A Three-Dimensional Model of Tonality

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Outline of the Talk

(1) Characteristic functions of pcsets and the DFT

(2) Diatonicity, Triadicity, and Dyadicity in tonal music

(3) The tonal plane: $\mathrm{Ph}_2 + \mathrm{Ph}_3 - \mathrm{Ph}_5 = 0$

(4) Examples: Corelli, Mozart, Chopin, Stravinsky

(1) Characteristic Functions of pcsets and the DFT

Discrete Fourier Transform on Pcsets

Lewin, David (1959). "Re: Intervallic Relations between Two Collections of Notes," *JMT* 3/2.

——— (2001). "Special Cases of the Interval Function between Pitch Class Sets X and Y." *JMT* 45/1.

Quinn, Ian (2006–2007). "General Equal-Tempered Harmony," *Perspectives of New Music* 44/2–45/1.

Amiot, Emmanuel (2013). "The Torii of Phases." *Proceedings of the International Conference for Mathematics and Computation in Music, Montreal, 2013* (Springer).

Yust, Jason (2015). "Schubert's Harmonic Language and Fourier Phase Spaces." *JMT* 59/1.

Characteristic Functions

The *characteristic function* of a pcset is a **12-place vector** with 1s for each pc and 0s elsewhere:

(1,	0,	0,	0,	1,	0,	0,	1,	0,	0,	0,	0)
C	**C♯**	**D**	**E♭**	**E**	**F**	**F♯**	**G**	**G♯**	**A**	**B♭**	**B**

Characteristic Functions

By allowing other integer values, the characteristic function can also describe **pc-multisets**

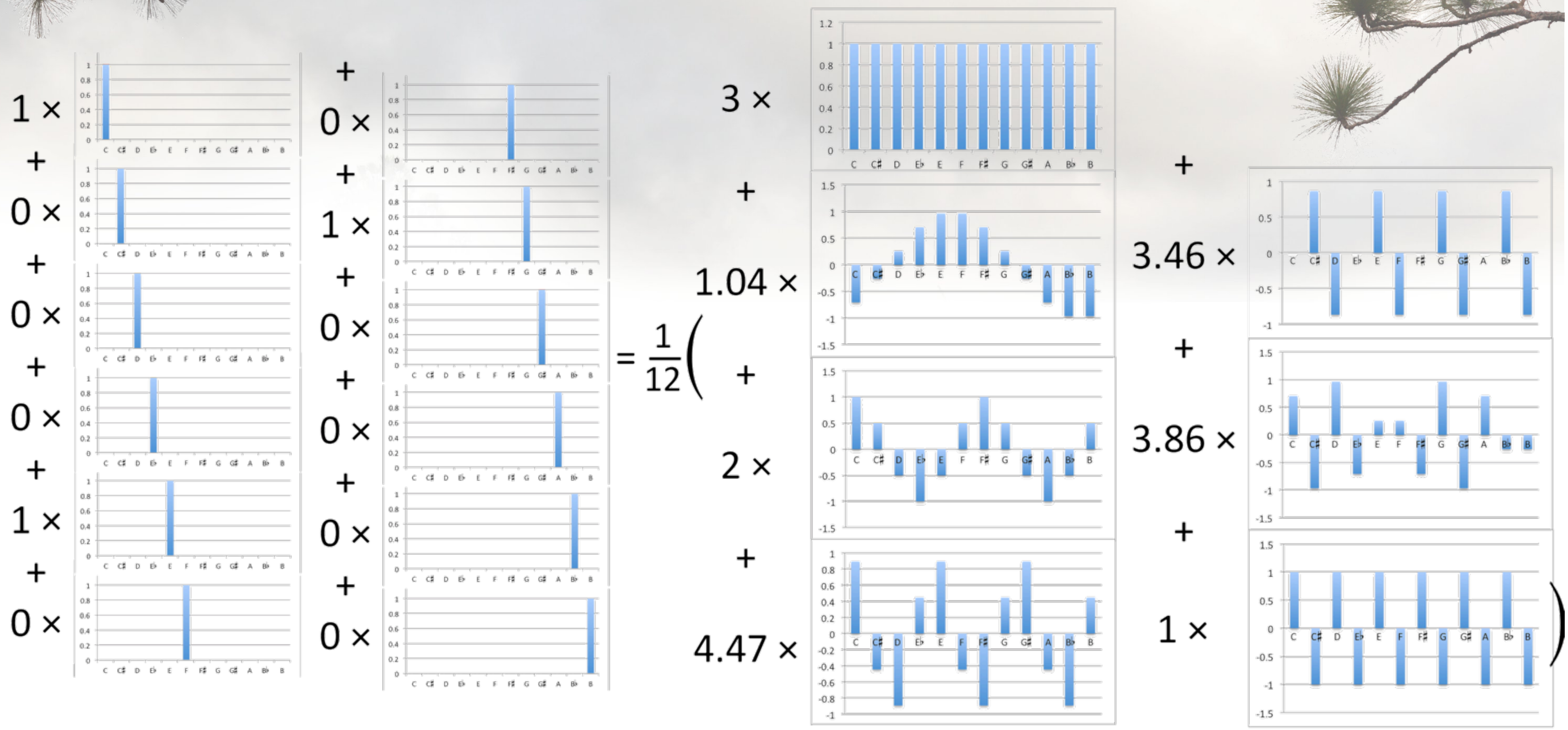
(2,	0,	0,	0,	1,	0,	0,	1,	0,	0,	0,	0)
C	**C♯**	**D**	**E♭**	**E**	**F**	**F♯**	**G**	**G♯**	**A**	**B♭**	**B**

Characteristic Functions

And using *non-integer* values, the pc-vector can describe **pc-distributions**

(2,	0,	0.5,	0.25,	0,	1,	0,	1,	0,	0.25,	0.5,	0)
C	**C♯**	**D**	**E♭**	**E**	**F**	**F♯**	**G**	**G♯**	**A**	**B♭**	**B**

DFT Components

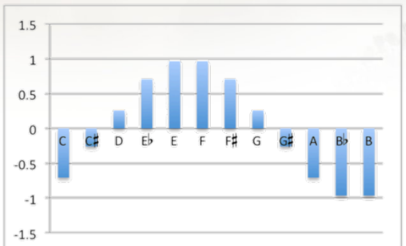
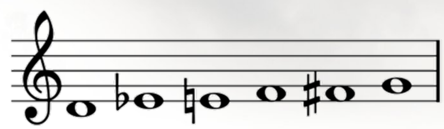


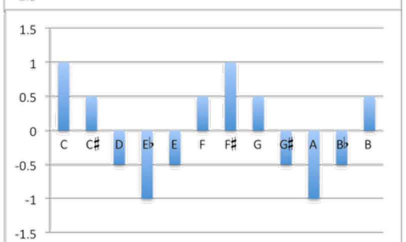
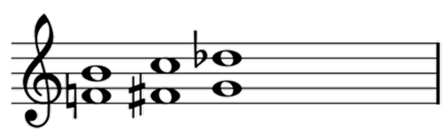
The DFT is a **change of basis** from a sum of pc spikes to a sum of discretized **periodic** (perfectly even) curves.

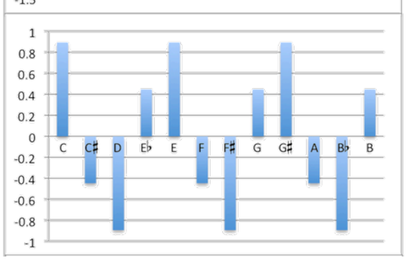
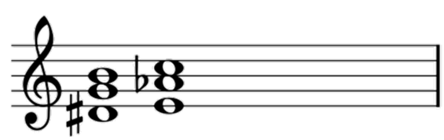
The **magnitudes** of DFT components contain precisely the intervallic information of the set. They are equivalent under transposition, inversion, *and Z-relations* (homometry).

DFT Components

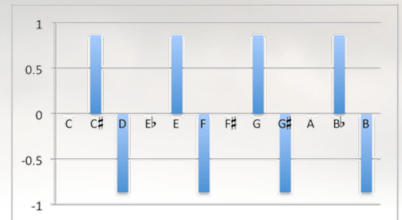
Component | **Prototypes**

1


Chromatic cluster

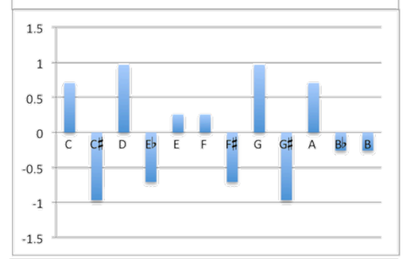
2


(012678) (Messiaen mode 5)

3


Hexatonic

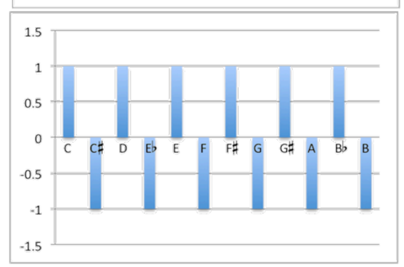
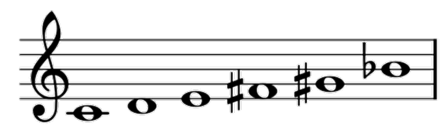
Component | **Prototypes**

4


Dim7th or Octatonic

5


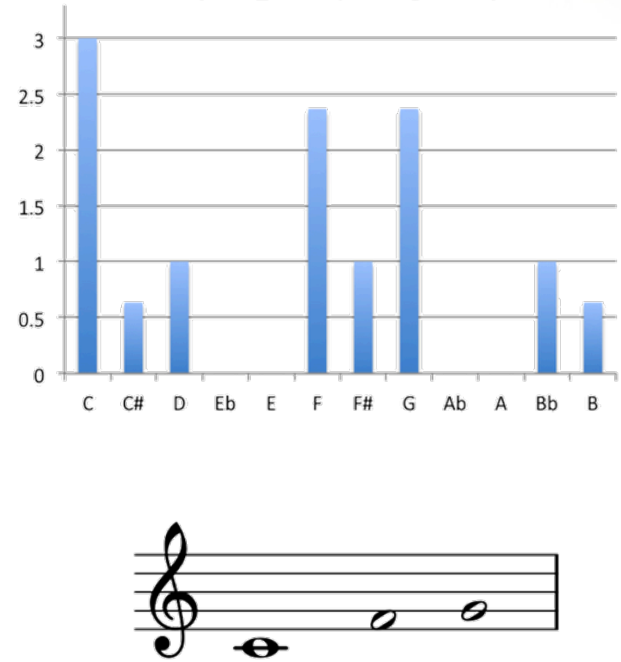
Circle-of-fifths cluster

6


Whole-tone scale

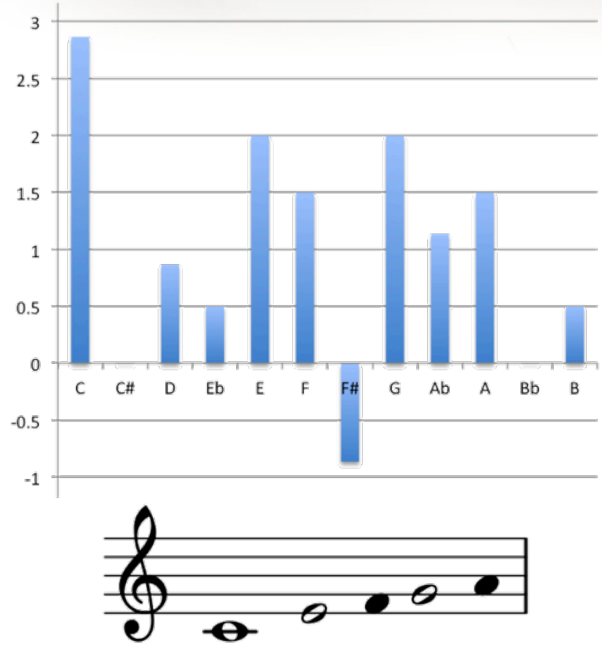
Quinn's *generic prototypes* are pcsets that maximize a given component by **approximating a sinusoidal function** of the given periodicity.

