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Meter networks: a categorical framework for metrical analysis

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This paper develops a framework based on category theory which unifies the simultaneous consideration of timepoints, metrical relations, and meter inclusion founded on the category **Rel** of sets and binary relations. Metrical relations are defined as binary relations on the set of timepoints, and the subsequent use of the monoid they generate and of the corresponding functor to **Rel** allows us to define meter networks, i.e. networks of timepoints (or sets of timepoints) related by metrical relations. We compare this to existing theories of metrical conflict, such as those of Harald Krebs and Richard Cohn, and illustrate that these tools help to more effectively combine displacement and grouping dissonance and reflect analytical claims concerning nineteenth-century examples of complex hemiola and twentieth-century polymeter. We show that meter networks can be transformed into each other through meter network morphisms, which allows us to describe both meter displacements and meter inclusions. These networks are applied to various examples from the nineteenth and twentieth century.

Keywords: meter; metrical layer; hemiola; transformational networks; binary relations; category theory;

2010 Mathematics Subject Classification: 00A65; 18A05;

1. Introduction

Recent decades have seen a flourishing of theoretical work on the topic of meter, and a foremost concern of this body of research has been finding a theoretical framework for the different kinds of metric conflict and play of meters that becomes common in European concert music of the nineteenth century and grows more complex in the twentieth century. One influential theoretical tradition is specifically focused on complex hemiola, and begins with Lewin's (1981) analysis of Brahms's *Cappricio*, Op. 76, no. 8, inspiring Cohn's (1992; 2001) theory of ski-hill graphs and metric spaces, later expanded to a theory of metric cubes by Murphy (2009). Also very widely influential is the concept of "metric dissonance" proposed by Krebs (1999), and its basic separation into two types, grouping dissonance and displacement dissonance. This theory has been applied to eighteenth century music by Mirka (2009), to nineteenth century art song by Malin (2006, 2010), and to rock music by Biamonte (2014), to name just a few examples. As is evident from all this literature, Cohn's theory lends itself more to mathematical treatment than Krebs's, but Krebs's has wider analytical application. That is because, according to Krebs's typology, Cohn's metric spaces only handle grouping dissonances, not displacement dissonances,

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and displacement dissonance is musically significant in many ways, and more common than grouping dissonance.

An important step in metrical theory, then, would be to wed the mathematical virtues of Cohn's theory with the analytical virtues of Krebs's, by further generalizing Cohn's approach so as to recognize displacement dissonance as well as other forms of grouping dissonance. Chung (2008) has proposed doing this using Lewin's (1987) group-theoretic formalism of generalized interval systems and transformations. In this paper, we propose a mathematical framework for this task based on the category of sets and binary relations, previously used by Popoff, Andreatta, and Ehresmann (2015, 2018) to generalize transformational theory for harmony.

Our strategy in the paper will be to introduce mathematical objects in stages, from simpler to more complex and powerful, while considering the musical implications of each, with reference to analytical examples, including a song by Brahms (Op. 106, no. 2), and Ligeti's Sixth Étude for piano. These examples are chosen to illustrate the necessity of certain mathematical tools for expressing significant musical claims about how they play of contrasting meters operates in these pieces. The category theoretic formalism is introduced in section three. The main concepts proposed here are *metrical relations*, defined in Section 2.1, *meter networks*, introduced informally in Section 2.4 and formally defined in Section 3, and morphisms of metrical networks, defined and exemplified at Section 3.3. Section 2 presents much of the theoretical content and analysis without requiring any background in category theory, while Section 3 presents the complete mathematical formalism assuming basic knowledge of category theory, including in particular 2-categories and lax functors.

2. Meters and metrical relations

We begin by developing basic concepts of meter. A theory of meter involves relationships at multiple levels. At the most basic level a meter consists of relationships between timepoints, which is some timespan or duration. A set of timepoints related by some regular duration is what Krebs (1999) calls a *metrical layer*. A *meter* is made up of (and defined by) multiple metrical layers. The most musically interesting questions arise at one more level of abstraction, in relating meters, and this will be our ultimate goal.

2.1. *Metrical layers as binary relations between timepoints*

Figure 1 shows the typical situation one imagines in connection with the concept of meter. The time signature indicates measures divided into two beats, which are then further subdivided by triplets. All timepoints belonging to the meter are articulated. The notational system introduced by Lerdahl and Jackendoff (1983), and adopted by Krebs (1999), shows each metrical layer by a row of equally-spaced dots. The top row shows the regularly spaced downbeats, the next row down the quarter-note beats, and at the lowest level the triplet eighths. The meter is *well-formed* per Lerdahl and Jackendoff because the dots of each row are a proper subset of those the next row below.

We first define timepoints as a set.

Definition 2.1 A *timepoint* t is an element of the set of the rational numbers $\mathbb{Q}$. The space of all timepoints is thus taken to be $\mathbb{Q}$ itself.

Here we model time using the rationals, $\mathbb{Q}$, rather than the usual choice of the real

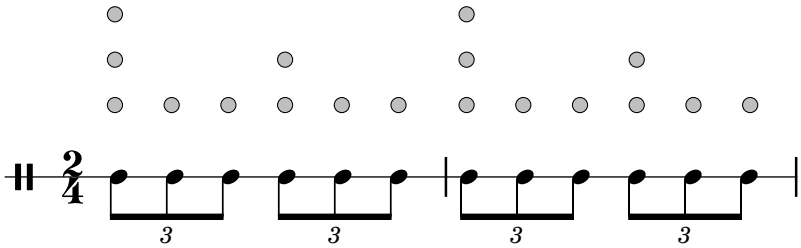


Figure 1. A simple regular metrical rhythm and its associated hierarchy of metric levels. The pulses of each metric level is shown by the gray dots above the rhythmic pattern.

numbers, $\mathbb{R}$. The theory presented here, like those cited above, deals essentially with notated rhythms, for which $\mathbb{Q}$ is the proper basis. This trend in music theory runs counter to one in music cognition which has recently moved in the direction of analyzing performed rhythms, for which time is modeled accurately by $\mathbb{R}$. See, for example, Polak and London 2014; Benadon and Zanette 2015; Danielsen 2018. However, the model of notatable rhythms in $\mathbb{Q}$ is also a useful idealization of perceived rhythms. Correspondingly, the metrical relations described below may be understood as idealizations of the neural oscillations underlying metrical perception.

The first task is to relate timepoints. The usual approach is to relate timepoints through *intervals*, the distance in time from one point to the other. This, for instance, is the strategy of Lewin (1987), whose timespan group is used by Yust (2018) to model metrical and rhythmic hierarchies. To define metrical layers according to this intervallic or timespan-based approach, one has to relate each time point to each successive one and extend this indefinitely in both directions. Our approach will instead be to define metrical layers directly as a relation. For this purpose we define relations in general using the set-theoretic approach.

Definition 2.2 Let X and Y be two sets. A *binary relation* $\mathcal{R}$ *between* X *and* Y is a subset of the cartesian product $X \times Y$. We say that $y \in Y$ is related to $x \in X$ by $\mathcal{R}$, which is notated as $x\mathcal{R}y$, if $(x, y) \in \mathcal{R}$. In the case $\mathcal{R}$ is a symmetric relation, we say that x and y are related by $\mathcal{R}$.

We define metrical layers using a *metrical relation*, such that the relation holds for any two timepoints in the given metrical layer.

Definition 2.3 Let $d \in \mathbb{Q}_{\geq 0}$. A *metrical relation* $\mathcal{M}_d$ on the set of timepoints $\mathbb{Q}$ is the reflexive binary relation defined on $\mathbb{Q}$ such that for $(t, t') \in \mathbb{Q}^2$, we have $t\mathcal{M}_d t'$ whenever $t' - t = kd$ with $k \in \mathbb{Z}$.

Note that a metrical relation $\mathcal{M}_d$ relates timepoints located at regularly spaced rational intervals d. We therefore consider only regular meters here. While it may in principle be possible to extend the approach to irregular meters, it would add a great deal of complexity. Metrical relations are unlike durational intervals in that they relate *all* timepoints belonging to a particular metrical layer, not only successive ones. It therefore is already an inherently metrical concept, where the more basic concept of time interval is not. Observe that the metrical relation $\mathcal{M}_0$ relates a timepoint t in $\mathbb{Q}$ only to itself, i.e. it is the identity function on $\mathbb{Q}$.

For the simple case of Figure 1, we could define three metrical relations, one for each metrical layer. Assigning the measure the value 1 (which will be our convention throughout without loss of generality) these are $\mathcal{M}_1$, $\mathcal{M}_{\frac{1}{2}}$, and $\mathcal{M}_{\frac{1}{6}}$. For any two downbeats, all

three relations hold. For a strong and a weak quarter-note beat, the relations $\mathcal{M}_{\frac{1}{6}}$ and $\mathcal{M}_{\frac{1}{2}}$ hold, but not $\mathcal{M}_1$. We could also define a hypothetical eighth-note layer implied by the time signature, $\mathcal{M}_{\frac{1}{4}}$, which would also hold for the beats, but not for arbitrary pairs of timepoints in the triplet layer.

A meter is typically defined as a set of two or more metrical levels that perfectly nest one another (as in Lerdahl and Jackendoff's well-formedness constraint). Figure 1, for example, shows a meter made up of three metrical layers. With metrical layers defined as a relation, the "nesting" property is an instance of *inclusion of binary relations*.

Definition 2.4 Let X and Y be two sets, and $\mathcal{R}$ and $\mathcal{R}'$ be two binary relations between them. The relation $\mathcal{R}$ is said to be *included in* $\mathcal{R}'$ if $x\mathcal{R}y$ implies $x\mathcal{R}'y$, for all pairs $(x, y) \in X \times Y$.

In the case of metrical relations, the following proposition establishes the conditions under which a metrical relation is included into another.

