Special Collections: Renewing Set Theory

Abstract

The discrete Fourier transform (DFT) on pcsets, proposed by David Lewin and advanced by Ian Quinn, may provide a new lease on life for Allen Forte's idea of a general theory of harmony for the twentieth century based on the intervallic content of pitch-class collections. This article proposes the use of *phases spaces* and Quinn's *harmonic qualities* in analysis of a wide variety of twentieth century styles. The main focus is on how these ideas relate to scale theoretic concepts and the repertoires to which they are applied, such as the music of Debussy, Satie, Stravinsky, Ravel, and Shostakovich. Diatonicity, one of the harmonic qualities, is a basic concern for all of these composers. Phase spaces and harmonic qualities also help to explain the "scale-network wormhole" phenomenon in Debussy and Ravel, and better pinpoint the role of octatonicism in Stravinsky's and Ravel's music.

Keywords

Post-tonal theory, Pitch-class set theory, Allen Forte, Scale theory, Scale networks, Interval function, Fourier transform, Phases spaces, Harmonic qualities, Diatonicity, Octatonicity.

Text

As music theory was rapidly growing as an academic discipline in the 1960s and '70s, Allen Forte responded to the pressing need for systematic theoretical approaches to twentieth-century music. This was a significant challenge, given that perhaps the most widely held aesthetic principles of modernism were freedom and stylistic diversity. Forte's insightful solution was a theory based on *interval content*, a property general enough to be significant to composers as divergent as Debussy, Stravinsky, and Webern. Forte's imperative to develop a theoretic framework that could encompass this range of compositional styles also led him to advocate a degree of literalness in analysis that contrasted with most previous approaches to analyzing the music of composers like Debussy and Stravinsky.

This article reevaluates Forte's idea of a general theoretical framework for twentieth-century harmony using the *discrete Fourier transform (DFT) on pcsets*. This method, I argue, accomplishes two of Forte's most basic original goals: first, it relates pcsets on the basis of interval content. Second, it embraces the kinds of observations that can be made from scale-theoretic methodologies while adhering to the literal parsing of the musical surface advocated by Forte.

In its initial manifestations (Forte 1964) the basic object of Forte's theory, the set class, was defined by its interval content, the interval vector. This changed in Forte's 1973 book, in response to a pointed critique from Clough (1965). Forte's book eclipsed the earlier article, so that now we understand equivalence under transposition and inversion as the essence of set theory and the interval vector as a secondary property of set classes.[1] The problem with

[1] As Ian Quinn has pointed out (in personal communication), Forte's original definition of set class is fossilized in a quirk of his now standard 1973 numbering system, because rather

interval vectors is that specific information about the original pcsets is non-recoverable from their interval content alone, limiting their usability, as Clough was able to show. Yet, the musical inscrutability of TI-equivalence in real musical situations left Forte's theory open to harsh criticisms from Perle (1990) and others.

In a series of important articles, Ian Quinn (2006–7) showed that the problem could be solved in a more elegant way through a proposal made by Lewin in a pair of short research notes in 1959–60. In these papers Lewin defines the *interval function*, which is a generalization of the interval vector (the interval vector is the interval function of a pcset to itself). Lewin hints in this article (which also anticipated Forte in identifying the Z-relation) at a special mathematical relationship between the interval function and the Fourier transform. Lewin's underlying methodology, the *discrete Fourier transform (DFT) on pcsets*, can isolate the interval content of a pcset in the Fourier *magnitudes*, yet it retains transposition- and inversion-dependent information in the Fourier *phases*. In contrast to Quinn, whose primary concern is the magnitudes, the present article focuses heavily on the meaning and analytical use of phases. The first part demonstrates how the DFT converts pitch-class information into six independent and musically meaningful *harmonic qualities* and introduces the idea of phase spaces, drawing upon recent applications of these to tonal music (Amiot 2013, Yust 2015c).

The remainder of the article focuses on viewing scale-theoretic analysis through the DFT, considering some of the most recent advances in understanding twentieth-century repertoires from a scale-theoretic point of view. Scale theory has a kinship with Forte's approach in the sense that both are concerned with collections. The primary difference between them lies in the interface between theory and analysis. Scale theory greatly limits the number of set types under theoretical consideration by applying criteria that distinguish a small number of scale-like collections (typically cardinality plus some kind of evenness conditions). An interesting formulation of scale theory roughly contemporaneous with Forte's development of set theory is given by Wilding-White (1961) who explicitly associates the term with a conceptual distinction between *gamut*, *scale*, and "formula," the relationship between these being one of containment, with gamut and scale playing the role of abstract collections of potential pitch material from which the literal pitch material (of the "formulas") are selected.

