Tonal Prisms: Iterated Quantization in Chromatic Tonality and Ravel's "Ondine"

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Maximally-even scales, first described by Clough and Douthett [1], and by now well known to music theorists, originated in a field of study, scale theory, that has largely remained theoretical and remote from music analysis.[1] Recent generalizations of the principles underlying maximal evenness, Douthett's [3] filtered point-symmetry (especially in Plotkin's work [4]), and Yust's [5] uniform progressions, have begun to derive tools for analysis, particularly in chromatic tonal music of the nineteenth century, from the principles of maximal evenness.

In parallel with this development, Tymoczko [6] has shown a wide array of analytical applications of the voice-leading geometries described in [7]. His work shares an underlying mathematical kinship with maximal evenness and filtered point-symmetry in that they all rely upon an underlying model of pitch that is inherently continuous, leading to well-formed concepts of distance between musical chords.

The present paper will bring these approaches together in an analytical setting, with a focus on second-order maximal evenness and its musical implications. I will reframe Douthett's filtered point-symmetry as iterated quantization in voice-leading geometries and consider the implications of the "classical" instance of second-order maximal evenness, diatonic triads, for the analysis of tonal music. I will then advance a novel filter configuration for understanding Ravel's use of acoustic scales in "Ondine," and analyse the piece in depth using a "scalar *Tonnetz*" that instantiates features of the second-order ME construction.

[1] The J.S. Bach examples in [2] are a notable exception.

(1) Diatonic triads and two P-R progressions

The concept of maximal evenness has always had the appearance of an economical explanation for common scales of five to eight notes. However, it has never been entirely clear exactly what kind of explanation it is supposed to be. It also does not quite hit the mark with three-note collections—the augmented triad, not major or minor triads, is maximally even. The concept of second-order maximal evenness, articulated in [1], [3] and [8], extended a kind of maximal evenness to major and minor triads as well as other relatively even chord types and scalar collections.

The description of diatonic triads in terms of second-order maximal evenness is roughly this: Take a 7→12 ME set (i.e. any diatonic scale); then, take a three-note ME subset of this scale, treating is as if it were a 7-equal universe. The result will be a major, minor, or diminished triad. Note that, unlike the ordinary ME property, the type of chord is not uniquely determined by the designation 3→7→12 ME set. Also, a given chord might be derived as second-order ME in multiple ways (a major triad, for instance, as a member of three possible diatonic scales).

Mathematically, the derivation of a second-order ME is essentially a process of *iterated quantization*. For the 3→7→12 example, take any perfectly even three-note and seven-note sets (and a fixed even twelve-note set). Move the notes of the three-note set to the nearest elements of the seven-note set ("*quantization*"), then move the resulting three notes to the nearest of the twelve (a second quantization). The definition of quantization implies a rounding function, rather than the floor function used by Douthett [3], but there is no essential mathematical difference between the two.[2]

One can visualize the process in three-note chord space [6–7] as shown in Ex. 1. The cubes running up the middle of the space connect points on the lattice of 12-equal chords near the centre of chord space, consisting of the perfectly even (augmented triads where pairs of cubes touch) and nearly even (major and minor) triads,[3] with one diminished triad thrown in. The space assumes octave and permutational (but not transpositional) equivalence, which means that the "A min" point on the upper right of the figure is the same as the one on the lower left—i.e., the real space is a twisted necklace of such cubes, which

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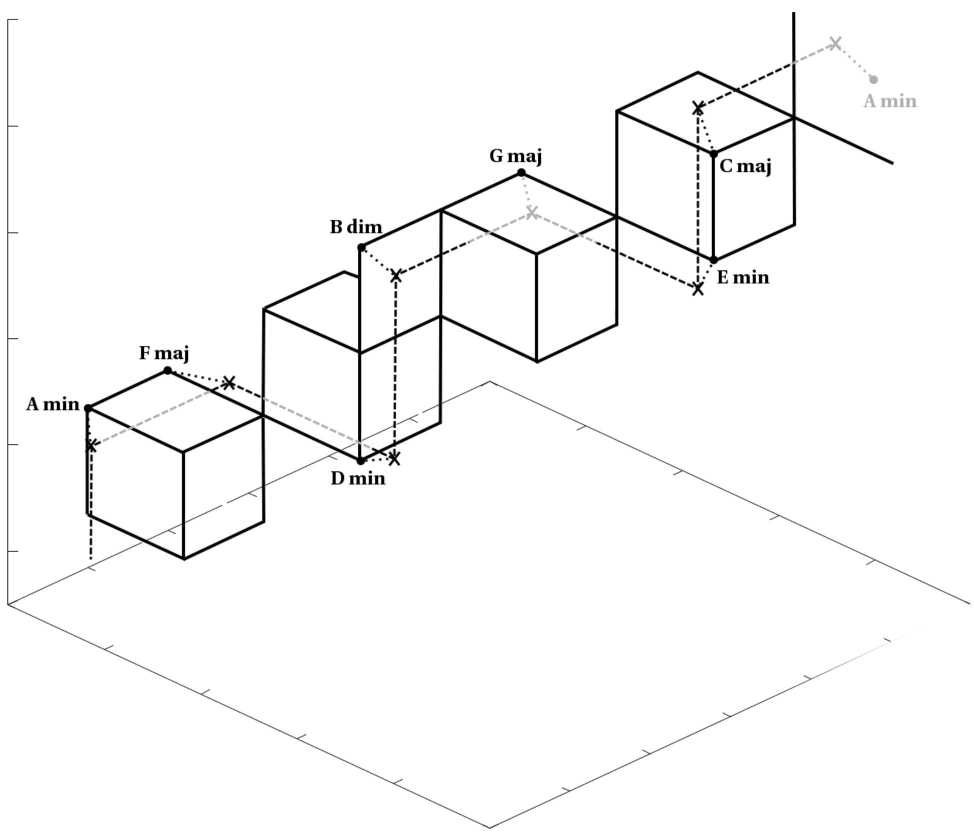
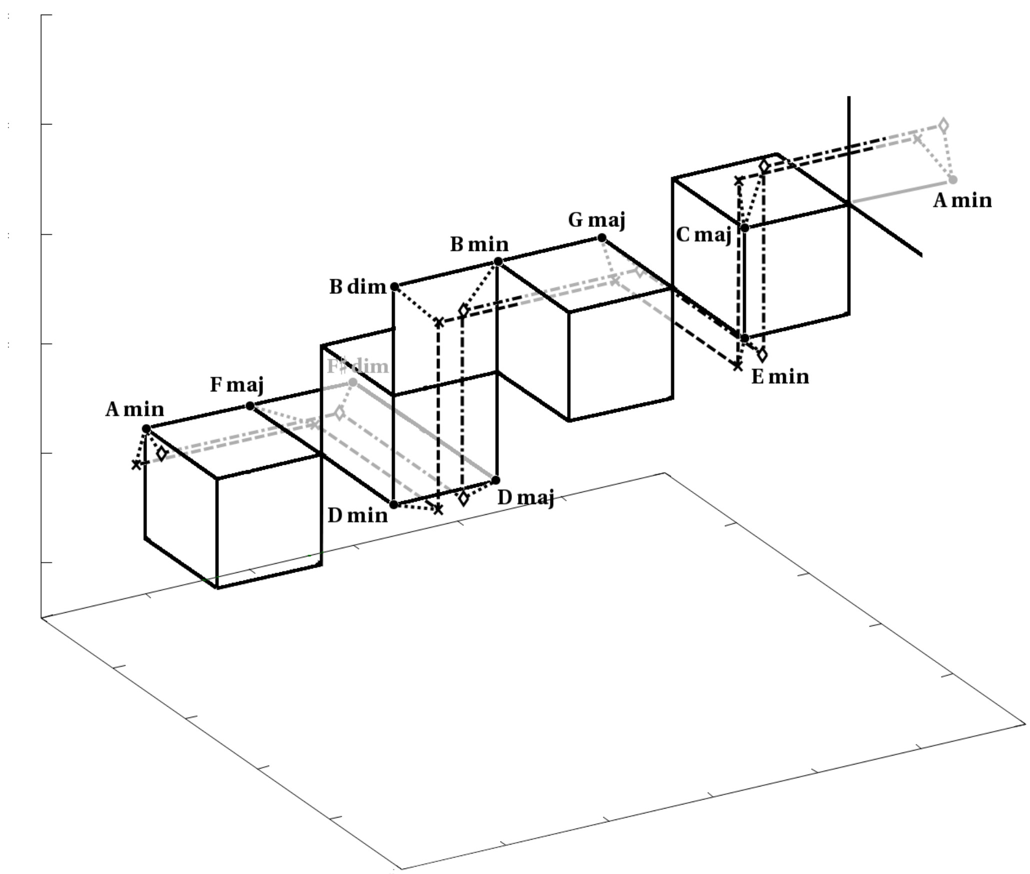
[2] Rounding is the same as adding a constant of 0.5 before applying the floor function. Note that this implies the round-up "tiebreaker rule." Tymoczko's contribution to this volume makes a case for rounding rather than using the floor function.

[3] See also the illustrations in [7, Fig. S5] and [6, pp. 101–106].

has been cut and unfolded for the purpose of visualization. The dashed line winding around the cubes touches all the maximally even triads in a lattice of 7-equal chords, connecting them via maximally smooth voice leading (third progressions). When each 7-equal triad is quantized (rounded to the nearest element of the twelve-tone lattice), the result is the set of seven triads belonging to one diatonic scale. Which diatonic scale depends on where the 7-equal lattice is positioned relative to the 12-equal lattice. For instance, if you shift ("microtranspose") the 7-equal lattice up by $^{1}/_{7}$-semitone (Ex. 2) four of the points still round to the same location, but the points that previously went to F major, D minor, and B diminished "flip" up to F♯ diminished, D major, and B minor. In other words, a shift of the 7-equal lattice by $^{1}/_{7}$-semitone has the effect of moving the set of resulting triads up one place in the circle of fifths. This microtransposition behaves in a way equivalent to Hook's [9] signature transformation s_1 acting on triads, but conceived specifically as a voice leading.