PROPOSITION 2.5 *Let $\mathcal{M}_{d_1}$ and $\mathcal{M}_{d_2}$ be two metrical relations. The metrical relation $\mathcal{M}_{d_1}$ is included in $\mathcal{M}_{d_2}$ if and only if there exists a positive integer u such that $d_1 = ud_2$.*

Proof. Assume that the metrical relation $\mathcal{M}_{d_1}$ is included in $\mathcal{M}_{d_2}$. This implies that for all timepoints $t \in \mathbb{Q}$, the image set $\{t + kd_1 \mid k \in \mathbb{Z}\}$ of t by $\mathcal{M}_{d_1}$ is included in the image set $\{t + k'd_2 \mid k' \in \mathbb{Z}\}$ of t by $\mathcal{M}_{d_2}$, i.e. for all $k \in \mathbb{Z}$, there exists $k' \in \mathbb{Z}$ such that we have $kd_1 = k'd_2$. In particular for $k = 1$, we have $d_1 = k'd_2$ for some $k' \in \mathbb{Z}$.

Assume now that there exists a positive integer u such that $d_1 = ud_2$. Let t be a timepoint in $\mathbb{Q}$, its image set by $\mathcal{M}_{d_1}$ is $\{t + kd_1 \mid k \in \mathbb{Z}\}$, which is $\{t + kud_2 \mid k \in \mathbb{Z}\}$, which is therefore included in the image set $\{t + k'd_2 \mid k' \in \mathbb{Z}\}$ of t by $\mathcal{M}_{d_2}$. ■

2.2. *Inclusion of metrical relations and ski-hill graphs*

We can represent a meter with a simple diagram on metrical relations, where an arrow indicates inclusion of metrical relations. A directed path in such a diagram corresponds to some meter type. For instance, the meter of Figure 1 can be shown with the following diagram.

$$\mathcal{M}_1 \longrightarrow \mathcal{M}_{\frac{1}{2}} \longrightarrow \mathcal{M}_{\frac{1}{6}}$$

Cohn (2001)'s ski-hill graphs are instances of this kind of network. Consider, for example, Cohn's example from Brahms's first Violin Sonata in Figure 2. Multiple metrical relations can be used to describe this measure.

- All of the downbeats (although the example shows only one) relate by $\mathcal{M}_1$.
- The low bass notes (G, E) and the high Gs in the piano right hand are in the relation $\mathcal{M}_{\frac{1}{2}}$.
- The beginnings of each slur in the violin are related by $\mathcal{M}_{\frac{1}{3}}$.
- The beginning of each small downward arpeggiation in the piano right hand are related by $\mathcal{M}_{\frac{1}{4}}$.
- All of the notes in the piano left hand are related by $\mathcal{M}_{\frac{1}{6}}$.
- All of the notes in the measure are related by $\mathcal{M}_{\frac{1}{12}}$.

We can include these all in one diagram with arrows showing inclusion, as follows.

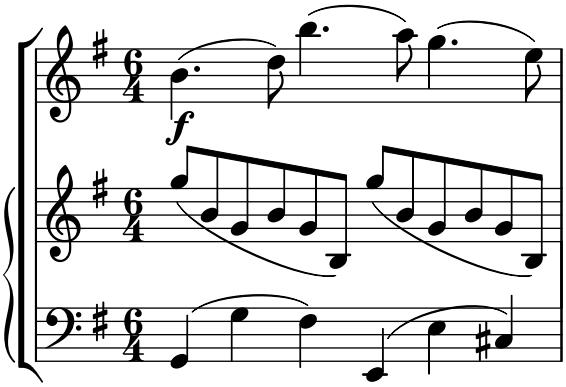


Figure 2. Brahms Violin Sonata, Op. 78, m. 235.

$$\begin{tikzcd}
 & & \mathcal{M}_{\frac{1}{3}} \arrow[dr] & & \\
\mathcal{M}_1 \arrow[urr] \arrow[r] & \mathcal{M}_{\frac{1}{2}} \arrow[r] \arrow[dr] & & \mathcal{M}_{\frac{1}{6}} \arrow[r] & \mathcal{M}_{\frac{1}{12}} \\
 & & \mathcal{M}_{\frac{1}{4}} \arrow[urr] & &
\end{tikzcd}$$

There are three paths from $\mathcal{M}_1$ to $\mathcal{M}_{\frac{1}{12}}$ in this network, each of which corresponds to a meter characterizing one part in the passage. The violin corresponds to the path,

$$\mathcal{M}_1 \longrightarrow \mathcal{M}_{\frac{1}{3}} \longrightarrow \mathcal{M}_{\frac{1}{6}} \longrightarrow \mathcal{M}_{\frac{1}{12}} \quad ,$$

The piano right hand corresponds to

$$\mathcal{M}_1 \longrightarrow \mathcal{M}_{\frac{1}{2}} \longrightarrow \mathcal{M}_{\frac{1}{6}} \longrightarrow \mathcal{M}_{\frac{1}{12}} \quad ,$$

And the piano left hand corresponds to

$$\mathcal{M}_1 \longrightarrow \mathcal{M}_{\frac{1}{2}} \longrightarrow \mathcal{M}_{\frac{1}{4}} \longrightarrow \mathcal{M}_{\frac{1}{12}} \quad .$$

This network is equivalent to Cohn's ski-hill graph for the passage. In general, we can say that a ski-hill graph is a particular instance of an inclusion diagram of metric relations, with the additional characteristic that the graph has one source node, $\mathcal{M}_d$, and one sink node, $\mathcal{M}_e$, such that $d = 2^a 3^b e$ for non-negative integers a and b. Murphy (2009)'s metric cube of dimension n can similarly be defined as an inclusion diagram on $\mathcal{M}_{2^a 3^b k}$ for all non-negative integers a, b such that $a + b \leq n$ and an arbitrary constant k.

This example from Op. 78 is tidy in that all of the metrical layers inferred from the score actually coincide in timepoints where the inclusion relations exist. In principle this need not be the case, so these diagrams, though they represent meters, are somewhat more abstract. For instance consider mm. 7–8 the Capriccio that Lewin (1981) originally used to illustrate complex hemiolas in Brahms, given in Figure 3. In m. 7, there is a prevailing half-note beat that persists from the beginning of the piece, but throughout all of this music the melodic notes of right hand (in the distinct higher register) are

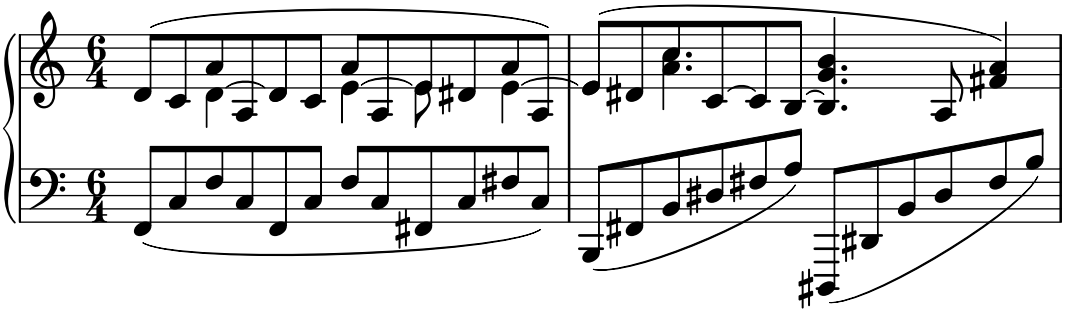


Figure 3. Brahms Capriccio, Op. 76, No. 8, mm. 7–8.

displaced by a quarter note from the bass notes and harmonic changes. In m. 8, which Lewin describes as a metrical transition, a half-measure arpeggiation begins in the left hand which will continue for four measures, but the right hand continues the pattern from before. The result is a metric conflict which could be described with the following diagram.

$$\begin{tikzcd}
 & & \mathcal{M}_{\frac{1}{3}} \arrow[dr] & & \\
\mathcal{M}_1 \arrow[urr] \arrow[r] & \mathcal{M}_{\frac{1}{2}} \arrow[r] & & \mathcal{M}_{\frac{1}{6}} \arrow[r] & \mathcal{M}_{\frac{1}{12}}
\end{tikzcd}$$

The left hand bass notes are in the relation $\mathcal{M}_{\frac{1}{2}}$ while the right hand melodic notes are in the relation $\mathcal{M}_{\frac{1}{3}}$. The inclusions $\mathcal{M}_1 \to \mathcal{M}_{\frac{1}{2}}$ and $\mathcal{M}_1 \to \mathcal{M}_{\frac{1}{3}}$ are both valid, but if we think of one as representing the left hand meter and one the right hand, the right hand timepoints in the $\mathcal{M}_1$ relation do not coincide with the left hand notes in the $\mathcal{M}_1$ relation. They are displaced from one another.

Technically, this flexibility with respect to displacement is a property of ski-hill graphs, but one that is usually glossed over with the implicit assumption that all metrical layers align, as in the Op. 78 example. The next section provides an example to illustrate the shortcomings of a theory of hemiola abstracted from timepoints, and the need to be able to address displacements in coordination with grouping dissonances.

2.3. *Meter displacement and hierarchies in Brahms's op. 106/2*

Figure 4 shows measures 5 to 8 of Brahms's song "Auf dem See," (Op. 106/2, 1886). A key musical idea of this song is a metrical dialogue between vocal line, written with the utmost rhythmic simplicity to reflect the youth and naïvité of the narrator, and the right hand of the piano. Both metrical displacement and hemiola are crucial to this dialogue, so that while the hemiola can be shown with a ski-hill diagram, this misses crucial aspects of the metrical relations, which are essential to interpreting Brahms's text setting strategy.

