THE STEP-CLASS AUTOMORPHISM GROUP IN TONAL ANALYSIS

Until recently, researchers who have dealt formally with tonal hierarchy (prolongation) have considered only models in which the objects of the hierarchy are musical events (where a musical event might be a chord or a note in a particular voice).[1] In contrast, Yust (2006) proposes a concept called "dynamic prolongation" in which the objects of tonal hierarchy are *motions between* events rather than events themselves. The events in the model of Yust (2006) are chords made up of harmonically related pitches from several voices. In the present study I develop a different approach to dynamic prolongation. Rather than expressing harmonic relations between notes by grouping them into chords, we can treat harmonic relations as intervals and mix them in a hierarchy with melodic motions. This creates a model that can posit long-range harmonic relationships and blur the boundaries between intervals of harmonic and melodic significance.

A particularly beautiful possibility opened up by this approach is to consider intervals in terms of step-class values and to view prolongational relationships between intervals in terms of step-class symmetries. A step-class is the residue modulo seven of the number of diatonic steps that separate two notes (referred to as "unison," second," third," etc.). The group of step-class intervals is therefore isomorphic to the integers under addition modulo seven ($\mathbf{Z}_7$).

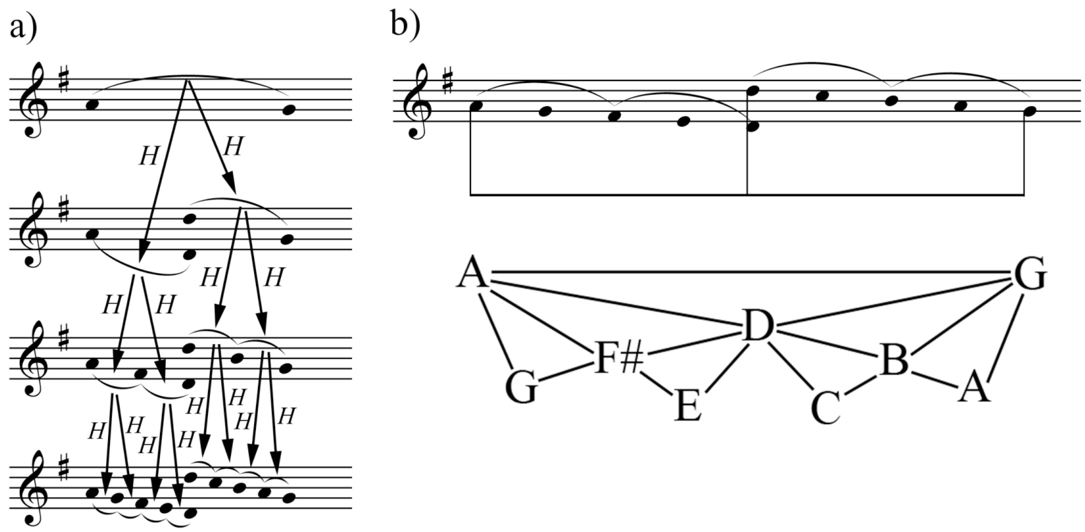
The automorphisms (i.e. symmetries) of $\mathbf{Z}_7$ (isomorphic mappings of $\mathbf{Z}_7$ to itself) can be thought of as multiplications modulo 7, and form a group ($Aut(\mathbf{Z}_7)$) isomorphic to $\mathbf{Z}_6$, which is in turn isomorphic to the direct product $\mathbf{Z}_2 \times \mathbf{Z}_3$. Step-class inversion, which I'll denote I, represents the order-two component of step-class interval automorphism group. The "halving" operation, H, which takes fifths to thirds, thirds to seconds, seconds to fifths, and so forth, is a representative order-three operation. Together these two operations generate the group, $\{1, H, H^{1}, I, HI, H^{1}I\}$.

