Decontextualizing Contextual Inversion

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Abstract. Contextual inversion, introduced as an analytical tool by David Lewin, is a concept of wide reach and value in music theory and analysis, at the root of neo-Riemannian theory as well as serial theory, and useful for a range of analytical applications. A shortcoming of contextual inversion as it is currently understood, however, is, as implied by the name, that the transformation has to be defined anew for each application. This is potentially a virtue, requiring the analyst to invest the transformational system with meaning in order to construct it in the first place. However, there are certainly instances where new transformational systems are continually redefined for essentially the same purposes. This paper explores some of the most common theoretical bases for contextual inversion groups and considers possible definitions of inversion operators that can apply across set class types, effectively de-contextualizing contextual inversions.

Keywords: Pitch-class set theory · Contextual inversion · Neo-Riemannian theory · Transformational theory.

1 Standardizing Contextual Inversion

Contextual inversion was first defined by David Lewin and applied in various ways in many of his analyses [21, 22]. It has been an important analytical resource to many theorists in a variety of analytical contexts. Exemplary analysis using contextual inversions, as well as references to many other applications, can be found in articles by Lambert [17] and Straus [24] and Kochavi's dissertation [15].

Contextual inversion is most simply defined group-theoretically as an operation that maps pitch-class sets to their inversions and commutes with transpositions. Traditional inversion operations, defined as reflections of the pitch-class circle over some axis, do not commute with transpositions. This is often a desirable property; one normally considers, e.g., C major → F minor and D major → G minor to be the same type of progression, but as traditional inversions they are not, because the axis of inversion changes (C/F♯ in the first progression, D/G♯ in the second). The advantage of traditional inversion operations is that they are readily defined in the same way for all pitch-class sets.

There are two kinds of contextual inversion, those that apply to ordered and to unordered pitch-class sets. In the former category are serial operations like Lewin's RICH operation. These are contextual inversions because they commute with transposition, but since they are typically defined by drawing upon aspects of the ordering, standardization across pitch-class sets is unproblematic. For instance, one can define a contextual inversion that uses the first note of the series as a common tone. This is, for instance, how Stravinsky often derives an I-form of a twelve-tone series from a P-form. This inversion readily applies to any kind of series, regardless of how long it is or what pitch-class set it is. Contextual inversion on serially ordered trichords has already been treated in excellent work by Fiore, Noll, and Satyendra [8–10] and for twelve-tone music by Hook and Douthett [14]. The present work addressed the harder problem standardizing contextual inversion on unordered sets.

Another method of turning contextual inversions like P, L, and R into global operations is to redefine them using multiplication by *spectral units* as Amiot [3] does. Spectral units derived from a contextual inversion, however, do not necessarily consistently act as inversions, and in fact they can have infinite order (as do spectral units defined from the neo-Riemannian inversions). Nonetheless, Amiot's approach is related to the idea of directed inversions suggested below.

The difficulty of standardization might be understood as a virtue of contextual inversion. Many of Lewin's analyses [22] illustrate this well: a special inversion operator is defined for use only in the analysis at hand, requiring a mix of theoretical and analytical reasoning that itself serves as a crucial stage of the analytical process. The counterargument to this anti-standardization stance is that the same kinds of reasoning may frequently reappear, so that not only is it efficacious to establish a single standard, but it also advances the theoretical project by making the conceptual links across analyses apparent.

Common-tone content is frequently used as a basis for defining contextual inversions. The most prominent example of this is the paradigmatic contextual transformations, the Neo-Riemannian P, L, and R operations, which are defined as the inversions that preserve two common tones [6, 13, 21]. They have also been described, however, as minimal voice-leading transformations [6, 7, 27]. These two descriptions only happen to coincide for this particular set class but lead to distinct generalizations. I will consider each possibility below.

There are two principal criteria for a good standardization,

1. It applies to a large number of set classes
2. It is meaningful

Of these, (2) is the most important. The primary value of contextual inversions, after all, is that they are often more meaningful as operations than standard inversions, and therefore the theoretical meaning of the operation is foremost. Criterion (1) is perfectly satisfied by a standard that applies to all set classes, which is easily possible if one completely ignores (2). For example, we could define an inversion around the first pitch class of a set's normal order. Since the normal order, however, is an essentially arbitrary convention, such an operation is of little value.

