inflections: Following up on *JMM*'s Special Issue on Mathematical Theories of Voice Leading

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Abstract: *JMM*'s recent special issue (7/2) reveals substantial common ground between mathematical theories of harmony advanced by Tymoczko, Hook, Plotkin, and Douthett. This paper develops a theory of scalar inflection as a kind of voice-leading distance using quantization in voice-leading geometries, which combines the best features of different approaches represented in the special issue: it is grounded in the concrete sense of voice-leading distance promoted by Tymoczko, invokes scalar contexts in a similar way as filtered point-symmetry, and abstracts the circle of fifths like Hook's signature transformations. The paper expands upon Tymoczko's "generalized signature transform" showing the deep significance of generalized circles of fifths to voice-leading properties of all collections. Analysis of Schubert's *Notturno* for Piano Trio and "Nacht und Träume" demonstrate the musical significance of inflection as a kind of voice leading, and the value of a robust geometrical understanding of it.

Keywords: Voice leading, quantization, geometry, scales, filtered point-symmetry, neo-Riemannian theory, signature transformations, Schubert, enharmonicism, chromatic harmony

The recent John Clough Memorial Conference in New Haven (June 2012) revealed a convergence of interest in mathematical music theory of the relationship between voice leading, quantization, and scalar context. The recent special issue of the Journal of Mathematics and Music (vol. 7/2) [1–5], assembled through the commendable efforts of Marek Žabka, reflects the different directions from which theorists have arrived at this nexus. Each represents a well established theory: Julian Hook's transformational approach to diatonic contexts, Dmitri Tymoczko's voice-leading geometries, and Jack Douthett's and Richard Plotkin's theory of filtered point-symmetry. In each paper these authors maintain the integrity of their own theoretical approaches while noting the often extensive overlap with the others.

The purpose of this response is to pick up important threads left by the special issue on the topics of voice leading, quantization, and scalar context, and begin to develop an approach that draws on the best insights of each theory. In the process I will reinforce some of the central arguments of the different contributions to the issue, including

- Douthett and Plotkin's argument for scalar context as a feature of voice leading,
- Tymoczko's argument that theories of quantization and scalar context should be developed from within voice-leading geometry because of its concreteness and generality, and
- The point demonstrated in Hook's paper as well as my own that using music analysis as a context for developing mathematics-based theories best reveals their significance and musical implications.

The response is in four parts. The first part derives surfaces in voice-leading geometries that resemble Plotkin's [6] charts of filtered point-symmetry configurations, using quantization. This shows that filtered point-symmetry is consistent with measures of voice-leading distance, and also that it implies a distinct type of voice leading by scalar inflection not immediately evident in basic chord space. Parts 2 and 4 address the topics of harmonic distance through analyses of Schubert's *Notturno* for Piano Trio and the song "Nacht und Träume" using the quantization-based voice-leading space developed in part 1, showing that enharmonicism creates a topologically distinct class of harmonic paths in these spaces. The third part further explores the topic of generalized signature transformations introduced in Tymoczko's paper. The mathematical results of this section show that pure voice-leading concerns advocate for the same abstraction of the circle of fifths that characterizes the analytical approach advanced in part two.

(1) Geometric reformulation of filtered point-symmetry configuration spaces

Plotkin and Douthett's paper [2] uses coordinate spaces developed by Plotkin [6] to show the behavior of filtered point-symmetry configurations. In these spaces, which I will refer to as *configuration spaces*, one axis represents the rotation of the beacon and the other the coordinated rotation of beacon and filter. Configuration spaces are a convenient way to explore all possible states and behaviours of the system of rotating rings.

The contributions to the joint issue begin to connect the dots between configuration spaces and voice-leading geometry, but because the relationship is so close, it is possible to draw it with an even greater degree of specificity. Tymoczko's paper,[3] as well as my own,[4] points out that filtered point-symmetry can be reformulated as iterated quantization in voice-leading geometries. This implies that configuration spaces can be interpreted as networks of voice leadings on ME chords prior to quantization. We can take this a step further by extracting a surface from chord space that embeds this network in a continuous geometry.

For the two-filter case, $n \rightarrow d \rightarrow c$, which is the only one I will consider here, there are two kinds of connections between chords in the configuration space, or in its associated network. Vertical connections, which I call *position changes*, are rotations of the ME(n, d) chord restricted to the ME(d, c) scale defined by the filters. Horizontal connections are *microtranspositions* of the ME(n, d) chord.

In the 3→7→12 space, ME(3, 7) is the 7-equal triad. The position changes are given by voice leadings $(-1\tfrac{5}{7}\ 0\ 0)$ and $(0\ 0\ 1\tfrac{5}{7})$.[1] Note that $1\tfrac{5}{7} = \tfrac{12}{7}$ is the 7-equal step. Horizontal motions are voice leadings of $\pm(\tfrac{1}{7}\ \tfrac{1}{7}\ \tfrac{1}{7})$. The resultant progressions are quantizations of these voice leadings (see Fig. 1). For more on quantized voice leadings, see [7].

As my previous paper [4] points out, the microtranspositions generate Hook's [8] signature transformation group on triads. Hook's group, however, does not have a precise equivalent to position change, which in the geometry is a shortcut to the mediants, a distinct path from the transposition by $3\tfrac{3}{7}$. This reflects a difference of voice leading, between moving one voice by step and moving all voices by third.

[1] As a convention I label triads and voice leadings on triads in root-third-fifth

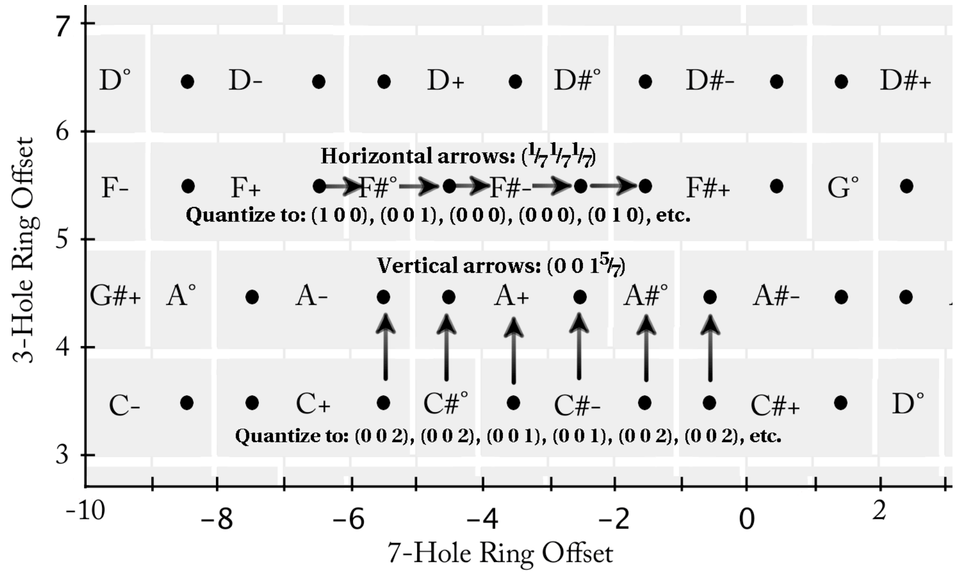

Figure 1: Configuration space for 3→7→12 filtered point-symmetry. Dots are added to show a full grid of possible positions in the space. Each position corresponds to a transposition of the 7-equal triad in $^1/_7$ increments. Quantization of these triads induces a quantization of the voice leadings between them. Horizontal arrows between adjacent points are transpositions by $^1/_7$, ($^1/_7$ $^1/_7$ $^1/_7$) voice leadings, between 7-equal triads before quantization, which quantize to (0 0 0), (1 0 0), (0 1 0), or (0 0 1) depending on the position in the space. Similarly, vertical arrows are (0 0 1$^5/_7$) before quantization, and (0 0 1) or (0 0 2) after.

Figure 2 shows these voice leadings between 7-equal triads in three-note chord space, labelled with the 12-equal triads that result from quantizing them. The line of transposition is densely populated, because the 7-equal triads appear in increments of $^1/_7$. This line circles chord space three times, creating a triangular prism that makes a 120º twist each time it circles the space (a similar geometry as the border of the entire space; see [9–10]). The voice leadings of (–1$^5/_7$ 0 0) and (0 0 1$^5/_7$) are the minimal voice leadings that jump across the prism from one point on the line of transposition to another. A series of seven such voice leadings circles the space while winding around the prism, touching all of the triads of a 7-equal lattice (which quantize to the all triads of a given diatonic scale). In chord space, these voice leadings change direction by 90º at each iteration, reflecting the fact that different voices are moving. For a space to include these voice leadings and be continuous, it must include the chord-types in between the 7-equal chords (which are all chords of the form (0 3$^3/_7$ x), $6^6/_7 < x < 8^4/_7$). Therefore, the space of interest is actually the surface of this twisted prism.