DFT Components

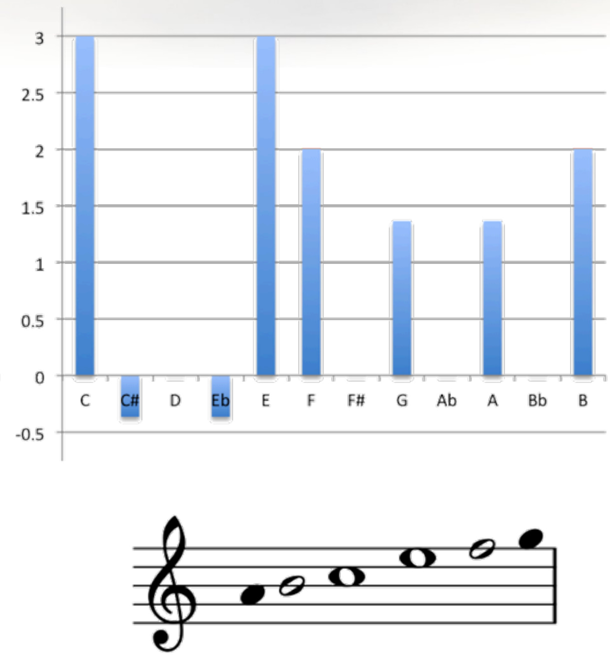
$f_2 + f_5$
$(Ph_2 = 0, Ph_5 = 0)$



$f_3 + f_5$
$(Ph_3 = 0, Ph_5 = 1)$



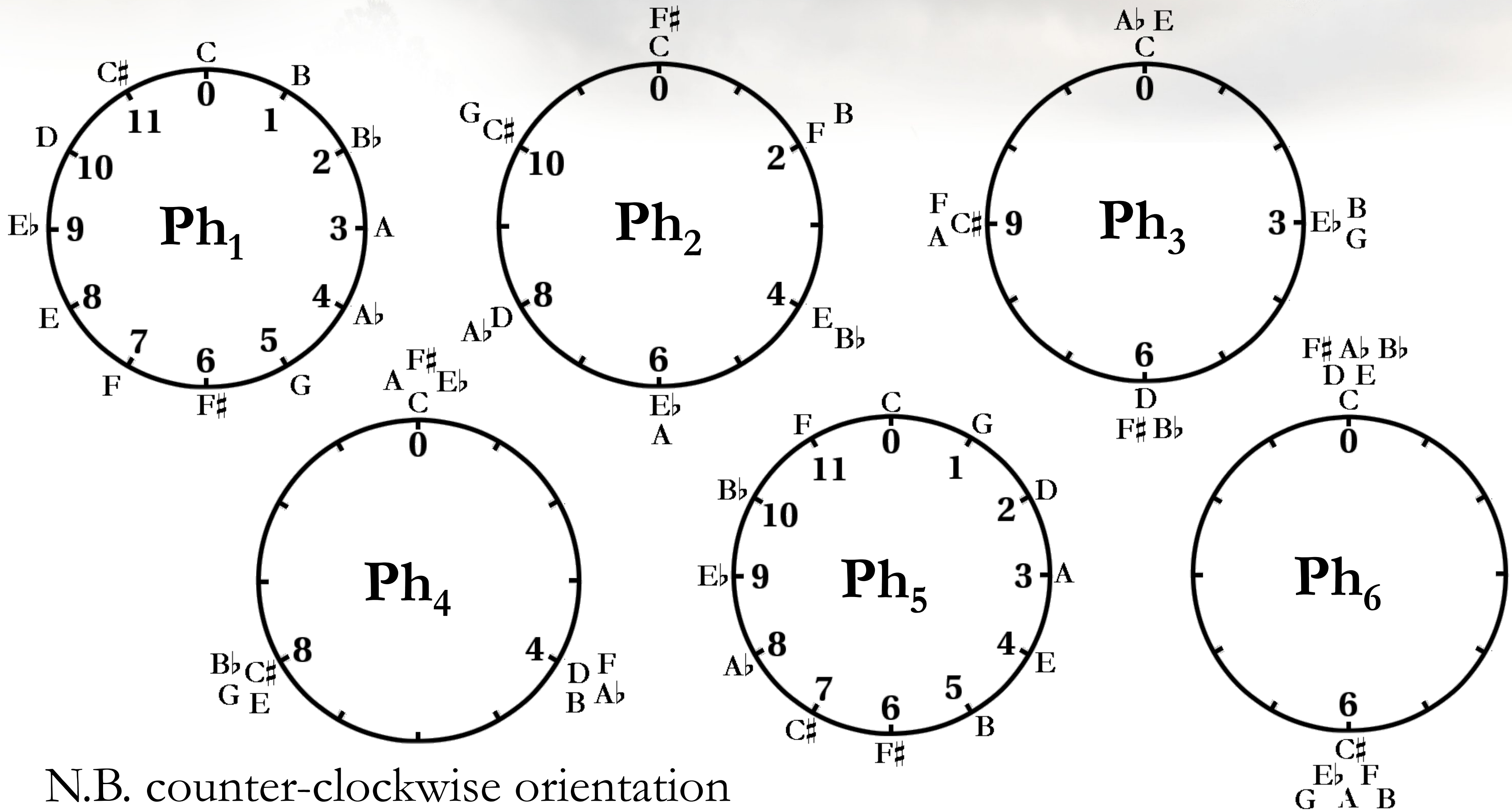
$f_2 + f_3 + f_5$
$(Ph_2 = 2, Ph_3 = 0, Ph_5 = 2)$



We can also define prototypes for mixtures of components
by approximating sums of sinusoidal functions

One dimensional phase spaces

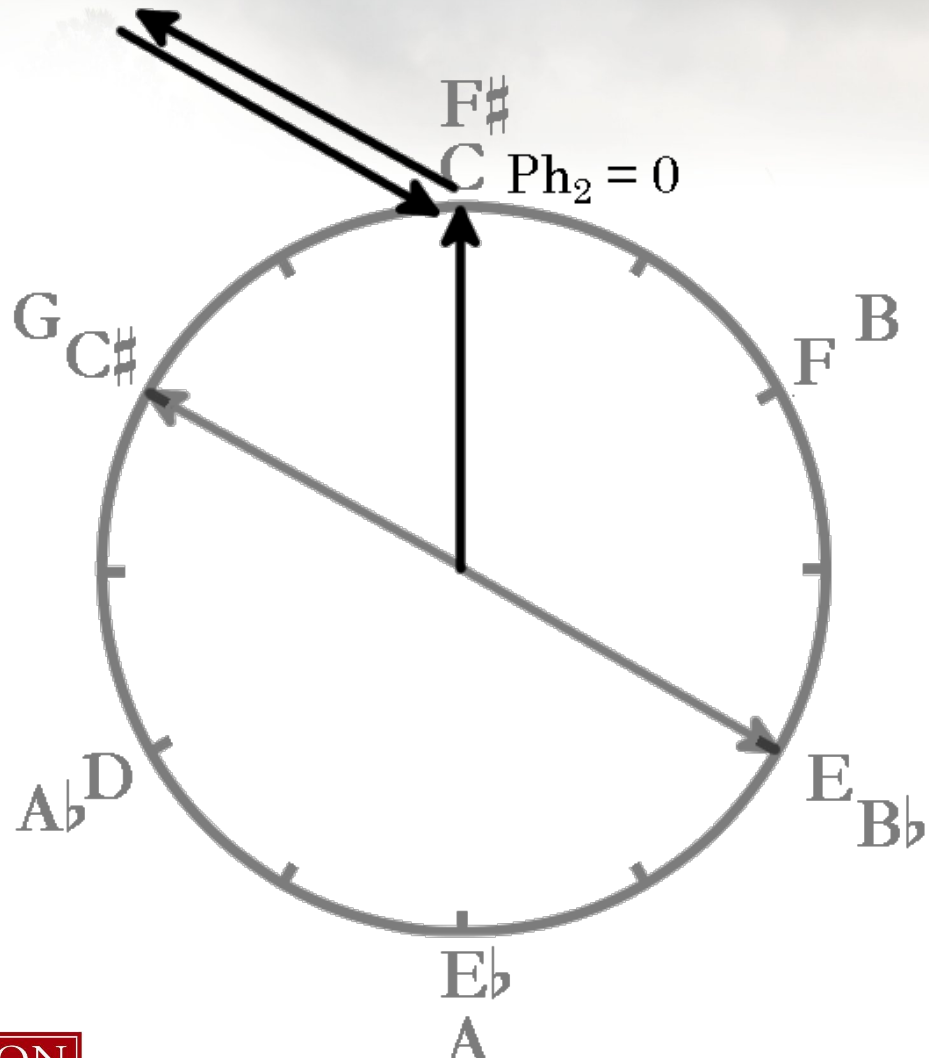
One-dimensional phase spaces are Quinn's *Fourier balances*, superimposed *n*-cycles created by multiplying the pc-circle by *n*.



N.B. counter-clockwise orientation

One dimensional phase spaces

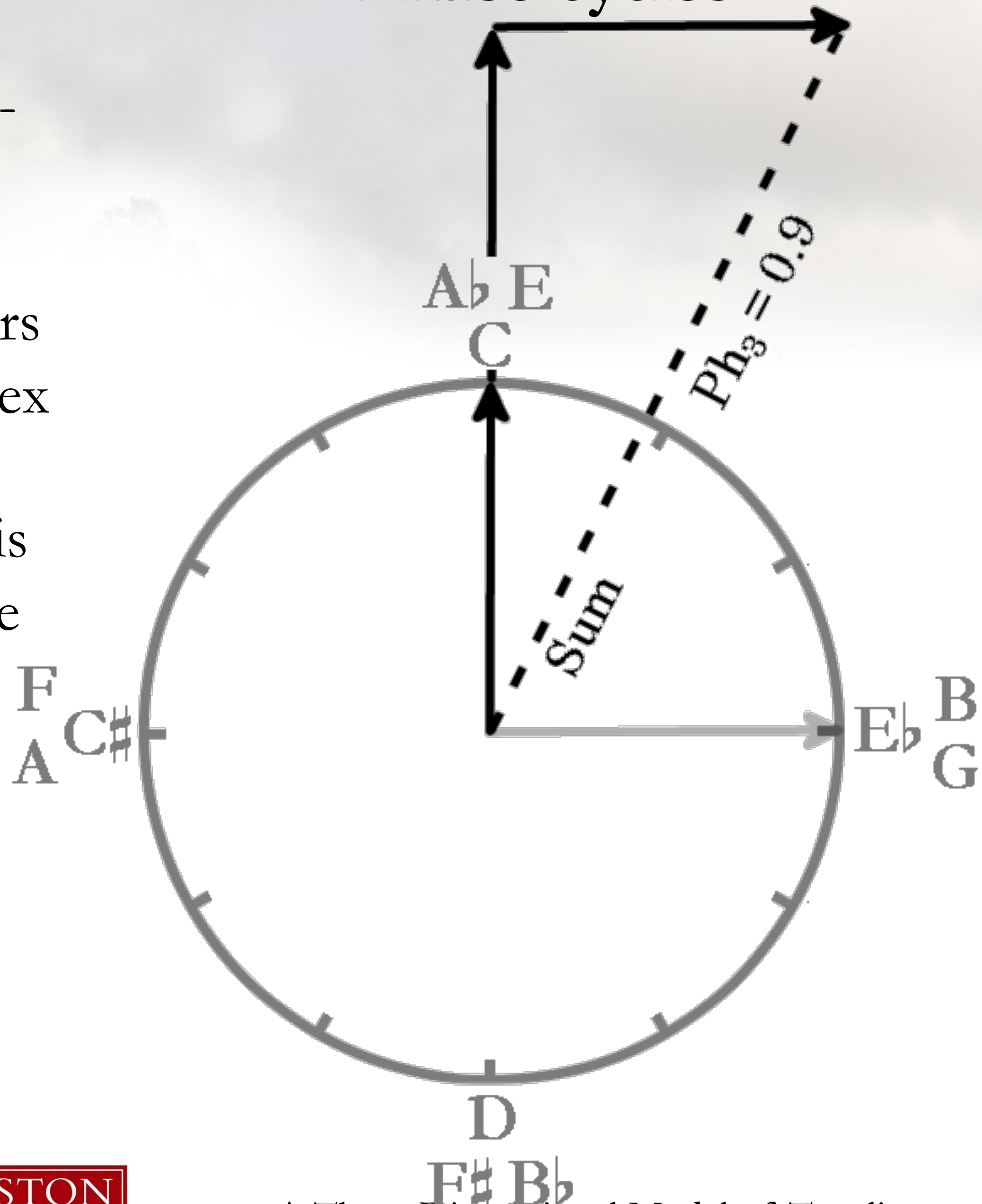
DFT components can be derived by adding vectors in the complex plane. The phase space is the unit circle in this plane.