Straus [24] takes on the task of satisfying (1) perfectly while preserving some of the common-tone meaning of the neo-Riemannian transformations, by generalizing the two-common-tone property of P, L, and R to all trichords. An important flaw of this strategy was identified, however, by Fiore and Noll [8], which is that there is not a consistent group structure on P, L, and R so defined. For instance, the neo-Riemannian PL is an order 4 operation, PR is order 3, and LR is order 12, while for (016), Straus's PL, PR, and LR are orders 12, 12, and 2 respectively. Therefore, despite the nomenclature, these are not really the same operations from one trichord-type to another. Furthermore, Straus has to lean on the normal-order convention to satisfy criterion (1), leading to a degree of arbitrariness in which operation is called by which name. For instance, the operation that maps (012) to itself is P, whereas for (027), it is L.

It turns out to be quite difficult to satisfy criterion (1) perfectly without compromising on (2). Rather than pursue that holy grail here, I will instead prioritize criterion (2) and explore multiple possible kinds of contextual inversion that partially generalize and relate to voice-leading and common-tone properties.

2 Voice-Leading Standards

One potentially useful property in contextual inversions is minimal voice leading. Minimal voice leading by itself, however, is certainly not sufficient to define a contextual inversion on all set classes, since multiple voice leadings may be equally small, an example being the 1-semitone voice leadings given by the P and L operations on major and minor triads. Also, "minimal voice leading" is vague, since there might be a number of relevant metrics. Tymoczko and Hall [12, 25] propose some limits on possible voice-leading metrics, but suggest that a number of metrics, in particular the L_p-norms, could be used. The choice between these will often change which voice leading would be considered minimal.

As an example consider the tetrachord {0147}. It has a unique maximal common-tone inversion: {1478}. However, considered as a voice leading, (C,D♭,E,G) → (A♭,D♭,E,G), this moves a single voice by the large distance of 4 semitones. On one commonly used voice leading metric, L_1, or the taxicab metric, there are a number of other inversions that have exactly the same size voice leading, such as (C,D♭,E,G) → (C,E♭,F♯,G), which moves two voices by two semitones, (C,tD♭,E,G) → (B,D,E♯,F♯), which moves two voices up by semitone and two down by semitone. In fact, ten of the twelve inversions have a voice leading with this same distance of 4 on the L_1 metric. On an L_2 (Euclidean) or L_∞ metric, the smallest voice leading is the one that preserves zero common tones (moving every voice by one semitone), and there are three such inversions for each (0147).

Regardless of the voice-leading metric chosen, the minimal voice leading will often be achievable in multiple ways. Therefore minimal voice-leading does not generalize well as a contextual inversion standard. It is well defined for relatively few set classes, and it is difficult to predict which set classes it applies to.

Another voice-leading property of potential interest is sum class. It has been effectively applied in analysis by Cohn [7] and play an important role in recent theory of voice leading proposed by Dmitri Tymoczko [28].

A sum class standard satisfies criterion (2) effectively. If two sets have the same sum class, this means that there is a balanced voice leading from one to the other (and the converse is also true). Sum classes specify cross-sections of voice-leading spaces oblique to the line of transposition, and therefore are basic to the theory of voice-leading geometry.

When cardinality shares a factor with twelve, however, transposition by that factor preserves sum class. Therefore, only when cardinality is co-prime to twelve is there a unique inversion with the same sum class.

For sets of cardinality five or seven, there is a transposition (T_5 or T_7) that changes the sum class by 1, meaning that all transpositions have a unique sum class. This means that there is always exactly one inversionally related set in the same sum class, so that a sum class standard for contextual inversion is defined in just these cases, but not for other cardinalities. Thus, it is easy to know which set classes this inversional standard applies to, but ones that tend to be of the most analytical interest (trichords and tetrachords) are not included.

A balanced-voice-leading inversion will map inversionally symmetrical collections to themselves. But it is not necessarily a minimal voice-leading standard in any sense in other instances. Consider, for example, the pentatonic scale {CEFGB}. The balanced voice-leading inversion of this is {C♯DF♯GB}, a voice leading of (1, −2, 1, 0, 0). However, the smallest voice leadings are not balanced: (C,E,F,G,B) → (C,E,F,A,B) moves one note up by 2 and (C,E,F,G,B) → (C,E,F♯,G,B) moves one note up by 1.