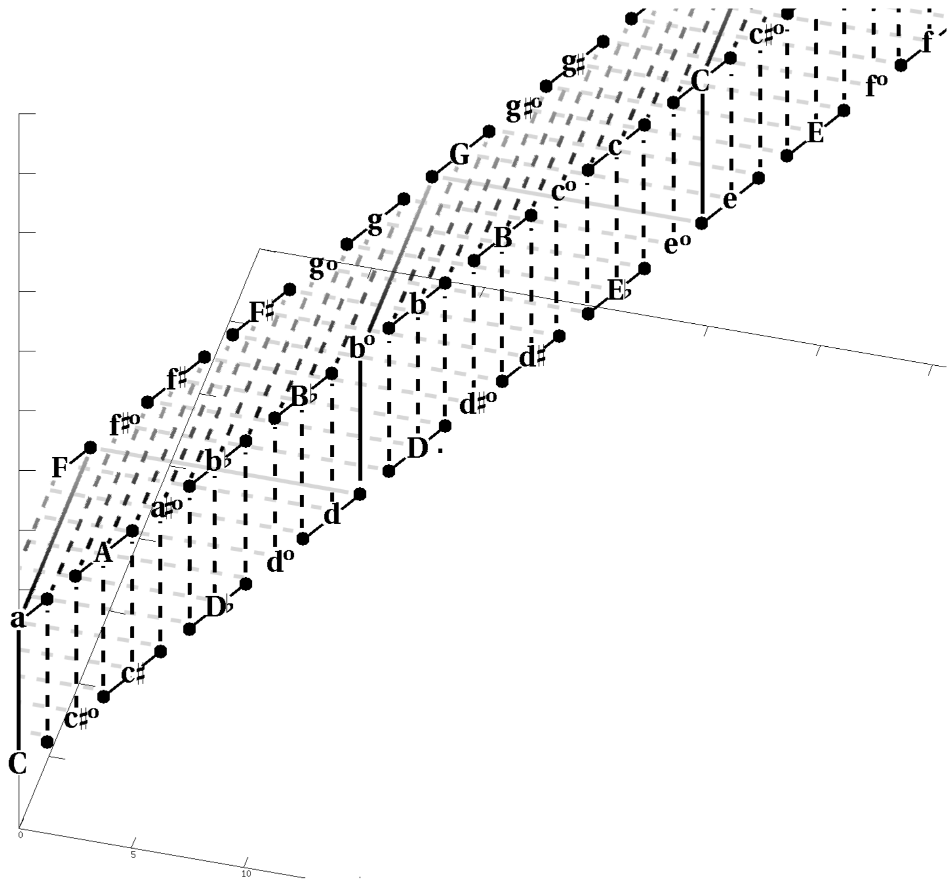

Figure 2: 7-equal triads in 3-note chord space, labelled by the 12-equal triads that result from quantizing them. Lines show $(0\ 0\ 1\tfrac{5}{7})$ voice leadings. Solid lines connect points on the 7-equal lattice that quantize to C major.

Fig. 3 shows this surface unfolded, with the 7-equal thirds-progressions unkinked and laid out vertically. The regions show the result of quantization for every position in the space. Because there are chord types other than 7-equal, some of the regions (shown in gray) quantize to augmented triads, but the line of 7-equal triads that winds around the space does not pass through these regions. The layout of Fig. 3 mostly reflects voice leading distances accurately, but there are some distortions that result from flattening out the sharp angles of the surface.[2] Ignoring

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[2] These distortions mostly affect distances involving chord types *in between* diatonic triads, because in the flattened space, these paths are forced to go through the corners of the prism (the line of transposition for 7-equal triads) instead of cutting corners through the middle of chord space. The same discrepancy emerges

the difference in the way regions are drawn, Fig. 3 is simply a skewed version of the configuration space. The skew improves the correlation of the space with voice-leading distances—for instance, F minor does appear closer to C major than F major.

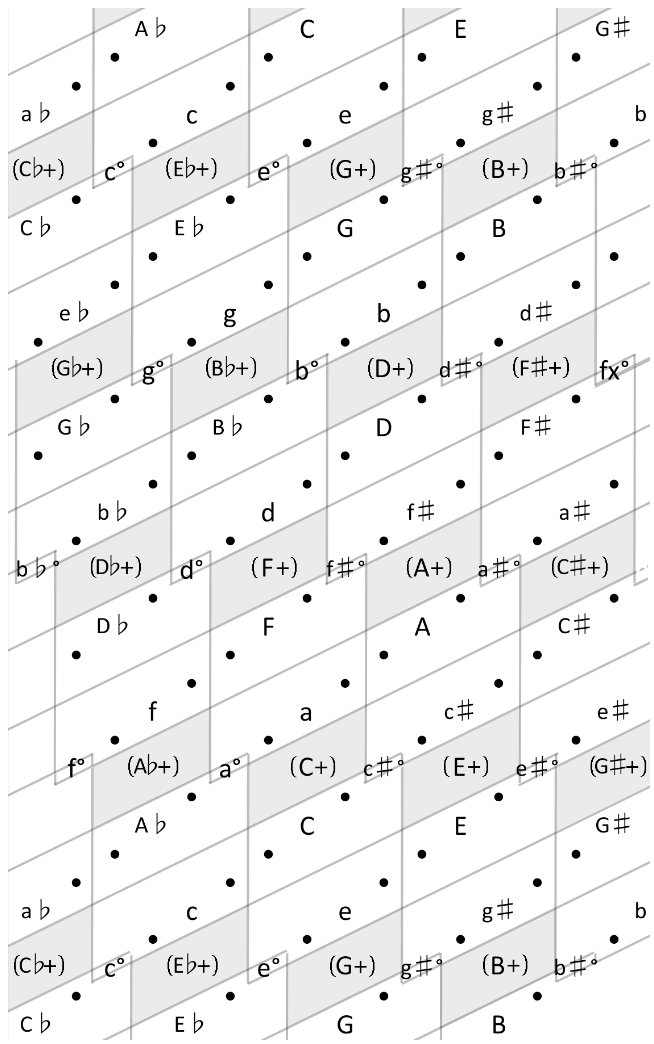

Figure 3: The geometry of 7-equal triads with position changes on the vertical axis and microtranspositions on a diagonal. Regions show the result of quantization. Grey regions, which quantize to augmented triads, contain no 7-equal triads. The pre-quantization chord types in these regions result from treating the position changes as continuous.

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when a path between 7-equal triads is relatively inefficient in the horizontal dimension.

The same type of space can be constructed for any analogous two-filter configuration space; the topology of the space is always the same (excluding degenerate cases), although the change of scaling and skew between the axes can make the space look substantially different. Fig. 4 shows the network of 7→18→12 ME scales from the “Ondine” analysis in my paper construed as a voice-leading space. The microtransposition distances are larger because of the common factor between 12 and 18, while the position changes are smaller because of the large cardinality of the intermediate lattice (18-equal). Notice how the distances on this chart more closely reflect the literal voice-leading distances between collections (such as fifth-related diatonic collections) than the network of microtranspositions and position changes.