This way of defining scale theories, as theories that require referential collections (scales) as abstract intermediaries between gamut and musical surface when applied in analysis, is especially inclusive.[2] It embraces Tymoczko's (2004, 2011) voice-leading based scale theory, Hook's (2011) spelled heptachords, my own (Yust 2013b) application of second-order maximal evenness to Ravel, Marek Žabka's (2014) *Tonnetz*-based scale theory, and the many discussions of whole-tone and octatonic collections in twentieth-century music—e.g., Van den Toorn (1983), Baur (1999), Antokoletz (1993), Tymoczko (2004), to name just a few.

Part two presents a DFT-based evenness condition that resembles the many that have been suggested in the scale theory literature. Section 2.2–2.3 relate the Fourier property of

than renumber, Forte tacked on the new set classes from each Z-related pair at the end of the list.

[2] Drawing upon the distinction made by Tymoczko (2011), one might say that this definition prioritizes the idea of scale as "macroharmony" over that of scale as a ruler.

diatonicity to Tymoczko's scale networks for heptatonic scales in analyses of Satie and Debussy. Section 2.4 offers a new perspective on the matter of diatonic-octatonic interaction in Stravinsky. Part three extends the application of DFT to whole-tone and octatonic materials in Debussy, Shostakovich, and Ravel, relating Fourier phase spaces to scale networks through mathematical observations on maximal intersections centered on Lewin's interval function.

1. Fourier Components and Phase Spaces

1.1: Phase Spaces

By now there is a large body of work describing the DFT on pcsets from a variety of angles (Lewin 2001, Quinn 2006–2007, Callender 2007, Amiot 2009, Amiot and Sethares 2011, Yust 2015a, b, c). Rather than retread all of that ground, I will introduce the procedure in a slightly different way here by beginning with *phase spaces*. The connection to Fourier analysis may not be immediately apparent, but it plays an important behind-the-scenes role where I draw upon two of its fundamental theorems: in the next section, the reversibility of the DFT and, in part three, the convolution theorem.

Figure 1 shows the one-dimensional phase spaces, which are equivalent to Quinn's (2006–7) "Fourier balances." All are topological circles, and every pitch class has a position in each of them. Phase space 1 is the (reflection of the) familiar pc-circle. Each other space can be derived by multiplying phase space 1 by 2, 3, 4, 5, or 6. The result is a superimposition of all interval cycles of a given index (see Perle 1985, Cohn 1991). Positions in the space correspond to the angles of lines extending from the origin to a point on the circle (relative to 0, which is placed on the y-axis according to the music-theoretic—not the mathematical—convention). Mathematical convention would measure this angle in radians, denoted φ_n for position in phase space n. For present purposes, though, a normalization to the pc-circle (a value between 0 and 12 rather than 0 and 2π), denoted by Ph_n, is more useful.

Any pitch-class set can be located in any of these phase spaces by taking the *circular average* of its constituent pitch classes. A circular average is a way of averaging points around a circle that does not depend upon the arbitrary choice of where 0 is located (which would affect a simple average of angles). To calculate a circular average we embed the space in a two-dimensional Cartesian plane (as a unit circle), and add the vectors corresponding to each pitch class (i.e., the vector from the origin to the location of the pitch class), as illustrated in Figure 2. The direction of the resulting vector is the position of the pcset in the phase space, or, simply, its *phase*.