3 Common-Tone Standards

Following Cohn [6], theorists often think first of maximal common-tone preservation as a way of defining contextual inversions. The inversions would then define a kind of proximity as reflected in the Tonnetz, or Cohn's generalization of it. The difficulty is that the maximal common-tone preserving inversion is rarely unique. Inversionally symmetrical trichords have a single maximal common-tone preserving operation, but other trichords have at least three inversions that preserve two pitch-classes (four if one of its intervals is a tritone). For this reason, Straus [24] defines three contextual inversions for all trichords. But defining multiple contextual inversions poses an additional danger: for two operations to be understood as the same when applied to different sets, they must generate the same groups. This is sometimes possible but limits generalizability. For instance, we could define two trichord inversions, I and J, that preserve a dyad other than ic1 or ic5. This can then only apply, however, to trichords with exactly one ic1/ic5 interval, i.e. (013), (014), (025), and (037), which is rather limited. To define three such operations, we can at best generalize over two trichord types, which is hardly a generalization.

We should therefore take a step back and investigate the phenomenon of common-tone preservation more systematically. Lewin observed the importance of common tones and treated them as a special case of his interval function [18, 21], and also connected the interval function to the discrete Fourier transform (DFT) [19, 20]. Specifically, the interval function is a cross-correlation and the number of common tones is its zeroeth entry. By the convolution theorem, which states that convolution of sets is equivalent to multiplication of their DFTs, we can derive the following expression for the number of common tones between a set A, and some inversion IA:

$$\frac{1}{12}\sum_{k=0}^{11}|f_k(A)|^2\cos(\varphi_k(A)-\varphi_k(\mathrm{I}A)) \tag{1}$$

Here, $f_k(A)$ refers to the kth Fourier coefficient of A and $\varphi_k(A)$ refers to its phase. When A is clear from context we can write simply f_k and φ_k. For a given pair of pitch-class sets, we can use δ_k to indicate the phase difference.

Equation (1) implies that a large number of common tones results when the phase values are close together, particularly on the larger DFT components. We may use this fact to define contextual inversions that relate to the sharing of common tones by using *distances in phase spaces*. Following [29] I will use the convention $\mathrm{Ph}_k = \frac{u}{2\pi}\varphi_k$, where u refers to the universe (division of the octave), assumed to be 12 unless otherwise stated.

For example, Table 1 gives magnitudes of each component for major/minor triads, then cosines of phase differences for some inversions from C major.

Table 1. DFT for major and minor triads: phase differences and common tones from C major

		f_1	f_2	f_3	f_4	f_5	f_6	CTs
Mag.^{2}		0.27	1	5	3	3.7	1	
Cos(δ_k)	E min	0.5	0.5	0.8	0.5	0.5	−1	2
	C min	0.5	0.5	0.8	0.5	0.5	−1	2
	A min	−0.87	−0.5	0.6	0.5	0.87	1	2
	G min	−0.87	−0.5	−0.6	0.5	0.87	1	1
	F min	0	1	0.6	−1	0	1	1
	G♯ min	−1	−1	0.8	−1	−1	−1	0
	B min	0	1	−0.6	−1	0	1	0
	D min	1	−1	−0.8	−1	1	0.5	0

According to (1), the last column of the table can be calculated from the previous six. For instance, for C major – E minor:

$$\begin{aligned}\frac{1}{12}(9(1)+0.27(0.5)+1(0.5)+5(0.8)+3(0.5)+3.7(0.5)+1(-1)\\ +3.7(0.5)+3(0.5)+5(0.8)+1(0.5)+0.27(0.5))=2\end{aligned} \tag{2}$$

Notice that there are twelve terms to the calculation, the first representing the zeroth coefficient, which is simply the square of the cardinality, and the last five being the same as the ones corresponding to f_1-f_5. Therefore, the common tones between C major and E minor are primarily attributable to their proximity in Ph_3, and secondarily Ph_4 and Ph_5. The calculation for C major – A minor on the other hand is:

$$\frac{1}{12}(9(1) + 0.27(-0.87) + 1(-0.5) + 5(0.6) + 3(0.5) + 3.7(0.87) + 1(1) \\ + 3.7(0.87) + 3(0.5) + 5(0.6) + 1(-0.5) + 0.27(-0.87)) = 2 \tag{3}$$