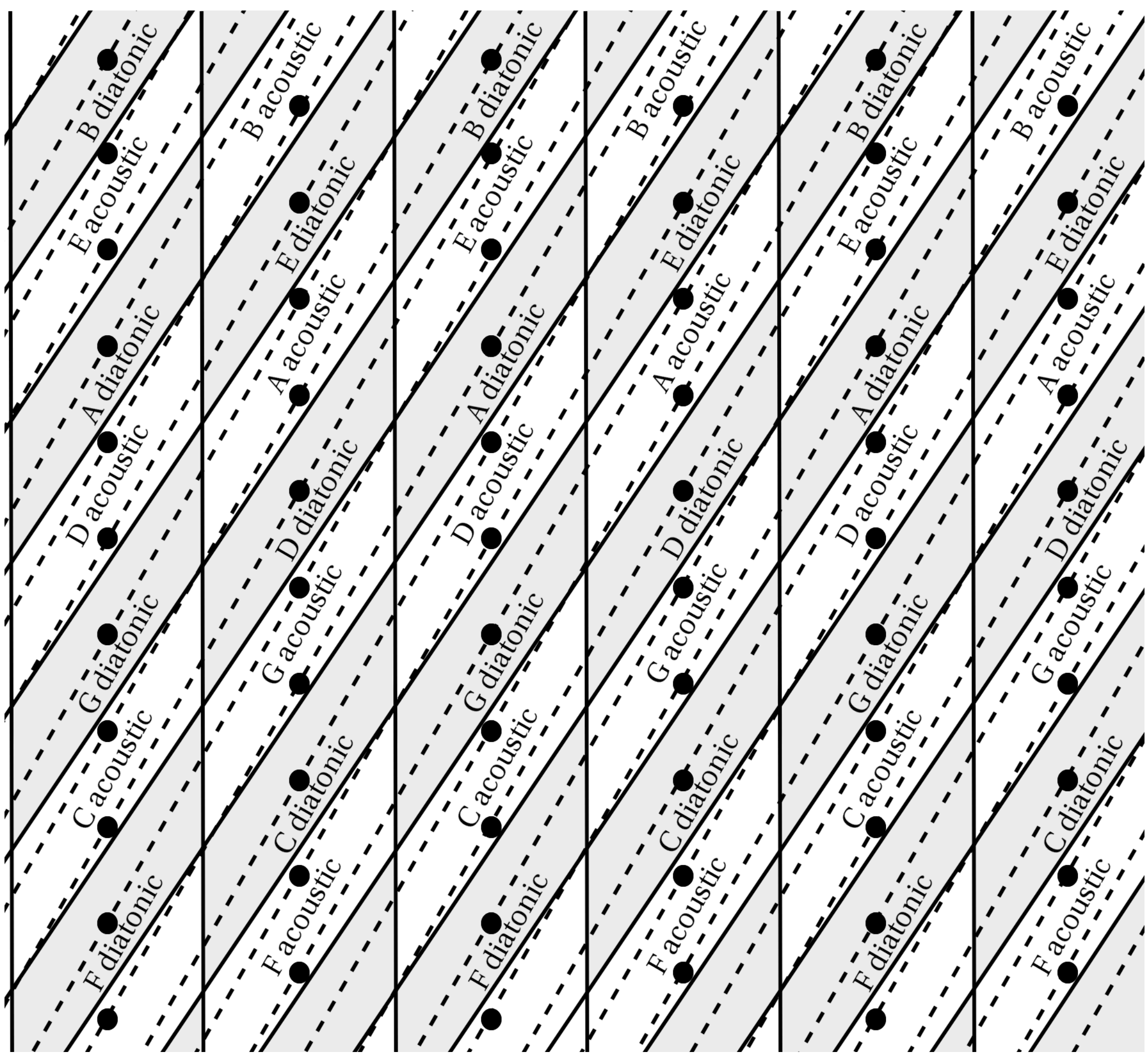

Figure 4: The voice-leading space for ME(7, 18) collections with regions showing quantization to 12-equal. Microtranspositions are shown by dashed lines.

Similar spaces can be constructed for any chord type, not just maximally even chords, and therefore are more general than filtered point-symmetry configuration spaces. Figure 5 shows a space for quantizations of neutral triads, (0 $3\frac{1}{2}$ 7), which resembles the space for 7-equal triads, but more regular and without diminished triads. Neutral triads are not maximally even, so their position changes are not restricted to a maximally even scale, but they do realize a musically significant voice leading cycle, the mediant cycle or R-L cycle.

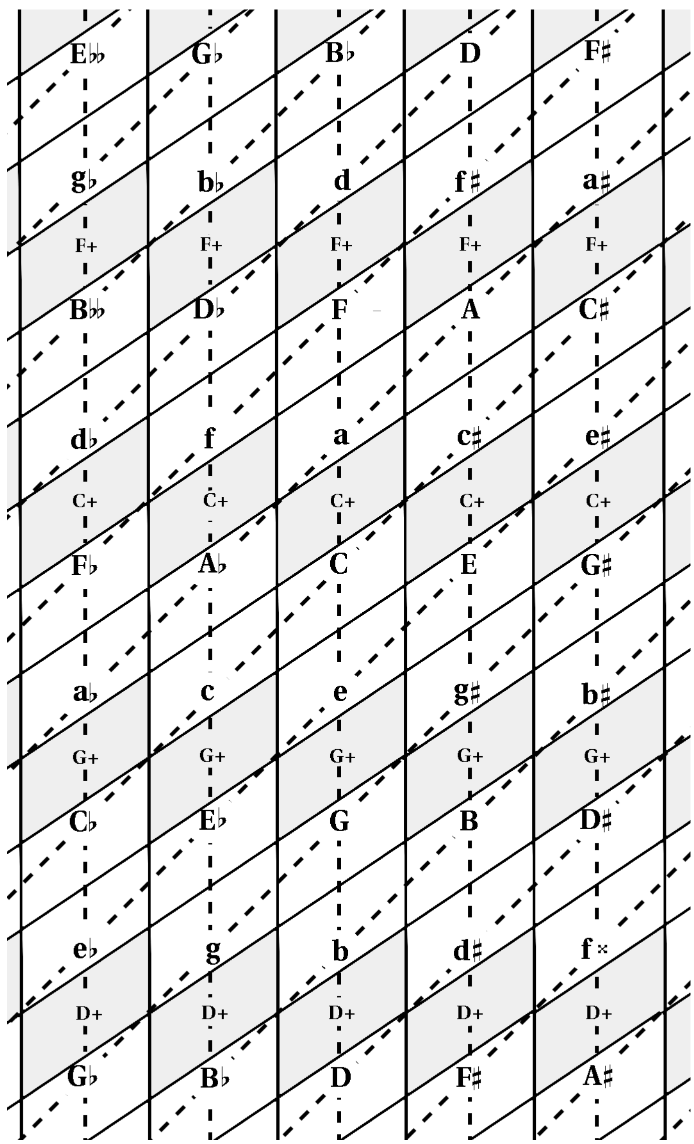

Figure 5: Voice-leading space for neutral triads, (0 $3\frac{1}{2}$ 7), with regions showing quantization to 12-equal. The dashed lines show microtranspositions and position changes.

Since these spaces are merely selections out of the larger voice leading space, what can they add to our theory of harmony that is not already implicit in three-note chord space? In the space of Fig. 3 literal voice-leading distance, distance in three-note chord space, is generally associated just with *vertical* displacement. The horizontal dimension layers a new kind of voice-leading distance, the distance associated with inflections caused by change in the underlying diatonic context, on top of literal voice-leading distances. The analyses below will show that this dimension of voice leading is incredibly significant in composers' expressive use of tonal harmony. Julian Hook points out in his paper [1],

> In chord space, motion is possible in any direction at all; filtered point-symmetry and signature transformations produce limited repertoires of musical phenomena but call attention to aspects of those phenomena (such as implications of diatonic structure) that are not apparent in the general geometric setting.

The space of Fig. 3 limits our purview from all three-note chords to those required for tonal harmony, but in doing so allows us to theorize the implications of diatonic structure to which Hook refers. Moreover, it shows that this added feature of tonal harmony can *itself be understood as geometric, and as a kind of voice leading on triads*. This means that aspects of raw voice leading and scalar contexts need not be addressed by disparate, quasi-independent theories, but can be treated as different aspects of a single voice-leading space on triads. The added layer of theory provided by quantization makes this possible.

2: Schubert's *Notturno* and the effect of distance in triadic progressions

As long as one focuses only on preferred progression-types between common triads and seventh chords, few significant distinctions between transformational theory, voice-leading geometries, and filtered point-symmetry will emerge. All of these theories highlight the same relatively parsimonious chord relationships. Plotkin and Douthett,[2] for instance, show that networks of parsimonious voice leading from Douthett and Steinbach [11] occur in configuration spaces for 3→8→12 and 4→9→12 filtered point-symmetry. However, this is because parsimonious voice leading is built into the mathematics of quantization, a point made apparent in [12]. Microtranspositions are always purely semitonal, and position changes of ME(n, d) always have $n - 1$ common tones when n and d are coprime. Therefore, the *Tonnetz* also occurs in 3→7→12, 3→10→12, and 3→11→12, and the "towers torus" can be found in 4→7→12 and 4→11→12. Similar observations can be made on microtransposition and position change generally, showing why, for instance, the voice-leading space of quantized neutral triads in Fig. 5 is a *Tonnetz* if we disallow the microtranspositions that move two notes at a time (the "slide" progressions, like C-c♯), as Plotkin and Douthett do in their 3→8→12 space.[3]

The distinctions between theories become more apparent in considering more specific conceptions of distance. Voice-leading geometry is founded on a robust and concrete notion of voice-leading distance. Tymoczko [13–14] has shown that this kind of voice-leading distance cannot necessarily be derived from transformational networks of parsimonious voice leading like the *Tonnetz*. In filtered point-symmetry, individual motions of a single ring produce clear distances, but how distances on different rings might combine is indeterminate (a point made in Plotkin and Douthett's [2] discussion of voice-leading geometry). The geometry in Figure 3 shows that 3→7→12 filtered point-symmetry is consistent with voice-leading distance, but the process of quantization from which this geometry is derived introduces a new kind of voice-leading distance by inflection. This distance by inflection corresponds to circle-of-fifths distance and is roughly independent of raw voice-leading distance on triads.

Voice-leading distance is valuable for its simplicity and broad relevance. However, because it is such a basic feature of chord progressions, plain voice-leading distance is not directly useful to composers as a way to communicate the poetic sense of distance. Composers like Schubert nonetheless do use harmony and voice leading to connote distance in very forthright ways, as I will argue in the following examination of Schubert's use of harmony in the *Notturno* for Piano Trio (D. 897).

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[3] The neutral triad and octatonic triad, (0 3½ 7) and (0 3 7½), are special as the two ways (other than (0 4 8)) to be halfway between major and minor in set-class space. That is, the midpoint of any strongly crossing-free voice leading between any major and minor triad is a neutral, octatonic, or augmented triad. See [9–10].