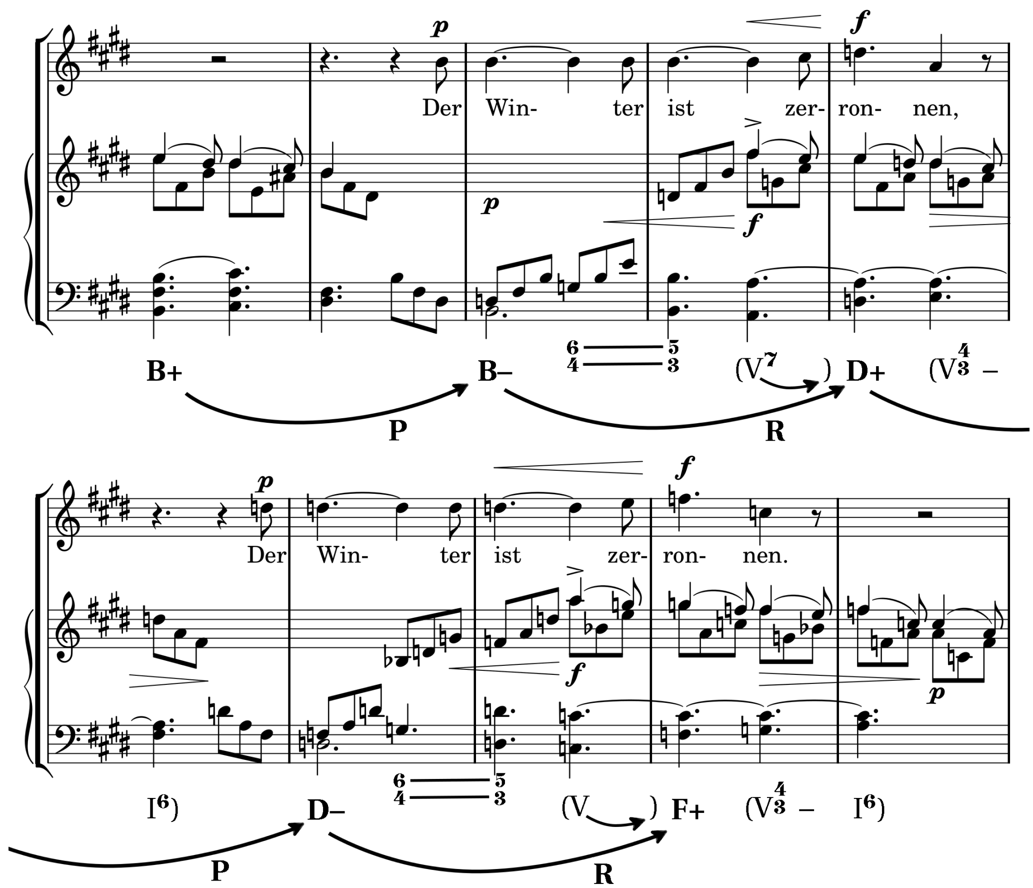
Example 1: A lattice of ME 7-equal triads quantized to a 12-tone lattice, resulting in the triads of C major.



Example 2: The lattice of 7-equal triads from Ex. 1 and the same shifted up by $^1/_7$-semitone, which quantizes to the triads of G major.

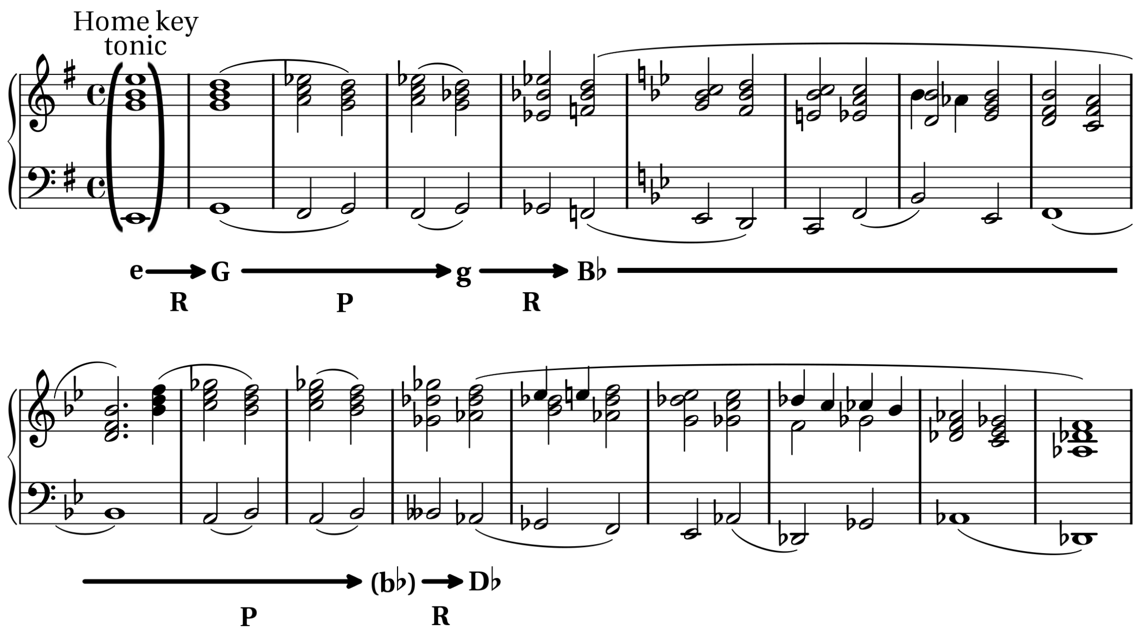
It is well-known that the 7→12 ME property explains the circle of fifths notion of key distance—that is, the fact that diatonic scales form a cycle consisting of single-semitone displacements. The 3→7 component of the 3→7→12 package has similar implications: triads within a given scale form a cycle consisting of single scale-step voice-leadings (the cycle of thirds). The analogy between these two kinds of voice-leading cycle and its implications for tonal harmony has also been described by Tymoczko [6, pp. 252–258].

At this point, a musical example will be helpful. I will give two: A passage from Brahms's song "Der Frühling" (Ex. 3) and from the development of Brahms's op. 38 Cello Sonata in E minor, first movement (Ex. 4). Both of these use the same modulatory recipe (a stock technique for a composer like Brahms) to make a quick, smooth traversal of a tritone key relationship: (1) change mode, (2) shift to the relative major, (3) repeat.



Example 3: Meas. 16–23 of Brahms' song, "Der Frühling," a straightforward example of the P-R modulatory scheme.

A neo-Riemannian description of the modulatory scheme as a P-R progression—following, e.g., [10–13]—reduces a relationship between keys to a progression between their tonic chords. The neo-Riemannian account highlights an important aspect of the scheme that makes it particularly useful in Brahms's compositional style: the maximal common-tone retention from one tonic chord to the next. However, because neo-Riemannian theories are based purely on raw voice leading relationships between triads, they somewhat impetuously scrap the distinction between chord and key. For these kinds of passages, transformational analysis necessarily leans on this commonplace distinction to separate tonic chords from non-tonic chords, but at the same time the theory delegitimizes keys by conflating them with their tonic triads. These tonic triads may or may not be literally juxtaposed (especially in the case of the R-related triads) and the voice leadings that, for Cohn ([12–14]), constitute the bulk of the theoretical underpinnings of the neo-Riemannian transformations, vary in their degree of literalness (although, interestingly, all are evident in some form in the Brahms examples). Cohn notes this problem of ontological status of keys and tonality in neo-Riemannian analysis in [13, ch. 8].[4]



Example 4: A harmonic reduction of the first part of the development section in Brahms's op. 38 Cello Sonata (mm. 91–107), showing the P-R modulatory scheme.

The Cello Sonata example in particular recommends some caution against allowing keys to slip through the sieve of one's theoretical apparatus. While Brahms prepares his *first* modulation in this case with a clear switch to the minor tonic, he revokes this clarifying minor tonic in its sequential repetition (mm. 100–107). The melodic repetition of the motivic $\flat\hat{6}$–$\hat{5}$, G♭–F, from the borrowed vii^{o7} is sufficient to bridge the modulation smoothly. The fact that the B♭ minor chord never appears literally does not fundamentally alter the nature of the modulatory process. This suggests that mode change is not as much a change of triad as it is a change of underlying scale, in line with the more traditional way of thinking. (The $\flat\hat{6}$–$\hat{5}$ linkage in fact prepares this transformation of *scales* in an obvious way.)

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[4] Cohn's solution is to recognize diatonic regions within the *Tonnetz* and confer a special status to progressions that "teleport" between the boundaries of this region, treating it loosely as a modular subspace. The mathematical foundations of this approach might be examined beginning from the special properties of diatonic subregions, which are far from being arbitrary selection from amongst many possible *Tonnetz* parallelograms (see [15]). This may result in some convergence between Cohn's approach and the present one.

Deriving diatonic triads via iterated quantization captures the insights of the neo-Riemannian perspective concerning the special voice-leading properties of Brahms's P-R modulatory scheme while also re-introducing a role for scales, which provides some additional appreciation for the properties of the P-R routine. The parallel relationship is special in that it changes the underlying scale without a change of tonic triad, while the relative key changes the tonic triad without a change in the underlying diatonic scale. Using quantization, we can say that P transposes the 7-equal triad by a small amount prior to quantization, shifting the lattice (and hence the scalar context) along with it, as in Example 2 (except down by $^3/_7$ instead of up by $^1/_7$). I will call this *microtransposition*, a voice leading created by quantizing a fractional transposition. The R progression is a move within a given 7-equal lattice prior to quantization, as in Example 1. I will call this a *position change* because it is a way of creating an efficient voice leading, moving one voice at a time, by rotating the position of the maximally even chord. Quantization thus decomposes the voice leadings into a triadic and a scalar component, showing that the P-R routine is a special type of progression that restricts itself to one "dimension" at a time. Where Brahms performs the two transformations at roughly the same time (as in moving directly from B♭ major to D♭ major in the Cello Sonata) we can nonetheless distinguish them by voice-leading types (the diatonic B♭–A♭ and the chromatic D–D♭).[5]

This "decomposition into components" can also be derived directly from Douthett's [3] J-function (limiting ourselves to the n = 2 case):

$$J_{d_0,d_1,d_2}^{m_1,m_2} = \left\{ \left\lfloor \left(d_0 \left\lfloor \frac{d_1(k)+m_2}{d_2} \right\rfloor + m_1\right)/d_1 \right\rfloor \right\}_{k=0}^{d_2-1}$$

The J-function generates a set by applying the floor function twice. Beginning from a perfectly even set of cardinality d_2, the floor function maps each note to the nearest lower value on a d_1-even scale. It then repeats this process mapping notes to a d_0-even scale. The numbers d_0, d_1, and d_2 do not fully determine the result, however, because each of the perfectly even sets can shift relative to one another. Their relative positions are set by real-valued parameters m_1 and m_2. The value m_1 sets the position of the *filter*, the perfectly even set of size d_1, relative to the fixed d_0 values (the integers 0–11). In the diatonic triads example, m_1 determines the position of the diatonic scale around the circle of fifths, starting with D♭ diatonic for $m_1 = 0$, A♭ diatonic for $m_1 = 1$, etc. Adding 1 to m_1 creates a microtransposition, which, in the 3→7→12 case, behaves like Hook's [9] signature transformation s_1: the diatonic framework moves up one place on the circle of fifths while the notes of the triad