The order-three operations of this group are of particular interest as prolongational operations on intervals. This is illustrated in Example 1a, which shows intervals at various voice-leading levels with, slurs. Moving from one level to the next, the operation H relates a prolonged interval to two motions that divide it in half. Three types of basic prolongational units result from the operation: the prolongation of a second by the framing fifths of a dominant–tonic progression, the triadic filling-in of a fifth, and the filling-in of a third with a passing motion.

The relationships between step-class motions in Example 1a take the form of a binary tree. We can also view this structure as a maximal outerplanar graph, or "MOP," as in Example 1b, with notes as vertices and intervals of the hierarchy as edges. (Yust 2006 deals extensively with mathematical properties of MOPs and their musical consequences.) The staff above shows the MOP shows intervals in a modified Schenkerian notation that shows third-motions with slurs and fifth-motions with beams.

Using step-class automorphisms as prolongational transformations suggests the following interpretation of the resulting MOP: step-class intervals define a space that can occupy a number of orientations. The different orientations define familiar musical perspectives on step-class space: a melodic space where notes a second apart are close

[1] Rahn 1979, Smoliar 1980, Lerdahl and Jackendoff 1983, Cohn and Dempster 1992.



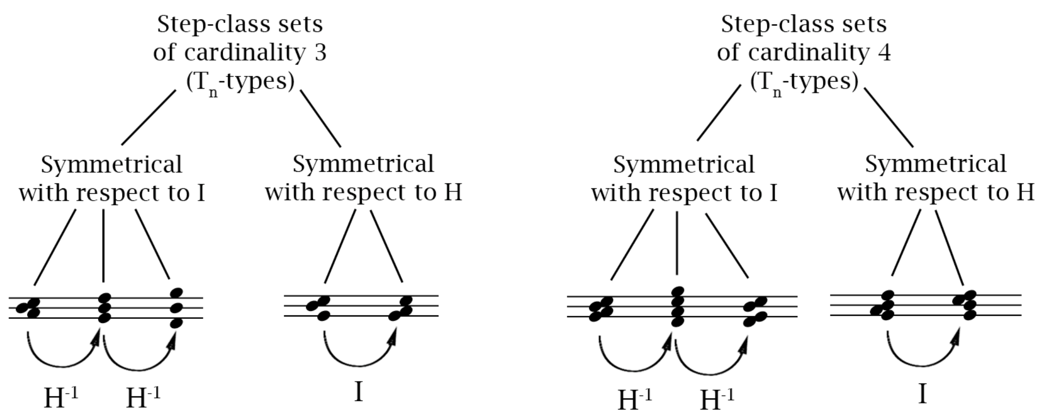
Example 1: H *as a prolongational operation*

together, a space of thirds that defines harmonic closeness, and a fifths-space that relates harmonies and keys. The presence of a particular step-class motion in a piece of music can be seen as an *orientation* of musical space in terms of that motion. That is, while the listener experiences that motion, thinking at the prolongational level where it occurs means thinking of a corresponding orientation to step-class space. The listener then moves conceptually between different prolongational levels by means of automorphic transformations of the space. The simplest relationship between levels of musical motion is the automorphism *H*, which defines a symmetrical binary filling-in of an interval.