The progressions that evoke a sense of harmonic distance are not systematically larger in voice-leading space; instead, Schubert actually *minimizes* literal voice-leading distances while maximizing circle-of-fifths distances. Tymoczko [10,p.214–216] makes a similar observation in his analysis of a chromatic progression from Schumann's "Chopin" movement from *Carnival*, which simultaneously creates a sense of tonal distance and reduces the size of literal voice leadings.

The theme of Schubert's *Notturno* establishes a languid harmonic atmosphere from the beginning, seemingly in no hurry to move away from tonic harmony, much less the tonic key. The first phrase of the theme (Fig. 6) nonetheless includes one striking non-diatonic progression. Though hardly remarkable as a harmony, the V^7/V in m. 7 is strikingly juxtaposed with IV in an unexpected articulation of beat two, highlighting the chromaticism A♭–A♮, and the chromatic mediant relationship between the two harmonies. The second departure from diatonicity comes at the beginning of the second phrase (following a reorchestrated repeat of the first), shown in Fig. 7. Schubert replaces the earlier IV–V^7/V progression with a mode-mixture iv, resolving to V^7 in the next measure. As a literal distance between triads, the progression I– $iv^\flat$ is actually smaller than the diatonic alternative, I–IV. Yet it creates a palpable feeling of distance, and effect that Schubert emphasizes with the sudden drawback in dynamic as the strings take a long two-measure breath.

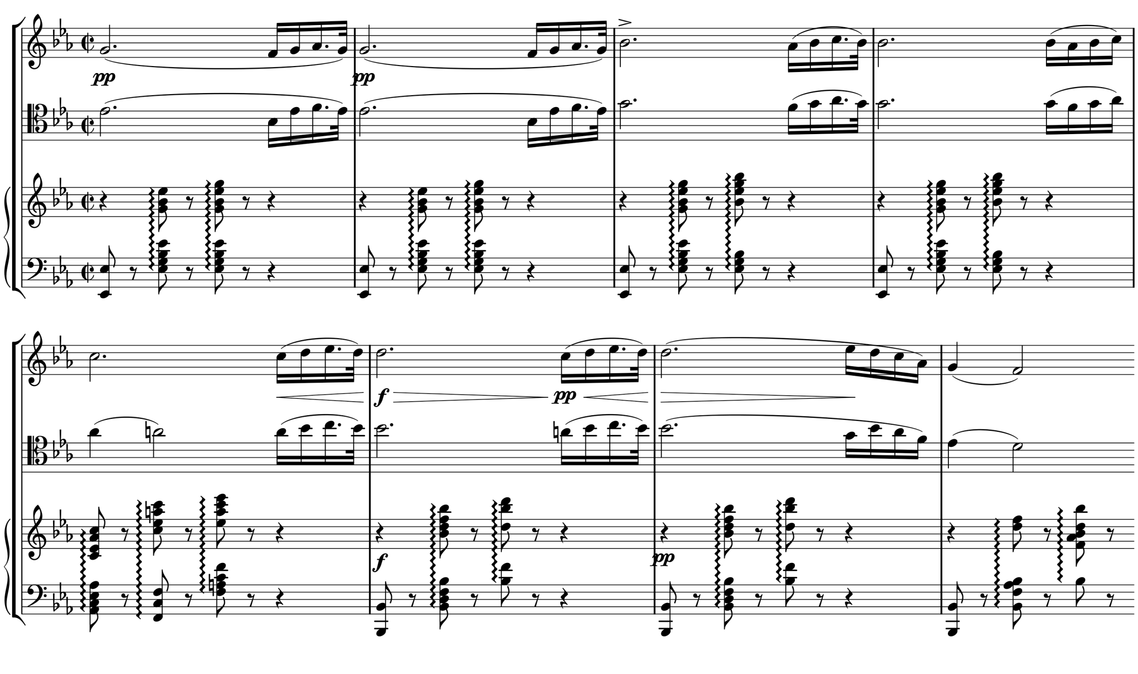

Figure 6. Schubert *Notturno* for Piano Trio, mm. 3–10

Figure 7. Schubert *Notturno,* mm. 21–28

One way to explain this impression of distance through reference to voice-leading spaces is to associate it with voice-leading distance between scalar collections in seven-note chord space. Yet the literal "macroharmonic" change (using Tymoczko's term; see [10,p.4,155–164]) is exactly the same in the I–iv$^{\flat}$ progression as it is in the earlier IV–V^7/V progression, a one-semitone shift (E♭ major–B♭ major vs. E♭ major–E♭ harmonic major). Therefore, to see this "tonal distance" as a kind of voice-leading distance between scales involves postulating additional implied voice leadings. A simple approach would be to assume diatonic contexts, which are related by circle-of-fifths distances—in this case (D→D♭, G→G♭, C→C♭). Tymoczko's method [10,p.246–252] is to reformulate traditional key distances by averaging voice-leading distances to and from the three traditional minor scales. Accordingly, Schubert's iv$^{\flat}$ might suggest E♭ minor, which has an average distance of 2 from E♭ major. Tymoczko's method involves an additional level of abstraction (note, for instance, that one of the E♭ minor scales does not contain a C♭ at all), but key distances still generally follow a circle-of-fifths pattern.[4] As we will see in

[4] The main differences being that between major and minor keys, the circle-of-fifths distances are counted from major–supertonic (or minor–subtonic) but with a

the next section, this relates to a general feature of seven-note chord space and is not dependent on the particular choice of representative minor scales, except insofar as they tend to be relatively even. Both methods therefore posit a larger distance by suggesting an "implied" G→G♭ essentially on the grounds that G♭ is closer than C♭ on the circle of fifths.

We can also interpret the tonal distance of Schubert's iv$^{\flat}$ more directly as a voice-leading distance on triads using the quantization-based spaces defined above. The solid arrows in Figure 8 show the two predominants in the voice-leading space for 7-equal triads. E♭ major–A♭ minor is shorter than E♭ major–A♭ major and about the same as A♭ major–F major, reflecting literal voice leading distances. The important difference in the progression to A♭ minor is that it involves a *lateral* excursion from the E♭ diatonic axis much larger than the one for V/V. Generally speaking, literal voice-leading distances are associated with vertical distances in the space, the lateral shifts modifying the vertical distances via the tilt in the line of transposition for 7-equal triads. However, the traversal of lateral space, associated with circle-of-fifths distance, produces the most musically impressive effect.

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penalty of 1.3 applied to major–supertonic and major–minor dominant (and their inverses) making relative keys (at $1 < 1.3$) the closest and parallel keys ($2 < 2.3$) closer than major–minor dominant. Also, between minor keys, small and large distances are attenuated, ranging from 2.33–4.67 rather than 0–6.

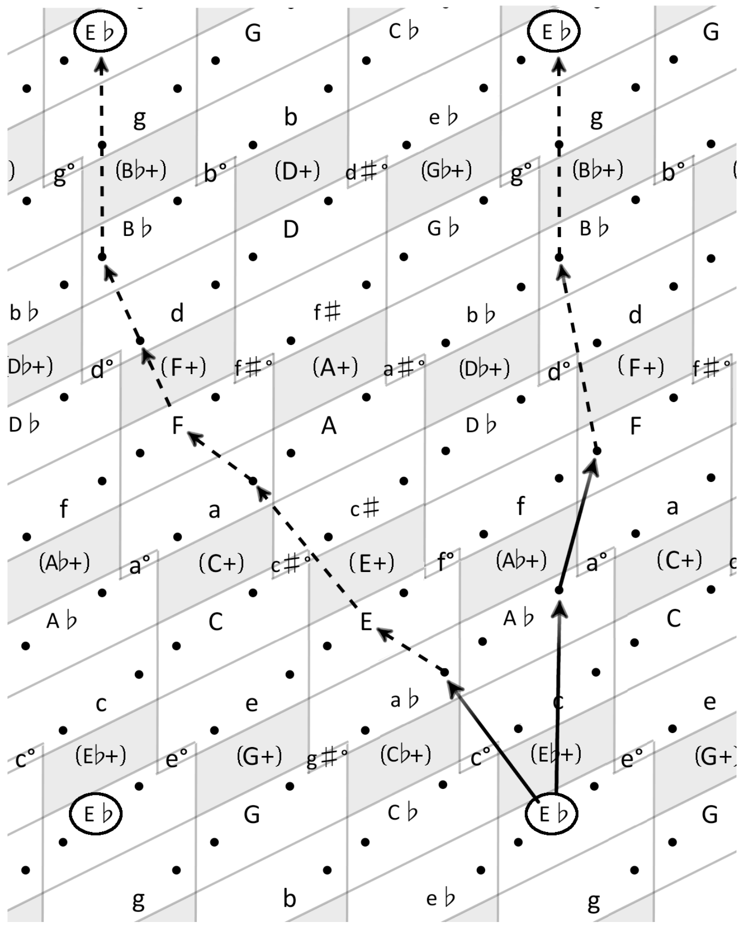

Figure 8. Voice-leading cycles from the main theme of Schubert's *Notturno* in the space of Fig. 3. Solid arrows show how the predominant harmonies Schubert uses relate to the tonic. Dotted arrows show the progressions that complete each phrase, the first phrase moving directly up while the second moves up and to the left. Circled E♭s all represent the same point in an unfolded torus.