Deriving phases for a C major triad by circular averages

Phase cycles

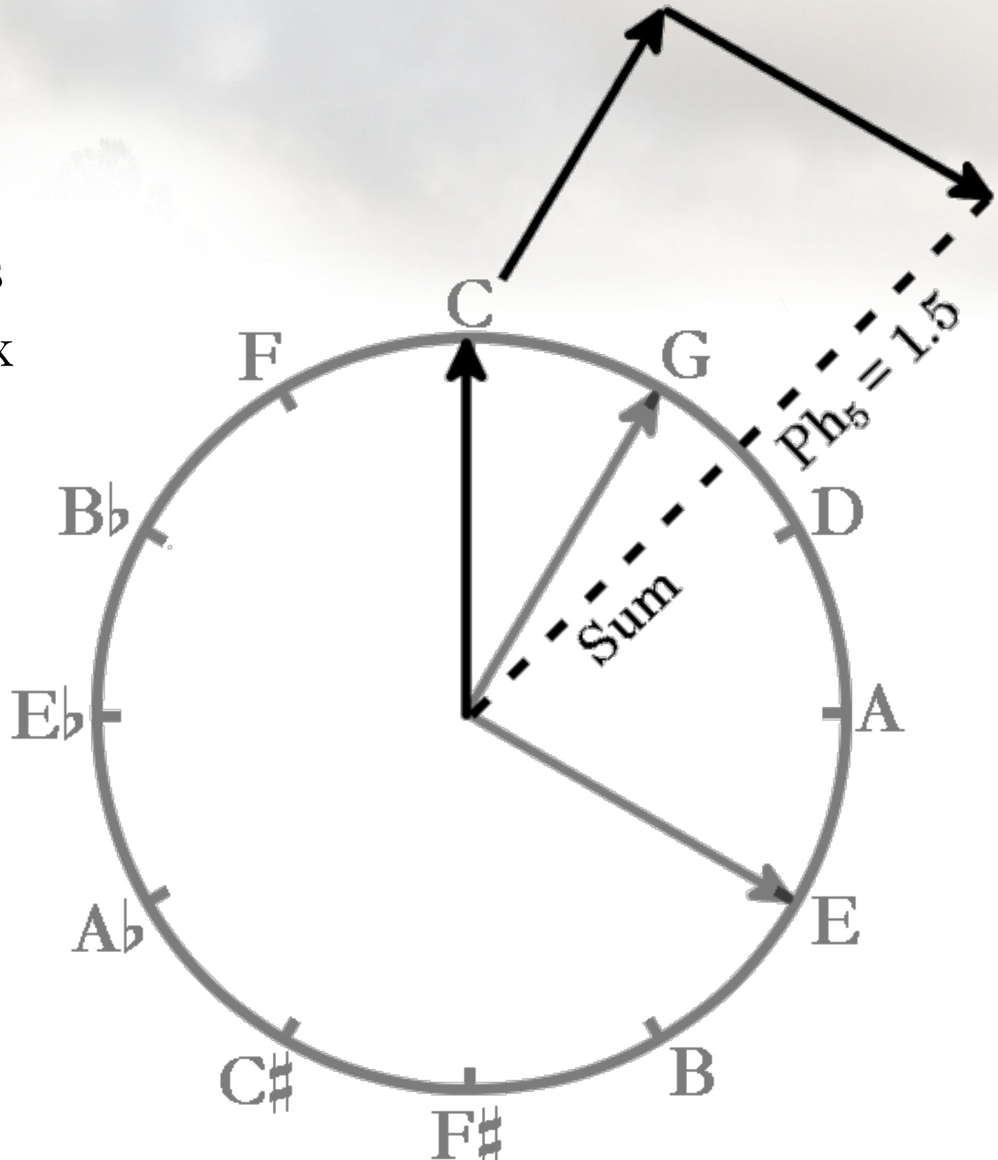
DFT components can be derived by adding vectors in the complex plane. The phase space is the unit circle in this plane.



Deriving phases for a C major triad by circular averages

Phase cycles

DFT components can be derived by adding vectors in the complex plane. The phase space is the unit circle in this plane.

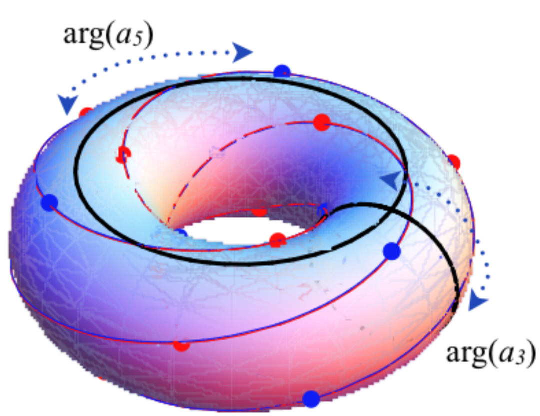


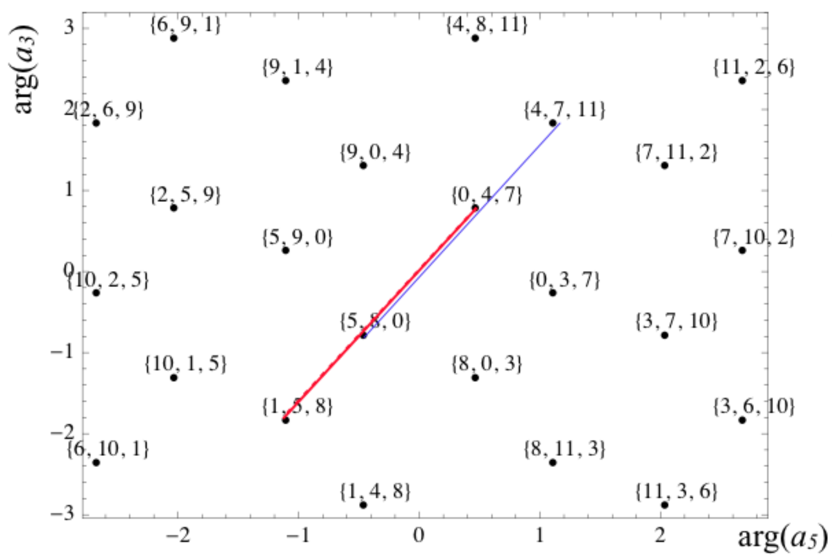
Deriving phases for a C major triad by circular averages

Two-dimensional phase spaces

A two-dimensional phase space tracks the phases of two components, and is topologically a *torus*.

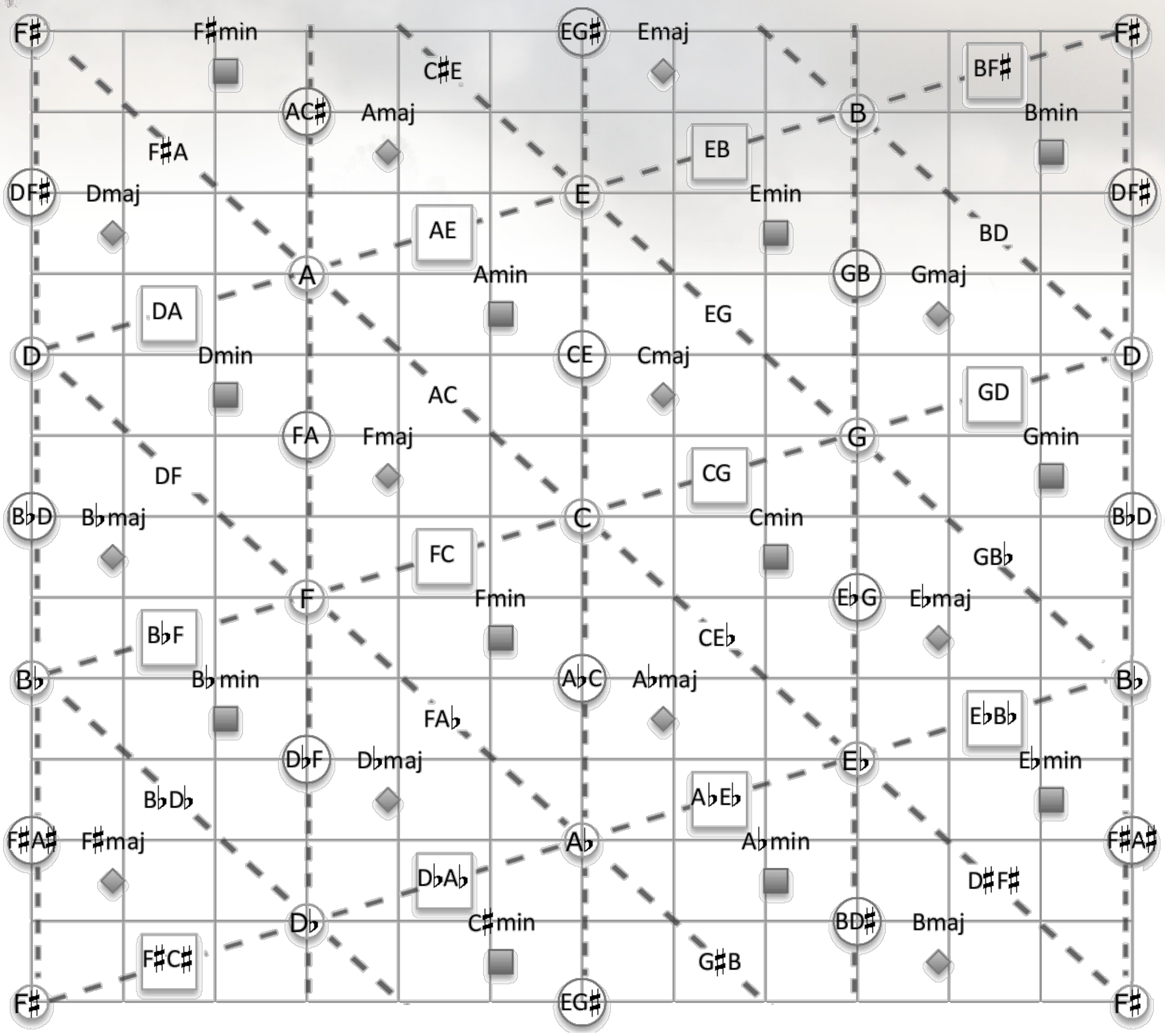
Amiot (2013) and Yust (2015) use $\mathrm{Ph}_{3\text{–}5}$-space to describe tonal harmony.