Although the poem is about a love affair, it is not addressed to the beloved, but to the lake that carries the boat they ride in. While the lake is described in a great deal of detail – and ultimately as both heaven and an Eden beyond words, thought, and time – the beloved is, incredibly, not described at all. Brahms was perhaps inspired most directly by the lines "Deine Wellen leuchten / Spiegeln uns zurück / Tausendfach die feuchten / Augen voller Glück!" and their decription of the lake reflecting the lover's

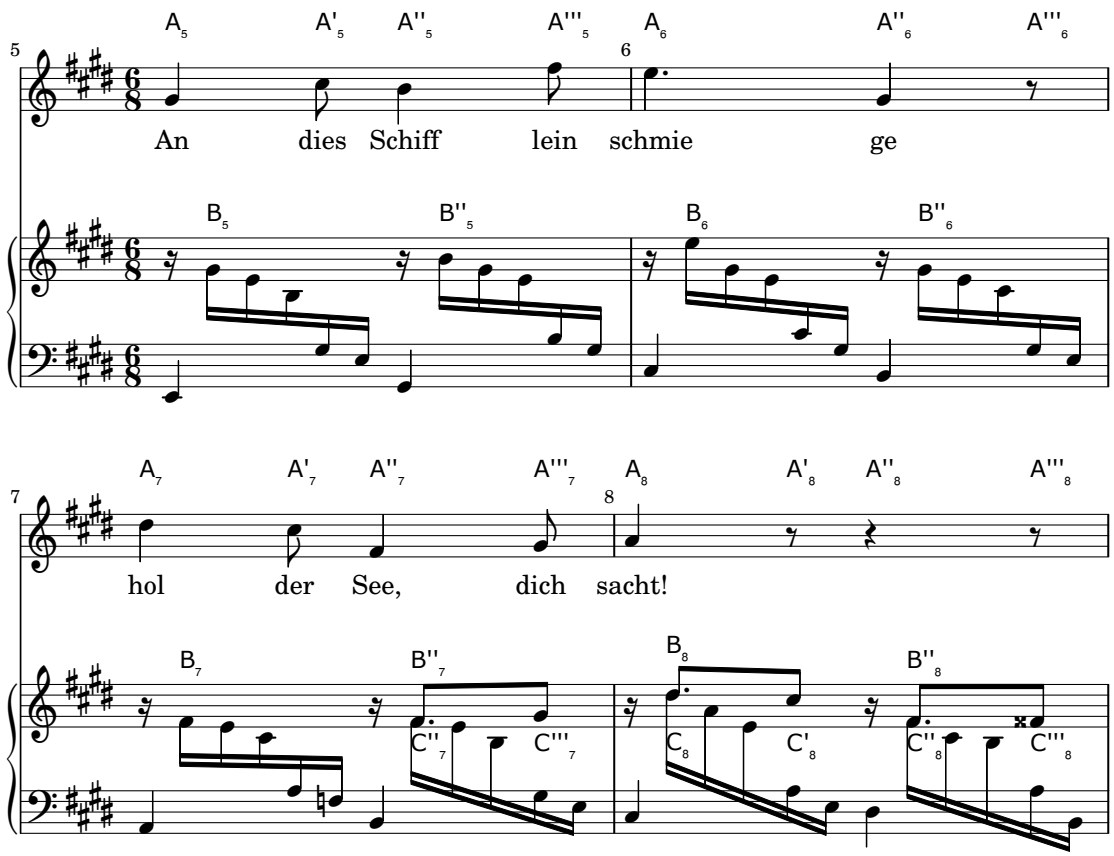


Figure 4. Brahms' op. 106/2, mm. 5–8. Specific timepoints have been labelled with letters, the superscript indicating their place inside a bar and the subscript indicating the bar number.

own feelings back to him (we even have the image of the lake being made up entirely of tears of happiness). The lake is then a character interacting with the narrator, and can be represented by the piano, but, in reality, is not in true dialogue (after all, we are in a place beyond human speech), but simply a reflection of the narrator's solipsistic love affair.

To metaphorically transfer these ideas into music, Brahms begins with a wave-like texture in the piano, with long arpeggiations. But, crucially, the piano does not really have its own melodic line, but rather follows the vocal line (notes marked A and A'' in Figure 4) by beginning each arpeggiation at a delay of a sixteenth note (notes marked B and B''). Brahms uses a trick that we might surmise he learned from Schumann: preparing the emergence of an alternate meter by a displacement that we initially might interpret as an innocuous textural feature, what Krebs (1998) would call a "preparation of a metrical dissonance." The example from Brahms's "Auf dem See" is very similar to examples from Schumann's *Dichterliebe*, Op. 48, and *Liederkreis*, Op. 39 analyzed by Malin (2010). A displacement caused by delayed doubling in an arpeggiated figure eventually takes on a metrical life of its own, becoming a metrical dialogue between singer and piano.

We could describe the hemiola between the voice and piano using the following inclusion diagram, as we have for the previous examples.

$$\begin{tikzcd}
 & & & \mathcal{M}_{\frac{1}{6}} \arrow[dr] & \\
\mathcal{M}_1 \arrow[r] & \mathcal{M}_{\frac{1}{2}} \arrow[r] \arrow[urr] & \mathcal{M}_{\frac{1}{4}} \arrow[r] & & \mathcal{M}_{\frac{1}{12}}
\end{tikzcd}$$

But this would not show that the half-note and downbeat layers of the two meters are not aligned. What is unique, and ingenious, about the hemiola in "Auf dem See" is Brahms's combination of it with the Schumann-esque displacement. The on-beat notes of the vocal line are delayed by a sixteenth note in the piano (for example the notes marked A_7'' and B_7'' in Figure 4), but the vocal line has a consistent quarter–eighth rhythm (for example, the notes marked A_7'', A_7''', and A_8 in Figure 4), and in measure seven the piano begins to double the offbeat note as well as the on-beat note, but the offbeat note is not delayed (for example, the note marked C_7'''' in Figure 4). The result is a change of meter in the piano's "reflection" of the vocal melody. We could simply say that Brahms combines displacement and grouping dissonances, but this description omits the crucial fact that the two meters share a timepoint, specifically the offbeat note of the melody. This is essential to the sense that the piano meter is derived purely from timepoints that already existed within the vocal meter, and hence represents the narrator's conversation not with a true interlocutor but a distorted reflection of his own thoughts.

2.4. *Timepoint-based analytical diagrams*

The example from Brahms's Op.106/2 illustrates that the significance of combinations of displacement and grouping dissonance has specifically to do with how they interact on timepoints. It is specifically the abstraction from timepoints that makes inclusion diagrams on metrical relations an inadequate tool for this purpose. We therefore require diagrams on timepoints, where the arrows of the diagram can show metrical relationships.

The diagram in Figure 5(a) is what we develop as a *meter network* in the next section, a digraph whose nodes are labeled with timepoints (or sets of timepoints) and arrows labeled with metrical relations. The diagram includes timepoints from the second half of bar 7 to bar 8 of the Op. 106/2 example involved in the main 6/8 meter of the vocal part, and the alternate 12/16 meter of the piano's echo. The $\mathcal{M}_{1/6}$ arrows show the eighth-note layer of the 6/8 meter, and the $\mathcal{M}_{1/4}$ arrows show the dotted-eighth layer of the 12/16 meter. The $\mathcal{M}_{1/2}$ arrows show the beats of each meter. Finally, the diagram includes relationships between the meters, the fact that they share a sixteenth-note pulse with $\mathcal{M}_{1/12}$, and the identity of the weak-beat timepoints, with $\mathcal{M}_0$.

This diagram therefore shows the essential elements of the metrical analysis: the presence of two metrical relations that are not inclusion related, $\mathcal{M}_{1/6}$ and $\mathcal{M}_{1/4}$, divides the diagram up into two meters. The $\mathcal{M}_{1/2}$ and $\mathcal{M}_{1/12}$ relations show the higher and lower metrical layers, common to both meters, with one forming a connection between the layers in the form of a shared pulse. Finally, the diagram shows that the identified ($\mathcal{M}_0$-related) timepoints are not part of the upper level of each meter as indicated by the $\mathcal{M}_{1/12}$ relations.

The diagram looks superficially similar to a diagram of time translations,[1] like Figure 5(b), so let us consider carefully what is different about the two. The meter network places timepoints on metrical grids, but does not depend upon specific durational intervals. For instance, Figure 6 labels some timepoints in the piano introduction of the Brahms song and shows two meter networks with the same labels on the arrows. The arrow labels

[1] In mathematical terms, we consider in this example the additive group $(\mathbb{Q}, +)$ acting on the set of timepoints by temporal translations.

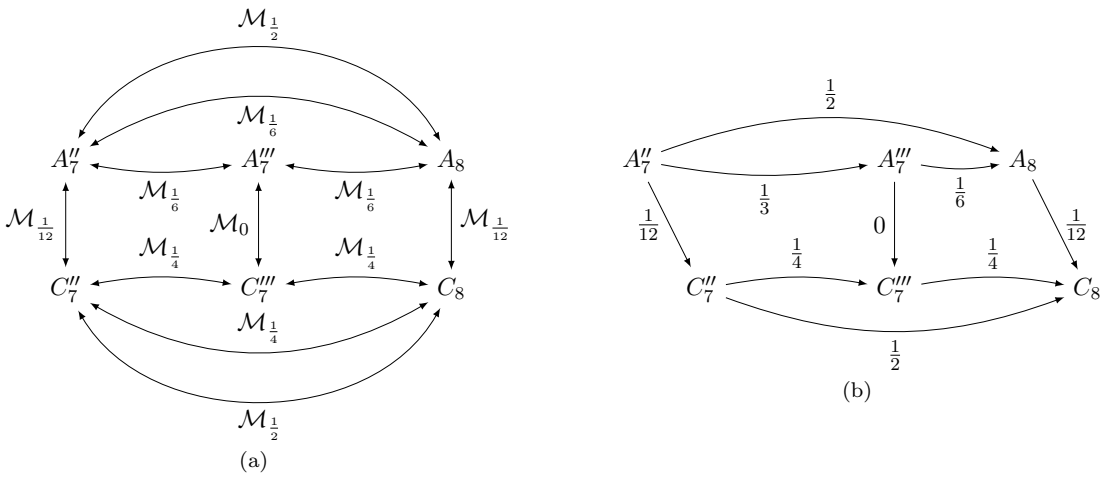


Figure 5. (a) An analytical diagram for measures 7-8 of Brahms's Op. 106/2. Nodes of the diagram are labelled with timepoints, and arrows between them are labelled with metrical relations. (b) An analytical diagram of the same measures in which the arrows are instead labelled with temporal translations.