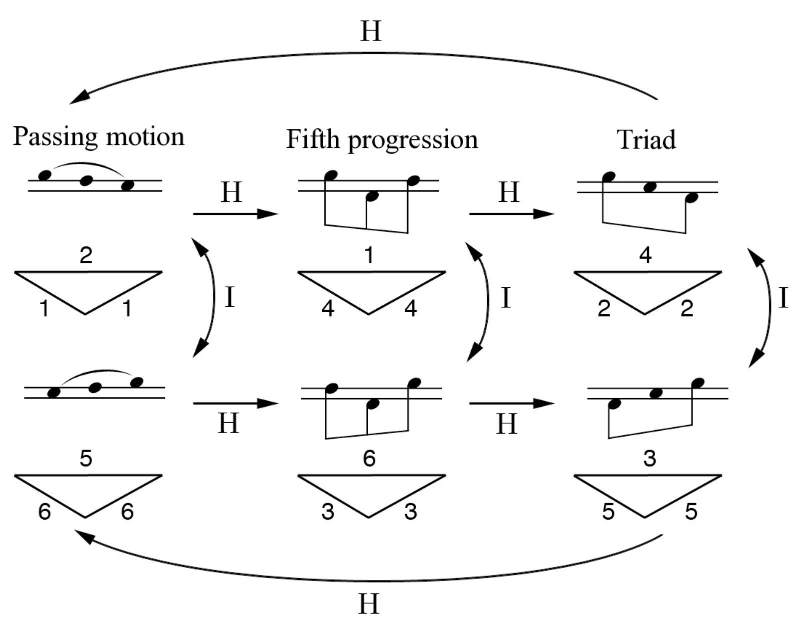
For example, a diatonic sequence by fifths once initiated implies continuation through further motion by fifth. But after hearing a couple repetitions of the sequence the listener can mentally move to a higher level of musical organization, hearing a larger sequence by second that suggests continuation by second. By hearing at various such levels at once the sequence by fifths sounds not like a homogenous motion that could start and end at any time, but at certain points gives a sense of completion of larger-scale motions from earlier tonal events in the piece and initiation of motions that may receive later completion.

The interval hierarchy includes not only such melodic motions, however, but also includes intervals that are found in the music as harmonic relationships at various levels. We are motivated to mix these interval types because the familiar structures that result from prolongational step-class automorphisms include passing figures, which are best expressed melodically, triads, which can be realized vertically, and fifth-progressions, which can come about through a combination of vertical and melodically articulated intervals. The total ordering of pitches implied by MOPs combined with the interpretation of intervals in the MOP both as melodic and harmonic intervals evokes Heinrich Schenker's important idea of horizontalization or unfolding—that intervals of harmonic significance can be transformed into melodic intervals and thereby subjected to melodic elaboration. In fact, it takes the idea one step further by projecting a direction onto undirected foreground harmonic intervals so that intervals that remain vertical on the musical surface are horizontalized in the analysis.

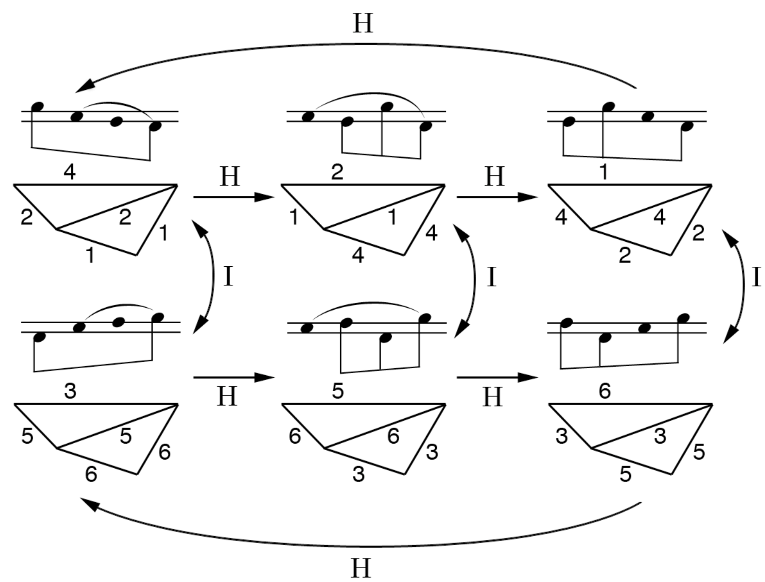
The interaction between unordered step-class sets and ordered, prolongationally-interpreted sets elevates the musical meaning that we can extract from these prolongational structures. Example 2 classifies the small number of transpositional types of step-class sets of order three and four. (Sets are shown on a generic staff to abstract them from particular pitch-class instantiations). The trichords symmetrical with respect to inversion function as the basic units of prolongational structures and provide an immediate musical interpretation of the structures in terms of the familiar musical-theoretic entities of passing motions, triads, and fifth-progressions. Figure 3 shows prolongational interpretations of these trichords and how they are transformed by the step-class automorphisms.