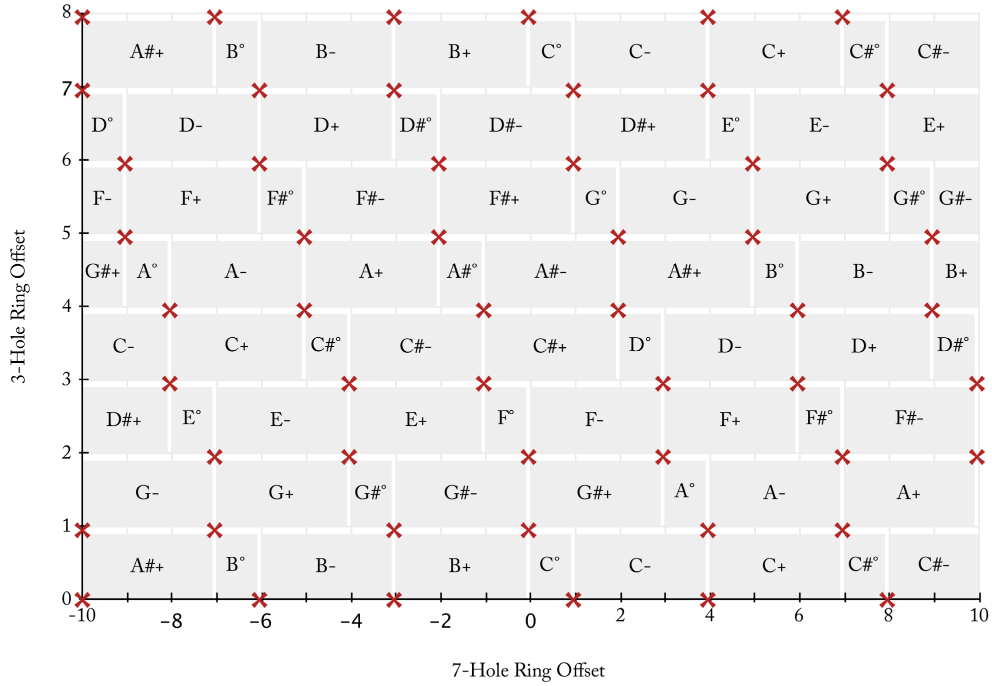
[5] See, e.g., Kopp's [16] detailed discussion of the voice-leading attributes of chromatic mediant relationships.

remain in place except where one is inflected by the new accidental. The value of m_2 determines the position of the *beacon*, the perfectly even set of size d_2, relative to the filter. Adding 1 to m_2 creates a position change, which is a descending-third progression within a fixed diatonic framework in the 3→7→12 case.

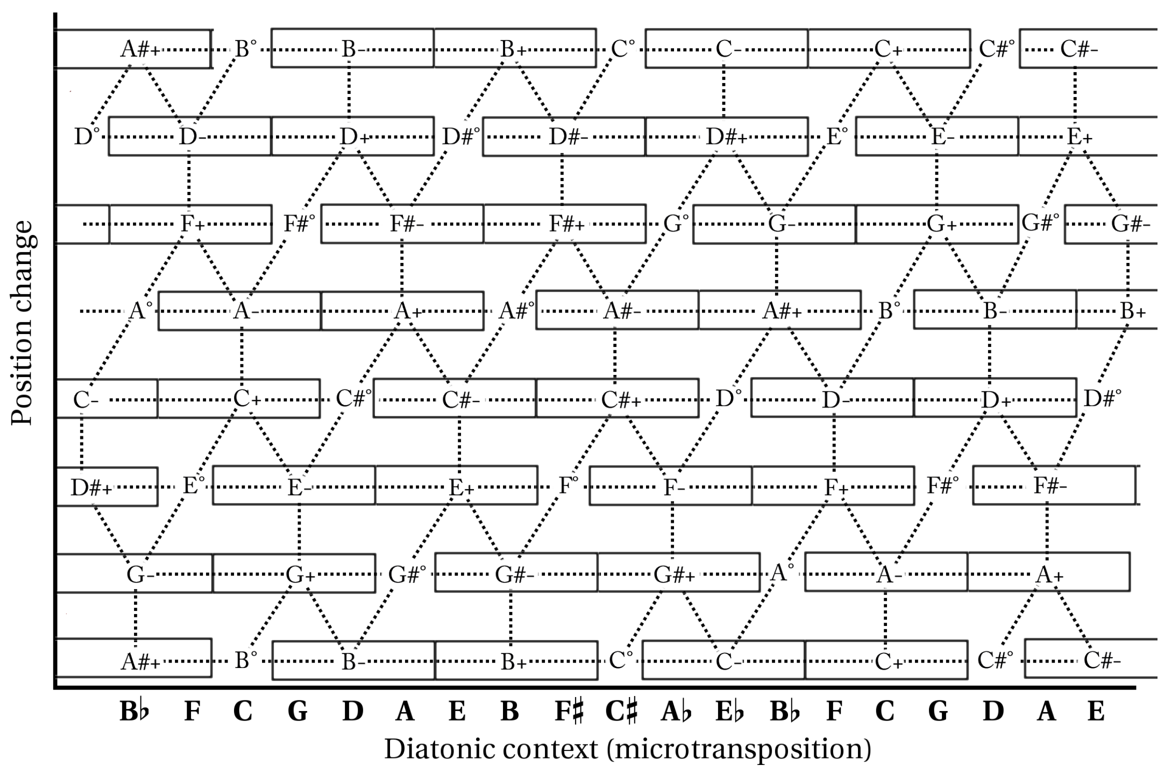
Technically, the J-function generates unordered sets and therefore does not track specific voice leadings. However, reframing the function as iterated quantization shows that voice leadings are a natural byproduct of the way that the J-function generates sets. This explicit voice-leading characterization of continuous changes in J-function parameters can also be derived by labelling beacon-holes in filtered point-symmetry. It is equivalent, however—and often useful—to treat second-order ME sets as the quantization of some maximally even set in a voice-leading space. The terms microtransposition and position change should be interpreted specifically in this sense: as types of quantized voice leadings.[6]

Because two-stage iterated quantization parameterizes voice leadings in terms of microtransposition and position-change components, it can be conveniently visualized in two dimensions. Plotkin [4] uses such coordinate spaces extensively; Example 5 reproduces his illustration for 3→7→12 triads. When thinking in terms of voice leading, however, it is better to construe the space as a network of microtransposition and position changes, as in Example 6, because while microtransposition is continuous, position changes are discrete voice leadings. Also, because they combine different kinds of voice leading, larger distances in Plotkin's charts, or in these networks, do not reliably reflect voice-leading distances (either pre- or post-quantization): motions along the NW–SE diagonal are smaller than similar motions on the SW–NE diagonal. The same is true of the *Tonnetz*, for similar reasons (see [6, pp. 412–417] and [17]). Because microtransposition is continuous, nodes of the network in Ex. 6 represent segments of the line of transposition that quantize to the same triad. Major and minor triads constitute larger segments, as shown by the boxes. Each vertical connection represents position changes between some range of transpositions. The spatial layout is retained from Plotkin's charts because, though not a feature of the network as such, it is significant for two reasons: first, it aligns position changes that belong to a given scalar context. Second, although we should not read too much into distances on the chart, directions are clearly meaningful. The slope of a line drawn arbitrarily over the network corresponds to an average circle-of-fifths distance per position change. This is important for the discussion of anchored progressions below.

[6] These terms are not restricted to quantizations of ME sets, either. I do not consider quantizations of non-ME chords in this paper, but they do appear frequently in [5].

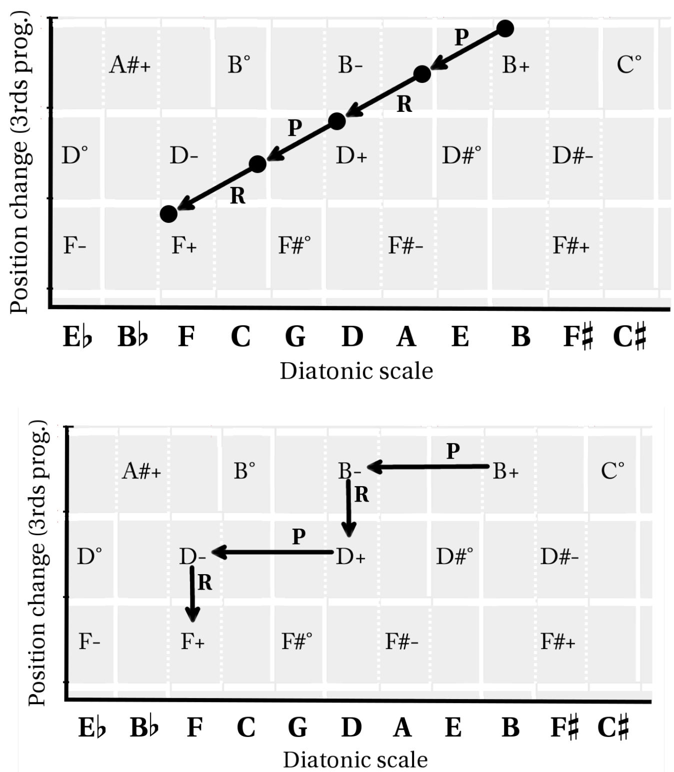


Example 5: Graph of 3⟶7⟶12 from Plotkin [4, p. 76]. X's indicate that this kind of "king's move" in the space in non-parsimonious (because it executes microtransposition and position change simultaneously.) Plotkin's "7-hole ring offset" is m_1 *and his "3-hole ring offset" is* m_2 *in Douthett's J-function.*



Example 6: The graph of Ex. 5 as a network

Douthett [3] adds another constraint to progressions derived from filtered point-symmetry with his dynamical systems, which I will refer to as *second-order uniformity* (although second-order uniformity is technically less restrictive than Douthett's construct because it allows for free definition of the initial set). Second-order uniformity results when the m_1 and m_2 values are linear functions of a single variable, t, and the progression consists of a series of integer values of t. (Restriction to integer values of t corresponds to what Douthett calls a "stroboscopic portrait.") This means that the progression results from repeatedly applying a single vector in a coordinate space like Ex. 5, so that the microtransposition per position change is constant. Example 7 shows Brahms' P-R progression from "Der Frühling" according to Douthett's [3] second-order uniform derivation of P-R (shifted slightly to the left for clarity). It is impossible to get P and R from the same vector in such a way that each chord is a tonic, because second-order uniformity requires that the diatonic contexts proceed at a uniform pace around the circle of fifths. In Ex. 7(a), the points for minor chords are to the right, making them ii or iv chords. A different trajectory, (b), results from using the key centres implied in the musical example. The P and the R progressions of (b) involve vectors of different direction and size, navigating the space in a city-block fashion.



Example 7: Brahms' P-R progression as a path in Plotkin's space, as (a) a second-order uniform progression and (b) a progression of two components (within key progressions and key changes).

The property of second-order uniformity is ingenious in that, where it applies to a given sequential progression, it is a highly non-trivial and therefore may reveal something about the progression that can have significant analytical consequences. This fact is evident from Yust's [5] application of a more restrictive principle of (first-order) uniformity to chromatic sequences. First-order uniformity refers to a progression that is evenly paced prior to a single quantization. Yet, it also allows for a free definition of the initial set where Douthett's J-function requires that the initial set be perfectly even. Therefore, one can derive, e.g., a diatonic thirds progression as (first-order) uniform by defining the initial set as a 7-equal triad, $(0\ 3\tfrac{3}{7}\ 6\tfrac{6}{7})$, rather than deriving the triad through another step of quantization.