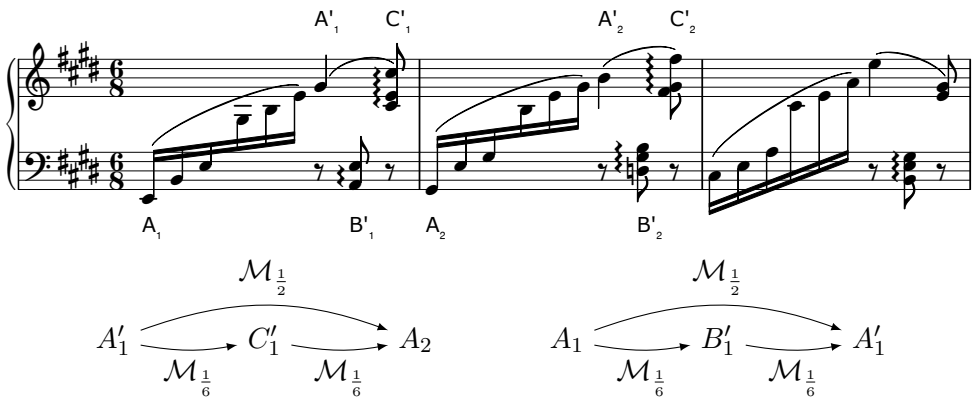


Figure 6. Brahms, "Auf dem See" (Op. 106/2), mm 1–3, and two meter networks with the same arrow labels but referring to different timepoints.

do not constrain the timepoints to have any specific intervallic relationship, or even the temporal ordering of the points. What they do constrain are the kinds of metrical grids defined by the timepoints: all three points fall on a consistent eighth-note ($\frac{1}{6}$) grid, and the points connected by $\mathcal{M}_{\frac{1}{2}}$ define a dotted half-note grid within this eighth-note grid. The meter network therefore expresses a higher level of generality than a timespan diagram, but with more specificity than a ski-hill graph or metrical inclusion diagram. For this reason, the analytical diagram in Figure 5(a) is more regular than the one in Figure 5(b).

As the definition of meter networks presented in the next section will show, the nodes can refer to sets of timepoints, not only individual timepoints, as is the case in a timespan diagram. For instance, we can study the set $\{A_i, i \in \{5, 6, 7, 8\}\}$ of timepoints of notes marked labelled with A in Figure 4, and similarly for the A''_i and A'''_i timepoints. We can then reassign the nodes of Figure 5(a) to such sets, as shown on Figure 7. Note that arrows between sets are partial functions, which are a particular case of binary relations.

The diagram in Figure 5(a) readily divides into two meters, and each of those meters into metrical levels. This is not true of all meter networks though. Consider a meter network to describe the first part of Ligeti's Sixth Étude, "Automne à Varsovie." Figure

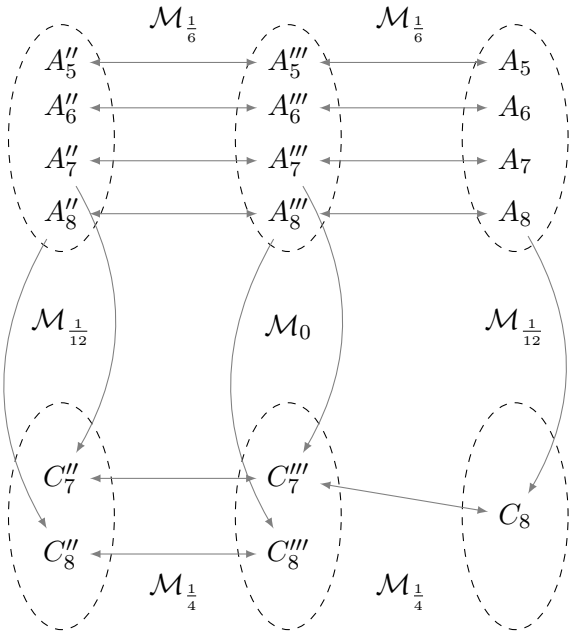


Figure 7. An analytical diagram for measures 5-8 of Brahms's Op. 106/2. Nodes of the diagram are sets of timepoints, and arrows between them are partial functions labelled with metrical relations. We omit here the arrows labelled with $\mathcal{M}_{\frac{1}{2}}$ in Figure 7 for clarity.

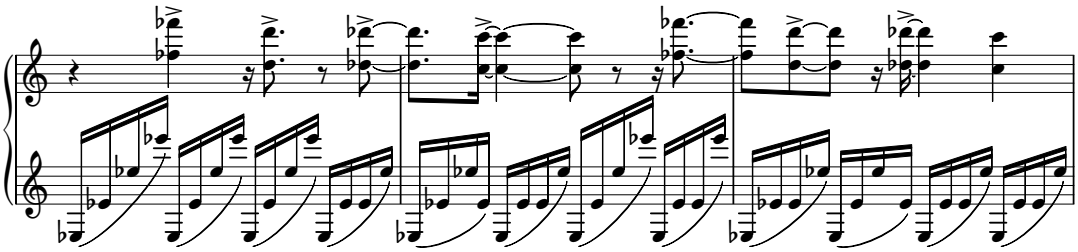


Figure 8. Ligeti, Ètude no. 6, "Automne à Varsovie," mm. 2–4.

8 shows the beginning of the piece, where Ligeti establishes two layers based on meters in a 5:4 relationship, an ostinato of octave leaps on E♭, and an upper line in octaves descending by semitone (the "Lamento" motif). This texture continues up to measure 9, where the ostinato suddenly shifts back by a sixteenth note, and simultaneously down by semitone to D, as shown in Figure 9. Figure 10 charts the rhythms of the entire passage through measure 12 in simplified form. The parts are grouped into four sets of points: the ostinato from measure 1–9 (A_0), from measure 9–10 (A_1), and 11–12 (A_2), and all of the points corresponding to the upper line B. Three individual timepoints are also isolated from set B: B_0 in m. 2, B_1 in m. 9, and B_2 in m. 10.

Figure 11 shows a meter network on all of these points. We can identify two meters based on the non-inclusion-related $\mathcal{M}_4$ and $\mathcal{M}_5$ relations, with the points B_0, B_1, B_2 shared between these. However, there are actually three $\mathcal{M}_4$ layers that only need to be on a common sixteenth-note grid. Again, the exact relationship of A_0, A_1, and A_2 is not specified, although the network shows it originating from the B layer via the intermediary roles of the B_0, B_1, and B_2 timepoints. This precisely expresses the effect of the passage: at the moments corresponding to B_1 and B_2, the solid ground of the A_0

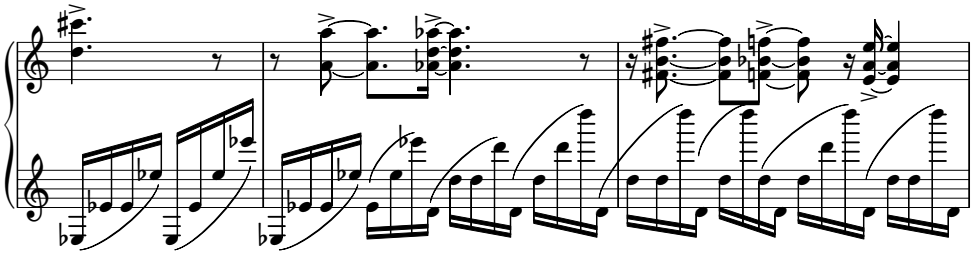


Figure 9. Ligeti, Ètude no. 6, “Automne à Varsovie,” mm. 8–10.

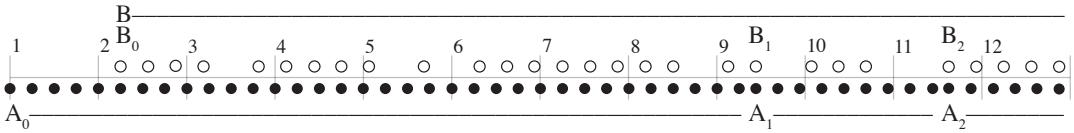


Figure 10. Ligeti, Ètude no. 6, “Automne à Varsovie,” rhythmic summary, mm. 1–12

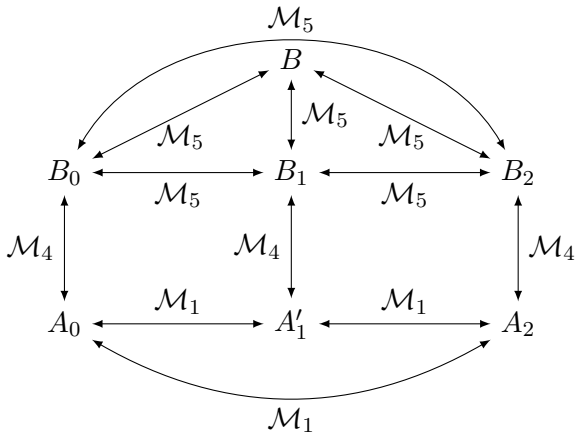


Figure 11. An analytical diagram for measures 1–12 of Ligeti’s Étude no. 6. Here the unit chosen is the sixteenth-note pulse, since the barlines are not intended to convey any musically significant metrical information.

ostinato shifts under the pressure of the B layer, where the sharing of timepoints shows the implied causal relationship. We could go a step further in relating this rhythmic design metaphorically to the program of the piece: the Lamento line represents Polish people, its 5:4 metrical relationship to the ostinato and descending chromatic line evoking their suffering. Eventually the weight of this suffering overtakes the seemingly immovable, autocratic, A_0 metrical layer. This is represented rhythmically by the A_1 layer responding to a B-layer timepoint, and by the semitone descent from the E♭. Ultimately, as the piece proceeds, this first dislodging of stability unravels precipitously into metrical chaos. What is important about the rhythmic design is not so much the exact durational relationships between elements of the A_0, A_1, and A_2 lines. It is the dislocation between them as 4-layers, and the source of that dislocation in the persistent 5-layer represented by B.[2]

In the example from Ligeti, the level of abstraction of the meter network is well-tuned to what we want to say about the piece. The precise displacement between the A_0, A_1, and A_2 layers is less important than their relationship to the B layer through

[2]A more complete analysis of dislocations, polyrhythms, and form in this Étude can be found in Taylor (1997).

specific timepoints. In the Brahms example, however, there is a specific displacement that is important to our account of the piece. The piano's alternate meter relates to the main meter in two ways. The upper metrical layer first emerges as a sixteenth-note displacement of the half-note layer of the main meter, while at the same time the weak beats (A''' and C''') coincide. We are able to specify the identity of the weak beats with the meter network, but not the precise displacement of the upper layer. In the next section, we will reformulate this network using morphisms of meter networks, which will allow us to split this meter network into separate meter networks corresponding to each metrical layer, and show inclusion relations as morphisms between these, and also show specific displacements.