Example 2: A classification of step-class sets



Example 3: Automorphic transformations of step-class trichords



Example 4: Automorphic transformations of step-class tetrachords

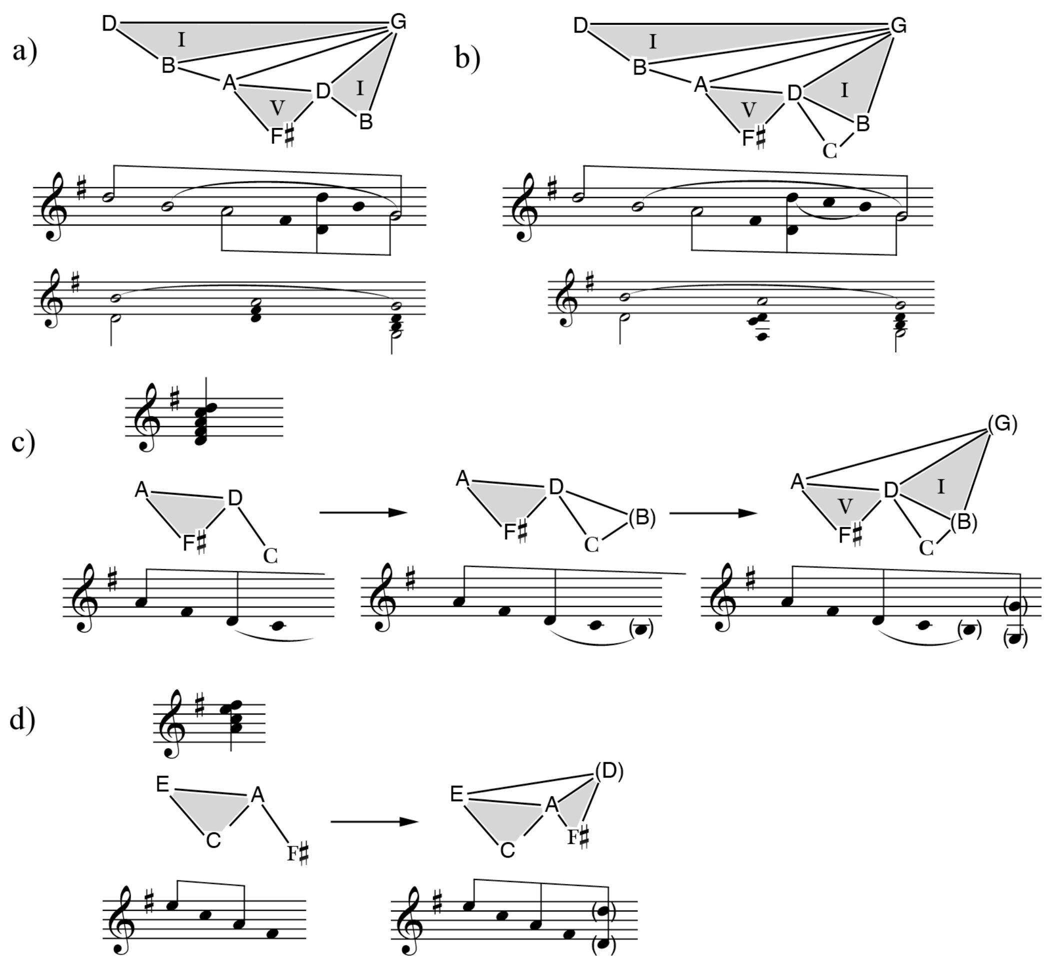
On the other hand, the step-class tetrachords that play a significant direct role in prolongational structures are not the more familiar inversionally-symmetrical sets but the *H*–symmetrical sets. These sets function as minimal *tonal regions* because they include exactly one instance of each type of inversionally-symmetrical trichord, and because a step-class tetrachord of this sort can prolong itself indefinitely due to its symmetry with respect to *H*. Example 4 shows prolongational instantiations of these tetrachords and their transformations under the step-class automorphisms. Note that *H* and H^{-1} don't change the tetrachord but reorder it to change the background interval and the types of trichords that make up the prolongational structure. The possibility of such prolongational reorientations allows these tetrachord types to prolong themselves indefinitely.