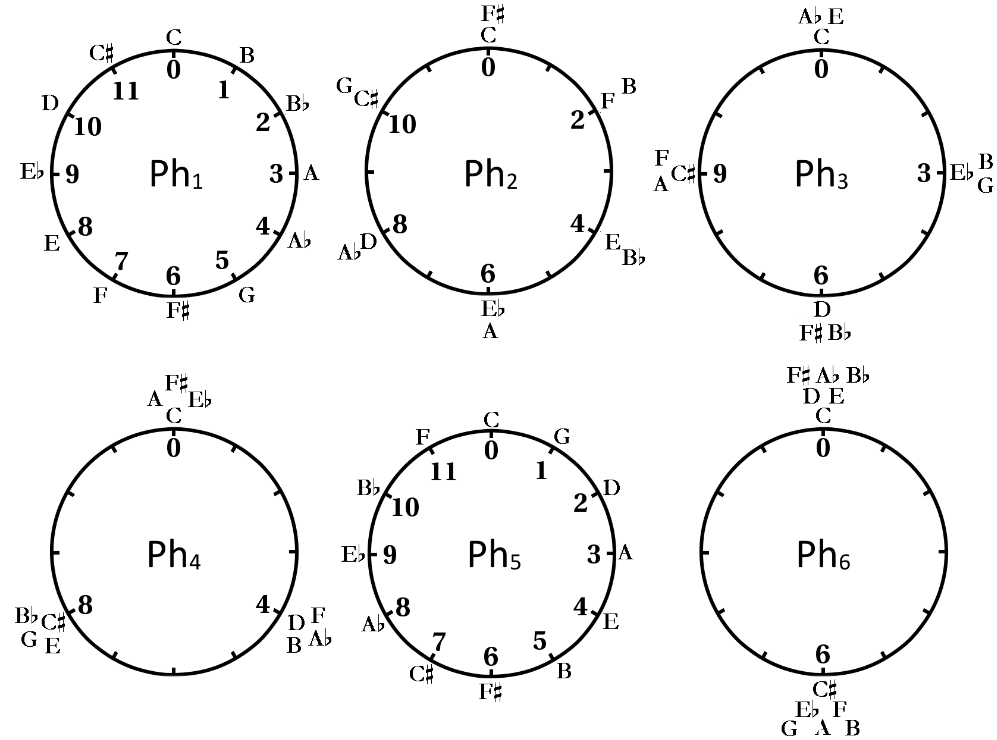

Figure 1: The six one-dimensional phase spaces

There is also a *magnitude* for each pcset in each of these spaces, which is the length of the vector created when determining the circular average of pitch classes in the set. The magnitude of an individual pitch class is 1. Magnitudes can be 0, in which case the phase is undefined. The vector for a pcset in phase space *n*, both magnitude and phase, is denoted f_n, and the magnitude by itself is $|f_n|$.

By locating a pcset in all six phase spaces, then, we derive 12 values, Ph_n and $|f_n|$ for $1 \leq n \leq 6$. The full DFT, however, is a 12-place vector with magnitudes and phases for each component: $\langle\langle f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11} \rangle\rangle$. The zeroeth component is simply the cardinality of the pcset (with $\mathrm{Ph}_0 = 0$), and f_{12-n} are equal to f_n in magnitude and opposite in phase.[3] All the non-trivial information in the DFT, then, is contained in components 1–6.

Higher-dimensional phase spaces are created simply by combining two or more one-dimensional phase spaces, resulting in a toroidal topology. A two-dimensional phase space using components 3 and 5 ("$\mathrm{Ph}_{3,5}$-space") is introduced in Amiot 2013 and Yust 2015c as a way of representing spatial features of tonal harmony. Analyses below use two-dimensional $\mathrm{Ph}_{4,5}$- and $\mathrm{Ph}_{5,6}$-spaces.

[3] This could be demonstrated by constructing phases spaces for components 7–12, which would simply be a reflection of phase spaces 1–6 around an axis through 0.