3. A categorical approach to meter networks

In the previous section we have informally introduced diagrams to represent metrical relations between different timepoints, in which the nodes are labelled with timepoints and the arrows with metrical relations. Furthermore, the above discussion on metrical displacements suggests that we can describe our Brahms example using such *meter networks*. In this section, we provide a unifying mathematical framework behind these diagrams, using constructs from category theory. This allows us to properly define meter networks, and network morphisms, which we will apply to describe the remaining hemiola relations in the measures 7–8 of Brahms's Op. 106/2. We assume in this section that the reader is familiar with basic notions of category theory such as functors and natural transformations.

3.1. *Algebraic properties of metrical relations*

All binary relations have a natural algebraic structure given by the operation of composition. This is defined generally as follows.

Definition 3.1 Let X, Y, and Z be sets, $\mathcal{R}$ a relation between X and Y, and $\mathcal{R}'$ be a relation between Y and Z. The *composition of* $\mathcal{R}'$ *and* $\mathcal{R}$ is the relation $\mathcal{R}'' = \mathcal{R}' \circ \mathcal{R}$ defined by the pairs (x, z) with $x \in X$ and $z \in Z$ such that there exists at least one $y \in Y$ with $x\mathcal{R}y$ and $y\mathcal{R}'z$.

For metrical relations, the composition operation has an immediate musical interpretation as the earliest point where a meter involving $\mathcal{M}_{d_1}$ and $\mathcal{M}_{d_2}$ could come together in a diagram like those described in the previous section. The following result makes this more precise ("gcd" stands for "greatest common divisor," and we adopt the convention that $\gcd(0, x) = x$ for all x in $\mathbb{Q}_{\geq 0}$).

Proposition 3.2 *Let d_1 and d_2 be elements of $\mathbb{Q}_{\geq 0}$. We have*

$$\mathcal{M}_{d_2} \circ \mathcal{M}_{d_1} = \mathcal{M}_{gcd(d_2,d_1)}.$$

Proof. Let t be an element of $\mathbb{Q}$, and $\mathcal{M}_{d_1}$ and $\mathcal{M}_{d_2}$ be two metrical relations. The image set of t by the relation $\mathcal{M}_{d_1}$ is the set $\{t + kd_1 \mid k \in \mathbb{Z}\}$. Therefore, the image set of t by the relation $\mathcal{M}_{d_2} \circ \mathcal{M}_{d_1}$ is the set $S = \{t + kd_1 + k'd_2 \mid k \in \mathbb{Z}, k' \in \mathbb{Z}\}$. Let $d = \gcd(d_2, d_1)$. Then $d_2 = ud$ and $d_1 = vd$ with u and v being coprime integers. We then have

$$S = \{t + (ku + k'v)d \mid k \in \mathbb{Z}, k' \in \mathbb{Z}\},$$

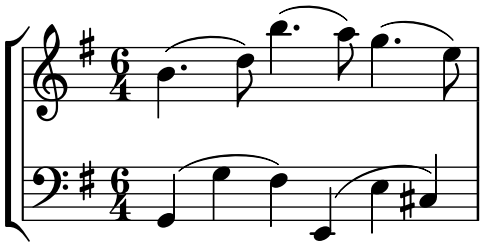


Figure 12. Brahms Violin Sonata, Op. 78, m. 235, violin and piano left hand.

and since u and v are coprime, this is equal to

$$S = \{t + k''d \mid k'' \in \mathbb{Z}\},$$

thus proving the proposition. ■

COROLLARY 3.3 *Any metrical relation $\mathcal{M}_d$ is idempotent.*

Proof. For all d in $\mathbb{Q}_{\geq 0}$ we have $\gcd(d, d) = d$, thus proving the proposition. ■

As an illustration, consider the example from Brahms's Op. 78 given above in Figure 2. The violin and piano left hand are reproduced in Figure 12. Say we have some timepoint such as A, shared between two meters. If there is a B related to A by $\mathcal{M}_{\frac{1}{3}}$, as in the violin, and a C related to A by $\mathcal{M}_{\frac{1}{2}}$, as in the piano left hand, then B relates to C by $\mathcal{M}_{\frac{1}{3}} \circ \mathcal{M}_{\frac{1}{2}} = \mathcal{M}_{\frac{1}{6}}$. Therefore the metrical relation $\mathcal{M}_{\frac{1}{6}}$ is where the violin and piano left hand meters come together.

The presence of the composition law turns the set of all metrical relations into a commutative monoid notated M, as the proposition below shows.

PROPOSITION 3.4 *The set of all relations $\mathcal{M}_d$ with $d \in \mathbb{Q}_{\geq 0}$ forms a commutative monoid M isomorphic to $\mathbb{Q}_{\geq 0}$ equipped with gcd as the operation.*

Proof. Metrical relations $\mathcal{M}_d$ with $d \in \mathbb{Q}_{\geq 0}$ can be uniquely identified with d, and their composition is given by the gcd operation. The identity element is 0, and the gcd operation is known to be associative and commutative, thus proving the proposition. ■

Although metrical relations $\mathcal{M}_d$ can be uniquely identified with positive rational durations $d \in \mathbb{Q}_{\geq 0}$, we emphasize that the resulting monoid M should not be assimilated to the usual notion of a group of multiplicative durations, as encountered in Lewin's (1987) timespan group for example. Here, each rational duration d represents an associated pulse, and the corresponding metrical relation $\mathcal{M}_d$ relates multiple timepoints of $\mathbb{Q}$ at the same time. The resulting composition law (gcd) is distinct from duration multiplication, and non-identity elements of M are not invertible.

In the previous section, we have seen that elements of M can also be partially ordered by inclusion. In fact, the consideration of both the above composition law and the ordering given by inclusion can be understood from a categorical point of view. It is known that sets and binary relations between them form a 2-category **Rel** defined as follows.

Definition 3.5 The 2-category **Rel** is the category which has sets as objects, binary relations as 1-morphisms between them, and inclusion of relations as 2-morphisms between

relations.

Thus, the monoid M can be viewed as subcategory of **Rel**, i.e. as category with a single object (corresponding to $\mathbb{Q}$), whose 1-morphisms are the metrical relations $\mathcal{M}_d$, and whose 2-morphisms corresponds to inclusion of metrical relations. Observe that for any two parallel binary relations (1-morphisms of **Rel**) there is at most one 2-morphism between them: the 2-category **Rel** is in fact a 2-poset, i.e. a category in which the hom set between any two objects has the structure of a partially ordered set (poset). The monoid M inherits the same structure: it can then immediately be seen that Cohn's ski-hill graphs corresponds to sub-posets of the poset structure of the 2-morphisms between metrical relations.

The monoid M has in fact some additional structure, as the following proposition shows, whose proof is immediate.

PROPOSITION 3.6 *The set of elements of M is closed under the operation of intersection of binary relations. For any two metrical relations $\mathcal{M}_{d_1}$ and $\mathcal{M}_{d_2}$, we have*

$$\mathcal{M}_{d_2} \cap \mathcal{M}_{d_1} = \mathcal{M}_{lcm(d_2,d_1)}$$

where "lcm" stands for "least common multiple."

COROLLARY 3.7 *The set of elements of the monoid M has the structure of a join-semilattice. Inclusion of metrical relations (the 2-morphisms of M) corresponds to the partial order induced by the lcm monoid operation of the join-semilattice.*

Note that the two monoid operations of composition and intersection of metrical relations correspond to the two monoid operations associated with the lattice structure on $\mathbb{Q}_{\geq 0}$ given by gcm and lcm. From a musical point of view, the interpretation of the intersection metrical relation $\mathcal{M}_{d_2} \cap \mathcal{M}_{d_1}$ is immediate: it relates timepoints common to two metrical layers, i.e. it is the regular pulse of smallest duration common to both pulses.

In categorical terms, the definition of the metrical relations and of the monoid M gives us a functor $S\colon M \to$ **Rel**, which will be an important element in the definition of meter networks. Note that the consideration of such a functor $S\colon M \to$ **Rel** precisely fulfills our attempt to take into account both meter interaction and timepoints at the same time: on one hand, the algebraic structure of the monoid M describes both the composition and the inclusion of metrical relations, while on the other hand, the image of the single object of M by S is the set of timepoints $\mathbb{Q}$ on which these metrical relations are based. The interplay between these two aspects is shown in the next section.