In the first phrase the theme does not reveal itself to be anything more than a pleasant triviality. Even the mode mixture at the beginning of the second phrase is at best a decorative stroke of color. In the conclusion of the phrase, however, Schubert reveals that he has been surreptitiously laying the groundwork of a larger harmonic strategy. In the progression that leads up to the final cadence, Schubert opens up the space between the previous two chromatic predominants, iv$^{\flat}$ and V/V, with the uncanny harmonic sequence E♭ maj.–A♭ min.–E maj.–A min.–F maj. In Cohn's [14] transformational terminology, the progression is N–L–N–L. The "N" ("*Nebenverwandt*"–see [15–16]) progression most often appears in nineteenth century harmony with the sense of a V–i progression in some minor key, but here it is introduced as I–iv$^\flat$. Either way, loosed from its tonal moorings, as it is here, it often alternates with a major-third mediant ("L") progression to create the type of sequence we find in the *Notturno*. What is remarkable about the sequence, especially as Schubert uses it here, is its rapid traversal of chromatic space. The straightforward earlier appearances of V/V and iv$^\flat$ emphasized their status as sharpward and flatward shifts, respectively. Yet the sequence moves from the flat-side harmony (iv$^\flat$) to the sharp-side one (V/V) by means of a precipitous *flatward* harmonic cascade. The dashed lines in Fig. 8 track this progression and compare it to the more tonally-centred progression of the first phrase. Both progressions cycle from tonic to tonic by means of a continually ascending voice leading (vertical in the space). Yet the second phrase also traverses the space laterally at the same time, making a complete tour of the circle of fifths. The effect of the progression is as if a window has suddenly been opened onto a chromatic universe. The initial reticence of the theme to introduce significant chromaticism, followed by the palpable feeling of distance created by the initial flatward move (to iv$^\flat$) masterfully sets up this moment as a revelation of sublime depth. Like an adept landscape painter Schubert places a yardstick in the foreground and then extends a line to the horizon. (On the sublime in late eighteenth and early nineteenth century thought as it relates to music, see [17–18]).[5]

Geometrically, the two cycles in Figure 8 can be described as *non-homotopic*, meaning that, if you imagine each as a rubber band wrapped around the torus, there is no way to stretch or twist one to get it to match the other. Both paths circle the space vertically, meaning they consist of continually ascending voice leadings. But the path for the second phrase also tours the space laterally, meaning that it requires an enharmonic respelling. Thus, this kind of voice-leading space shows that essential enharmonicism is not only a feature of seven-note chord space, but can also be interpreted as an essential topological property of a surface in three-note chord space. This property is not exclusive to the space of 7-equal triads (the paths are similar, for instance, when plotted in the neutral-triad space of Fig. 5). However, in diatonic triad space, it corresponds directly to the circle of fifths.

[5] That this enharmonic progression is focal event of the theme is confirmed by the fact that the subsequent internal theme in E major appears within a hugely expanded repetition of what is effectively the same progression, with E major approached via $\flat\widehat{6}$ in mm. 31–32, modulating to F major in mm. 65–66, which eventually functions as V/V in m. 78 shortly before the return.

The musical implications of homotopy in this space are further explored in the analysis of “Nacht und Träume” below.

In his critique of filtered point-symmetry, Tymoczko [3] says that it “seems much more likely that listeners respond directly to voice-leading distances than that they hypothesize the existence of perfectly even continuous chords only very imperfectly realized in equal temperament.” In the *Notturno*, however, the effect of distance is not reducible to literal voice-leading distances, either between scales or chords. And yet that does not mean we cannot use voice-leading spaces to explain how the effect of distance is achieved—we can. To do so requires another level of abstraction—either in the way voice leading relates to musical surfaces (allowing for implied voice leadings in seven-note chord space), or within the model of triadic voice leading itself, using quantization of 7-equal triads to “factor” the raw voice-leading distances and isolate the diatonic-inflection component. The advantage of the latter is that both kinds of voice leading—literal triadic voice leading and circle-of-fifths inflections—exist in a single space, rather than in quasi-independent theories on three-note and seven-note chords. Regardless of which approach one prefers, the next section shows that both lead to a similar abstraction of the circle of fifths.

(3) Generalized signature transformations and basic voice-leading cycles

Tymoczko’s contribution to the special volume [3] shows that generalized circle-of-fifths distances, what were described in my contribution as special voice-leading cycles derivable via quantization, undergird the voice-leading relationships between all chords of a certain cardinality embedded in a universe of another cardinality. This surprising fact is implied by the “generalized signature transforms” that he introduces [3,p.133–139], although to see why requires a little more exploration of how these work. This extension of Tymoczko’s results leads to a more abstract conception of the circle of fifths, the same abstraction that allows us to use it as a kind of “inflection distance” in the analysis above. The special role of the circle of fifths for all 7-in-12 collections, not just the diatonic, was previously noted by Hook [19], and the abstraction of the cycle is also implied by Hook’s signature transformations [8]. Tymoczko’s paper and the following extension connect these insights specifically to voice-leading geometry.

According to Tymoczko, the basic insight of Hook’s [8] signature transformations is that the circle of fifths is not generated by simple transpositions of the diatonic scale by fifths, but a single-step voice leading made by balancing this transposition with a rotation in the opposite direction. The circle of fifths also generalizes to any ME(d, c) set, when gcf(d, c) = 1. Because all voice leadings are single-step this generalized circle of fifths is a perfect proxy for voice-leading distance. I will call this the "basic voice-leading cycle for $d \rightarrow c$" or "$d \rightarrow c$ cycle." Tymoczko then points out that such a cycle can be constructed for *any* cardinality-d chord in a cardinality-c universe using the same transposition-plus-rotation as for ME(d, c). The relative evenness of the chord determines how closely these "generalized signature transformations" reflect actual voice-leading distances.

The behavior of a generalized signature transformation (ST) can be better understood by decomposing it into three voice leadings, $X^{-1} \circ C \circ X$, where X makes the chord maximally even and C is the ordinary signature transformation. Because this decomposition is a conjugation, the nth iteration of ST is just $ST^n = X^{-1} \circ C^n \circ X$. Figure 9 shows such decompositions for the harmonic minor scale, where X is of size 1. Since X and C are both of size 1, the total voice leading size[6] of C is $n + 2$ unless one of the C's cancels out X or X^{-1}. As Figure 9 illustrates, for the harmonic minor this cancellation will happen for all and only $n \geq 3$.

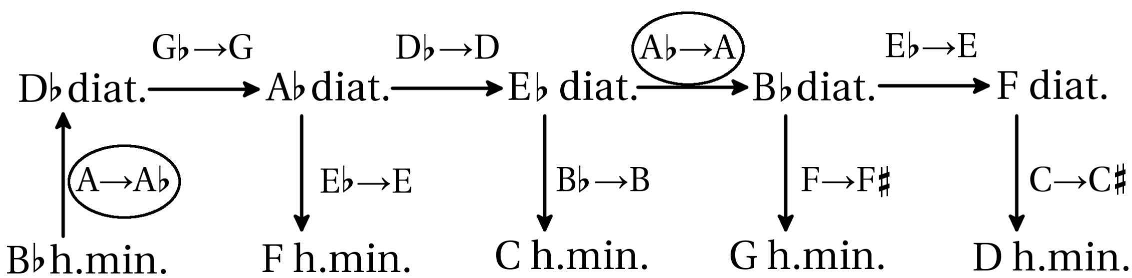

Figure 9: A decomposition of generalized signature transform voice leadings for harmonic minor scales. The top line is a segment of the (7, 12)-cycle, and the bottom line arranges harmonic minor scales in generalized-signature-transformation order. Circled voice leadings cancel one another out.

Table 1 gives the voice-leading sizes for generalized signature transforms (ST) on acoustic scales, harmonic minor scales, harmonic minor ♭2, and "Dom♯9♯11" scales. (Hook [18] refers to the latter two as "Neapolitan minor" and "sub-octatonic" heptachords).

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[6] Using a city-block metric of voice-leading size.

Table 1: Voice-leading sizes for n iterations of the generalized signature transform (transposition by fifth and rotation by four scale steps) for four scale types.