from Amiot, MCM 2013 proceedings

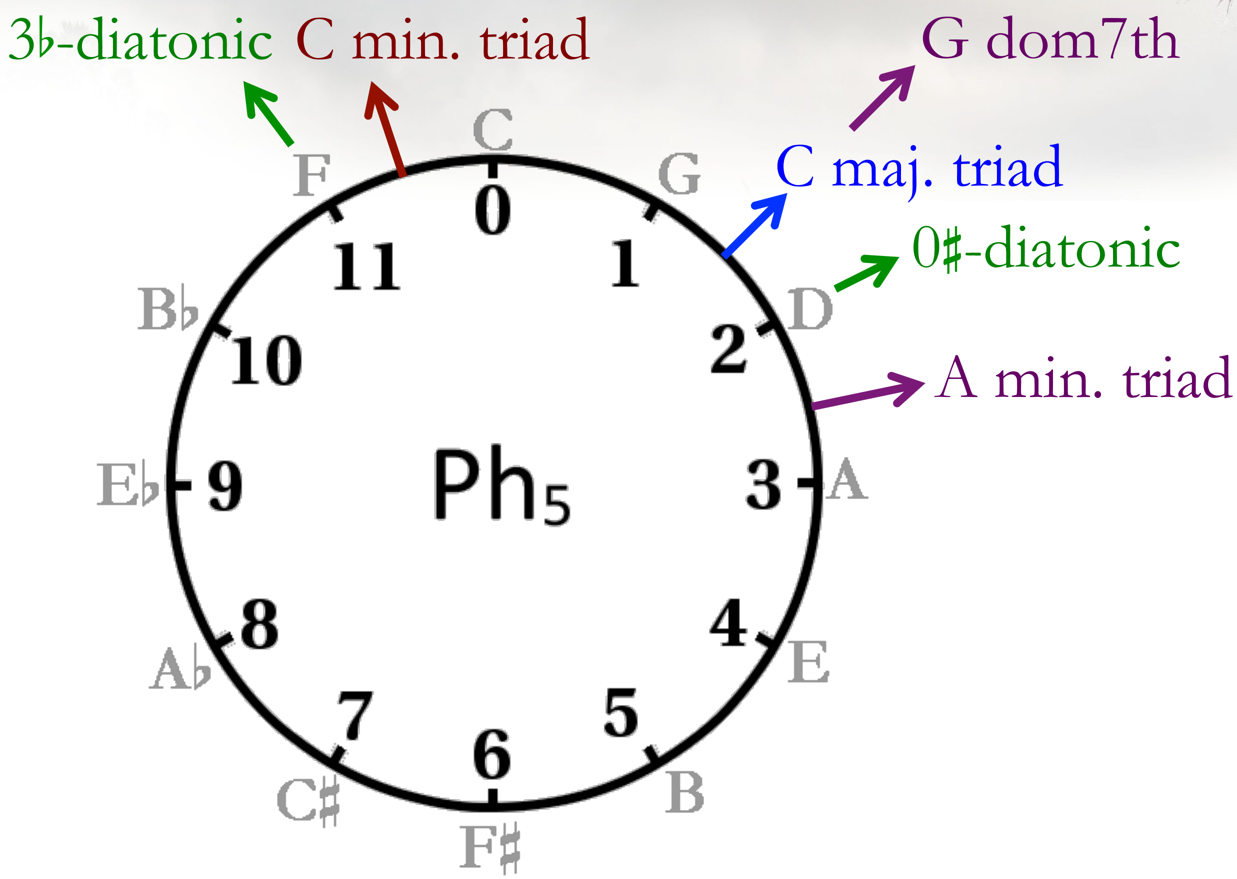
Two-dimensional phase spaces



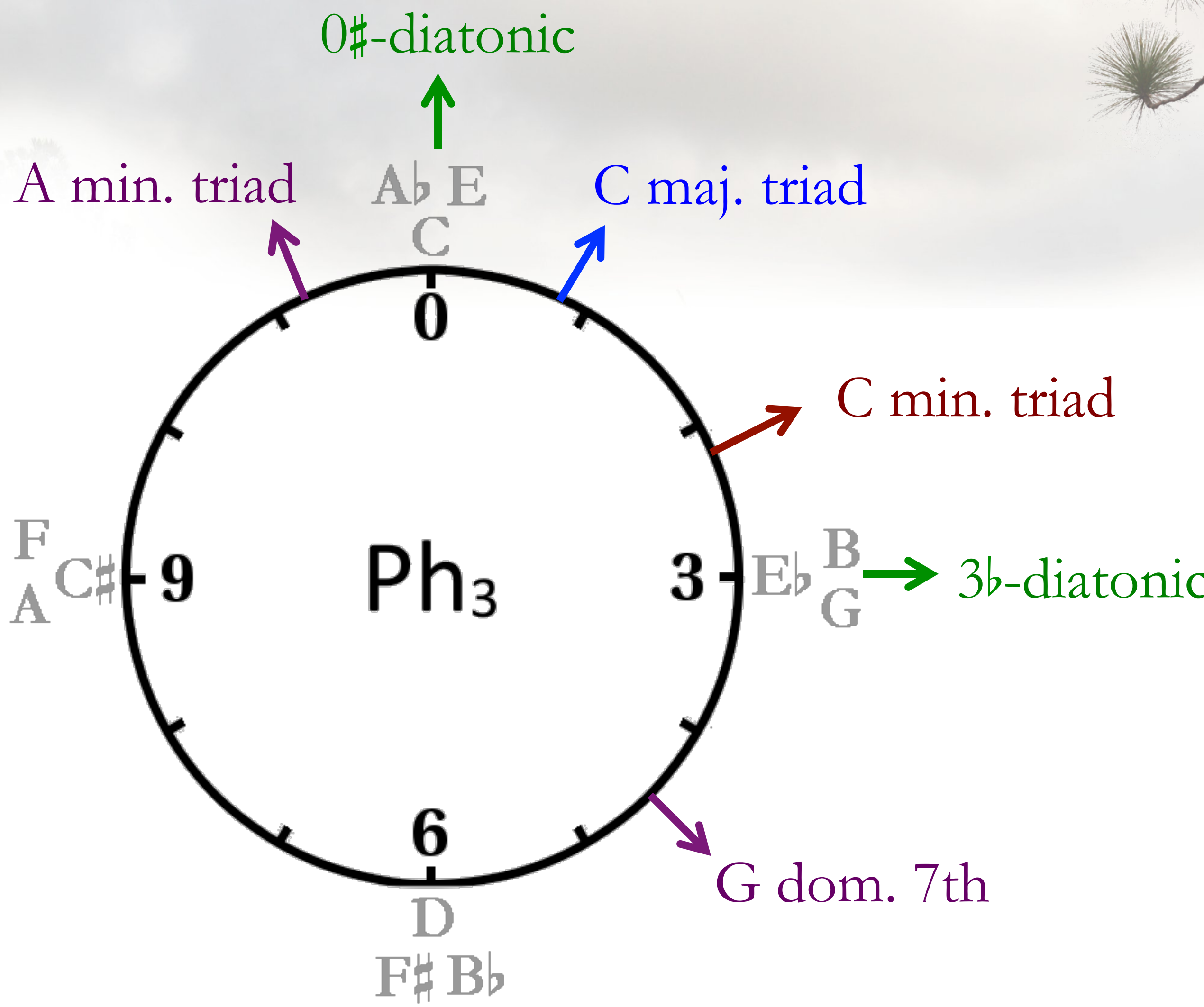
Pcs, consonant dyads and triads, and *Tonnetz* in $\mathrm{Ph}_{3,5}$-space, from Yust (2015) (*JMT* 59/1)