When applying such mathematical concepts in music analysis, however, we face the fact that analysis is only useful to the extent that it is explanatory. Uniformity reflects a special property of a limited class of musical phenomena, and applying it willy-nilly wherever a certain set of parameters happens to work out dilutes its explanatory value. The fact that there is a second-order uniform description of the P-R progression does not seem to have any direct casual relationship to its appearance in nineteenth century harmony, because the second-order uniform description requires the diatonic contexts to be different than those implied by musical examples. (Such progressions typically appear as they do in the Brahms examples, with each chord of the P-R sequence treated as a local tonic).[7]

However, P–R exhibits a *different* special property when derived via iterated quantization. Plotted as in Example 7(b), the two components of the progression isolate position change (R) and microtransposition (P). This property nicely captures the sense that the P-R modulatory scheme isolates minimal instances of two distinct types of voice leading, change-of-scale voice leadings and within-scale ones, and it reinforces the argument that the second-order ME construction adds explanatory power by making diatonic context a feature of triads.

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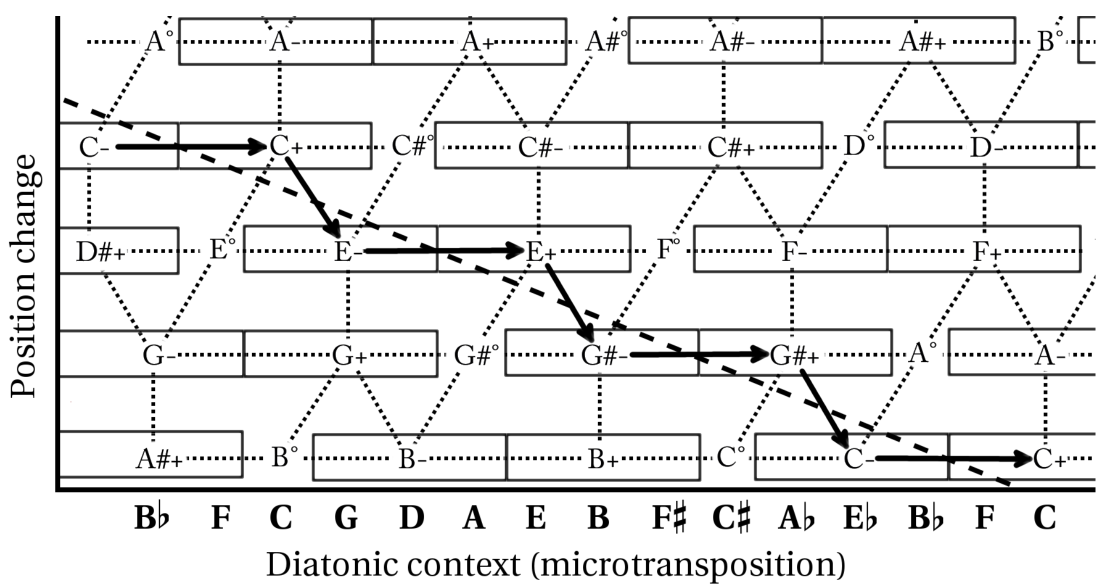
[7] In their contribution to this issue Plotkin and Douthett also derive the P-R cycle using an 8-hole filter. This similarly has little obvious explanatory connection to the Brahms examples and others like it beyond the feature of voice-leading parsimony, because there is no apparent use of octatonic scales in the passages. One caveat is that this 3→8→12 space may have some relevant application to tonal passages whose harmonic logic is governed more by enharmonic reinterpretations of diminished seventh chords than diatonic contexts.

A special type of second-order uniformity occurs when the pace of position change and microtransposition exactly cancel one another out in terms of the overall voice-leading trajectory of the maximally even chord prior to quantization. As Example 8 illustrates, this kind of progression, which I will call an *anchored progression*, can produce one of Cohn's [14] hexatonic cycles,[8] and exhibits the property of voice-leading balance (balance between upward and downward voice leadings). The slope of the dashed line in Example 8 reflects the fact that a $3 \rightarrow 7 \rightarrow 12$ anchored progression includes exactly one position change for each $\frac{4}{7}$ of microtransposition (because $\frac{4}{7} \times 3 = 1\frac{5}{7}$, the size of the 7-equal step). The term "anchored progression" reflects the fact that the progression quantizes the maximally even sets closest to a given point at the centre of chord space,[9] so that this point acts like an anchor restricting the progression to a limited range of what Cohn [13] calls *voice-leading zones* as the key moves around the circle of fifths. Cohn demonstrates some far-reaching music-analytical implications of voice-leading zones and voice-leading balance in [13]. Section 2.4 below will also apply these concepts in an analytical setting.



Example 8: A hexatonic cycle as an anchored progression of diatonic triads.

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[8] Note that anchored progressions on triads generalize hexatonic cycles, including, e.g., varieties with diminished triads.

[9] This fixed point is equivalent to a fixed beacon position in filtered point symmetry. Douthett [3] shows how dynamical systems with a fixed beacon can generate hexatonic cycles.

(2) Acoustic Collections in Ravel's "Ondine"

(2.1) "Ondine": Harmony and Thematic Design Ravel describes the pieces in his collection *Gaspard de la Nuit* as "poems for piano," each drawing its inspiration from a poem of Aloysius Bertrand. "Ondine" describes a water nymph hallucinated by a sailor awoken in the night by a storm at sea. The poetic imagery, centred on the water, gradually shifts from the glittering of the sea-spray in the moonlight, to the violence of the storm, whipped up by an angry sea-god, culminating in the Ondine's proposal to take the sailor to her palace made of water. When the sailor refuses, the sea swells with the Ondine's indignation.

Ravel conveys this imagery musically in the first half of the piece by using sonorities based on the acoustic scale, whose magical quality evokes the fantastical nature of the Ondine. At the midpoint of the piece (mm. 45–51) Ravel transfers the melody into a bass register to mark the point where the angry sea god is introduced, "thrashing the croaking water with a green branch of alder."[10] In a subsequent development, the new melody transfers back up into the upper register uniting the personae of the Ondine and her father. The violence of the middle section ebbs when the melody from the first part of the piece returns (mm. 68–71 / 80–87) representing the Ondine's proposal.[11]

The first gesture of the piece (Ex. 9) is an accompanimental ostinato that evokes the glittering of the beads of water in the sailor's window with the twinkling minor second, G♯–A, over a B acoustic-scale "II" chord.[12] Ravel floats a melody, marked "sweet and expressive," within the cracks of the accompaniment. Its characteristic descending leap perhaps represents the Ondine's salutation, "hark, hark." The piece is defined by this melody, sustained in long continuous stands of developing variation, in gradually evolving contexts of increasingly technical and registrally sprawling accompaniment patterns that fill out the rich extended altered-dominant sonorities based on acoustic-scale or diatonic collections.

Ravel's use of scalar collections has previously been described by Tymoczko [20] (see also [6], pp. 309–311) Tymoczko's analysis notes the importance of the acoustic collection

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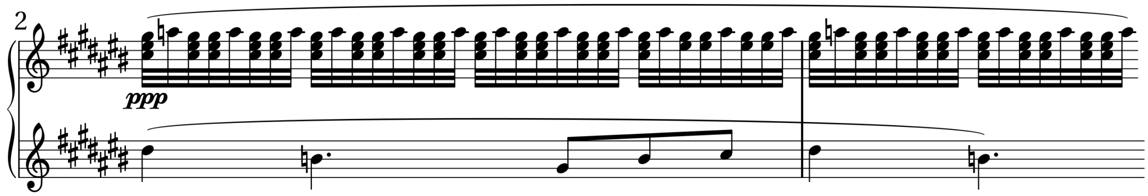
[10] Translation Mattias Müller, from the Henle edition.

[11] Other analysts, such as Howat [18], characterize this design in sonata-form terms.

[12] Bhogal [19] brings out the interesting metrical aspects of this accompaniment as it develops over the course of the piece.

and identifies its appearances in different modes. My analysis here focuses only on collections, setting aside questions of tonal centre or harmonic root. (Indeed, such identifications of tonal centre are often elusive in this music. For instance, while my ear, focusing on the melody, strongly hears Example 9 as B-centred, others such as Tymoczko, focusing on the right-hand accompaniment, hear this as the C♯ mode of B acoustic.)[13]



Example 9: "Ondine," mm. 2–3.

A harmonic strategy that Ravel uses throughout the piece is evident in mm. 2–10 (Ex. 10). The collectional "harmony" progresses at two levels. At the local level, Ravel often oscillates between two collections on the last beat of a measure. In Ex. 10 the secondary collections are indicated in parentheses. The collection that occurs in the larger first part of each measure usually persists and changes only every three measures or so. Thus, in mm. 2–10, there is a larger-scale progression, B acoustic → G♯ acoustic → C♯ diatonic, but also local oscillations to C♯ major in the context of G♯ acoustic and B acoustic in the context of C♯ major. The progression resembles ocean waves or tides with the way it superposes small regular fluctuations over longer ones. The larger-scale progression matches the phrasing: mm. 2–3 is the Ondine's signature melodic idea, mm. 5–6 develop the idea, and mm. 8–9 restate it up a whole tone, but in a diatonic context.

The three collections that Ravel uses in mm. 1–10 are closely related on a voice leading metric (see [22] and [6]). The ideal voice leading from B acoustic to C♯ major is {A→A♯, B→B♯}, and from C♯ major to G♯ acoustic is simply {C♯→C𝄪}. These collections

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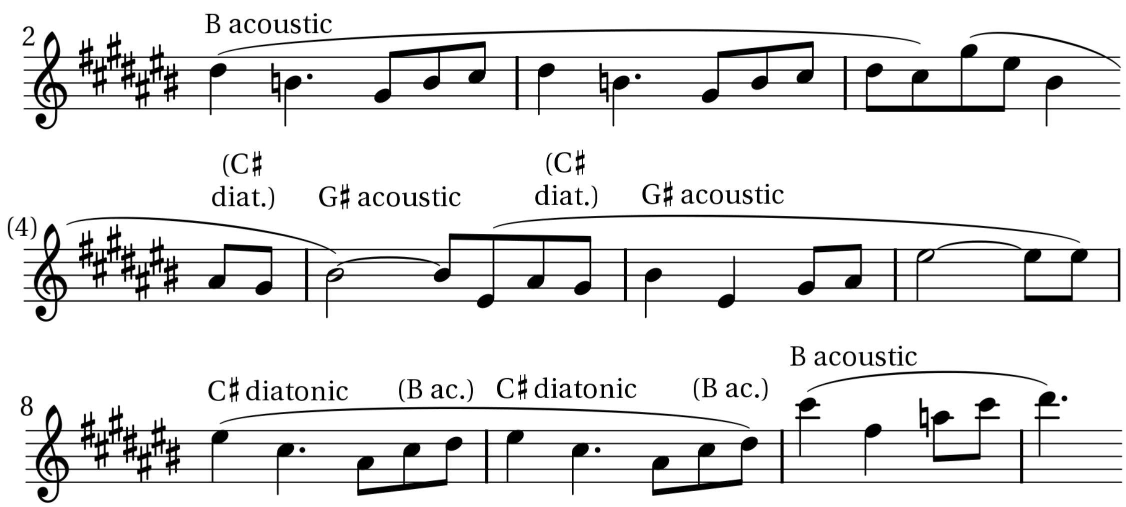
[13] Throughout this analysis I use note-names to indicate the transpositions of scalar collections (such as "B acoustic" or "C♯ diatonic") according to musical convention, without intending to imply anything about perceived tonic or root. More neutral nomenclature is available, most notably Hook's method [21] of naming spelled heptachords, which uses circle-of-fifths positions. I adhere to the conventional nomenclature here only to make the analysis easier to follow.