The common tones here are attributable more to a close diatonic relationship, represented by the high $\cos(\delta_5)$, and less to the triadic similarity, $\cos(\delta_3)$, although generally f_3, f_4, and f_5 remain dominant. Note that the large difference in $\cos(\delta_1)$ is relatively immaterial given the small value of $|f_1|$ for triads. The C major – G minor relationship ("fifth-change") is similar in most respects to the C major – A minor (relative) relationship, but the reversal of the triadic proximity results in one fewer common tone.

Distances in phase spaces will typically not be completely generalizable because some set classes will either have undefined phases for certain Fourier coefficients, or will have equivalent distances with multiple inversions. For instance, we can see in Table 3 that the P relation of triads (e.g., C major – C minor) cannot be distinguished from the L relation (C major – E minor) at all using phase-space distances alone. Many kinds of inversion can nonetheless be defined that generalize to a large number of set classes. Consider, for example:

1. Minimum phase distance on a single component, such as Ph_1 or Ph_5
2. Minimum distance in a two-dimensional, or higher-dimensional, phase space, such as $\mathrm{Ph}_{2/3}$
3. Minimum distance in a two-dimensional, or higher-dimensional, phase space, with values weighted by their coefficient size for the given set class.

For 12-tET, criterion (1) only works on Ph_1 or Ph_5, because when coefficient number k divides 12 (the size of universe), multiple transpositionally related chords will have the same phase values. All of these types of inversion, when applied to inversionally symmetrical sets, will map the set class to itself.

Of the 2-common-tone inversions, C minor (P) and E minor (L) can only be distinguished by taking into account direction of phase change. The other 2-common-tone inversion, A minor (R), is a nearest neighbor in a $\mathrm{Ph}_{3/5}$ space, the tonal phase space defined by Amiot and Yust [1, 30]. Or, an inversion maximizing a weighted sum $\sum |f_k|^2 \cos(\varphi_k)$ for just the high-numbered components $f_3 - f_6$, would also specify the relative relation for triads, and be generalizable to other set classes. A different two-dimensional phase space, $\mathrm{Ph}_{2/3}$ would give C major – F minor as a minimum-distance inversion (if the dimensions are equally weighted). Just looking at a single phase value, Ph_1 or Ph_5, a 0-common-tone inversion, C major – D minor, would be the minimum-distance inversion. This inversion essentially splits the difference between the 2-common-tone inversions that are relatively close in both Ph_1 and Ph_5 (C minor and E minor), falling halfway between them on both the pitch-class circle and circle of fifths.

Defining an inversion that balances on either Ph_1 or Ph_5 has the advantage of being a relatively simple contextual inversion to understand across set classes, and can be defined for all set classes except only those that are *perfectly balanced*, or zero-valued on f_1 and f_5 [16]. In fact, complete systems of contextual inversions I_x^{ph1} and I_x^{ph5} can be defined where x indicates the change in phase (possibly non-integer valued) for the given inversion. The disadvantage is, as the previous example illustrates, the inversion that minimizes the phase change is not always actually high in common tones, because it is only one out of six distinct coefficients determining the total common tones in Equation (1).

One interesting fact relevant to one-dimensional phase proximity is the following:

Proposition 1. *If set A has $f_1 \neq 0$ and an inversion $\mathrm{I}A$ with the same Ph_1 value (hence an integer value) then $\mathrm{I}A$ also has the same Ph_5 value as A, and the converse is also true.*

Proof. Assuming A and $\mathrm{I}A$ have the same Ph_1, then the pitch-class multiset sum of A and $\mathrm{T}_6(\mathrm{I}A)$ has a zero-valued f_1, because their f_1 values will be equal and opposite. As Amiot [2] shows, $f_1 = 0$ implies $f_5 = 0$ (and vice versa). Therefore A and $\mathrm{T}_6(\mathrm{I}A)$ must also have opposite Ph_5 (since they have equal $|f_5|$ and sum to $f_5 = 0$), so $\mathrm{Ph}_5(A) = \mathrm{Ph}_5(\mathrm{I}A)$. The same argument works for the converse.