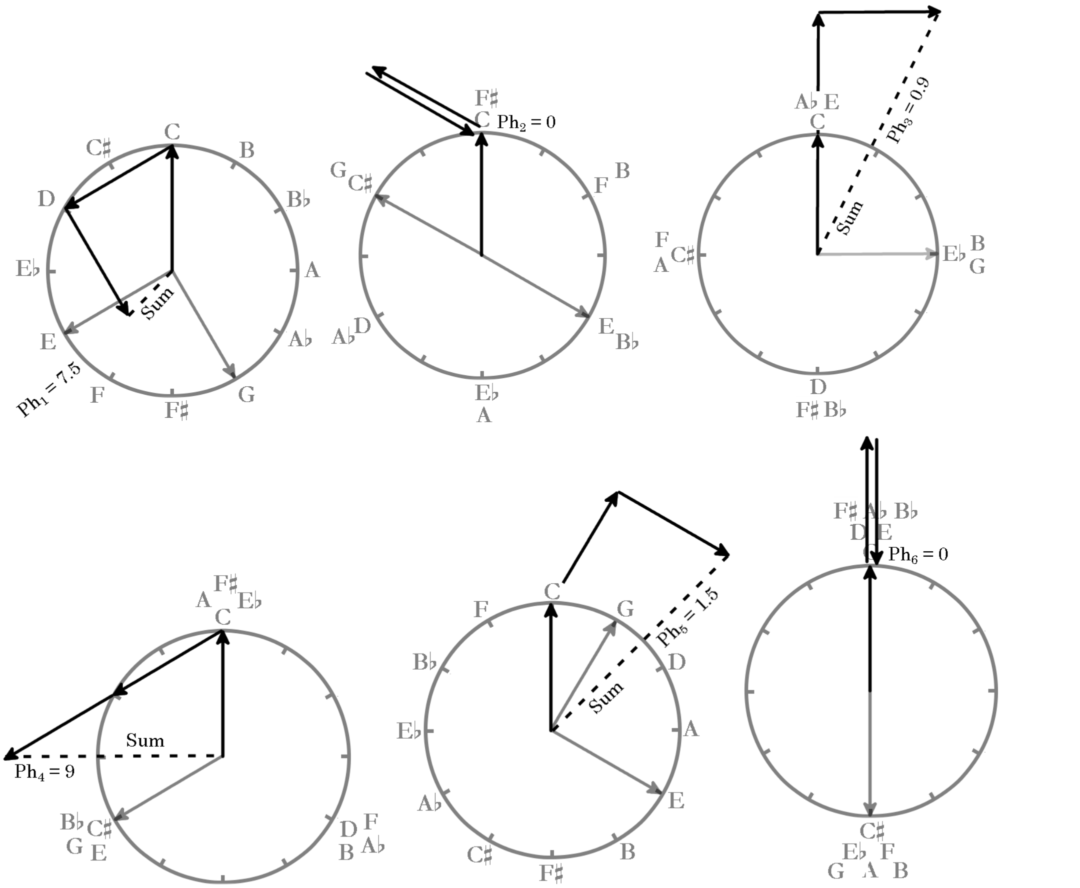

Figure 2: Deriving Fourier components for a C major triad by adding vectors in a one-dimensional phase space embedded in the plane

1.2: Fourier Components as Harmonic Qualities

An important property of the DFT is its reversibility, guaranteed by one of the fundamental Fourier theorems. This means that given only the DFT of a pcset, the pcset itself can be recovered. This is best understood by thinking of the DFT as a *change of basis* meaning that it represents the same object (a pcset) with a different set of twelve quantities. The "raw" pcset is given by a twelve-place vector with 1s in the places corresponding to pitch classes in the set and zeroes elsewhere. For instance, the C major triad has the following pc-vector: (1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0). A pc-vector can also represent a pc-multiset if it uses integers larger than 1, or a pc-distribution with non-integer values.

Imagine that we were given the full DFT of a C major triad (as derived in Fig. 2) but were unaware what pcset these values came from. We can recover the pcset as follows:

(1) Draw an axis in each phase space corresponding to the phase on the given component, and project each pc onto this axis, taking the distance from the origin (positive in the same direction as the phase, negative in the opposite direction), as illustrated in Figure 3.

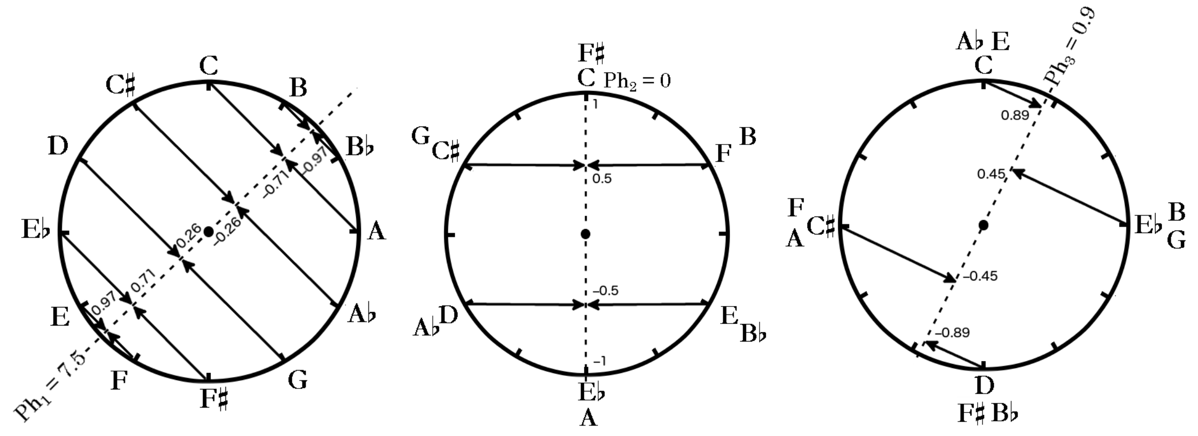

Figure 3: Geometric representation of the cosine functions for reversing the DFT of a C major triad (components 1–3)

These distances define an n-periodic cosine function peaking at 1 for Ph_n (if any pcs happen to coincide with this point) and with minimum value –1 for $Ph_n \pm 6$.

(2) Weight each of these cosine functions by the DFT magnitudes, and add them together componentwise (i.e., a separate sum for each pc), including f_0 (which is a constant value equal to the cardinality) and complements (which give the same values as components 1–5). Divide the totals by 12.

This recovers the original pc-vector, as Figure 4 shows.