Example 5 shows a few possible prolongational relationships between tonal regions. Example 5a shows how a region can right-prolong itself through prolongational reorientation, and Examples 5b–d show possible relations between a region and its dominant, supertonic, and mediant regions. Note that there are two possible pitch-class manifestations of the step-class set for the basic tonal region: one major and one minor. This is because the fifths occurring in the set are always perfect fifths, and the remaining step-class not related to the others by fifth can be a minor or major third with respect to the set's triad. The shading of triads in the MOP facilitates reading of the structures and will be used as a convention for the remaining examples.

The method of step-class symmetries also generates an interesting approach to the prolongational interpretation of seventh chords (an inversionally-symmetrical type of step-class set). Example 6a shows the typical configuration of a dominant-tonic progression, and Example 6b shows how the addition of a seventh to the dominant may be prolongationally interpreted in terms of a passing motion.

The nature of the dominant seventh chord by itself implies this sort of interpretation, in fact. Consider such a chord presented in isolation, as in Example 6c.

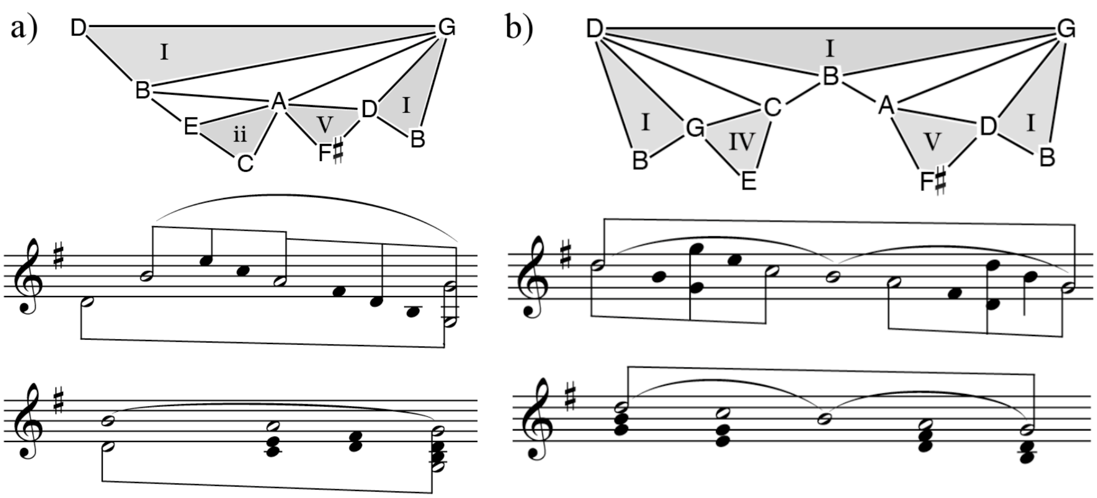
Example 5: Relationships between basic tonal regions