3.2. *Meter networks*

Consider the diagram in Figure 13(a), which describes a single metrical relation between two half-bar beats in the example of Figure 1. This diagram seems rather intuitive at first sight, and recalls the diagrams commonly encountered in transformational music analysis, for example using the T/I group or the neo-Riemannian *PRL* group. Upon closer inspection, it is however flawed: indeed, the element $\mathcal{M}_{\frac{1}{2}}$ is a binary relation and as such the timepoint $\frac{1}{2}$ cannot be unambiguously considered as being the image of 0 by $\mathcal{M}_{\frac{1}{2}}$. Unlike with usual musical transformations, which are functions between sets, the notion of "image of an element by a relation" corresponds to a set which is not neces-

sarily a singleton. Recent work on the categorical formalization of Klumpenhouwer networks and musical diagrams (Popoff, Andreatta, and Ehresmann 2015; Popoff et al. 2016; Popoff, Andreatta, and Ehresmann 2018) has shown however that the diagram in Figure 13(a) is meaningful when considered as the simplified representation of a categorical construction called *relational poly-Klumpenhouwer network (PK-net)*, which generalizes Klumpenhouwer networks (Lewin 1990; Klumpenhouwer 1991) in a relational setting. To avoid confusion with Klumpenhouwer networks and their generalizations (which are based on pitch-class set theory and usually involve the T/I group or the neo-Riemannian *PRL* group) and given the specific scope of this paper, we will call such categorical objects *meter networks*.

This categorical construction is represented by the general diagram in Figure 13(b), in which Δ, $\mathbf{C}$ are categories, R, S, and F are functors, and ϕ is a left-total lax natural transformation (see definition below). We refer the reader to Popoff, Andreatta, and Ehresmann (2018) for a detailed exposition of this construction and only recall in this introduction the parts relevant to the scope of this paper.

Coming back to our example above, the category $\mathbf{C}$ is the monoid M of metrical relations along with the functor $S\colon M \to \mathbf{Rel}$ which encodes its action on the set of timepoints. The image set of the timepoint 0 by the metrical relation $\mathcal{M}_{\frac{1}{2}}$ is the set $\{k/2 \mid k \in \mathbb{Z}\}$ of timepoints in $\mathbb{Q}$. We can construct a relational PK-net with an *ad hoc* lax natural transformation that would keep only the timepoints 0 and $\frac{1}{2}$, as shown on Figure 13(c). Based on this example, we can detail the role of each element in the diagram of Figure 13(b). The category Δ defines the general form of the diagram: it indexes groups of timepoints linked by morphisms (in this case two objects with one non-trivial morphism f between them). The functor R translates this category to $\mathbf{Rel}$, in which each image set $R(X) = \{x\}$ and $R(Y) = \{y\}$ is a singleton, and in which $R(f)$ is the obvious relation between them. The functor F gives a label to the non-trivial morphism f: its image by F is the metrical relation $\mathcal{M}_{\frac{1}{2}}$. Finally, the purpose of the lax natural transformation ϕ is to label the elements in the singletons $R(X)$ and $R(Y)$ with timepoints in $\mathbb{Q}$, enforcing the condition that x and y should be related by $\mathcal{M}_{\frac{1}{2}}$. Notice that while the metrical relation $\mathcal{M}_{\frac{1}{2}}$ is reflexive by definition, the meter network we construct shows a directed arrow between the timepoints 0 and $\frac{1}{2}$ which originates from the chosen category Δ and its unique non-trivial morphism $f\colon X \to Y$. Our initial diagram in Figure 13(a) is then a simplified representation of this categorical construct.

To give a general definition of meter networks, we first recall some categorical notions related to the category $\mathbf{Rel}$, which were already introduced in Popoff, Andreatta, and Ehresmann (2018). Since $\mathbf{Rel}$ is a 2-category, the notion of a *lax functor* to $\mathbf{Rel}$ has to be introduced to account for the possible 2-morphisms between relations. In the general case, the comparison natural 2-cells of a lax functor are required to satisfy three coherence diagrams. Since there is at most one 2-cell between any two morphisms in $\mathbf{Rel}$, all diagrams of 2-cells in $\mathbf{Rel}$ commute. It follows that the definition of lax functor can be simplified, since the requirements on the coherence diagrams are automatically satisfied. The notion of a lax functor can thus be defined more precisely as follows.

Definition 3.8 Let $\mathbf{C}$ be a category. A *lax functor* F from $\mathbf{C}$ to $\mathbf{Rel}$ is the data of a map

- which sends each object X of $\mathbf{C}$ to an object $F(X)$ of $\mathbf{Rel}$, and the identity morphism id_X of X to the identity morphism $\mathrm{id}_{F(X)}$ of $F(X)$, and
- which sends each morphism $f\colon X \to Y$ of $\mathbf{C}$ to a relation $F(f)\colon F(X) \to F(Y)$ of $\mathbf{Rel}$, such that for each pair (f, g) of composable morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ the image relation $F(g)F(f)$ is included in $F(gf)$.

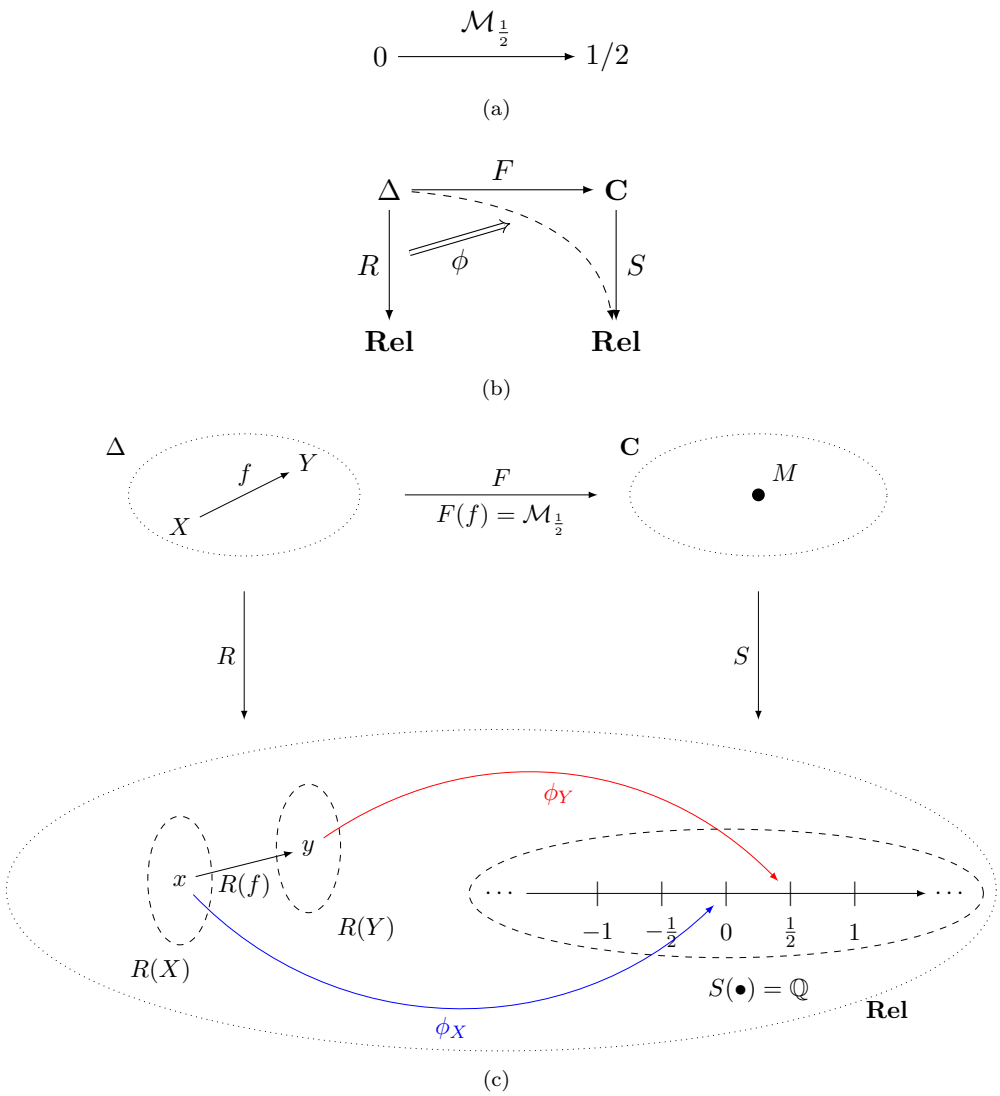


Figure 13. (a) A simple meter network. (b) Diagrammatic representation of a relational network : Δ and $\mathbf{C}$ are categories, F, R, and S are functors, and ϕ is a left-total lax natural transformation. The dashed arrow represents the composite functor SF. (c) Diagram showing the constitutive elements of the relational network (R, S, F, ϕ) behind the diagram (a). The functor F sends the objects X and Y to the monoid M of metrical relations considered as a single-object category, whose image $S(\bullet)$ by the functor S is the set of timepoints $\mathbb{Q}$. The components ϕ_X and ϕ_Y of the lax natural transformation ϕ pick out the two timepoints 0 and $\frac{1}{2}$.

A lax functor will be called a 1-functor when $F(g)F(f) = F(gf)$.

Given two lax functors F and G to **Rel**, the notion of a *lax natural transformation* η between F and G has to be introduced to similarly account for the possible 2-morphisms between relations. Since the necessary coherence diagrams are automatically satisfied in **Rel**, this notion is defined as follows.

Definition 3.9 Let $\mathbf{C}$ be a 1-category, and let F and G be two lax functors from $\mathbf{C}$ to **Rel**. A *lax natural transformation* η between F and G is the data of a collection of relations $\{\eta_X : F(X) \to G(X)\}$ for all objects X of $\mathbf{C}$, such that, for any morphism $f : X \to Y$, the relation $\eta_Y F(f)$ is included in the relation $G(f)\eta_X$.

A meter network is then defined as follows.

Definition 3.10 Let $\mathbf{C}$ and Δ be small 1-categories, and R a lax functor from Δ to $\mathbf{Rel}$ with non-empty values. A *meter network* is a 4-tuple (R, S, F, ϕ), in which $S\colon M \to \mathbf{Rel}$ is the functor induced by the monoid of metrical relations M, F is a functor from Δ to M, and ϕ is a lax natural transformation from R to SF, such that, for any object X of Δ, the component ϕ_X is left-total.