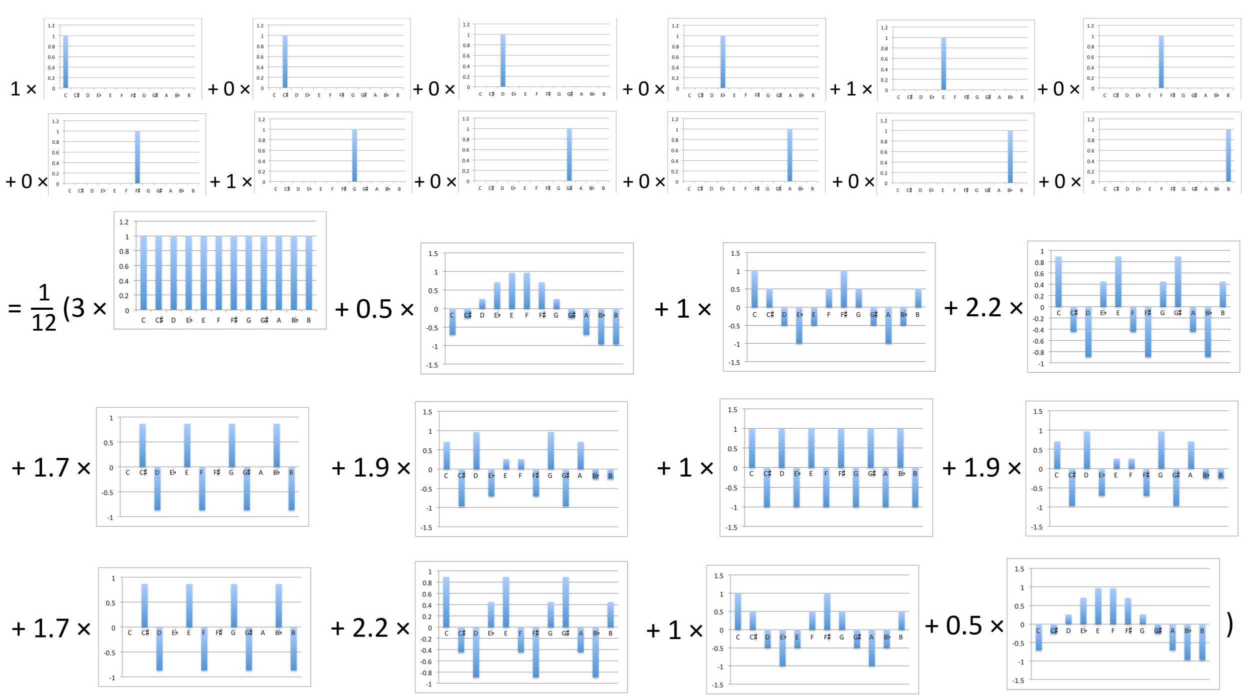

Figure 4: DFT of a C-major triad as a change of basis

We can think of the DFT, then, as a way of converting between two representations of the same object. One is the raw pc-vector, which simply tells us how much of each pc is present. The other, the DFT, represents the pcset (or multiset) as a sum of periodic functions. The reversibility of the process means that there is no information loss, and the values for Fourier components 1–6 are *independent*. That is, it is possible, in principle, to vary any one without affecting the others, so long as we allow for possible non-integer values in the pc-vector.

One virtue of the DFT is that it accomplishes Forte's goal of *isolating the interval content* of the set, which is contained entirely in the magnitudes of the components. Transposition, inversion, and the Z-relation affect only the phases. At the same time, preserving the phase information, the "residue" of interval content, as it were, resolves Clough's (1965) qualms with the interval vector as equivalence relation, the most important of which is the inscrutability of inclusion relations. As Section 3 will demonstrate, the DFT—and specifically *phase space distances*, one of the main analytical tools explored below—has a special relationship with common pc-content, a generalization of inclusion.

Another important virtue of the DFT is that its quantities relate closely to well-known musical concepts. For instance, f_5 is large for sets that are closely packed on the circle of fifths—i.e., diatonic subsets—and Ph_5 indicates which diatonic collection(s) it is a subset of. Therefore, we can refer to f_5 as the *diatonicity* of a pcset. By a similar reasoning, f_4 may be understood as the *octatonicity* of a pcset, and so on. Each component defines a distinct *harmonic quality*.

Quinn (2006–7) captures these harmonic qualities through the notion of prototypes, as shown in Figure 5. Notice that some prototypes are better representatives of the given component than others. A pc cluster is the best pcset representative of component 1, but a better representative would be a multiset sum of all clusters with a common center: (EF) + (D♯EFF♯) + (DD♯EFF♯G) + ... The same can be said for component 5, using the circle of fifths. Component 2 also has no perfect pcset representative. Hexatonic, octatonic, and whole-tone collections, on the other hand, are perfect representatives of components 3, 4, and 6 (though not at every phase: note that the hexatonic in Figure 5 should be weighted more towards (CEG♯).)