This proposition can be generalized to any f_j and f_k in any universe (u), by replacing T_6 with $\mathrm{T}_{u/2}$, or, if u is odd, transferring into universe $2u$ by oversampling and using T_u. In combination with the common-tone formula in (1), this implies that if u is prime, then only inversionally symmetrical sets have integer-valued Ph_k for any k (because integer-valued Ph_k would mean that there is some inversion $\mathrm{I}A$ with *all* phase values equal to those of A, which means that the number of common tones is equal to the cardinality of A, and $\mathrm{I}A$ and A are therefore the same set).

As a case study, let us consider defining contextual inversions for major and minor triads that also can also be applied to dominant and half-diminished sevenths. Table 2 lists the phase differences for all of the inversions that retain at least one common tone. It is hard to define a PLR-type system of transformations in a principled way for seventh chords because the 2-common-tone case is so common (including half of all the possible inversions) and the 3-common-tone case only occurs one way. Childs [4], for example, generates the contextual inversion group using all seven 2–3 common-tone inversions, which, from a group-theoretic perspective, is rather extravagant for a 24-element group that requires only two generators. All seven 2–3 common-tone inversions are also minimal voice leadings of two semitones (with the possible exception of the 3-common-tone inversion, which would be larger on many metrics).

Consider then, the following possible types of inversion:

1. J_0 inverts to preserve Ph_1 and Ph_5.
   J_0\{CEG\} = \{DFA\} and J_0\{CEGB♭\} = \{ACE♭G\}.
2. J_f inverts to maximize $\cos(\delta_2)+\cos(\delta_3)$.
   J_f\{CEG\} = \{FA♭C\} and J_f\{CEGB♭\} = \{C♯EGB\}.

Table 2. DFT for dominant sevenths and half-diminished sevenths: phase differences and common tones from C^7

		f_1	f_2	f_3	f_4	f_5	f_6		
Mag.^{2}		0.27	1	2	7	3.7	4		CTs
Cos(δ)	$E^{\varnothing 7}$	−0.87	−0.5	0	0.79	0.87	1		3
	$A^{\varnothing 7}$	1	−1	1	0.14	1	−1		2
	$G^{\varnothing 7}$	0.5	0.5	−1	0.79	0.5	-1		2
	$C^{\varnothing 7}$	0	1	0	0.14	0	1		2
	$F\sharp^{\varnothing 7}$	0	1	0	0.14	0	1		2
	$C\sharp^{\varnothing 7}$	−0.5	0.5	1	0.79	−0.5	−1		2
	$B\flat^{\varnothing 7}$	0.87	−0.5	0	0.79	−0.87	1		2
	$D^{\varnothing 7}$	−0.87	−0.5	0	−0.93	0.87	1		1

3. J_t inverts to maximize $\cos(\delta_3)+\cos(\delta_5)$.
   $J_t\{\mathrm{CEG}\} = \{\mathrm{ACE}\}$ and $J_t\{\mathrm{CEGB}\flat\} = \{\mathrm{ACE}\flat\mathrm{G}\}$.
4. J_h inverts to maximize $\sum_{k=3}^{6} |f_k|^2 \cos(\delta_k)$.
   $J_h\{\mathrm{CEG}\} = \{\mathrm{ACE}\}$ and $J_f\{\mathrm{CEGB}\flat\} = \{\mathrm{EGB}\flat\mathrm{D}\}$.

Two of these inversions, J_t and J_h, operate the same way on triads but differently on dominant seventh chords, while J_0 and J_t are equivalent on dominant sevenths but different on triads. These different kinds of inversions therefore give rise to distinct group actions when combined with contextual transposition to generate a 24-element contextual inversion group and applied across multiple set classes. However, they are all defined as involutions so that, when combined with contextual transpositions, they generate a group isomorphic to D_{12}, like standard inversions and the contextual inversion groups.