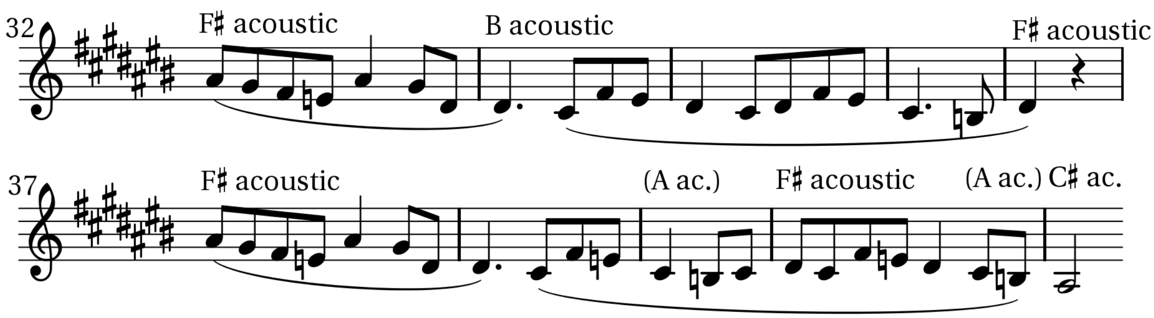
are therefore in close proximity in voice-leading space, and C♯ diatonic smoothens the transition between the acoustic collections as a voice-leading intermediary.



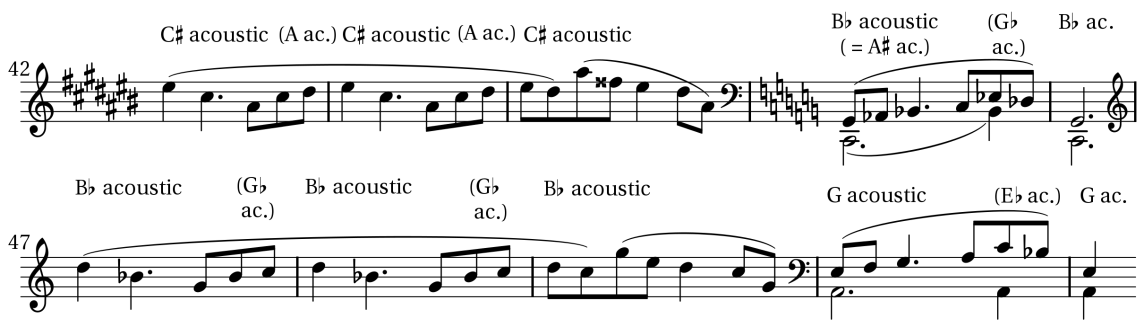
Example 10: The melody in mm. 2–10 of "Ondine," with the collection implied in the accompaniment indicated. Collections in parentheses are secondary. The motivic derivation of mm 5–6 from mm. 1–3 is also indicated.

The overall plan for "Ondine" depends crucially on this sense of distance between scalar collections, treated essentially as large-cardinality harmonies. A gradual increase in the distances between scalar collections represents the rising violence of the surging sea. Ravel layers the harmonic oscillations, superposing quick within-measure harmonic shifts upon larger-scale shifts in harmonic focus. Underlying this procedure is a way of relating scalar collections based on the whole-tone segments that occur within them, which is not perfectly reflected in simple voice-leading distances. A derivation of the collections via iterated quantization in the next section captures the special mathematical features of Ravel's voice leadings.

The first break in the piece's continuous melody occurs in m. 30. The following music is based primarily on the F♯ acoustic collection (Ex. 11). A harmonic lurch in m. 39 to A acoustic is the largest collectional shift (3 semitones) Ravel has used up to that point. This begins a process of increasingly wide harmonic oscillations which continues in mm. 42–51 (Ex. 12), where the new baritone melody first appears. The melody + accompaniment design of the passage and the tiered collectional oscillations are very similar to the opening of the piece in mm. 2–10. The main difference is that the frequent within-measure shifts are now between more distantly-related collections, C♯ acoustic and A acoustic, then B♭ acoustic and G♭ acoustic, inducing virtual sea sickness in the listener. Also, the focal collections (C♯ acoustic → A♯ acoustic = B♭ acoustic → G acoustic) change more frequently and involve more distantly related collections.

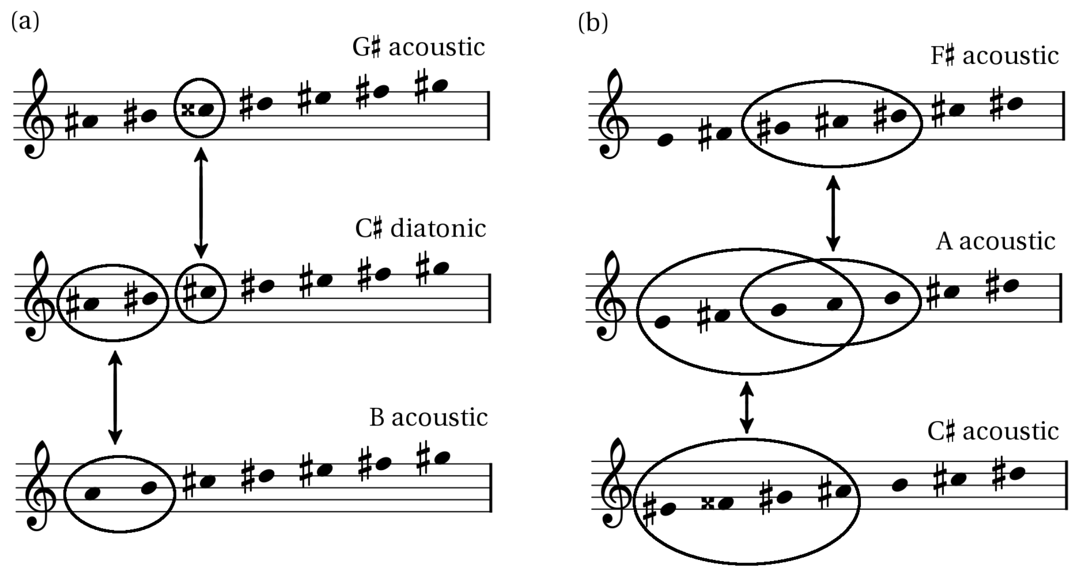


Example 11: The melody and scalar context of mm. 32–41 of Ravel's "Ondine." Note the use of A acoustic as the harmonic contrast in a predominantly F♯ acoustic context.



Example 12: The melody and harmonic collections of mm. 42–51 of "Ondine." The design is similar to mm. 1–15, but involves larger collectional shifts.

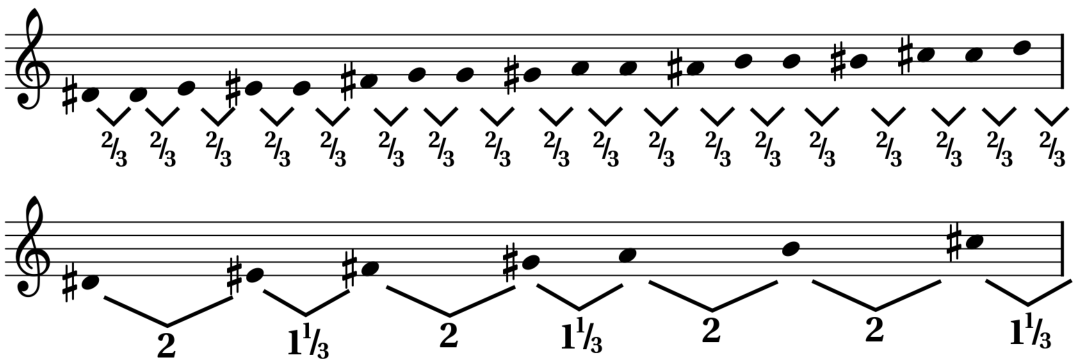
(2.2) A System of 7→18→12 ME Collections From a theoretical perspective, Ravel's choice of collections is remarkable not only for this ingenious compositional manipulation of raw voice-leading distances, but yet more so for the systematic nature of the voice leadings (see Ex. 13). The oscillations systematically increase in size over the course of the first part of the piece: from G♯ acoustic to C♯ diatonic (one semitone) in mm. 5–7, C♯ diatonic to B acoustic in mm. 8–10 (two semitones), F♯ acoustic to A acoustic in mm. 37–40 (three semitones), and A acoustic to C♯ acoustic in mm. 42–44 (four semitones). But, additionally, each of these involve the shift of a contiguous whole-tone segment or pair of adjacent whole-tone segments by a semitone. This added feature makes the size of the voice leading particularly palpable to the listener since it corresponds not only to a semitone count or a count of moving voices, but also more directly to the length of the scalar segment involved.



Example 13: Voice leadings between seven-note collections from "Ondine" (a) mm. 1–15 and (b) mm. 36–44. Each voice leading shifts a contiguous segment of the collection, of 1–2 notes in (a) and 3–4 in (b).

We can explain Ravel's scalar-segment voice leadings by deriving acoustic and diatonic collections via iterated quantization as 7→18→12 ME sets. This quantization is unusual in that it does not involve monotonically increasing cardinality values. The idea of ordering cardinality values freely in this way is perhaps more accessible to the "iterated quantization" way of thinking, but is nonetheless also a previously unexploited potentiality of second-order maximal evenness.

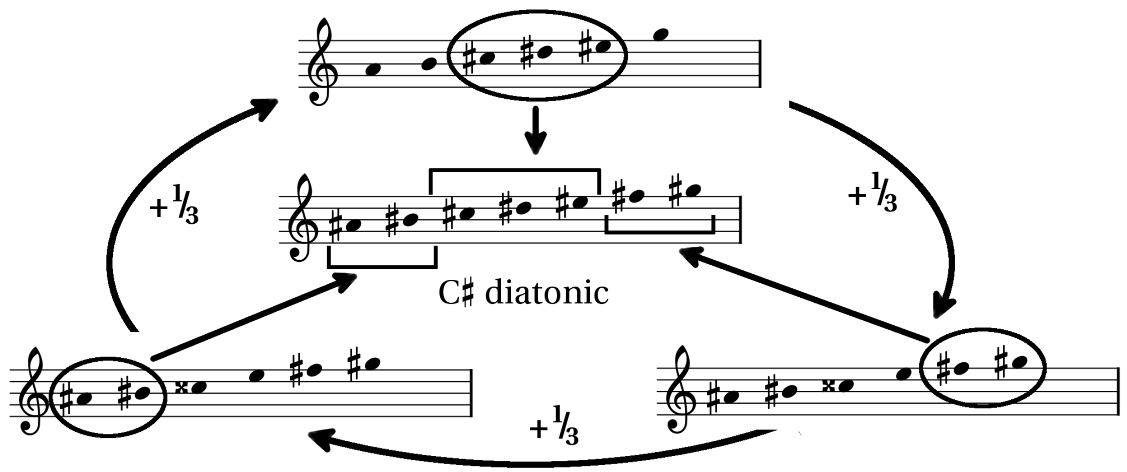
The 18→12 ME filter scale, because of the reversed order of the cardinality values, includes *doubled* pcs (see Ex. 14). The doubled pcs belong to one whole-tone collection, so the 18→12 ME is a set of *three* whole tone collections, two of which are identical after quantization. The 7→18→12 quantization produces an acoustic collection $^2/_3$ of the time, and a diatonic collection the other $^1/_3$ of the time. This is apparent from the step pattern of the 7-note collection before quantization to 12-equal: 2-$1^1/_3$-2-$1^1/_3$-2-2-$1^1/_3$. One of the three intervals of size $1^1/_3$ quantizes to 2, while the others quantize to 1, and between these are whole-tone segments 2–3 notes in length. There are two ways to combine whole tone segments of length 2 or 3, giving an acoustic collection, and one way to combine the whole tone segments of length 2, giving a diatonic collection.