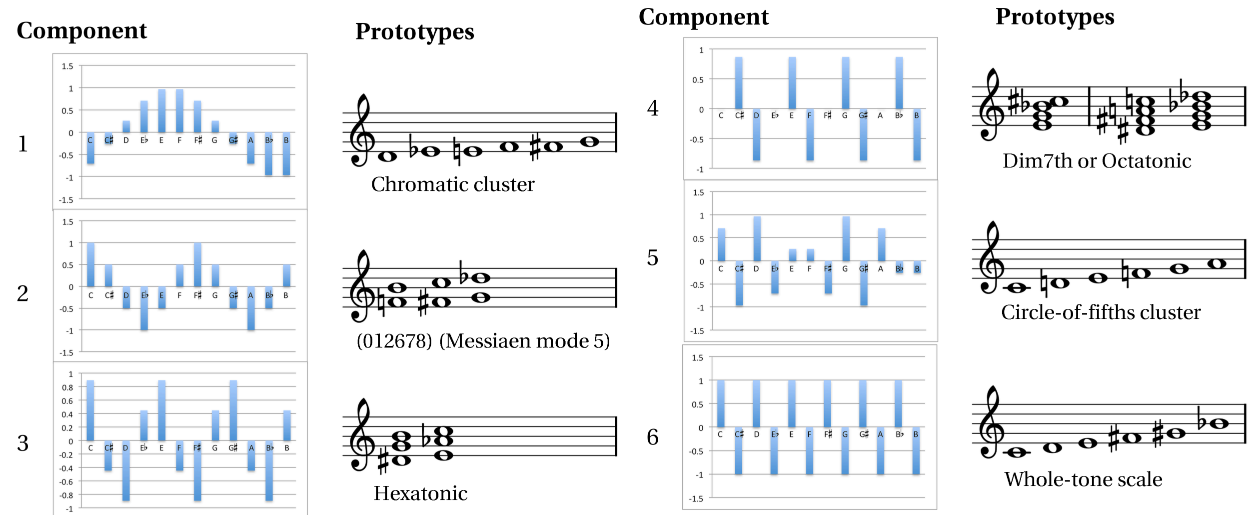

Figure 5: Prototypes for the six DFT components (based on the phase values of a C-major triad)

Quinn's generic prototypes are thus a good first approximation to each of the six harmonic qualities, which we may describe as (1) chromaticism, (2) quartal quality, (3) hexatonicity, (4) octatonicity, (5) diatonicity, and (6) whole-tone quality. All are readily recognized harmonic properties tying into a wealth of existing music theory and analysis except for f_2. The term "quartal quality" references the prototype of chords built from stacked tritones and perfect fourths. A brief analytical excursion in the next section will explore the meaning and use of f_2.

1.3 Quartal Quality and Crawford-Seeger's "White Moon"

Harmonies that exemplify quartal quality are hallmarks of early twentieth-century modernism, such as stacks of perfect and augmented fourths, Messiaen's modes 4 and 5 and their subsets (See Yust 2015a). A general avoidance of thirds and sixths (especially minor thirds and major sixths) is distinctive of quartal quality. Such thirdless harmony exhibits a characteristically ascetic dissonance.

As an example of how a composer may use f_2 in coordination with other harmonic qualities, consider Ruth Crawford Seeger's "White Moon" (*Sandburg Songs* no. 2). In the opening piano part (Figure 6), Seeger focuses on two harmonic qualities: she uses chromatic, high-f_1, harmony for its shimmering effect in the upper register. This quality dominates in the piano right hand and in the vertical combinations. However, in the harmonic content left hand, which plays the middle-register "melody," f_2 predominates. The third-avoidance of this harmonic material produces the effect of whiteness that conjures the subject of the poem.

Seeger marks the third line of the poem in her setting (Fig. 7, mm. 7–8) as especially significant through a striking new texture, a disjunct, declamatory vocal line punctuated by chords in the piano that use the lower register for the first time. The first of these is a whole-