(2) Diatonicity, Triadicity, and Dyadicity in Tonal Music

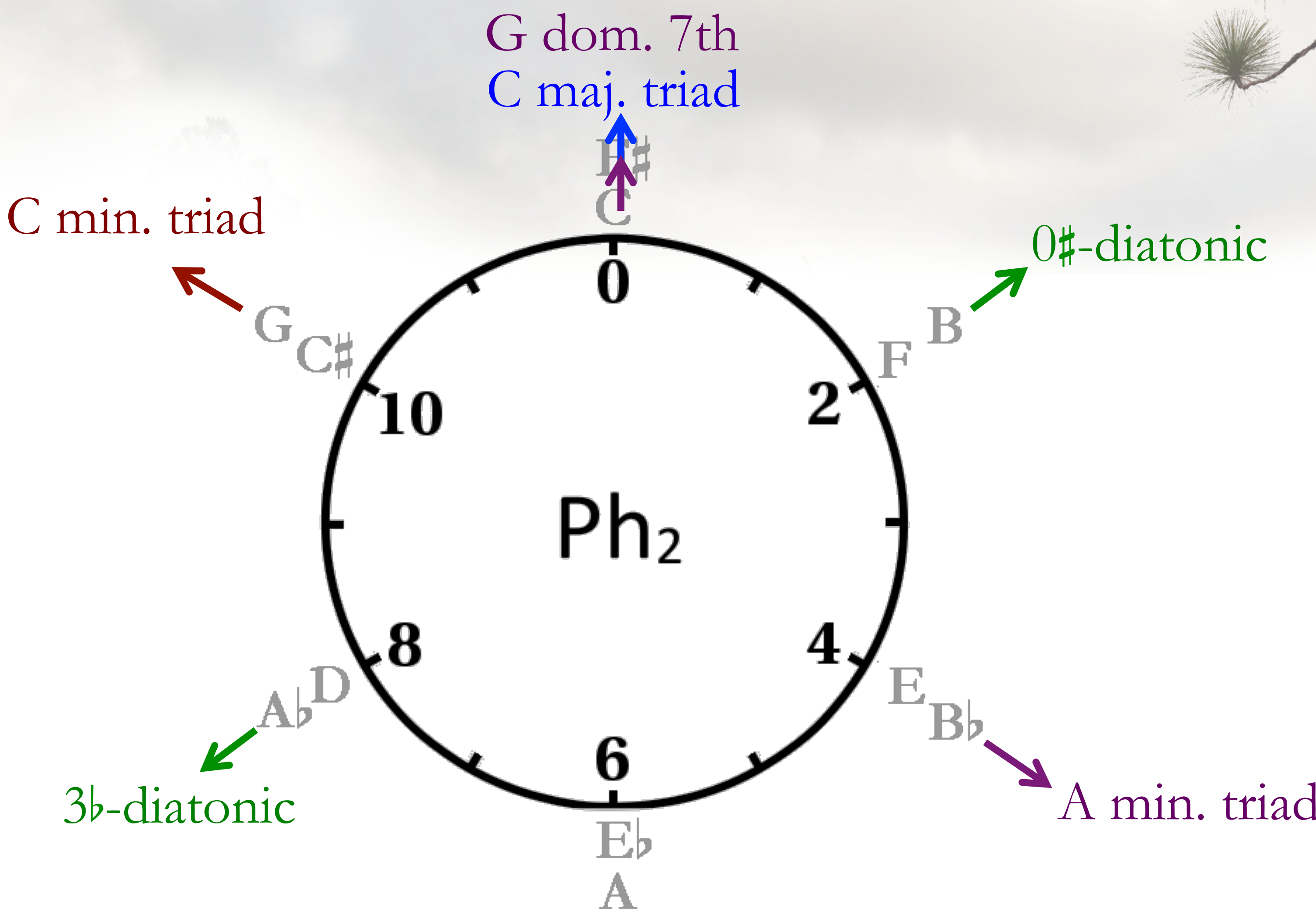
Diatonicity



Triadicity

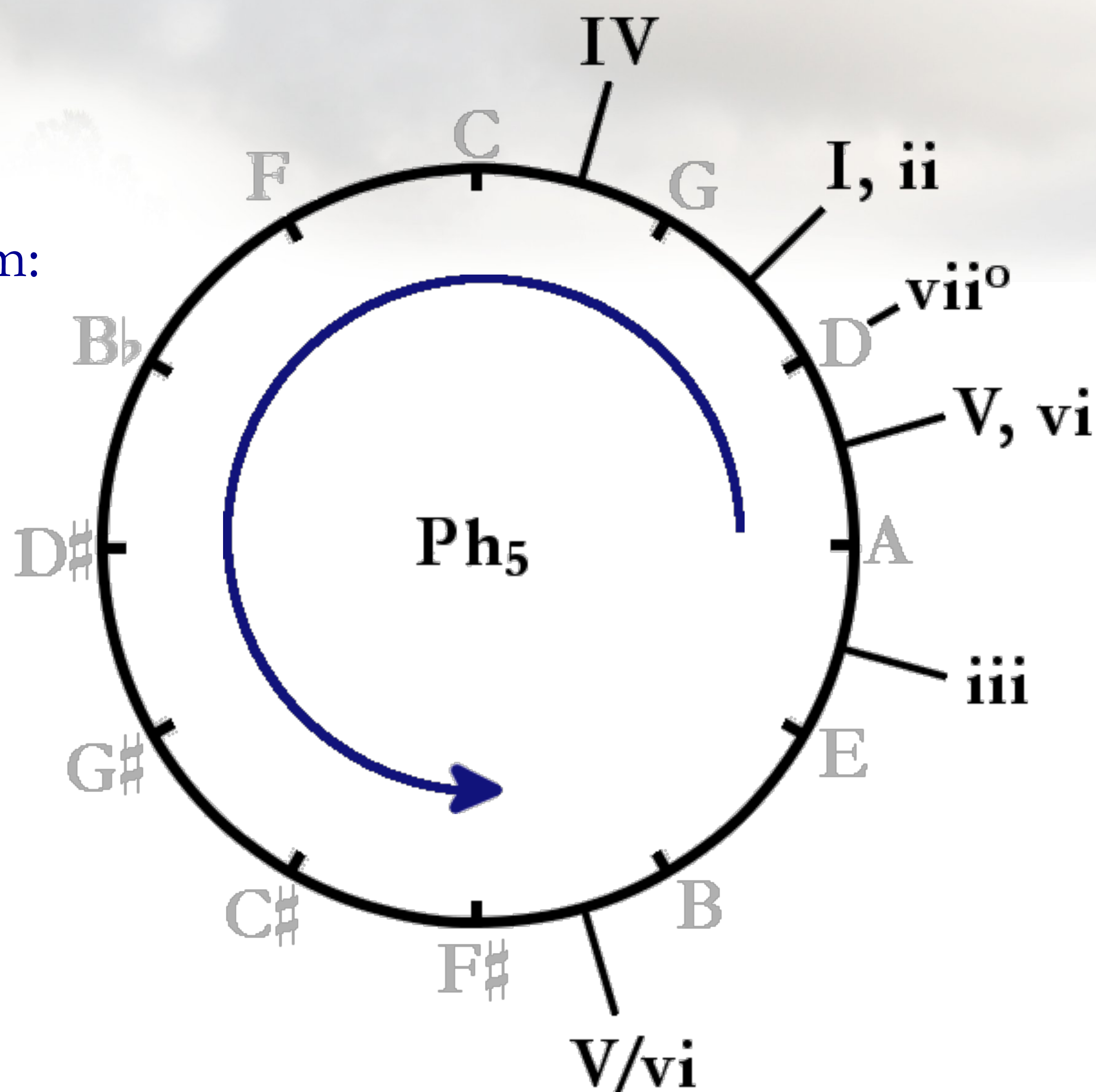

0♯-diatonic
A min. triad
A♭ E
C
C maj. triad
0
C min. triad
F
C♯
A
9
Ph$_3$
3
E♭
B
G
3♭-diatonic
6
G dom. 7th
D
F♯ B♭

Dyadicity


G dom. 7th
C maj. triad
F♯
C
C min. triad
0♯-diatonic
G
C♯
B
F
0
2
10
Ph$_2$
8
4
A♭
D
E
B♭
6
3♭-diatonic
A min. triad
E♭
A

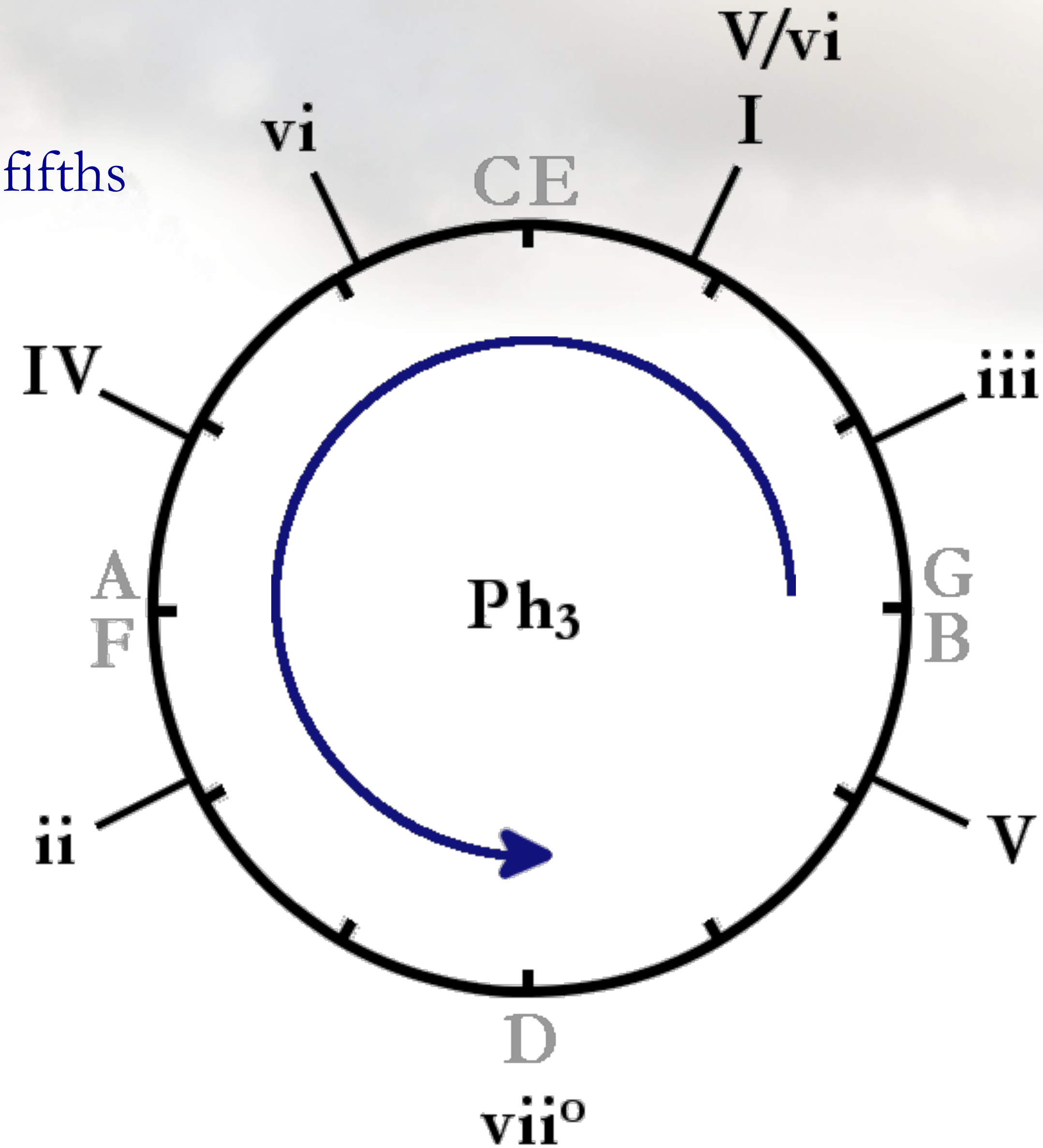
Triads in diatonic space

Ph_5-cycles indicate enharmonicism: They are only possible by going though other keys.



Triads in triadic space

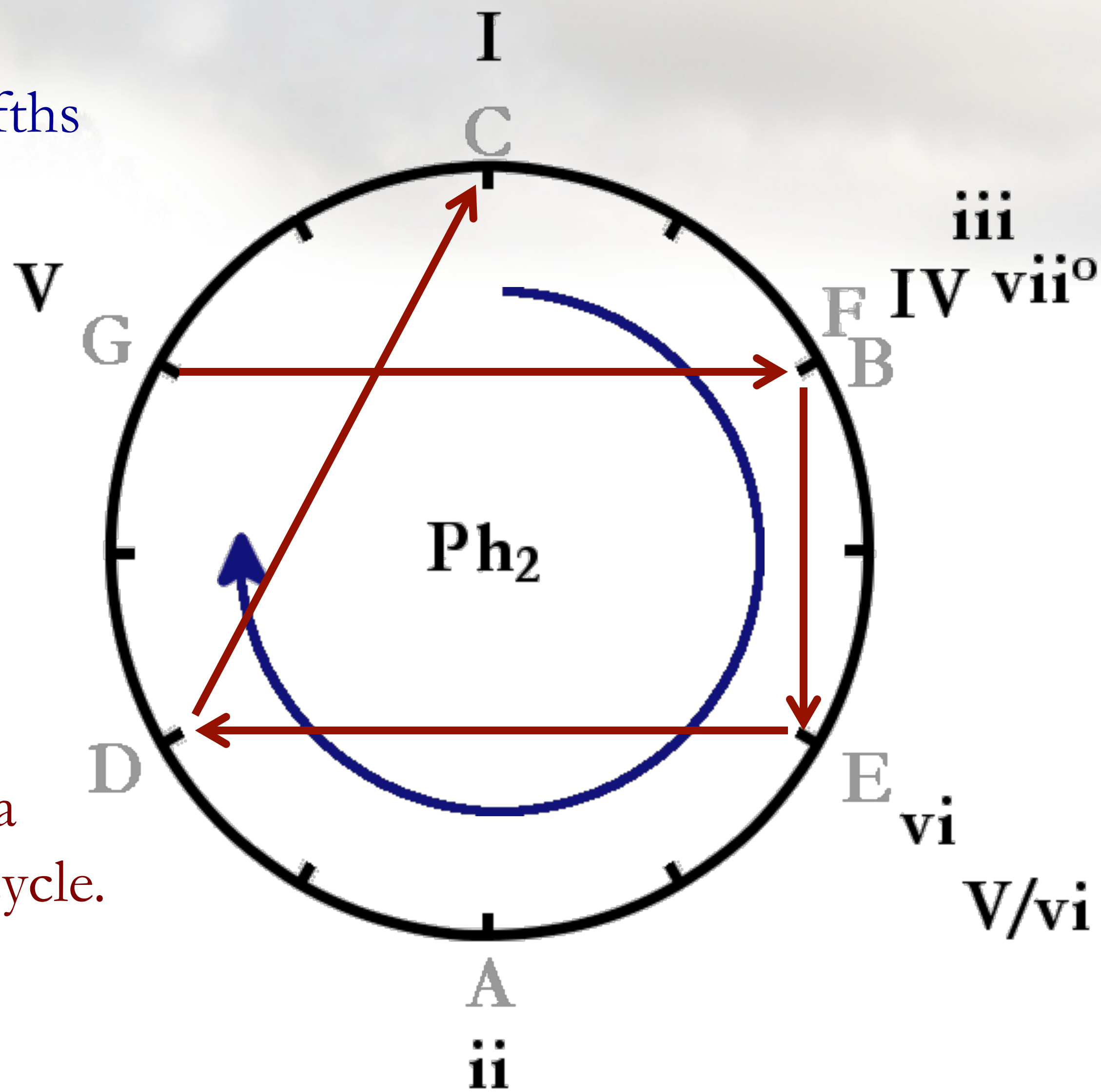
Descending fifths progression produces a negative Ph_3-cycle.