Example 14: B acoustic as 7→18→12 maximally even. The values below the staves represent interval values before the final quantization to 12-equal.

Example 15 visualizes the segmentation of C♯ diatonic by splitting up the three whole-tone collections of the 18→12 ME set. Voice leadings on the diatonic and acoustic collections, as second-order ME sets, are analogous to those of diatonic triads. Position changes for 7→18→12 collections, analogous to third progressions for diatonic triads, shift the 3–2–2 segmentation, moving a note from one whole tone plane into the adjacent one. This moves one note by one step ($^2/_3$) in the 18→12 scale, which means it moves up or down by semitone or not at all. The latter is an "invisible" position change and always occurs in acoustic collections. For instance, shifting the C♯ in Ex. 15 into the whole-tone set below changes it to C𝄪, giving a G♯ acoustic collection. Shifting the E♯ into the whole-tone segment above makes it an E♮, giving an F♯ acoustic collection.

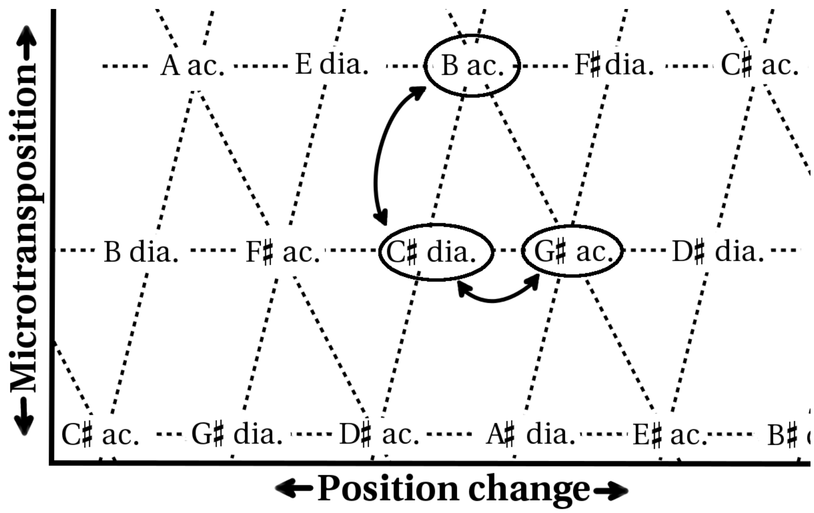
The other type of voice leading, microtransposition, is analogous to signature transformations of diatonic triads. In the 7→18→12 system, microtransposition moves a 2 or 3 note whole tone segment up or down by semitone. For instance, microtransposition down by $^1/_3$ for C♯ diatonic in Ex. 15 moves the segment A♯–B♯ down by semitone, resulting in a B acoustic collection. Microtransposition up by $^1/_3$ raises F♯–G♯ to F𝄪–G𝄪, giving D♯ acoustic.



Example 15: A 7→18→12 ME collection constructed out of segments from three whole-tone scales. Microtransposing the 7→18 collection by $^1/_3$ moves one of the three whole-tone scales by a semitone. A position change moves a note from one whole-tone scale to another.

These two voice leadings parameterize the system of second-order ME sets, so that, as with diatonic triads, they can be plotted on a two-dimensional network. Ex. 16 shows the collections used in mm. 2–10 of "Ondine" on this kind of "scalar *Tonnetz*." Horizontal links in the network correspond to position change and vertical links are microtranspositions (with the larger, three-note voice leadings represented by slightly longer diagonal lines). The three collections Ravel uses in the passage are maximally interrelated by these two types of voice leading. These horizontal and vertical axes are a 90º rotation of those used for diatonic triads above, because for these collections microtranspositions create larger voice leadings (2–3 semitones) than position changes (0–1 semitones). For diatonic triads, there are "invisible" microtranspositions (the same triad in different diatonic contexts) whereas acoustic collections have invisible position changes.

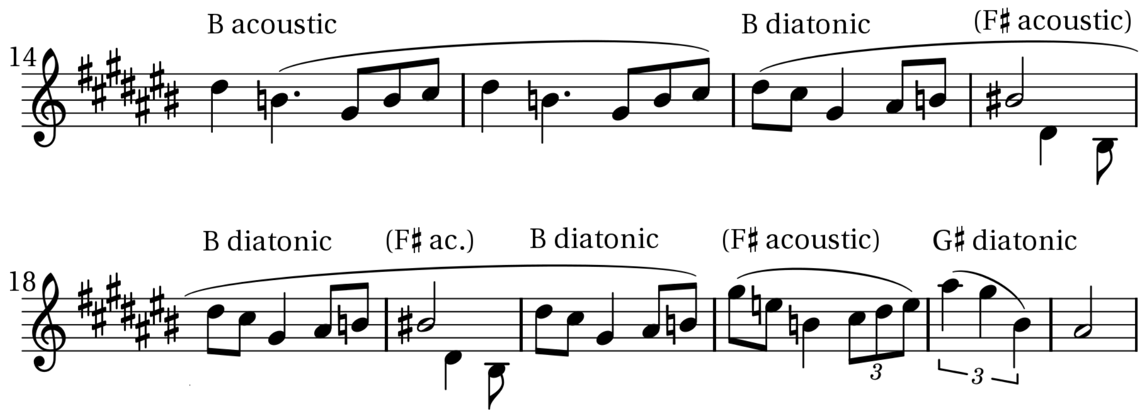
As with diatonic triads, the second-order ME construct provides a simplified system of voice leading, reducing a seven dimensional voice-leading geometry to a two-component parameterization. Tymoczko [6, 22] simplifies large-cardinality voice-leading spaces by extracting single-semitone voice leading lattices, drawing on the near-evenness of these collections. The closely related but distinct property of second-order maximal evenness identifies voice-leading cycles that are special in the same sense that third-progressions and signature transformations are special voice-leading cycles for diatonic triads. In the present example, the position changes create a cycle of single-semitone voice leadings that can also be found in Tymoczko's voice-leading lattice for seven-note collections [6, pp. 110]. The microtranspositions create voice-leading cycles that move larger segments of the collections because of the common divisors of 18 (the filter) and 12, and therefore do not appear in Tymoczko's voice-leading lattices. On the other hand, fifth-related diatonic collections, the centre of Tymoczko's voice-leading lattice, are far apart in our "scalar *Tonnetz*," requiring large position changes and microtranspositions that cancel one another out. (And, indeed, they play no significant role in "Ondine.")



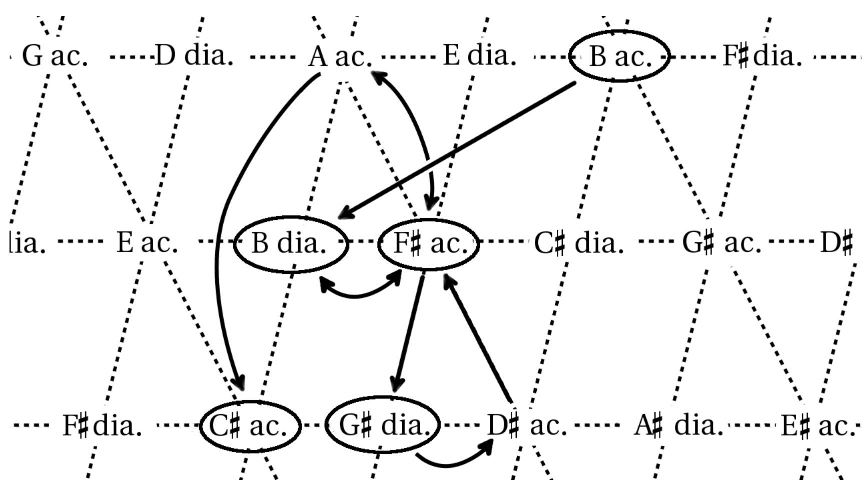
*Example 16: A plot of the collections used in mm. 1–10 of "Ondine" (see Ex. 10) on the 7→18→12 "*Tonnetz.*"*

In part 1 of this paper I pointed out that most of the explanatory power of iterated quantization for diatonic triads comes from the fact that the added mathematical machinery represents a significant feature of tonal music, the diatonic contexts that continually shadow the voice leadings between triads, and impart their own characteristic special voice-leading cycle (the circle of fifths) to them. An analogous point can be made about Ravel's acoustic-scale harmony. The 18-tone "scale" in Ex. 14 represents the shifting whole-tone planes underlying Ravel's collections which, like keys in tonal music, are manifest more in the movements between chords than as literal entities in themselves.

While the voice leadings characteristic of position change and microtransposition dominate the first part of the piece, there is one significant anchored progression in mm. 15–16 (see Exx. 17–18). This transition is marked in a number of ways, including the sudden change in the accompanimental texture to a new pattern of large wave-like arpeggios, the expansion of melodic ideas into two measures, and the foregrounding of a voice leading between collections (B–B♯) in the melody. The anchored progression in mm. 15–16 is a local manifestation of the large-scale harmonic plan, as we will see shortly.



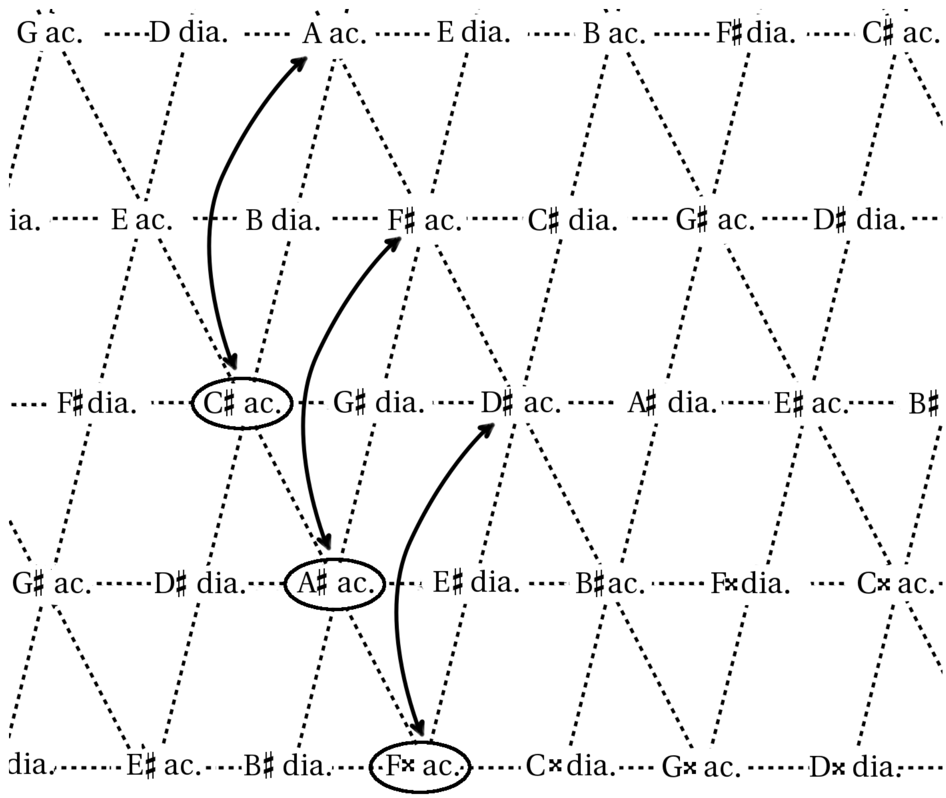
Example 17: The melody of mm. 14–23 of "Ondine" and the associated scalar contexts.