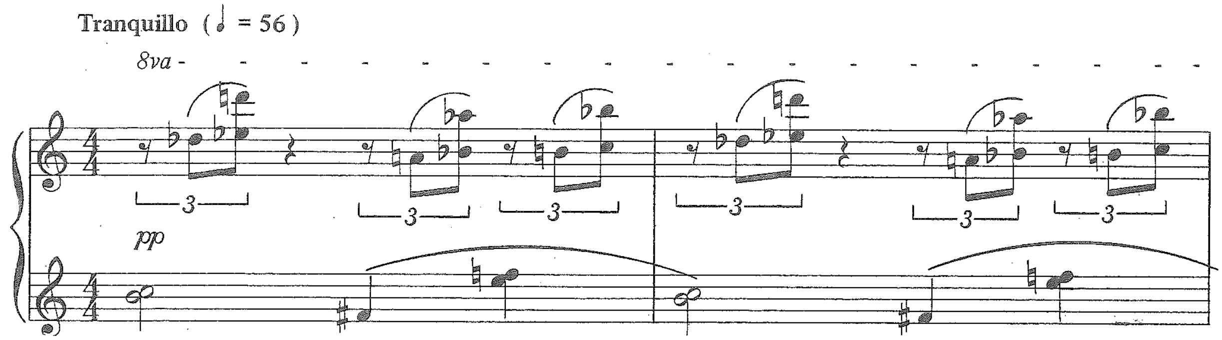

Figure 6: Ruth Crawford-Seeger, "White Moon," mm. 1–2

Figure 7: "White Moon," mm. 7–13

tone chord, a prototype of f_6. The second has a predominantly quartal (f_2) quality. Both contrast with the minor-third laden vocal line of the first phrase (Fig. 8) that evokes the human subjectivity of the woman described in the poem, who invests personal meaning in the objectively neutral, black-and-white landscape she witnesses in the nighttime scene. Seeger conjures the f_2 quality again at the end of the fourth line of the poem (Fig. 7, mm. 9–11).

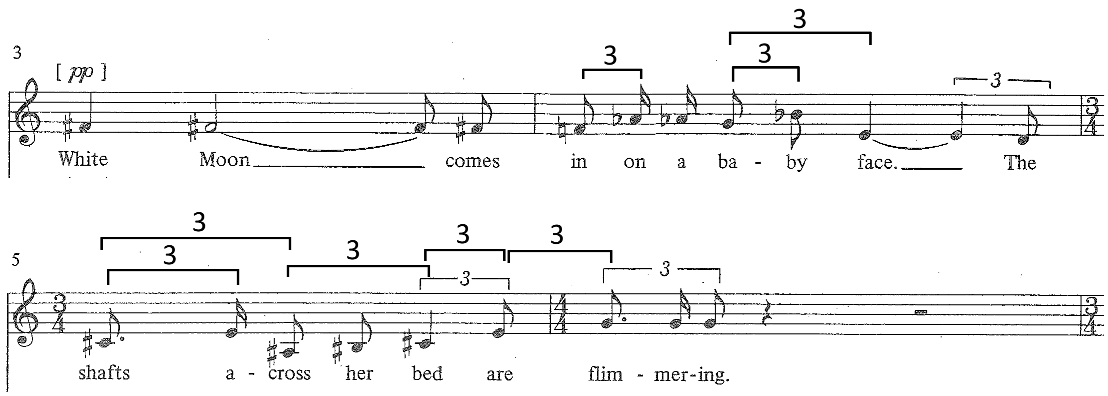

Figure 8: "White Moon," minor thirds in the vocal line of mm. 3–6.

Table 1 lists the pcsets involved and their squared DFT magnitudes. The size of components are expressed in squared magnitudes here and throughout the article, because they more comparable to familiar ic-vector values than raw magnitudes and therefore are more intuitive for analytical usage.[4]

Table 1: DFT magnitudes of collections in Crawford-Seeger's "White Moon"

	Pcset	$\lvert f_1\rvert^2$ $\lvert f_2\rvert^2$ $\lvert f_3\rvert^2$ $\lvert f_4\rvert^2$ $\lvert f_5\rvert^2$ $\lvert f_6\rvert^2$
Meas. 1–2, RH	(AB♭BCD♭EE♭)	⟨⟨ 13.9, 1, 1, 1, 0.07, 1 ⟩⟩
Meas. 1–2 LH	(EFF♯BC)	⟨⟨ 1, 13, 1, 1, 1, 1 ⟩⟩
Meas. 7 piano	(FGABC♯)	⟨⟨ 1, 1, 1, 1, 1, 25 ⟩⟩
Meas. 8 piano	(A♭AB♭D♭E♭E)	⟨⟨ 0.27, 9, 2, 3, 3.73, 7.5 ⟩⟩
Meas. 10–11	(GA♭CD)	⟨⟨ 0.27, 7, 2, 1, 3.73, 4 ⟩⟩
Meas. 12–13	(GA♭AD♭DE♭)	⟨⟨ 0, 16, 0, 0, 0, 4 ⟩⟩
Meas. 28, piano	(EF♯B♭C)	⟨⟨ 0, 4, 0, 4, 0, 16 ⟩⟩
Meas. 28–29, piano	(EFF♯GB♭C)	⟨⟨ 3, 7, 0, 3, 3, 4 ⟩⟩

The end of the song (Fig. 9) beautifully recalls the juxtaposition of f_6 and f_2 harmonic qualities in the setting of the third line, not by a succession of chords, but by a process of combination. The first part of the piano's concluding harmony is the conclusion of a process

[4] In fact, the squared magnitudes represent the *DFT of the interval vector*. This can be shown using the convolution theorem, which is discussed further in part 3.

that has been ongoing throughout the song, the gradual replacement of the shimmering ic1s of the piano opening by more "down to earth" ic2s. The text recounts a similar process, the descent of the moonlight to earth, or the conversion of an objective natural phenomenon (moonlight) into human meaning and beauty. The final, high-register ic2, (FG), however, converts this pure whole-tone chord (high-f_6), back to the characteristic f_2 quality from the opening. (Compare the last two entries in Table 1.) The resulting recollection of the shimmering moonlight provides an effective punctuation to conclude the song.