A different approach, which generates an inversion group of a distinct isomorphism class ($\mathbf{Z}_{24}$), is to define *directed* inversions, which go a particular direction in the phase spaces. For instance, let us use $\mathrm{Ph}_{3/5}$, but instead of simply looking for the *nearest* chord, let J_{t+} be the chord that is the nearest in the ascending direction in Ph_3 and Ph_5. Figure 1 shows the triads in this space, and a line that corresponds to semitone transposition. Such lines representing some T_x in some phase space are known as intervallic axes [30]. In fact, this particular axis reflects an important sequential procedure for Schubert and other composers [7, 30]. Because the triads fall close to the same line, J_{t+} is well-defined by proximity to it. This is not true of dominant and half-diminished sevenths, however, which coincide in the same point (via the inversion defined above as J_t), so J_{t+} is not well-defined for these.

The proximity of the chords to this line is related to a proposition proved in [30] and stated in a different form in [2] as proposition 6.8. Given any two inversionally symmetric sets, A, B, with well-defined phases in some phase space, we can draw a line connecting A to $\mathrm{T}_x(A)$ such that some transposition of B will fall on the midpoint of that line. Specifically, it will be a transposition of B stabilized by the inversion that maps A to $\mathrm{T}_x(A)$. Extending the line in either direction, then, all transpositions of A and B will fall regularly on such a line, ordered by T_x. The individual segments of such a line can be understood

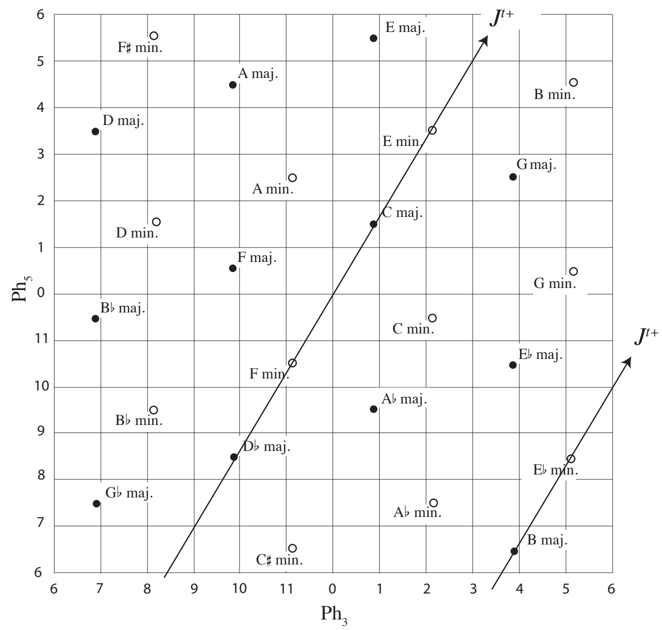

Fig. 1. J^{t+} in $\mathrm{Ph}_{3/5}$-space.

as inversions of one set over the center of symmetry of the other. The line in Figure 1 connects minor thirds and individual pitch classes with $x = 1$, such that adjacent minor thirds and singletons always combine to give a major or minor triad. One could define similar lines using the other ways of partitioning the triad into symmetrical subsets. A partition into a perfect fifth plus a singleton gives an operation that alternates parallel and slide relationships: C minor → C major → C♯ minor → C♯ major → . . . . A partition into major third plus singleton gives an operation that alternates relative and fifth-change relationships: C major → A minor → D major → B minor → . . . .[1]

Another form of directed inversion is Amiot's [3] multiplication by a spectral unit, which can be defined for any contextual inversion, and also generalizes to any relation between homometric (AKA Z-related) sets. The spectral units defined by neo-Riemannian operations happen to be of infinite order and when iterated beyond the initial two triads produce a series of pitch-class distributions that do not correspond to actual pitch class sets. However, these distributions could be correlated with triads or other pitch-class sets – seen this way, for in-

[1] The proximity of the triads to these lines can be calculated from the magnitude of the subsets on each Fourier component used to define the phase space, with perfect coincidence where the magnitudes are equal on each. Since all the subsets of major/minor triads are reasonably uniform in their $|F_3|$ and $|F_5|$, the triads fall quite close to all of these lines in $\mathrm{Ph}_{3,5}$-space.

stance, the spectral unit defined by the parallel operation approximates a P-L sequence ("hexatonic cycle": see [7]). Amiot, in [2] chapter 3, proposes study of the spectral units of finite order, and provides a useful mathematical classification of these, as well as a classification of all finite spectral units for $\mathbf{Z}_{12}$.