Observe that this definition allows for many different possibilities. For example, the images of objects of Δ by the functor R are sets in $\mathbf{Rel}$, which allows us to consider sets of timepoints, and not just single timepoints. The images of morphisms of Δ by the functor R are morphisms of $\mathbf{Rel}$, i.e. binary relations, which allows us to consider functions, partial functions, or more complicated relations between the different elements in the sets. We have seen such a case in Figure 7, where only the simplified representation of the whole categorical construction is shown.

3.3. *Morphisms of meter networks*

Having introduced meter networks, we can now define the notion of *morphism of meter networks*. We first recall the definition of inclusion of lax natural transformations between functors going to $\mathbf{Rel}$.

Definition 3.11 Let $\mathbf{C}$ be a 1-category, let F and G be two lax functors from $\mathbf{C}$ to $\mathbf{Rel}$, and let η and η' be two lax natural transformation between F and G. We say that η is *included* in η' if, for any object X of $\mathbf{C}$, the component η_X is included in the component η'_X.

A *morphism of meter networks* is then defined as follows.

Definition 3.12 A *morphism between two meter networks* (R, S, F, ϕ) *and* (R', S, F', ϕ') is a 5-tuple (I, χ, N, ν, η) where

- I is a functor $I\colon \Delta \to \Delta'$,
- χ is a left-total lax natural transformation $\chi\colon R \to R'I$,
- N is a functor $N\colon M \to M$,
- ν is a left-total lax natural transformation $\nu\colon S \to SN$, and
- η is a lax natural transformation $\eta\colon NF \to F'I$,

such that the lax natural transformation $\phi' I \circ \chi$ is included in $S\eta \circ \nu F \circ \phi$.

The constitutive elements of a meter network morphism and their interrelations are shown in Figure 14. Among such morphisms of networks, two particular cases are interesting, which we now detail here.

3.3.1. Automorphisms of the functor $S\colon M \to \mathbf{Rel}$

We consider the particular case where η is the identity natural transformation. A network morphism is then defined by $N\colon M \to M$ and $\nu\colon S \to SN$; i.e. it does not depend on any particular network but rather corresponds to transformations of the time points and the metrical relations associated with them. We will consider more specifically the automorphisms of the functor $S\colon M \to \mathbf{Rel}$, also called "complete isographies" in previous work (Popoff, Andreatta, and Ehresmann 2018), and show how the metrical transformations by displacement of Section 2.4 can be expressed as such automorphisms. We begin with

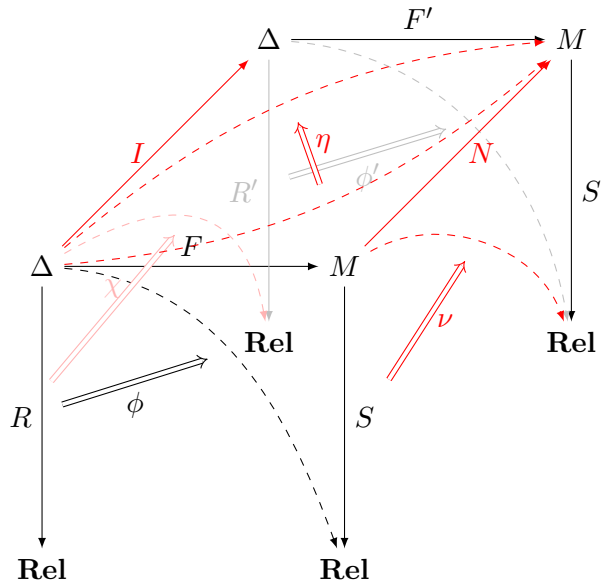


Figure 14. Diagram showing the constitutive elements of a meter network morphism (I, χ, N, ν, η) (in red) between two meter networks (R, S, F, ϕ) and (R', S, F', ϕ'). The dashed arrows represent composite functors.

a result regarding the automorphism group of M.

PROPOSITION 3.13 *The group $(\mathbb{Q}_{\geq 0}, \times)$ is a subgroup of the automorphism group of the monoid $M = (\mathbb{Q}_{\geq 0}, gcd)$.*

Proof. Let λ be an element of $\mathbb{Q}_{\geq 0}$. We have $\lambda \gcd(d_2, d_1) = \gcd(\lambda d_2, \lambda d_1)$ for all (d_1, d_2) in $(\mathbb{Q}_{\geq 0})^2$, and $\lambda.0 = 0$, thus proving the proposition. ■

This leads to the following proposition.

PROPOSITION 3.14 *The semidirect product $(\mathbb{Q}, +) \rtimes (\mathbb{Q}_{\geq 0}, \times)$ is a subgroup of the automorphism group of the functor $S\colon M \to \mathbf{Rel}$.*

Proof. Let N be an element of $(\mathbb{Q}_{\geq 0}, \times)$, i.e. an isomorphism of M characterized by $\lambda \in \mathbb{Q}_{\geq 0}$. Let ν be the equivalence (invertible natural transformation) between the functors S and SN defined by its unique component such that $\nu(t) = \lambda t + u$ with $u \in \mathbb{Q}$. Then for all $\mathcal{M}_d$ in M and for all $(t, t') \in \mathbb{Q}^2$ such that $t\mathcal{M}_d t'$, we have $\nu(t)\mathcal{M}_{\lambda d}\nu(t')$, and ν is therefore a valid equivalence. The automorphism (N, ν) of S is bijectively identified with the pair (λ, u), and it can be easily proved that composition corresponds to the structure of a semidirect product. ■

The structure of this subgroup corresponds (in an analogous manner to Lewin's (1987) time-span group) to temporal translations and dilations of time-points, and the corresponding temporal dilations of meters.

This result can immediately be applied to the Brahms Op. 106/2 example: consider the notes marked A''_7, A_8, B''_7, and B_8 in Figure 4. We can assume without loss of generality that A''_7 corresponds to the timepoint 0, in which case $B''_7 = C''_7 = 1/12$, $A'''_7 = C'''_7 = 1/3$, $A_8 = 1/2$, and $B_8 = C_8 = 7/12$. If we consider the affine map on $\mathbb{Q}$ defined by $\nu(t) = t + \frac{1}{12}$, then we have $\nu(A''_7) = B''_7$ and $\nu(A_8) = B_8$, and by the proposition above, these two timepoints are also related by $\mathcal{M}_{\frac{1}{2}}$. We can represent this transformation by the following diagram of meter networks.

$$\begin{tikzcd}
A''_7 \arrow[rr, leftrightarrow, "\mathcal{M}_{\frac{1}{2}}"] \arrow[d, "\nu(t) = t + 1/12"'] & & A_8 \\
B''_7 \arrow[rr, leftrightarrow, "\mathcal{M}_{\frac{1}{2}}"'] & & B_8
\end{tikzcd}$$

3.3.2. Network transformations by metrical inclusions

We now consider the particular case where N is the identity functor, and ν is an identity natural transformation. A network morphism is then defined only by $I\colon \Delta \to \Delta'$, $\chi\colon R \to R'I$, and $\eta\colon F \to F'I$. This last lax natural transformation allows metrical inclusion between meter networks.

As an example, let us return to the Brahms example of Figure 4 and define the following two meter networks.

- The first meter network (R, S, F, ϕ) is such that
  - the category Δ has only two objects X and Y and only two non-trivial morphisms $f\colon X \to Y$ and $f'\colon Y \to X$ between them, and
  - the functor F is such that $F(f) = F(f') = \mathcal{M}_{\frac{1}{2}}$, and
  - the functor R is such that $R(X) = \{x\}$, $R(Y) = \{y\}$,
  - the natural transformation ϕ is such that ϕ_X sends x to A''_7 and ϕ_Y sends y to A_8.
- The second meter network (R', S, F', ϕ') is such that
  - the category Δ' has three objects X', Y', and Z' and is generated by non-trivial morphisms $g\colon X \to Y$, $g'\colon Y \to X$, $h\colon Y \to Z$, and $h'\colon Z \to Y$ between them, and
  - the functor F' is such that $F'(f) = F'(f') = F'(g) = F'(g') = \mathcal{M}_{\frac{1}{6}}$, and
  - the functor R' is such that $R'(X') = \{x'\}$, $R'(Y') = \{y'\}$, $R'(Z') = \{z'\}$, and
  - the natural transformation ϕ' is such that $\phi'_{X'}$ sends x' to A''_7, $\phi'_{Y'}$ sends y' to A'''_7, $\phi'_{Z'}$ sends z' to A_8.

Then we have a morphism between (R, S, F, ϕ) and (R', S, F', ϕ') defined by the 5-tuple $(I, \chi, \mathrm{id}, \mathrm{id}, \eta)$ where

- the functor I sends X to X' and Y to Z', f to $h \circ g$, and f' to $g' \circ h'$, and
- the lax natural transformation χ sends x to x' and y to z', and
- the lax natural transformation η is defined by $\eta_X = \eta_Y = \mathcal{M}_0$. The lax condition is fulfilled since $\mathcal{M}_0 \circ \mathcal{M}_{\frac{1}{2}}$ is included in $\mathcal{M}_{\frac{1}{6}} \circ \mathcal{M}_0$.