Triads in dyadic space

Descending fifths progression produces a *positive* Ph_2-cycle.

Stepwise descent may also produce a positive Ph_2-cycle.



(3) The Tonal Plane,
$Ph_2 + Ph_3 - Ph_5 = 0$

$Ph_{2,3}$-space

Single pcs and diatonic scales:

$Ph_2 + Ph_3 - Ph_5 = 0$

($Ph_5 = 0, 1, 2, \ldots$)

Fifths:

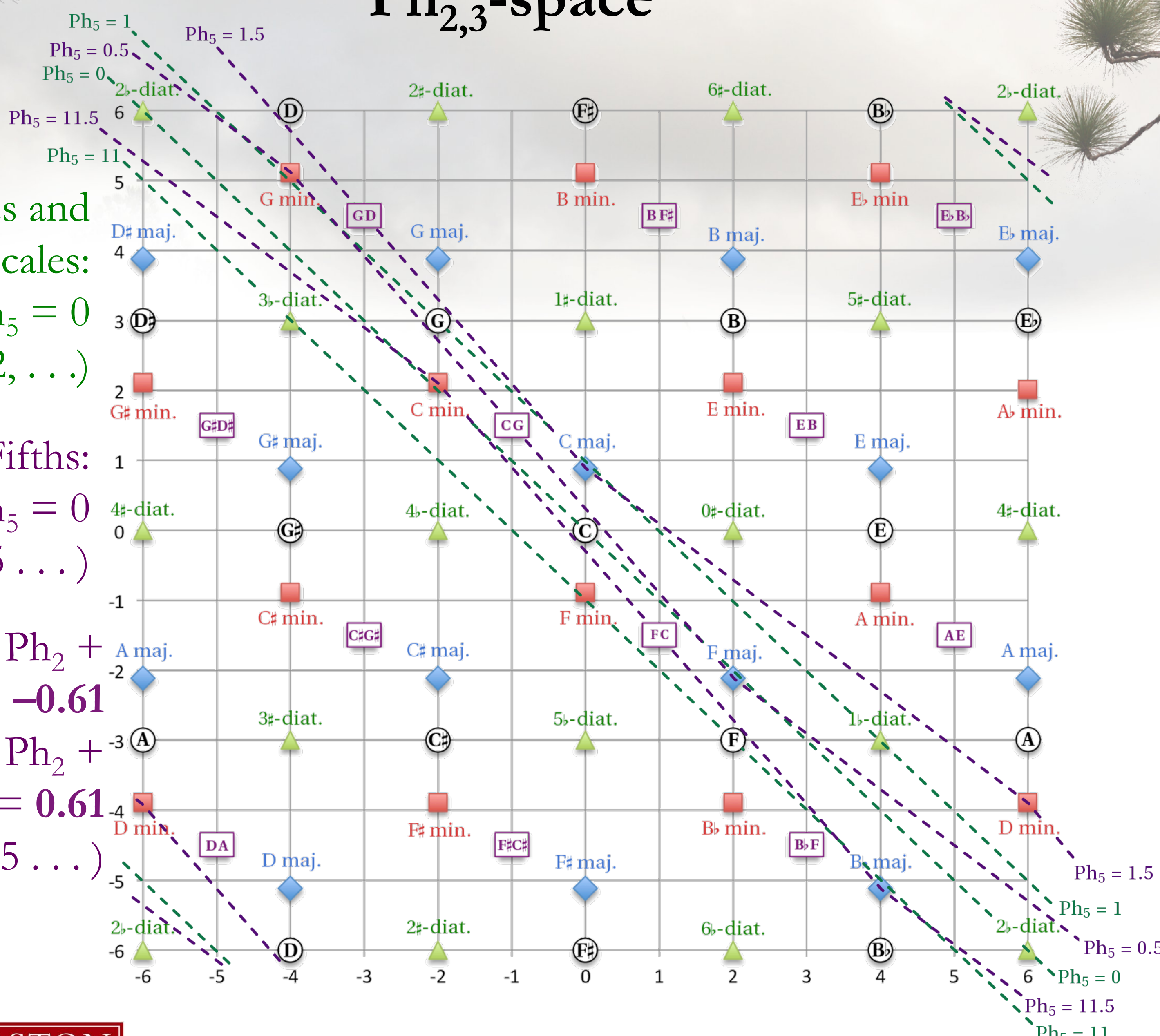
$Ph_2 + Ph_3 - Ph_5 = 0$

($Ph_5 = 0.5, 1.5 \ldots$)

Maj. Triads: $Ph_2 + Ph_3 - Ph_5 = \mathbf{-0.61}$

Min. Triads: $Ph_2 + Ph_3 - Ph_5 = \mathbf{0.61}$

($Ph_5 = 0.5, 1.5 \ldots$)



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The concept of key

Krumhansl-Schmuckler and Temperley key finding algorithms:

Based on pc-distributions. The most probable key is the nearest by Euclidean metric in 12-dimensional space of characteristic functions.

Tonal space transforms the basis of this 12-dimensional space and reduces to 3 dimensions.

The concept of key

In the smaller space of DFT phases, complex distributions are not necessary to represent keys. Each key can simply be represented by its tonic triad. The "tonal hierarchy" is a byproduct of periodicities.

		Krumhansl-Kessler	**Temperley**	**Triad**
Major	Ph_2	0.12	0.63	0
	Ph_3	0.16	0.65	0.89
	Ph_5	1.25	1.71	1.5
	$Ph_2 + Ph_3 - Ph_5$	–0.96	–0.43	–0.61
Minor	Ph_2	9.88	10.10	10
	Ph_3	1.62	2.24	2.11
	Ph_5	11.22	11.59	11.5
	$Ph_2 + Ph_3 - Ph_5$	0.28	0.75	0.61

(4) Examples:
Motion through tonal space

The following examples are generated by windowed DFT analysis of raw MIDI. Paths are tracked over time by showing the shortest path between successive points.

A big thanks to Richard Plotkin for helping to write the Python code that generated these graphs.