Directed inversions have the added advantage that they may be defined so that they compose consistently to the same transposition, so that multiple directed inversions may be combined in a single group. For instance, we might define an inversion J_{-5th} and J_{-semit} in $\mathrm{Ph}_{3/4}$ space, such that J_{-5th} projects the chords onto a line of descending fifths in the space, and J_{-semit} projects them onto a line of descending semitones, as shown in Figure 2. Since these compose to consistent (regular) transpositions, they can also be combined in a single group. These operations act as follows:

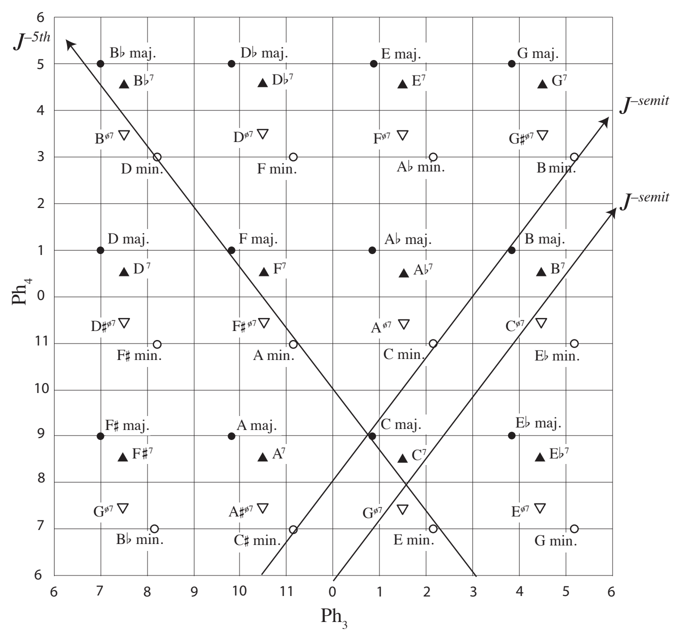

Fig. 2. J^{-5th} and J^{-semit} in $\mathrm{Ph}_{3/4}$-space.

$$\text{C major} \xrightarrow{J_{-5th}} \text{A minor} \xrightarrow{J_{-5th}} \text{F major} \xrightarrow{J_{-5th}} \text{D minor} \ldots$$

$$\text{C major} \xrightarrow{J_{-semit}} \text{C minor} \xrightarrow{J_{-semit}} \text{B major} \xrightarrow{J_{-semit}} \text{B minor} \ldots$$

$$\mathrm{C}^7 \xrightarrow{J_{-5th}} \mathrm{F}\sharp^{\oslash 7} \xrightarrow{J_{-5th}} \mathrm{F}^7 \xrightarrow{J_{-5th}} \mathrm{B}^{\oslash 7} \ldots$$

$$\mathrm{C}^7 \xrightarrow{J_{-semit}} \mathrm{C}^{\oslash 7} \xrightarrow{J_{-semit}} \mathrm{B}^7 \xrightarrow{J_{-semit}} \mathrm{B}^{\oslash 7} \ldots$$

4 Conclusion

This paper has taken an expansive approach to the issue of generalizing contextual inversions on unordered sets, in recognition of the fact that contextual inversion groups can only be defined to apply across all set classes by abandoning the premise that the operations have a meaningful identity that is preserved regardless of what set-type it acts on. Instead, by prioritizing the premise of defining inversions through some meaningful music-theoretic construct, we have proposed a number of possibilities, though none that is well-defined across all set classes. Of the two kinds of properties most often used to explain contextual inversions, voice leading and common tones, the latter, through its mathematical relationship to Fourier phase, is the most promising for defining inversions that can be applied widely, if not to all set classes. That a single obvious standard does not emerge from this investigation may in fact be a virtue: it preserves the aspect of contextual inversions that Lewin turned from a seeming flaw into an asset, the fact that they must be chosen carefully to serve a specific analytical purpose. This forces the analyst or theorist to invest the operation with meaning, rather than rely on conventions. The present work opens the possibility that this aspect of contextual inversions may coexist with the possibility of generalizing these meanings across set types.

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