We thus obtain the following diagram of meter networks, which shows from a metrical point of view how the half-bar duration delimited by A''_7 and A_8 is further divided by A'''_7.

$$\begin{tikzcd}
A''_7 \arrow[rr, leftrightarrow, "\mathcal{M}_{\frac{1}{2}}"] \arrow[d, dashed, "\mathcal{M}_0"'] & & A_8 \arrow[d, dashed, "\mathcal{M}_0"] \\
A''_7 \arrow[rr, leftrightarrow, bend left, "\mathcal{M}_{\frac{1}{6}}"] \arrow[r, leftrightarrow, bend right, "\mathcal{M}_{\frac{1}{6}}"'] & A'''_7 \arrow[r, leftrightarrow, bend right, "\mathcal{M}_{\frac{1}{6}}"'] & A_8
\end{tikzcd}$$

Note that a similar diagram of meter networks can be constructed to show how the duration delimited by $C''_7 = B''_7$ and $C_8 = B_8$ is further divided by C'''_7. This procedure gives a general way to represent a conventional metrical interpretation of a piece: represent each metrical layer with a meter network that uses just one metrical relation, then relate these layers by metrical inclusion morphisms. By relating one layer to multiple other layers with different meter relations, such a network can incorporate grouping dissonances. Displacements between layers, in the form of affine transformations, can also be added. This, then gives a complete means of representing the metrical situation underlying our analysis of Brahms's Op. 106/2, as shown in Figure 15.

This network shows the displacement relation between the $\mathcal{M}_{\frac{1}{2}}$ layers of the voice and piano arpeggiations with an affine transformation. It shows the hemiola by including two metrical inclusions from the $\mathcal{M}_{\frac{1}{2}}$ layers to $\mathcal{M}_{\frac{1}{6}}$ and $\mathcal{M}_{\frac{1}{4}}$ layers. Note that when showing this meter network under the simplified representation of Figure 15, it is implicitly assumed that morphisms between sets of timepoints are adequately chosen partial functions. Finally it uses another inclusion morphism to show the identity of the A'''_i and C'''_i timepoints. If we think of each sequence of metrical inclusion morphisms as a meter of the passage, which works given the constraints that allow meter networks to represent individual metrical layers, this last element is of particular hermeneutic interest. It means that the coincidence points of the two meters create a third meter in which the weak beats of the $\mathcal{M}_{\frac{1}{6}}$ and $\mathcal{M}_{\frac{1}{4}}$ layers become strong, and the strong layers become weak. This, we might say, is the alternate universe in which the lake no longer echoes the lover's words but sings in synchrony with him, without displacement, where he can exist eternally in a single moment. While it is unlikely to be experienced by the listener as a "real" meter, its existence is key to Brahms's metaphorical translation of poetic idea into music.

We can observe that in the particular case of this transformational diagram, each meter network morphism corresponds to an inclusion morphism (a 2-morphism of M) between the metrical relations characterizing each meter network. This means that to the diagram of meter morphisms of Figure 15 correspond a diagram in the semilattice of 2-morphisms of M, i.e. a ski-hill graph expressing the hemiola in this piece. This result is not general because an affine transformation can transform a metrical relation $\mathcal{M}_d$ into a metrical relation $\mathcal{M}_{\lambda d}$ without $\mathcal{M}_d$ being included in $\mathcal{M}_{\lambda d}$, and the same could happen with meter inclusion morphisms. Note however, the essential content of the network in Figure 15 that would be lost in summarizing it with its associated ski-hill graph. In particular, the two distinct versions of each meter would collapse into single $\mathcal{M}_{\frac{1}{2}} \to \mathcal{M}_{\frac{1}{6}}$ and $\mathcal{M}_{\frac{1}{2}} \to \mathcal{M}_{\frac{1}{4}}$.

A specific case of metrical inclusion is given by natural transformations ϵ from the identity functor $\mathrm{id}\colon M \to M$ to itself. The following proposition, which is immediate from the definition of natural transformations and from the commutative property of M, gives these natural transformations.

Proposition 3.15 *The set of natural transformations ϵ from the identity functor $id\colon M \to M$ to itself is in one-to-one correspondence with elements of M.*

The following proposition draws a link between these natural transformations and affine transformations.

Proposition 3.16 *Let ν be a natural transformation from S to S. There exists a natural transformation ϵ from $id\colon M \to M$ to itself such that ν is included in $S\epsilon$.*

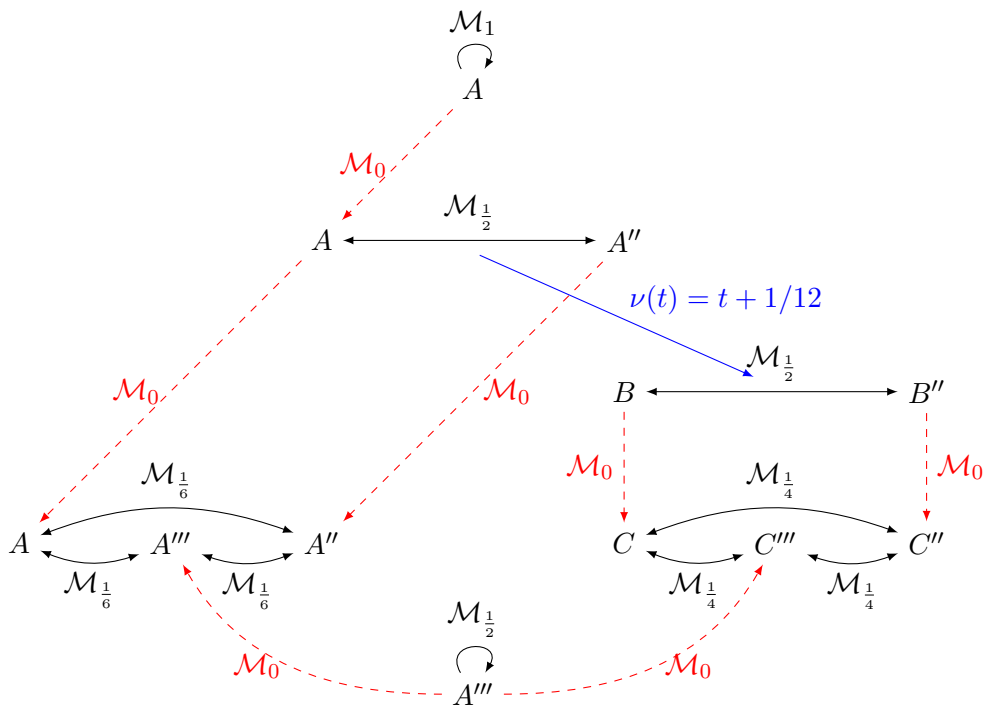


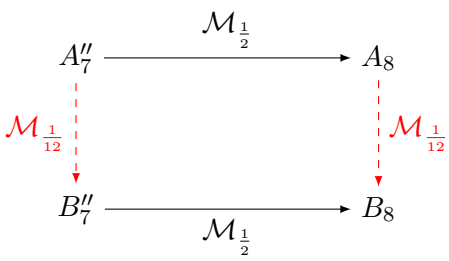
Figure 15. A transformational diagram for measures 5–8 of Brahms's Op. 106/2. The meter networks (in black) are transformed either through a metrical displacement transformation (blue), or through metrical division (red). The label A correspond to the set $\{A_i\}$ of timepoints of Figure 4, and similarly for the other labels. In this simplified representation, morphisms between sets are implicitly assumed to be adequately chosen partial functions.

Proof. Let ν be a natural transformation from S to S. From Proposition 3.14, ν correspond to an affine map on $\mathbb{Q}$ of the form $\nu(t) = t + u$ with $u \in \mathbb{Q}$. Since this function, considered as a particular binary relation, is included in the metrical relation $\mathcal{M}_u$, it suffices to take the natural transformation ϵ corresponding to $\mathcal{M}_u$ to prove the proposition. ■

This proposition shows that any meter network morphism defined in Section 3.3.1 by an affine transformation with $N\colon M \to M$ being the identity functor can be recast as a particular case of a meter inclusion morphism. Indeed, if $\nu \subset S\epsilon$, then

$$S\eta \circ \nu F \circ \phi \subset S\eta \circ S\epsilon F \circ \phi = S(\eta \circ \epsilon F) \circ \phi.$$

By taking the new natural transformation $\eta' = \eta \circ \epsilon F$, we obtain a meter network morphism such that $\phi' I \circ \chi$ is included in $S\eta' \circ \nu F \circ \phi$. For example the diagram of section 3.3.1 illustrating a meter displacement can be recast as the following diagram.



4. Conclusions

Much of recent metrical theory has focused on the ways that different meters and metrical interpretations can conflict and interact in music, as a crucial aesthetic resource spanning different styles and musical eras. The categorical definition of meter networks which we have presented in this paper takes an important step towards analyzing more complex instances of this kind of metrical interaction, allowing us to simultaneously consider relations between timepoints, metrical layers, and meter inclusion. It is the algebraic properties of the category **Rel** of sets and binary relations that make this possible. The subcategory M of **Rel** defined by metrical relations on the set of timepoints $\mathbb{Q}$ defines a functor $M \to$ **Rel** which encodes both the data of timepoints and metrical layers. Moreover, the 2-category structure of **Rel** gives M the structure of a 2-monoid in which the 2-morphisms encode the inclusion of metrical layers. We have defined meter networks through appropriate categorical constructions, showing metrical relations between timepoints or sets of timepoints with diagrams like those familiar from transformation theory. Meter networks relate through morphisms, which capture phenomena such as meter displacement and meter inclusion. In fact, meter network morphisms are not univocal and one can envision multiple possibilities. For example we can simultaneously represent displacement and inclusion through an appropriate choice of functors and natural transformations. Constructing these upon very general category-theoretic foundations offers a flexible theory that can be re-tuned in a variety of ways for use with different sets of theoretical premises and different analytical situations.

Higher categories are seldom encountered in mathematical music theory. In this work, the 2-category structure of M has immediate musicological relevance: the 2-morphisms correspond to inclusion of metrical relations, and the set of 2-morphisms of M even inherits the structure of a join-semilattice given by the lcm operation. The mathematically-inclined reader will have noticed that our definition of meter networks considers a 1-category Δ and a 1-functor F to M. As such, a meter network does not take into account 2-morphisms, i.e. it cannot represent meter inclusion by itself, which is instead represented through appropriate meter network morphisms. A natural continuation of this work would thus be to properly define higher-order meter networks from a categorical point of view, by requiring the category Δ to be a 2-category and adapting the remaining elements (the functors F, R, and the natural transformation ϕ) to this new definition.

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