		Voice-leading size at						
Type	Example	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
Acoustic	CDEF♯GAB♭	3 (n+2)	2 (n)	3	4	5	6	7
Harm. minor	CDE♭FGA♭B	3 (n+2)	4 (n+2)	3 (n)	4	5	6	7
Minor ♭2	CD♭E♭FGA♭B	3 (n+2)	4 (n+2)	5 (n+2)	4 (n)	5	6	7
Dom♯9♯11	CD♯EF♯GAB♭	5 (n+4)	4 (n+2)	3 (n)	6 (n+2)	5 (n)	6	7

The pattern for the harmonic minor, where ST^n has a voice-leading distance of $n + 2$ up to some value of n and reverts thereafter to n, holds for the acoustic and minor ♭2 scales as well. The only difference is that the acoustic reverts at $n = 2$, the harmonic minor at $n = 3$, and the minor ♭2 at $n = 4$. The patterns are similar because all of these scales are one semitone away from diatonic. The circle-of-fifths distance of the altered note from the diatonic scale determines the n at which the generalized signature transforms give the same voice-leading distance as diatonic ones. This fact generalizes to *any* set of generalized signature transforms with this property (with cardinality coprime to universe), because the sort of cancellation shown in Fig. 9 will happen precisely under the condition that n is equal to or less than the distance of the "altered note" of X^{-1} on the basic cycle. Notice that the choice of X does not matter.[7] G acoustic, for instance, can be thought of as "C Mixolydian ♯4" or as "C Lydian ♭7." Both of these involve altering a diatonic scale at a circle-of-fifths distance of two.

The result holds for any coprime d and c. Figure 10 presents examples where $c = 7$. The 2→7 cycle is the diatonic circle of fifths. Thirds and sixths (Fig. 10(a)) are one step away from maximally even at a distance of two on the circle of fifths. For example, C-E is one step from C-F, and E is two diatonic fifths below F (F–B–E). Therefore the voice-leading distance between thirds a fifth apart is $n + 2 = 3$ steps

[7] With one caveat, pointed out by Tymoczko: X must be strongly crossing free.

—as in (C, E)→(B, G)—while the distance between thirds a step apart is $n = 2$—as in (C, E)→(D, F). For diatonic stacked-fourths tetrachords, Fig. 10(b), the result is the same except that the 4→7 cycle is a diatonic thirds cycle. They are one step away from seventh chords at two places on the circle of thirds. For example, (C E G B) → (C F G B) is a one-step voice leading with F two thirds below C. Diatonic triads with doubled roots, Fig. 10(c), are also one step away from seventh chords, but at a greater 4→7 cycle distance of four.

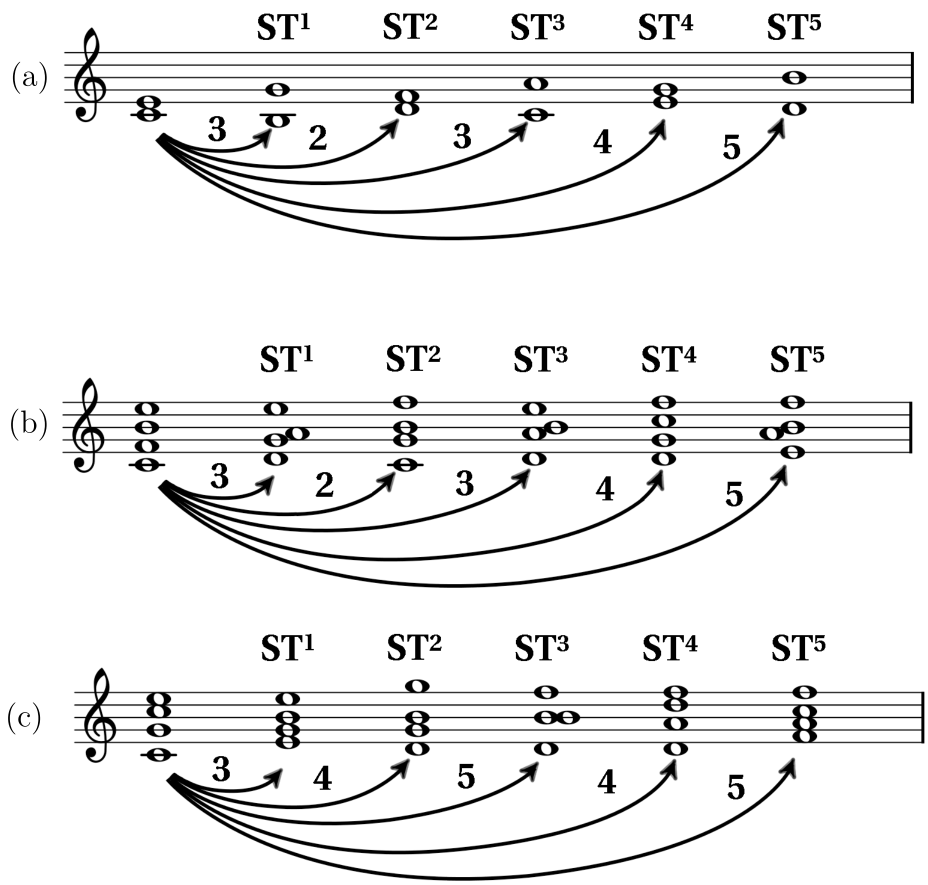


Figure 10: The sizes of different voice leadings, ordered by iteration of the generalized signature transform, on (a) diatonic third/sixth dyads, (b) diatonic stacked-fourth chords, and (c) diatonic triads with doubled roots.

This result also shows how the evenness of chords effects the fidelity of their ST cycle to true voice-leading distances. The chords that are one voice-leading step away from ME are not all themselves equally even. The more uneven they are, the fewer small voice leadings there are between their transpositions, according to the result outlined above. The relative evenness of chords one step away from ME corresponds to the distance of the altered note on the $d \to c$ cycle. For example, the harmonic major scale, as "major $\flat\hat{6}$," is less even than diatonic or acoustic scales because of the 7→12 cycle distance of $\flat\hat{6}$, and this is reflected in the sizes of voice leadings between harmonic major scales. Thus the sense of distance engendered by the C♭ in Schubert's *Notturno* is related to the macroharmonic progression in the sense that the circle-of-fifths distance of C♭ determines the relative unevenness of the resulting collection. In other words, E♭ major–E♭ harmonic major entails more "effort" as a move away from the center of the space whereas E♭ major–B♭ major is a move within its orbit.

The pattern for chords more than one step away from ME is more complex, but can be explained using the $d \to c$ cycle by extending the reasoning used above. For the Dom♯9♯11 scales the voice leading of X is size 2. For example, the G Dom♯9♯11 scale goes to C major under (A♯→A, C♯→C). A♯ is five places away on the circle of fifths from C major, and C♯ is two places away. The voice-leading size therefore changes from $n + 4$ to $n + 2$ at $n = 2$ and to n at $n = 5$. There is one other deviation from the pattern, however, at $n = 3$, where voice leadings from X and X^{-1} cancel one another out. This is determined by the circle-of-fifths distance between the two altered notes, A♯ and C♯ (three). The general size of any ST^n, then, is $n + 2a - 2b$ where a is the number of single-semitone moves in X at a basic-cycle distance $> n$ and b is number of basic-cycle differences equal to n between single-semitone moves in X.[8]

The result also extends to the non-coprime case using Tymoczko's observations about the generalized signature transformation for these cases. The important difference is that for $(d, c) = k$, the basic-cycle voice leadings are of size k. For instance, the 3→12 cycle consists of size-3 voice leadings: (C, E, G♯) → (C♯, F, A) → (D, F♯, A♯) → (D♯, G, B).

These results rely upon an abstraction of the circle of fifths and other basic cycles. Rather than being a specific voice leading in seven-note chord space, it is an ordered cycle of inflections, (. . . , F→F♯, C→C♯, G→G♯, . . .), conceived independently of the particular collection upon which the inflections act. As such the 7→12 cycle can even act on sets of a cardinality other than 7, such as triads.

[8] This result assumes that all the voice leadings in X are in the same direction, which is reasonable since it is always possible to choose X such that this is the case, although X may not be the most efficient voice leading to a ME set.

The same abstraction of the circle of fifths provides the foundations for Hook's [8] signature transformations, and also results from interpreting the horizontal dimension of diatonic triad space as a kind of voice leading by inflection, as in the analysis of Schubert's *Notturno* above.

(4) Spatial metaphor and non-homotopic paths in "Nacht und Träume."

The sense of distance expressed by the $\flat\hat{6}$ of Schubert's *Notturno*, according to the analysis in part two, admittedly involves some subjective inference. While that interpretation is unlikely to strain anyone's musical intuition and is supported by some circumstantial evidence in Schubert's expressive markings, such an interpretation would be even more secure in the presence of a text that explicitly evokes metaphors of distance. To that end I will piggyback on Carl Schachter's [20] widely admired analysis of Schubert's song "Nacht und Träume" with a reformulation that emphasizes voice leading and spatial metaphors. This analysis will draw analogies between enharmonic reinterpretation and "permutational reinterpretation" and discuss the musical implications of topological notions of equivalence of paths.