Example 18: The collections used in mm. 14–40 on the scalar "Tonnetz." *(See Exx. 17 and 11).*

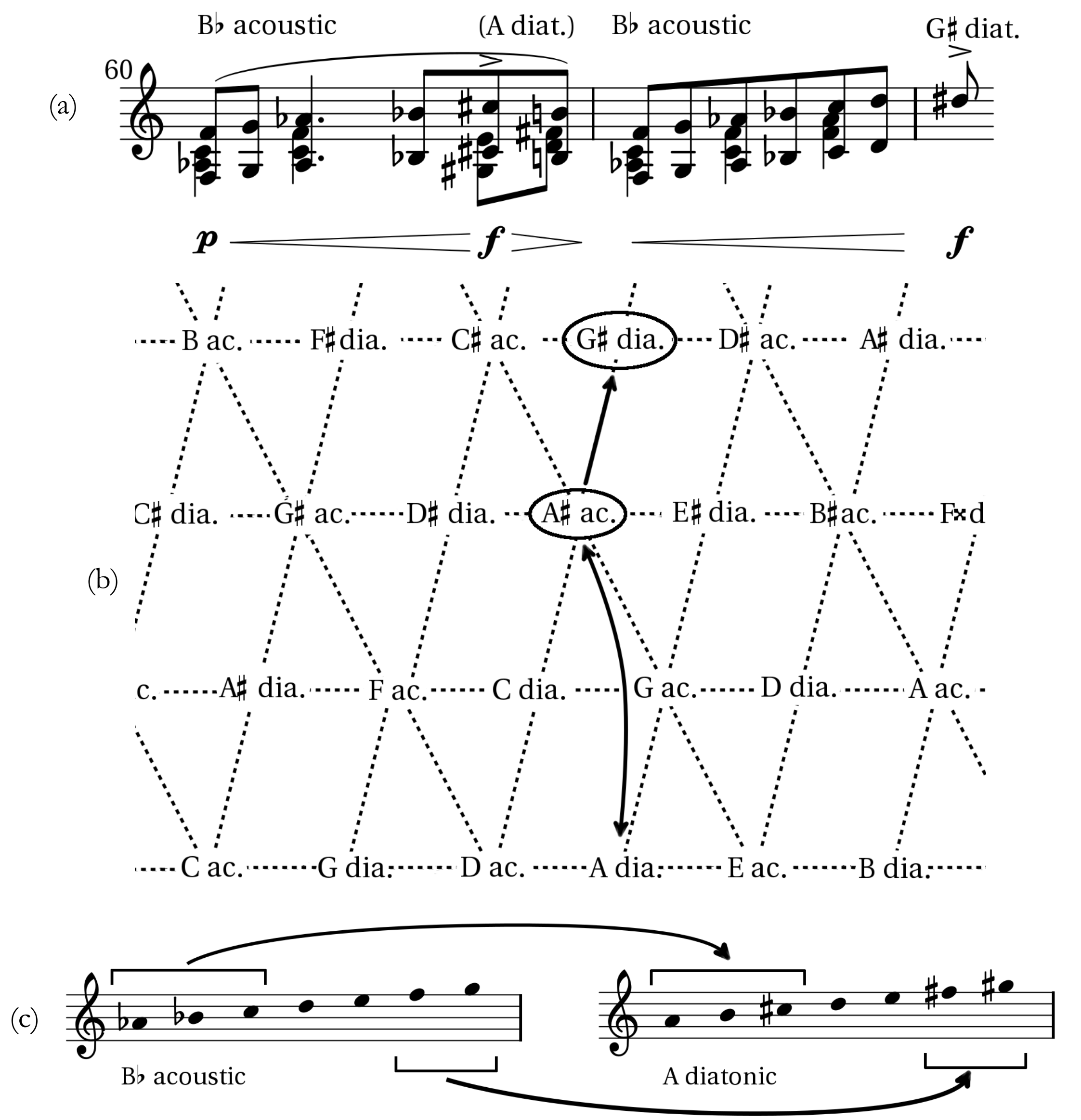
(2.3) Expansion of Voice Leadings When the middle section begins and Ravel introduces the sea-god theme, the gradual swell in voice-leading sizes begins in earnest. In mm. 42–51 (Ex. 12) the lurching four semitone shifts within measures combine two microtranspositions (Ex. 19). The shifts in focal collection in the passage are also microtranspositions (involving three-note whole-tone segments).

Ravel continues the process of voice-leading expansion in mm. 60–62 (Ex. 20). This is the moment where the sea-god melody ascends into the register that previously belonged to the Ondine's melody. The quarter-note harmonic lurch in m. 60 from B♭ acoustic to A diatonic combines vertical moves involving three-note and two-note whole tone segments.



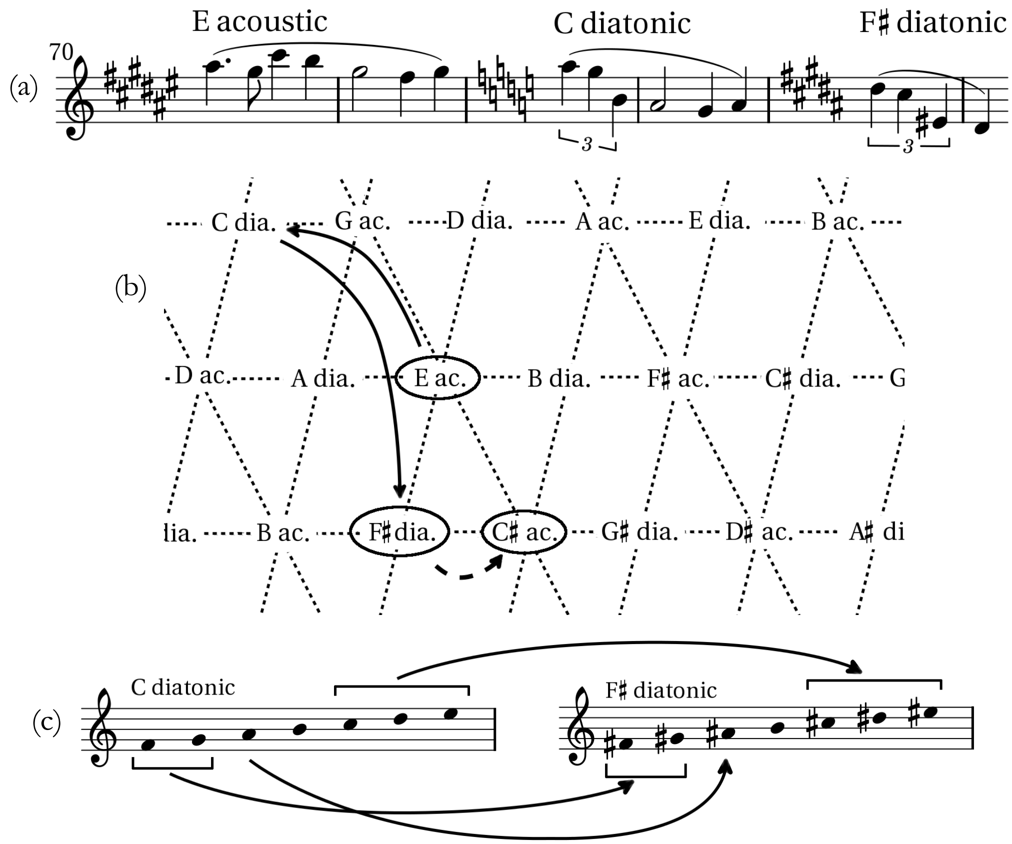
Example 19: The collections used in mm. 42–51. (See Ex. 12)

Ravel goes one step further, to six notes moving by semitone. as the culmination of the process of increasingly violent harmonic swings, at another remarkable moment in mm. 72–75, where the Ondine makes her proposal to the sailor (see Ex. 21). After a sequential tumult (mm. 66–67) a fragmentary version of the original melody reappears in E acoustic in mm. 68–71. In m. 72 there is a sudden shift in texture and tempo, with an expansive white key glissando. The slowing of tempo and melodic rhythm give the effect of water receding after a large wave. Ravel then transposes the melody and glissando by tritone in m. 74. The shift in collection, C diatonic to F♯ diatonic, requires a six-semitone voice leading, but, as the largest possible voice leading between such collections it entails a potential enharmonic ambiguity (all notes except F going down, or all notes except B going up). In this case, the preceding E acoustic collection provides some context to orient the six-semitone shift, which combines one position change with the kind of five-semitone shift used in mm. 60–61 (compare Ex. 20).

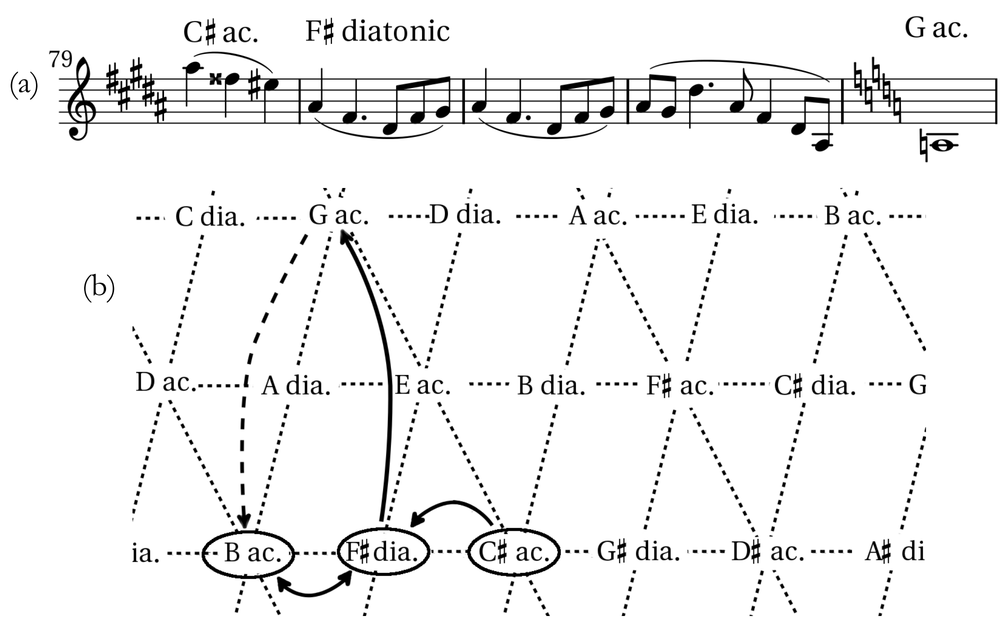


Example 20: (a) Melody and supporting collections in mm. 60–62, (b) a plot of the progression, and (c) the voice leading involved in the B♭ (=A♯) acoustic – A diatonic shift.