Example 6: Prolongational interpretations of dominants

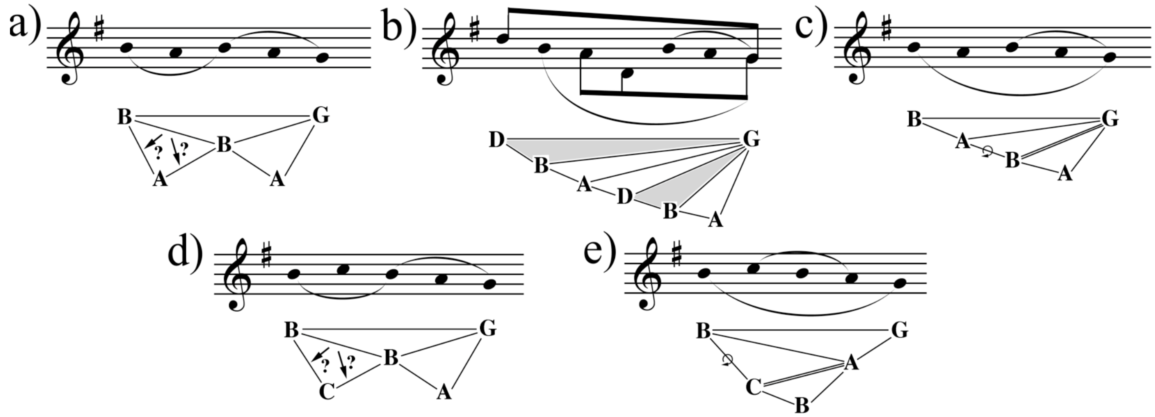
Though the chord contains two triads, only one is bounded by a perfect fifth, so this major triad stands out as a prolongational unit. Yet, there's no way for the remaining note, C, to make a complete prolongational structure with this triad, so it implies a continuation that will make a completion of the structure possible. A continuation of the stepwise motion D–C to B incorporates the note C into a prolongational unit, but the entire structure is incomplete and requires further continuation to the note G. In this sense, the dominant seventh chord by itself implies a continuation to its tonic. The (pitch-class) intervallic makeup of the chord is essential to this interpretation; Example 6d shows the simpler prolongational implications of a half-diminished seventh chord.

The theory of step-class symmetries also leads to new outlooks on harmonic elaborations of the basic dominant-tonic structure. As Example 7a shows, the supertonic triad is a natural expansion of the stepwise motion from the third scale degree in the basic structure. The subdominant triad, on the other hand, occurs more naturally in an expansion of stepwise descent from the fifth of the tonic mirroring the progression of dominant to tonic on the right, as shown in Example 7b.



Example 7: Prolongational interpretations of pre-dominant harmonies

There is one step-class interval that is not related to others through automorphism: the unison step-class interval. Therefore a theory in which harmonic intervals and melodic motions relate entirely through symmetries of the step-class group excludes the idea of repetition as a kind of motion or unison as a kind of interval. An analysis cannot express a melodic neighboring motion as a prolongation of a melodic repetition as in Example 8a, because the repetition doesn't relate to the step intervals through any automorphism. This idea can be replaced, however, with the more compelling notion of *embedded repetition of a prolongational structure*. When we observe a departure and return, we first put the departure into a larger context of *continuation*. For instance, the motion B–A, a departure from B, implies a continuation to G (which is provided in Example 8a). We then interpret the *return* as a delay of the continuation by repetition.