The central point of Schachter's analysis is that a G major passage in the middle of this B major song is built around an ascending chromatic passing tone, and therefore what appears to be a G, $\flat\hat{6}$, is really an F𝄪, $\sharp\hat{5}$. Figure 11 reproduces Schachter's harmonic summary showing the voice leading approaching and withdrawing from the G major section. For Schachter, this fragile tonicization of $\sharp\hat{5}$ evokes the ephemeral quality of dreams. He writes,

> The song embodies a musical symbol of dreams. The G-major section crystallizes around a most transitory musical event—a chromatic passing tone. Yet while we are immersed in it, it assumes the guise of that most solid tonal structure. Only at "wenn der Tag erwacht" does its insubstantiality become manifest; it vanishes, never to return except as an indistinct memory in the G♮'s of the coda.[20, p. 217]

One could also point to the depiction of physical expanse associated with night and dreams by the poem: "Holy night, you descend; / Dreams, too, float down, / Like your moonlight through space." The metaphorical gambit of Collin's poem is to lend materiality to intangibles, night and dreams, by portraying them moving through space. In this sense, it is the harmonic remoteness of the F𝄪 chord, a musical metaphor of space, that most directly unites musical and poetic imagery. Figure 12 uses diatonic triad space to show how F𝄪-major implies a much stronger sense of remoteness, because the spelling implies something to the right of B major rather than to the left.

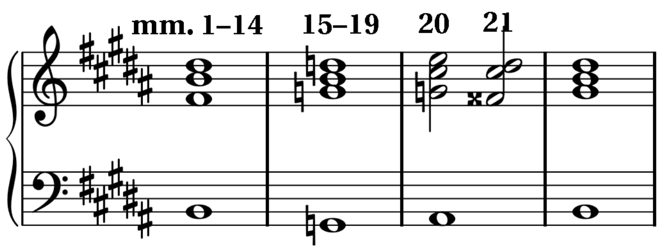


Figure 11: Schachter's summary of the harmony approaching and departing from the central G major section of the song.

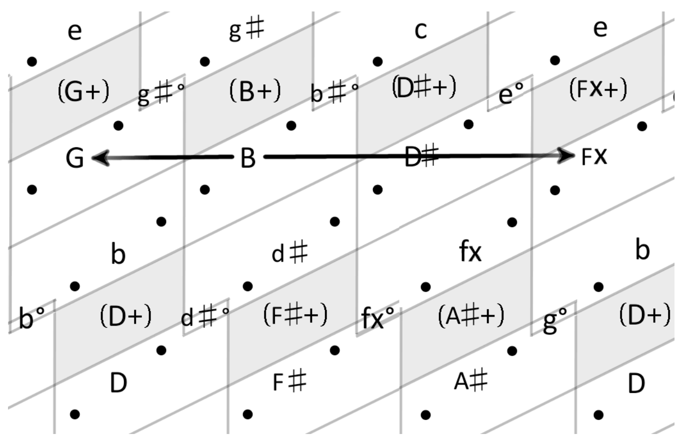


Figure 12: Distances of G major and F𝄪 major from B major.

There is a subtle paradox of musical language woven into Schachter's otherwise ingenious analysis of the song. According to Schachter, the *identity* of the note G, specifically the enharmonic identity, is contingent upon the note's behavior. Normally this would be an unproblematic assumption, when the gravitational force of the tonic key holds all harmonic events firmly in its orbit—a note approached as $\flat\hat{6}$ also resolves as $\flat\hat{6}$. However, Schubert approaches G major as if it is ♭VI and resolves as a ♯V. The idea that its identity is secured only by its resolution means that within our five measure experience of the G major tonality we cannot know the identity of the notes we are hearing, since the G♮ could—indeed is likely to—resolve otherwise. That is, if the point of enharmonic distinctions is that an F𝄪 might sound quite different, in a direct and immediate way, from a G (an easily demonstrable fact) then it can only do so on the basis of preceding context, not on what might happen in the future. This problem is not merely symptomatic of our more harmonic gloss on Schachter's analysis, either; it could just as easily be explained in purely melodic terms: if the F𝄪 does not become an F𝄪 until it ascends, then it can only obtain an identity after our literal experience of it is over.

The spatial metaphor helps to untangle this paradox of enharmonic ontology. As long as music avoids enharmonic cycles, enharmonic ontology is unproblematic—an F𝄪 is not only different from a G, it is very far away from it. However, Schubert's enharmonic tours reveal that enharmonic orientation belongs to *paths*, not points. For instance, G major, under ordinary circumstances, appears to be to the left of B major, making B major → G major a flatward move. However, in Schubert's progression, G major returns to B major via a series of flatward moves mediated by D♯ major (V/vi). According to this path, G major (or F𝄪 major) is to the *right* of B major. The chord itself is enharmonically indeterminate, but the enharmonic orientation of the paths (flatward) is definite.

The enharmonic tour is a geometrically special kind of path in the same sense that it is musically special. This is demonstrated by the topological principle of *homotopy* discussed in part 2 above. Consider three voice-leading trajectories that Schubert takes from B major to G♯ minor in the song. The first form the progression takes in the song is the most direct semitonal path, as shown in Fig. 13. The second path, summarized in Fig. 14, is a larger-scale harmonic progression from mm. 5–10 (and actually contains the progression of Fig. 13). This progression is harmonically conventional, but does have one remarkable feature: in order for the G♯ minor triad to appear in the correct position, a "permutational reinterpretation" must occur somewhere in the progression. In the harmonic summary this is placed on the half-cadential V chord.

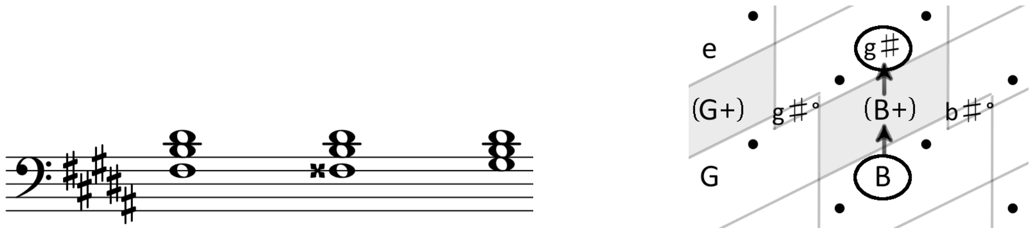


Fig. 13: A path from B major to G♯ minor from mm. 7–8.

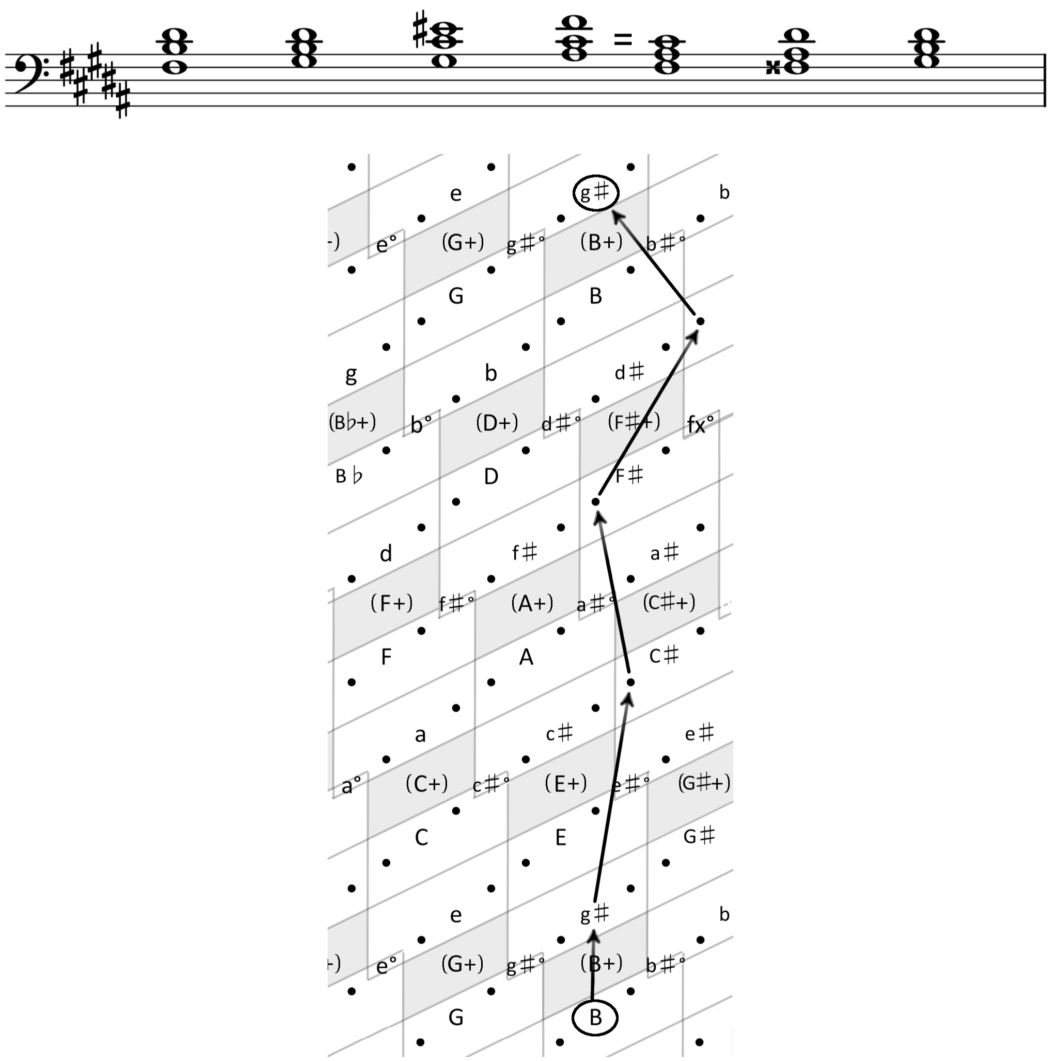
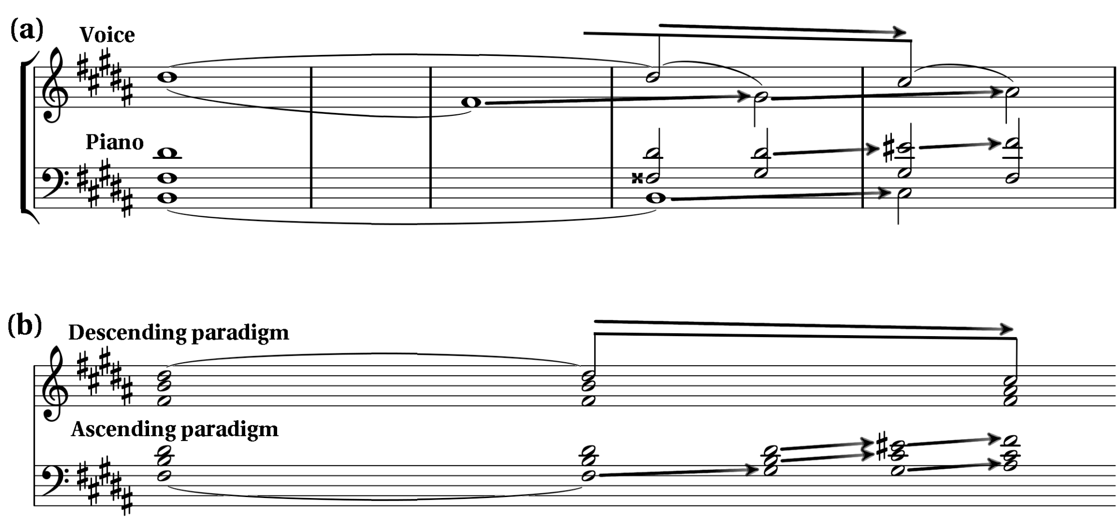