Chopin, Mazurka, op. 33/2

Example of an enharmonic cycle

D major

B♭ major

B♭ minor

D♭ major

Chopin, Mazurka, op. 33/2

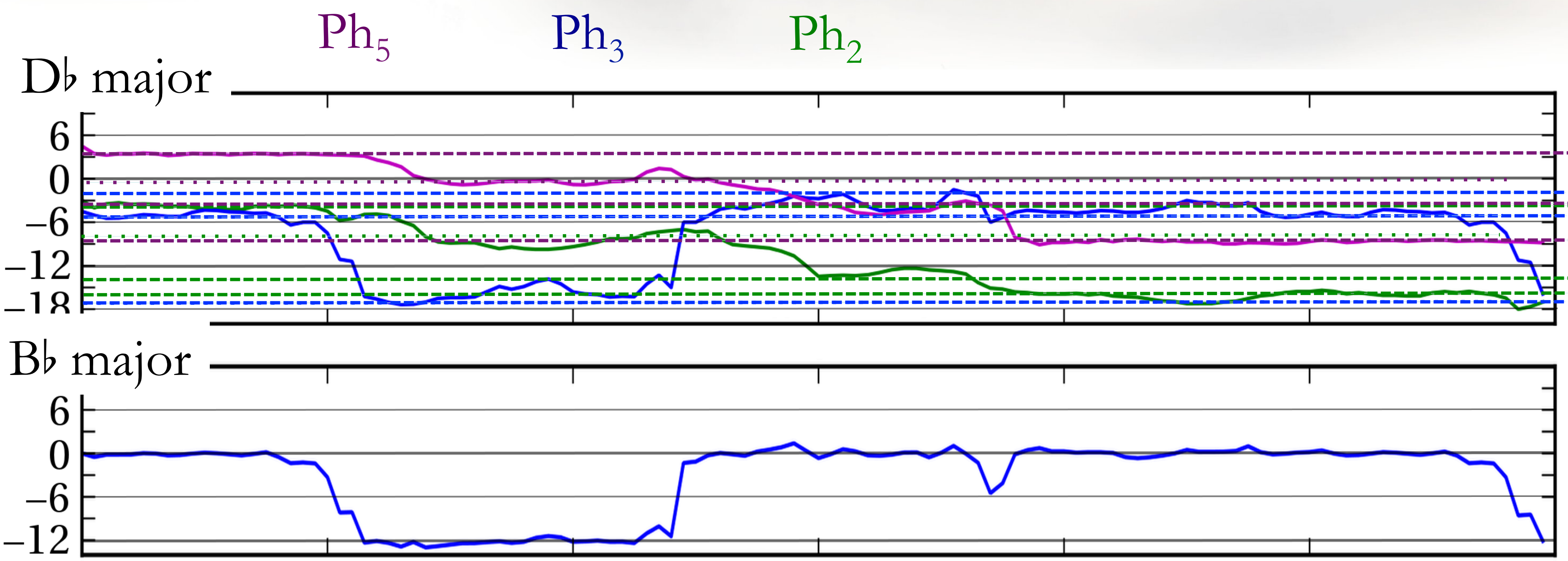
Example of an enharmonic cycle

F♯ minor

A major

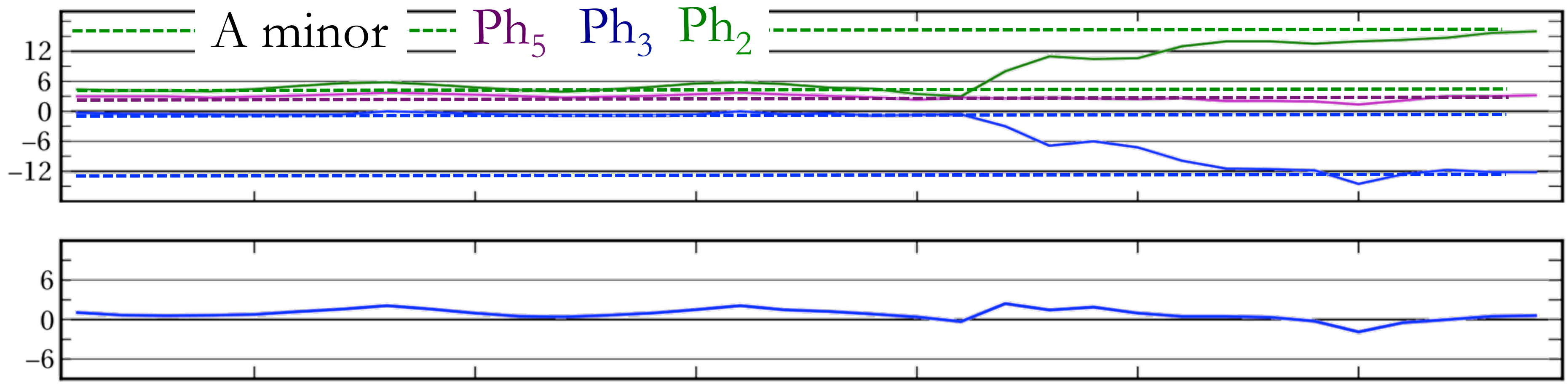
D major

Chopin, Mazurka, op. 33/2



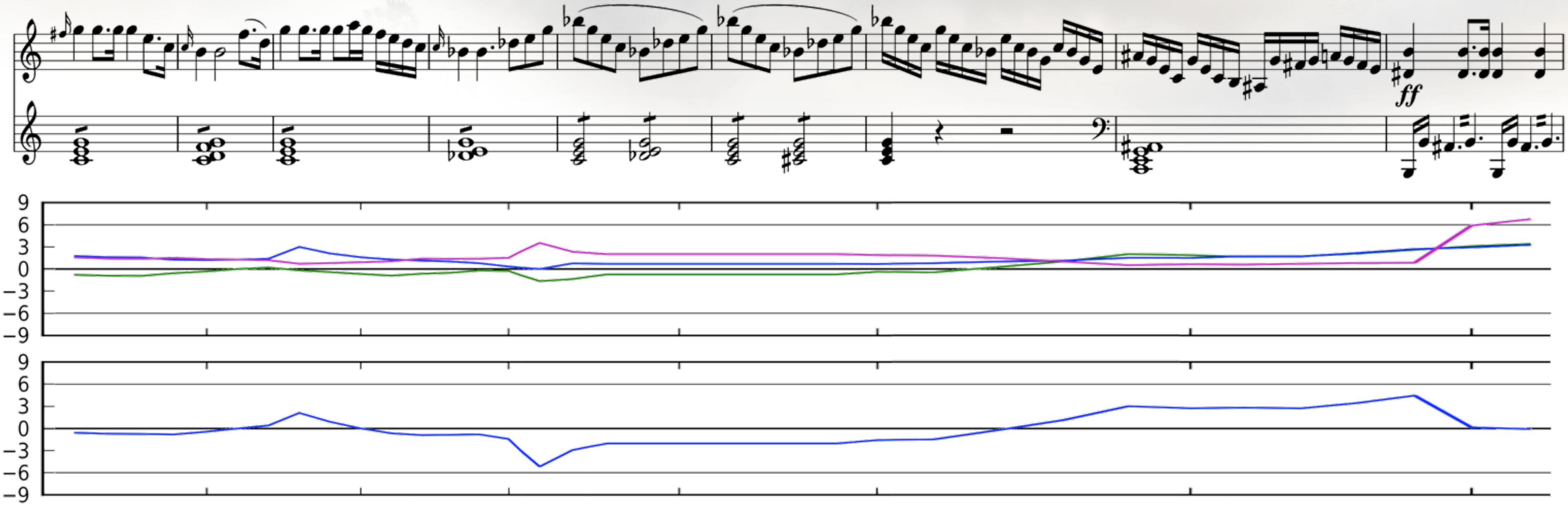
Tonality index: $Ph_2 + Ph_3 - Ph_5$

Mozart, K.310 main theme



Tonality index: $Ph_2 + Ph_3 - Ph_5$

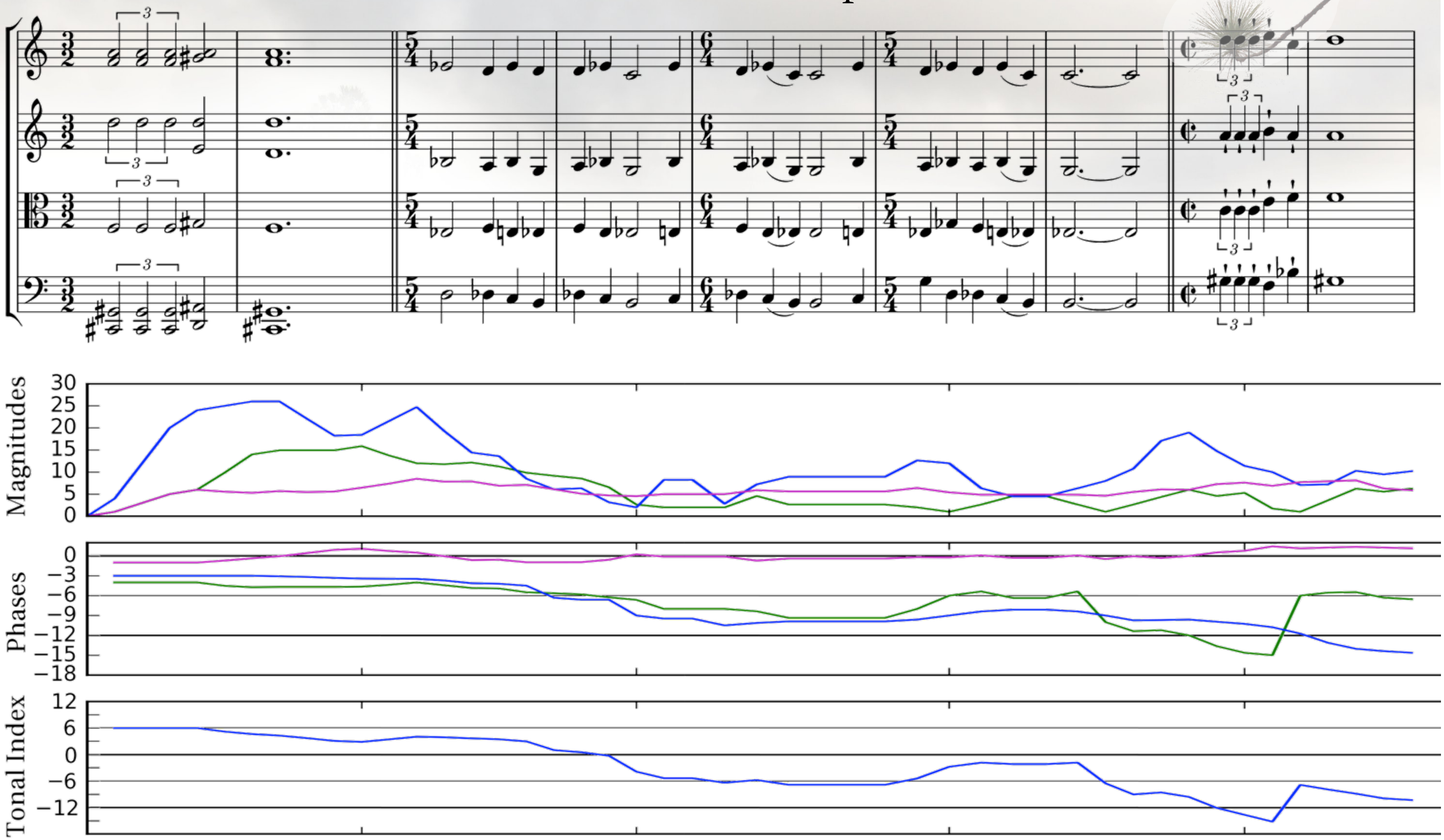
Mozart, K.310 development



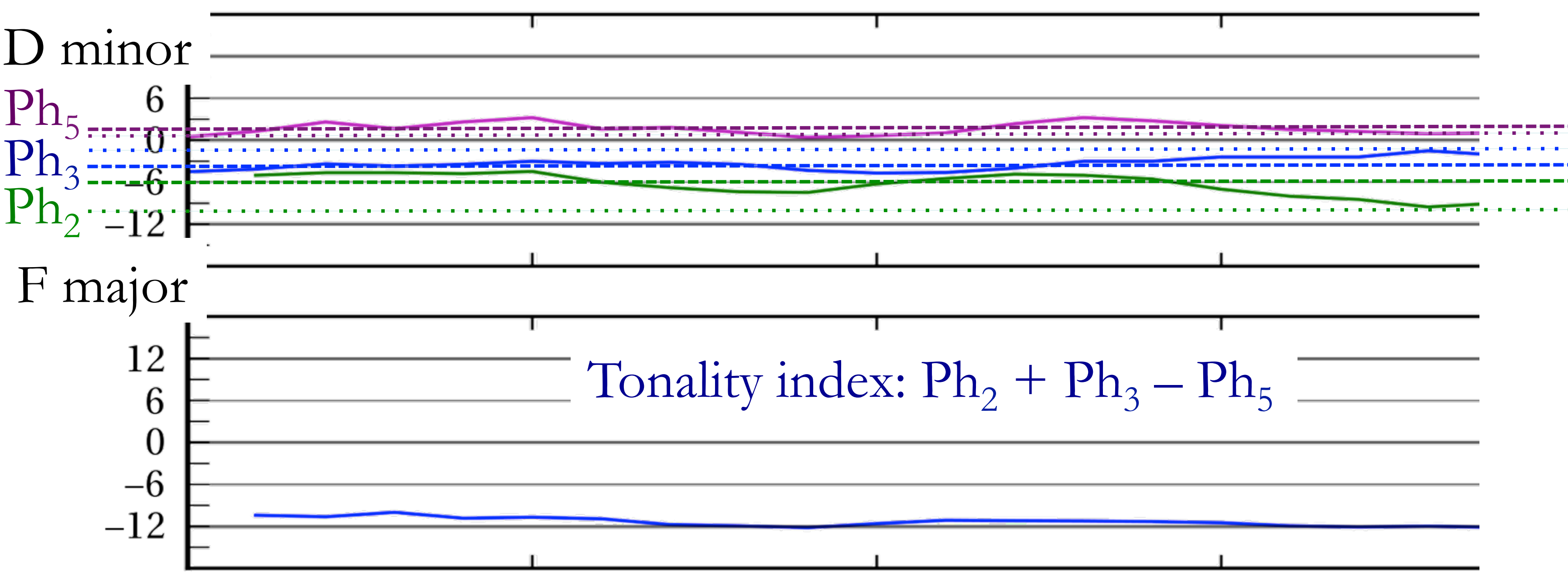
Tonality index: $Ph_2 + Ph_3 - Ph_5$

Stravinsky, Piece for String Quartet no. 3

A non-tonal example

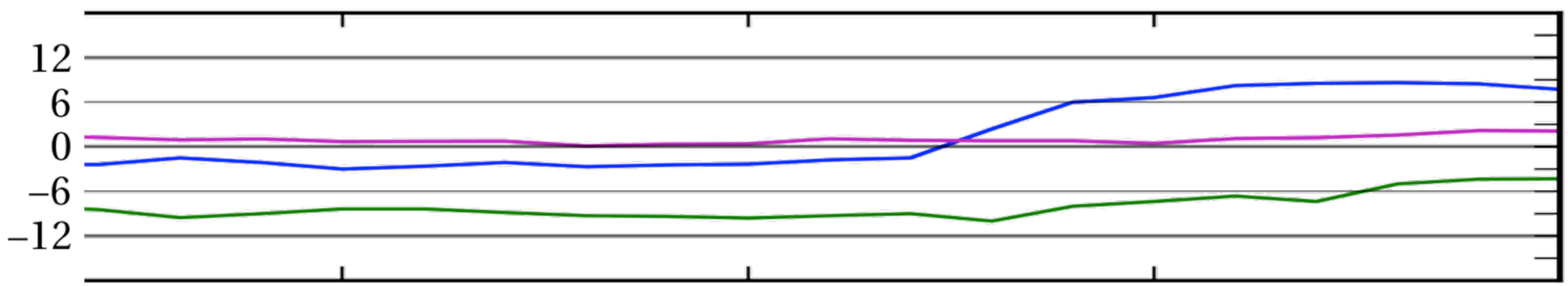


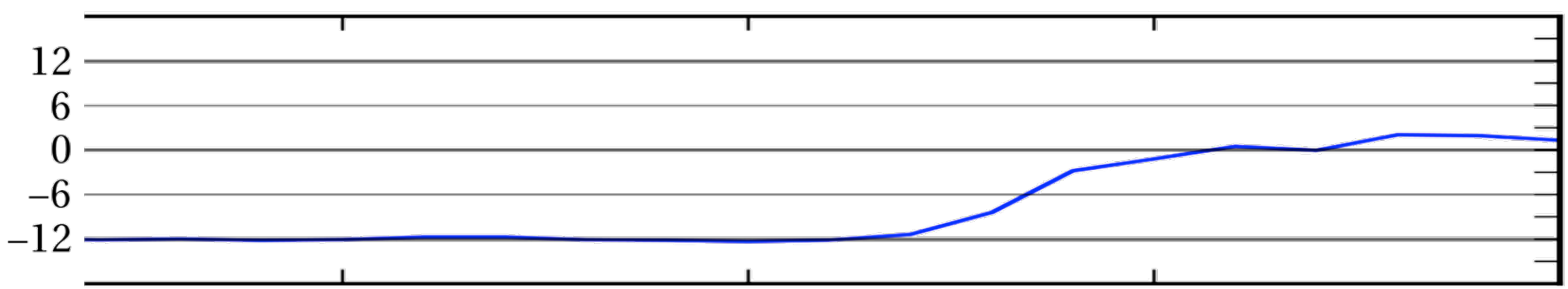
Corelli, Op. 4/8 Sarabande



Corelli, Op. 4/8 Sarabande

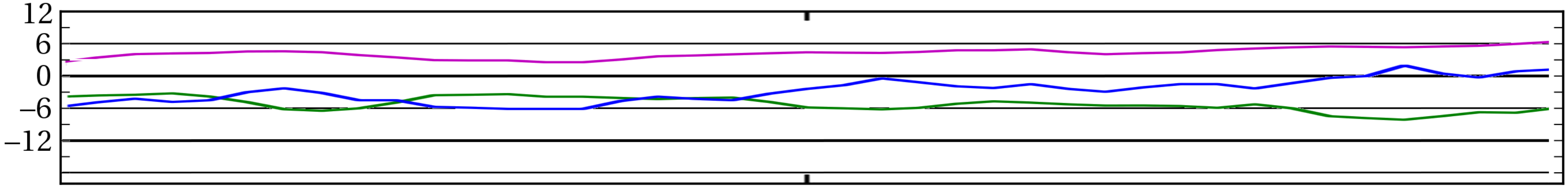
Ascending 5ths sequence



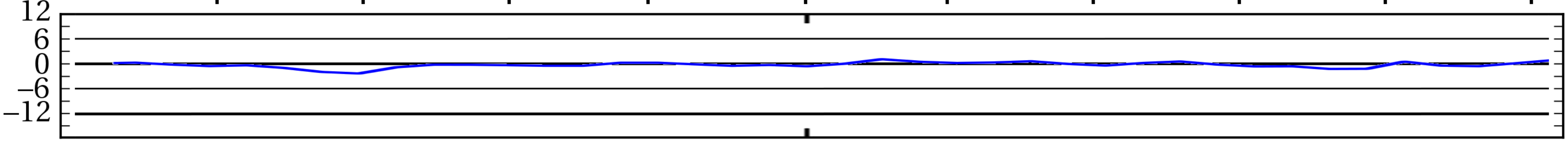


Corelli, Violin Sonata, Op. 5/1 mvt. 3

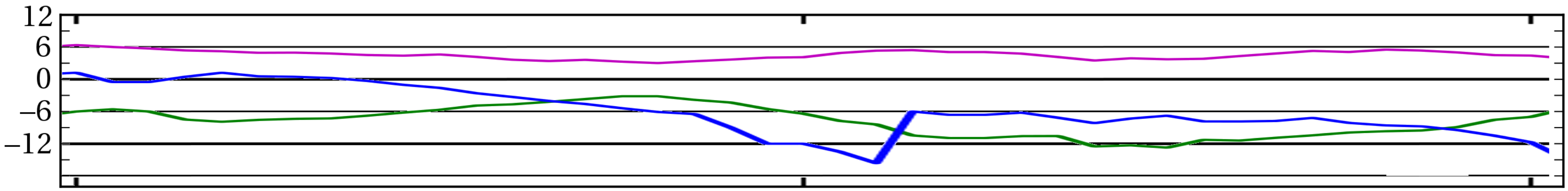
Corelli, Violin Sonata, Op. 5/1 mvt. 3



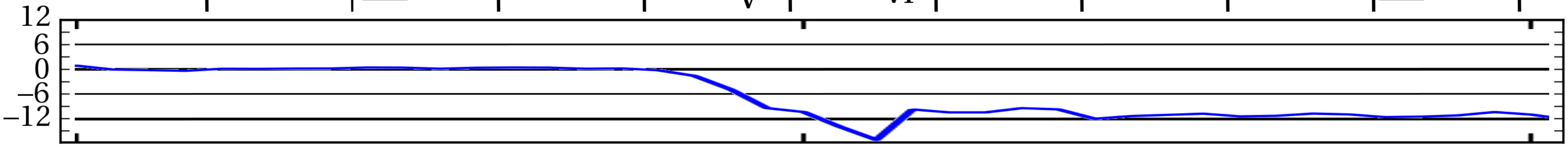
$\boxed{\text{D:}}$ I V^6 | vi V V | I^6 IV | V I | V^6 vi $\boxed{\text{A:}}$ V | I V^6 | vi I^6 IV V | I V^6 | vi $\boxed{\text{E:}}$ I | V^6 IV^6 |



Key changes gradually ascend by fifths