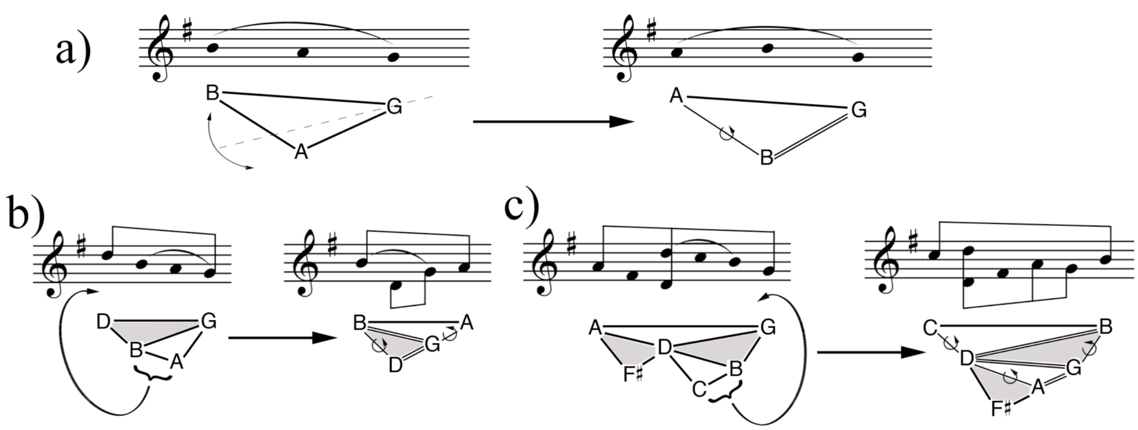


Example 8: Embedded repetitions and extensions of prolongational structures

In harmonic context, the best interpretation of the melodic motions shown in Example 8a would be in the form shown in Example 8b, a common type of background structure (as in a sonata first movement, for example). This analysis consists of the (right) embedded repetition of a tonal region, D–B–A–G, which can be accomplished entirely through symmetrical elaborations (i.e. H prolongations). Here, the motion B–A is part of a larger structure whose resolution to G is interrupted with a return to D–B.

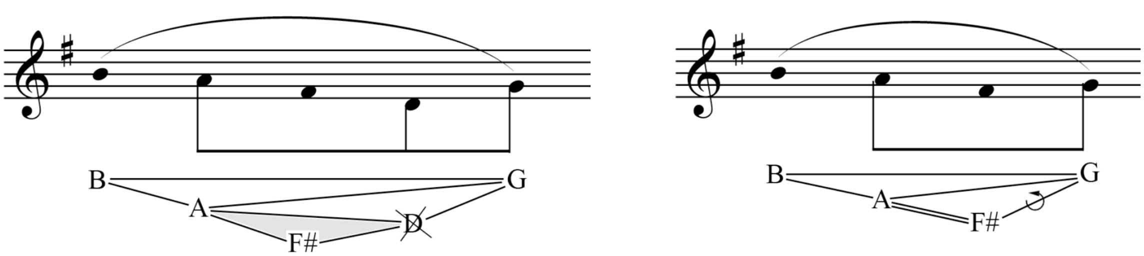
However, in certain cases the analyst may want to directly embed a repetition of a simple trichord structure, which one cannot do solely with symmetrical elaborations (H-prolongations). Example 8c accomplishes this with an asymmetrical prolongational unit where one motion is related to the prolonged motion by I (shown with a loop-arrow) and one is related to the prolonged motion by H^{-1} (shown with a double line). A different application of this sort of asymmetrical prolongation can replace the type of neighboring motion shown in Example 8d with a *prolongational extension* of a trichord (here a passing motion), as in Example 8e. In this case the motion B–C is not an initiation of a prolongational unit with an eventual continuation, but the reversal of a passing motion to A which then finds continuation.

We can understand prolongational extension better by considering the asymmetrical prolongation as an *involution* or *reflection* of the normal symmetrical type. Example 9a shows how an asymmetrical prolongation results from reflecting a normal passing figure along one of its more foreground motions (B–A), pulling the other foreground motion (A–G) in behind what would ordinarily be the background interval (B–G). The loop-arrow indicates the flipped interval while the double line indicates the exposed background interval. Examples 9b–c show how the idea of involution can be extended to larger prolongational structures, drawing any foreground interval into the background. The advantage of deriving asymmetrical prolongations in this way is that one can base an interpretation of the resulting structure on the symmetrical one that it involutes. For instance, the structure of Example 9c gives a prolongation of a descending minor second by making it the resolving seventh of a V^7–I progression.