Fig. 14: A path from mm. 5–10.

This sort of permutational reinterpretation, though ordinary, is nonetheless musically significant. The composer must draw upon some feature of musical texture or the unfolding of harmonies to achieve such a slight-of-hand. Schenker's principles of motion to an inner voice and reaching over, for instance, are essentially kinds of permutational reinterpretation. Schubert's method in "Nacht und Träume" is particularly sophisticated, drawing on the fundamental independence of voice and piano. Figure 15 shows the prominent voice-leading connections in the two parts. The vocal part expresses voice-leading connections at a more background level, which allows it to draw upon a more direct voice-leading connection between I and V whose most efficient voice leading descends. The ascending pattern appears in prominent connections in the chordal accompaniment in the piano, the bass, and inner-voice connections present in the vocal line. Although the descending paradigm is represented by only one voice-leading connection, it is an especially important one because of its prominence in the vocal part, its status as a structural line, and its alignment with the phrasing boundaries. Thus Schubert parallels the poem by juxtaposing an ascending schema (in the piano) with a descending one (in the voice) which are magically united, like the sleeper who, striving upward, finds himself immersed in dreams and night floating downward from the heavens, so that up becomes down and down up in an altered reality of directionless space.



(c)

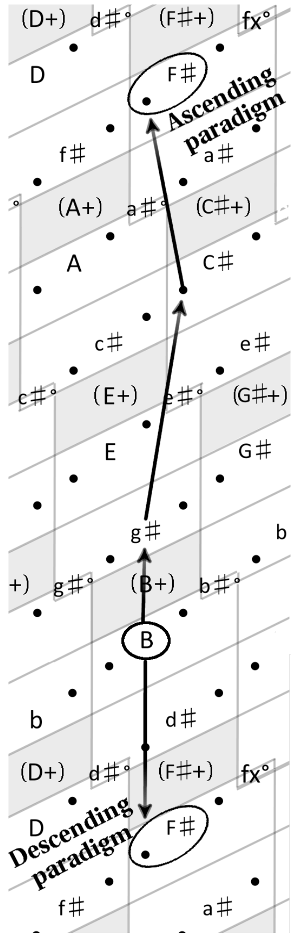


Figure 15: (a) A voice-leading summary of mm. 5–9, and (b–c) the two triadic paradigms they invoke.

The paths of Fig. 13 and Fig. 14 are non-homotopic. Because the vertical cycle of the space is created by the octave and permutational equivalence assumed in the underlying voice-leading space, this topological non-equivalence corresponds to the musical fact that one series of voice leadings can not be considered a simple embellishment of the other by neighboring motion.

The enharmonic cycle traced in the middle section of the song is non-homotopic with the other two in an analogous way, as shown in Fig. 16. The musical non-equivalence is not one of literal triadic voice leading, however. It *is* possible to consider this a neighbor-motion embellishment of the simple progression in mm. 7–8. (A point conveyed by Cohn's [15, 21] discussions of balanced voice leading in hexatonic cycles, which this progression closely resembles.) The topological uniqueness instead has to do with the essential enharmonicism in the progression of diatonic inflections, which, because they move consistently downwards, require a single enharmonic respelling to restore the sharps of the original key signature. *Where* that enharmonic identification occurs, however, is essentially arbitrary.[9]

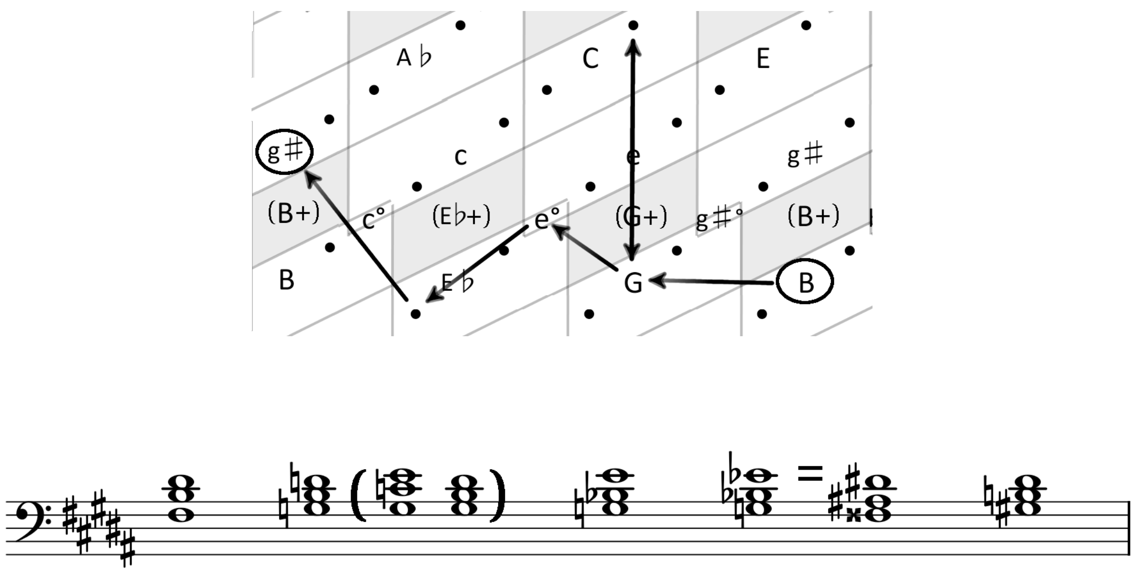


Fig. 16: A path from B major to G♯ minor from mm. 15–20.

We can make two broader conclusions about musical space from this analysis of "Nacht und Träume." First, metaphors of space can be elicited both by literal voice leading between triads as well as circle-of-fifths inflections in the scalar contexts implied by the harmony. Second, these two kinds of inflection are more effective for different kinds of spatial metaphor. Voice leading provides strong connotation of direction, ascending and descending, while tonal inflection is better at creating a sense of distance or depth.

Conclusion

This paper shows that the overlap between mathematical theories of harmony promoted by Tymoczko, Hook, Plotkin, and Douthett is extensive enough that it is possible to unify them in a theory of voice-leading quantization that retains the best features of each. This unified approach is (1) the theory of voice-leading

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[9] Cohn [21] makes a similar point about hexatonic cycles.

geometry with an added concept of voice leading by scalar inflection that uses the circle of fifths (or generalized circle of fifths) in the abstract sense promoted by Hook [8], or it is (2) a looser, more flexible theory of filtered point-symmetry adjusted to make literal voice-leading distances apparent. In either case we find that the unified theory is a natural outgrowth of each particular theory, especially when tested on the grounds of music analysis.

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