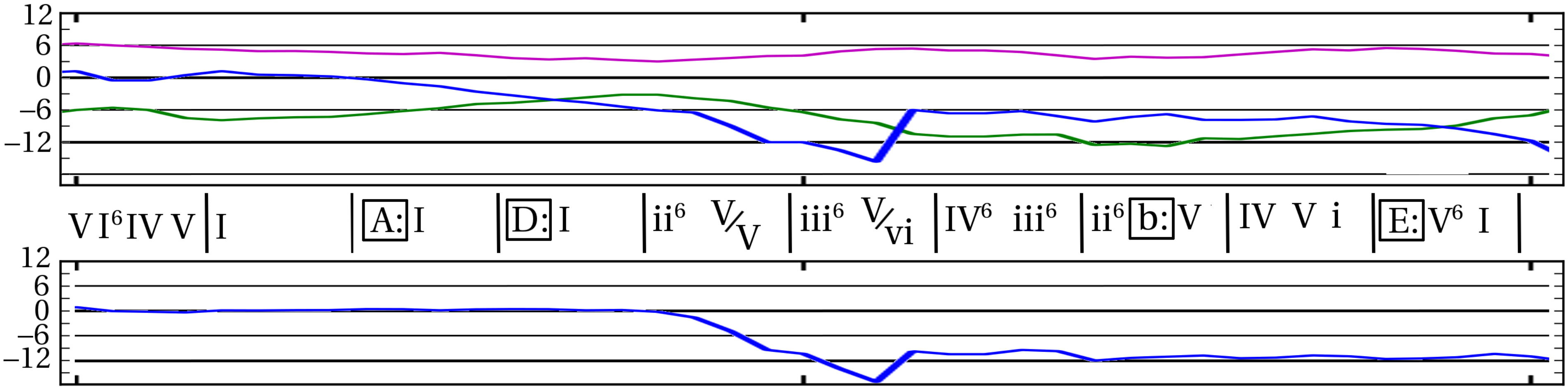
V I^6 IV V | I | $\boxed{\text{A:}}$ I | $\boxed{\text{D:}}$ I | ii^6 V/V | iii^6 V/vi | IV^6 iii^6 | ii^6 $\boxed{\text{b:}}$ V | IV V i | $\boxed{\text{E:}}$ V^6 I |

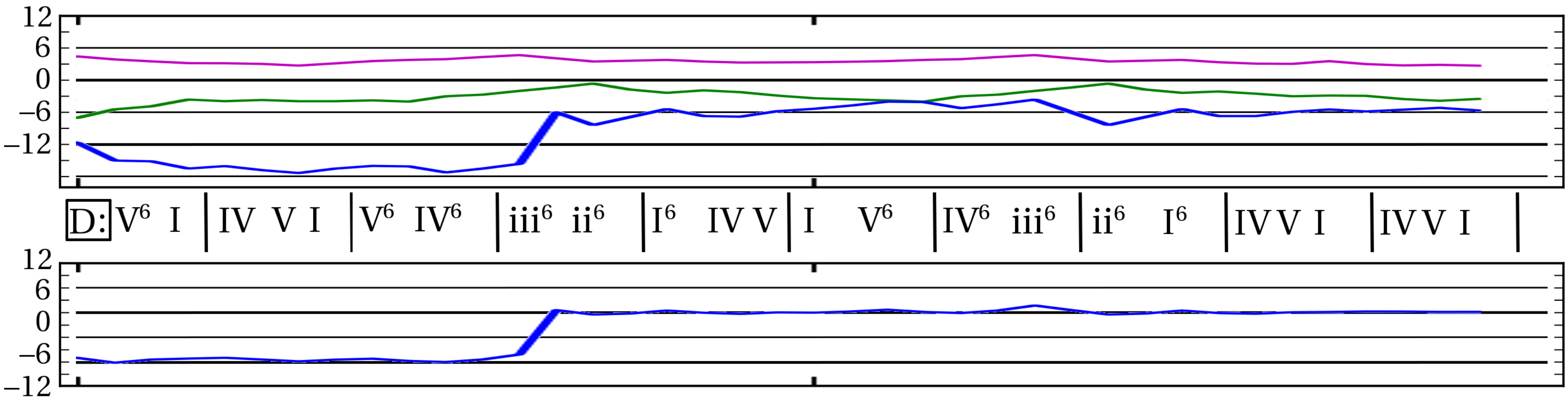


Sequence descending by fifths

Sequence ascending by step

Corelli, Violin Sonata, Op. 5/1 mvt. 3





— Sequence descending by step →

Conclusions

- In the three-dimensional phase space $Ph_{2,3,5}$, regions around a major or minor triad correspond to the typical pitch-class distributions of that triad's key.
- Typical pitch-class distributions of tonal music are concentrated around the plane defined by $Ph_2 + Ph_3 - Ph_5 = 0$ in $Ph_{2,3,5}$-space. Large deviations of this tonal index, $Ph_2 + Ph_3 - Ph_5$, from zero indicate tonally unstable or non-tonal harmonic states.
- Sequential progressions may cycle the space in different ways:

—Progressions by fifth move around the space within the tonal plane through complimentary motion of Ph_2 and Ph_3.

—Enharmonic major-third cycles also stay on the tonal plane through coordinated complimentary motion in Ph_2 and Ph_5.

—Stepwise sequences may go through tonally unstable regions when Ph_3 cycles independently of the others.