Example 9: I/H^{-1} *prolongations as a reflections or involutions of* H-*prolongations*

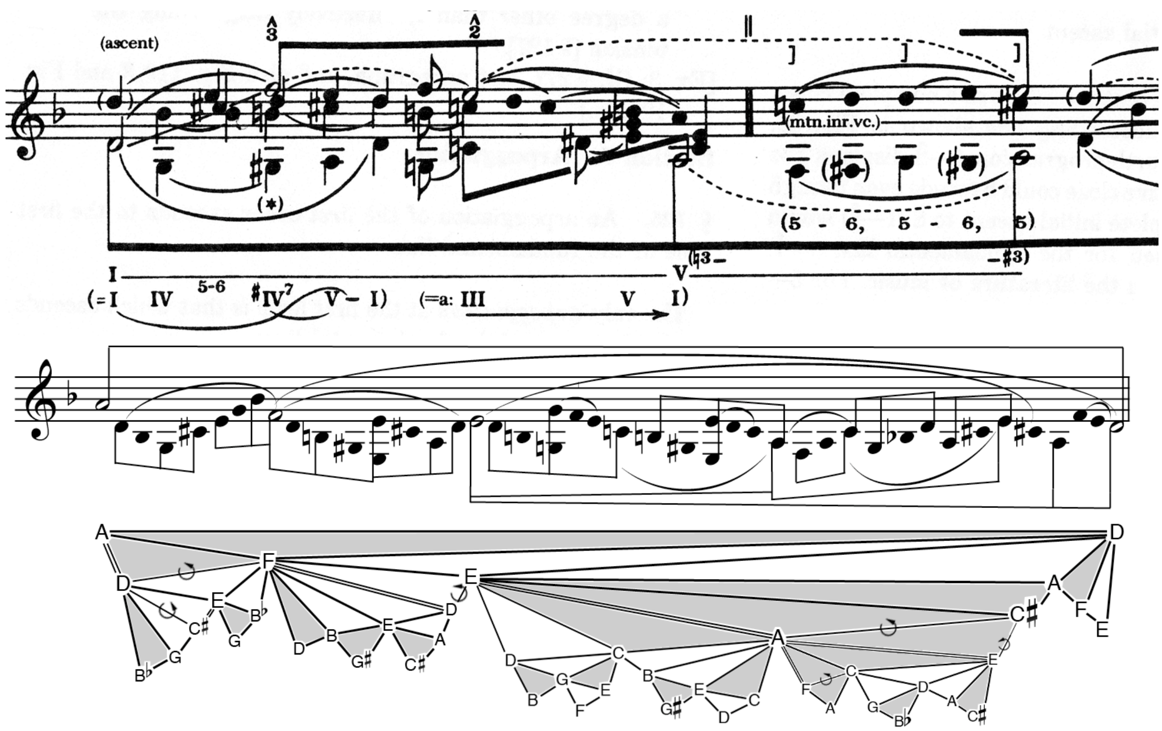
Another way to think of the I/H^{-1} prolongations is as an elision from a normal structure, which is illustrated in Example 10. Excising the root of the dominant, D, collapses two levels of structure into one (indicated by the double line from A to F# in the resulting structure), and turns the indirect leading-tone motion F#–G into an element of the prolongational hierarchy.



Example 10: The I/H^{-1} *prolongation as an elision*

Example 11 uses the tools developed so far to give a representation of Schenker's middleground analysis of the Largo of Beethoven's op. 10 no. 3 (excluding details of the recapitulation) from *Der Freie Satz*. The analysis begins (at the most background level) from the basic D minor tonal region, elaborated with the standard interruption structure. The resolving tonic of this background structure is pulled back to the left in two places. Bringing the tonic back as the prolongation of A–F provides the foundation for the initial ascent, and as the prolongation of F–E it establishes the initial D-minor tonality of the second theme.

The extended A minor region of the exposition occurs in an embedded A minor repetition of the background descent through the dominant (A major). The first part of this descent hosts the C major area of the exposition. (I depart from Schenker here in showing the dominant of C major following the arrival on E, m. 13, as it does in measures 14 and 16.) Schenker shows the startling F major area of the middle section initiating an ascending motion to regain the E of the fundamental line. The step-class symmetry analysis produces this ascending motion as an inversion of the preceding E–A motion. The chord F major is a triadic extension of the overriding A minor.



Example 11: Schenker's analysis of Beethoven's op. 10 no. 3, 2nd mvt., and a step-class automorphism analysis

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