At the end of the piece (Ex. 22), Ravel states the Ondine's melody once more, transposing the diatonic version (from mm. 8–9; see Ex. 11) to F♯ diatonic. Another sudden flatward shift mimics the previous one (from m. 72; see Ex. 21). After an unaccompanied return of the melody from mm. 10–14 transposed to G acoustic, and an explosive polychordal flourish, the opening B acoustic collection returns in the last few measures with an echo of the accompaniment pattern from mm. 1–3.



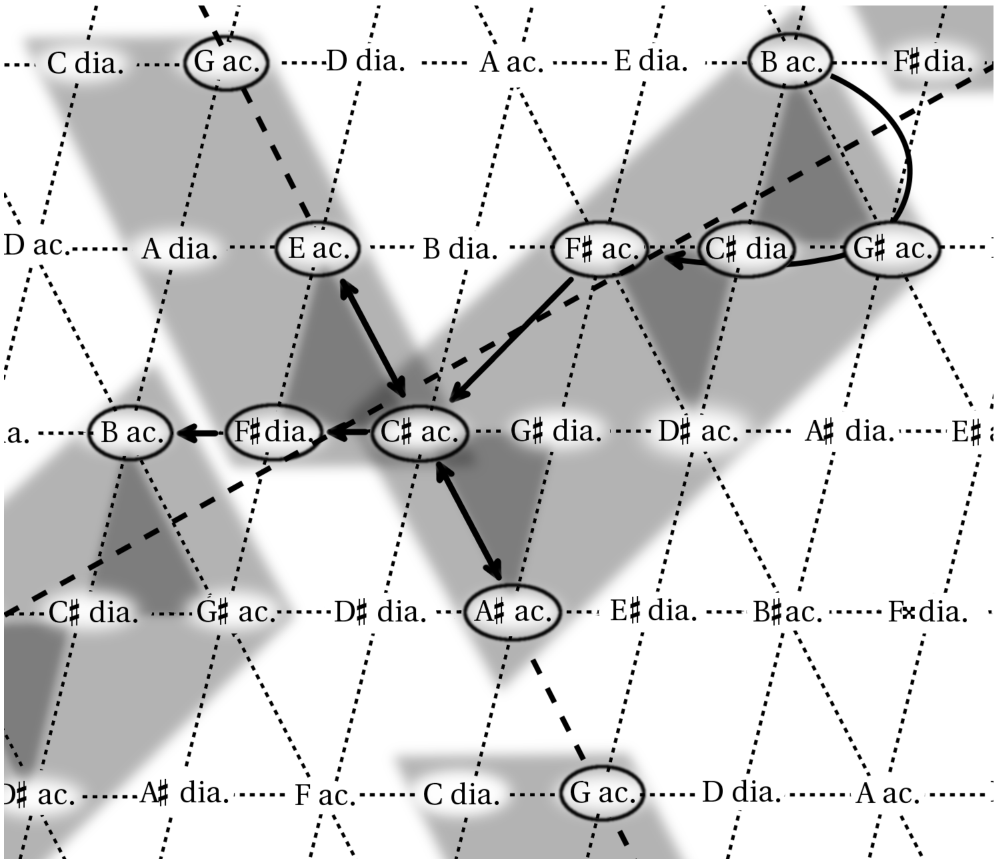
Example 21: (a) Melody and supporting collections in mm. 70–75 (b) a plot of the progression, and (c) the voice leading involved in the C diatonic – F♯ diatonic shift.



Example 22: Melody and supporting collections in mm. 79–83 of "Ondine."

(2.4) Large-Scale Harmonic Plan Summarizing the use of diatonic and acoustic collections that I have just surveyed over the entire piece, a directed traversal of the 7→18→12 network comes into focus (Ex. 23). The first part of the piece generally explores small regions gradually moving up by fifth, with the central collections going from B acoustic to F♯ acoustic (mm. 16–21, 32–40) to C♯ acoustic (mm. 41–44, 79). The process breaks off in mm. 45–78 but resumes at the end of the piece to return to B acoustic via F♯ diatonic (mm. 80–82). In the first part of the piece, significant thematic statements also occur in G♯ acoustic (mm. 5–7) and C♯ diatonic (mm. 8–9). This sequence of collections (B ac. – G♯ ac. – C♯ dia. – F♯ ac. – C♯ ac. – F♯ dia. – B ac.) is a *continuous anchored cycle*. This means that it can be produced by continuously transposing the 18-equal filter and quantizing all the maximally even sets on it that are closest to a fixed point at the centre of seven-note chord space. In fact, the collections that frame larger spans of music, F♯ acoustic and C♯ acoustic, are also the ones closest to the fixed point and therefore take up the longest portions of the continuous cycle.

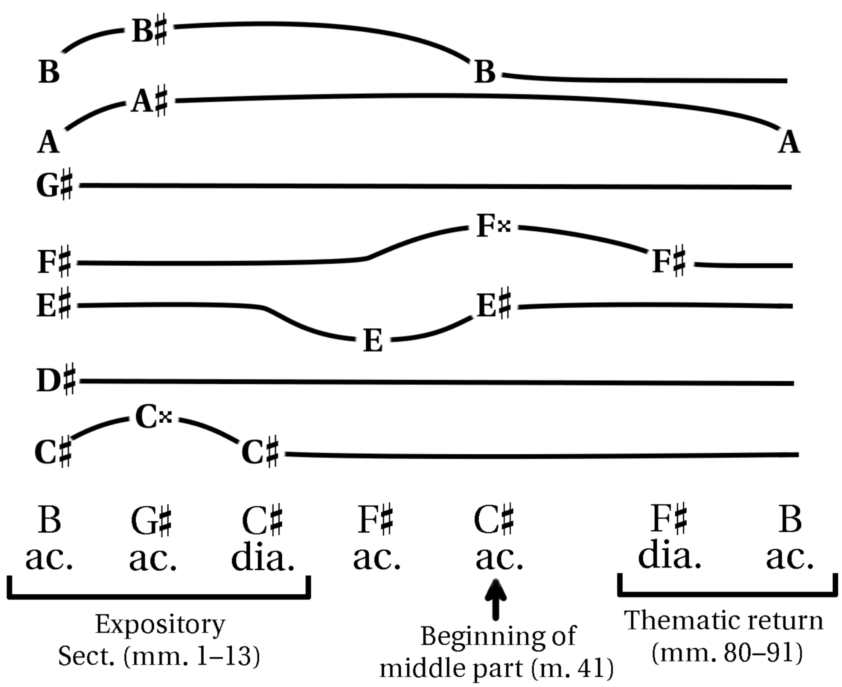
This process does not account for much of the music in the large middle section from m. 41 to m. 79. This section introduces the sea-god melody in m. 45, and includes the development of that melody (mm. 57–65). Its focal collections generally move along an axis perpendicular to the balanced voice leading axis—an axis of maximally directed voice leadings. All of the collections along the axis that passes through C♯ acoustic (the framing collection for this section) occur prominently, featuring thematic returns (A♯ acoustic in mm. 47–49, E acoustic in mm. 68–71, G acoustic in mm. 84–87). An interesting feature of these collections is that, as T_3-related acoustic collections, they all overlap in six out of seven notes with a common octatonic collection (Oct. I). This is, in fact, the one scalar collection that Ravel uses in a significant capacity in the piece which is not acoustic or diatonic, and it appears in the middle of this section (mm. 57–59) in a development of the "sea-god theme" just before this theme transfers into a higher register.



Example 23: A summary of Ravel's use of acoustic and diatonic collections in "Ondine." Circled collections appear at major returns of melodic material and moments of formal articulation, and serve as central reference points for substantial stretches of music. Shaded areas show regions of significant activity around these central collections. The dashed lines show an axis of perfectly balanced voice leading that defines the large-scale progression, and a perpendicular axis of maximally directional voice leading that dominates a large middle section of the piece, from mm. 41–79.

Having insisted, in the previous section, that mathematical concepts be applied in music analysis only to the extent that they are explanatory, I must hold myself to such a standard in this instance. Is the anchored progression in evidence here a convenient coincidence, or does it reveal some aspect of Ravel's compositional plan? We have already seen that the anchored progression occupies the first and last parts of the piece and is suspended in the middle section in favour of directed voice leadings along a perpendicular axis. These directed voice leadings are more conducive to the turbulent developmental quality of the middle section, so in this regard the compositional plan matches the outlay of harmonic collections on the 7→18→12 network.

The anchored progression should, and does, have a more specific significance, however. Like Cohn's [14] hexatonic cycles, anchored progressions are generally "toggling cycles," in which each voice toggles between notes a semitone apart, rather than moving consistently upwards or downwards. Example 24 shows the toggling voices of Ravel's anchored progression. The first move of the cycle, in the expository section of the piece, sets the drama into motion with a large initial upward shift that is not resolved until the end of the piece (because B acoustic is the furthest collection flatward from the anchor position). Each moment in the piece resolves a local displacement of notes from their original positions (C𝄪→C♯, E→E♯, F𝄪→F♯). Only the B→B♯ and A→A♯ dislocations from the beginning of the piece are more persistent. The resolution B♯→B marks the main formal division, the end of the first large part. The A♯ recedes only at the end of the piece, its return to A♮ drawing the piece to a close. The special significance of these two pitches is, indeed, evident at the outset: A is the characteristic note of the accompaniment while B is the focal point of the melody, and they are also flagged as the ones that deviate from the seven-sharp key signature. Where B returns in m. 41 it is the characteristic note of the accompaniment in C♯ acoustic, and the same is true of A at the end of the piece, where Ravel signals the final return with a return of the original accompanimental pattern.



Example 24: The toggling voices of the large-scale anchored progression in "Ondine."

Conclusion

As a branch of applied mathematics, the heart of mathematical music theory is in the implementation rather than the pure theory building. Music analysis is especially valuable for mathematical theories, not purely as verification procedure, but as a process that enriches theory with musical signification. In this paper I hope to have made a convincing case that we should appreciate interesting mathematical features of chords and scales, like maximal evenness, for how composers exploit them to shape works of art. In doing so do we infuse these mathematical constructs with musical meaning.

The two examples of second-order maximal evenness investigated here are quite dissimilar in that diatonic triads are basic to a broad range of tonal music, while the 7→18→12 construction of acoustic collections is unlikely to be applicable much beyond this one piece. Yet, the abstractions involved in relating these instances—such as the generalization of signature transformations as voice leadings inherited from voice-leading cycles of filter collections—and strategies for uncovering the meaning of unusual filter configurations, such as the 18→12 "scale," provide a guide to, and indicate the potential for, further analytical applications for other kinds of maximal evenness.

Acknowledgments

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