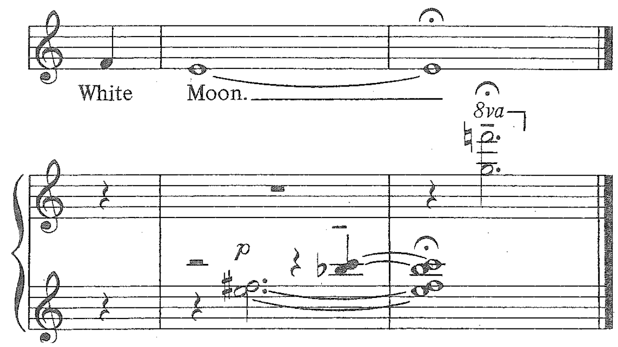

Figure 9: The ending of Crawford-Seeger's "White Moon," mm. 28–29.

2. Scale Theory and Diatonicity

In a review of Forte's *The Harmonic Organization of the Rite of Spring* that sparked a fiery exchange, Richard Taruskin (1979) took umbrage at Forte's neglect of possible remnants of tonal practice in Stravinsky's harmonic materials. Taruskin, reasonably, questions Forte's categorical separation of tonal and atonal repertoire and complains that "Forte fails to distinguish between atonal musics that originated in revolt against triadic functional tonality," i.e., Schoenberg, Berg, and Webern, "and those that are . . . rooted in it," particularly Stravinsky's. (118) Forte's (1985, 35–7) response is illuminating, justifiably protesting the violence that Taruskin's Roman numeral analysis of a passage from the *Rite* does to the musical surface, and the profusion of non-harmonic and omitted tones it presumes.

This dispute may appear to have little direct relation to the scale-theoretic approach to Stravinsky—octatonic-diatonic interaction—that Taruskin would soon, again contentiously, champion. (Taruskin 1986, 1987, 2011) Yet Forte's objections to Taruskin's harmonic analysis of the *Rite* apply similarly to virtually any scale-theoretic analysis, if perhaps not always so emphatically. In analysis, scale theory has a built-in circularity, asserting the primacy of certain collections (scales) for understanding the music, yet assuming such primacy to parse the music (into "harmonic" and "non-harmonic" tones in reference to often-incomplete scales). Straus (1984)—as well as Forte (1986, 324–5)—react to exactly this kind of circularity in Van den Toorn's (1983) theory. While it does not necessarily represent a logical fallacy (it could instead constitute a sort of bootstrapping process) such circularity does make scale-theoretic analyses difficult to evaluate.

Forte makes tellingly oblique reference to scale theory in his 1964 article on set theory, by citing Wilding-White 1961 (mentioned in the introduction above) in the following context:

> It is hoped that the theory presented here will be of use to the historian. Consideration of special properties of sets as well as of the special nature of their complexes and subcomplexes may suggest interesting hypotheses of historical development, style, and so on. (178)

For Forte, the *a priori* assumptions of scale theory belong exclusively to tonal analysis, so the set-theoretic properties of scales are of primary interest only for investigation of historical remnants of tonality in the twentieth century. Forte's set-theoretic analyses therefore collapse the three-tiered system of gamut–scale–musical surface to a two-tiered one, from gamut directly to the literal pitch material.

The reformulation of some scale-theoretic ideas using the DFT described below closes this gap. The DFT, like Forte's theory, can deal more directly with the musical surface, yet many of the harmonic qualities it identifies—diatonicity, octatonicity, whole-balance, hexatonicity—invoke the kinds of prototypes that scale theory takes as primary referential collections. It therefore validates many of the general ideas set forth in scale-theoretic analysis, yet with a procedure that is more easily evaluated and replicable. Sections 2.2–2.4 focus especially on diatonicity and diatonic phase spaces. Section 2.4 and part three consider the interactions of diatonicity with octatonicity and whole-tone qualities.

2.1: Scale Theory and Evenness

A foundational premise of scale theory is the limitation of significant collections to a handful of scales. Theorists have distinguished scalar collections by many different criteria, but for all of them *evenness* is a key consideration. We can define an evenness criterion using the DFT, not quite equivalent to any previously proposed method but essentially agreeing with all of them on a short list of important scalar collections. While it is hardly necessary to have a new evenness criterion, the one proposed here is useful for our subsequent discussion, because it draws precisely on those parts of the DFT (f_1 and f_2) that do not define harmonic qualities associated with concepts from scale theory. Since the DFT components are all independent, this gives a simple way to represent the assumptions of scale theory: ignore f_1 and f_2.

Even collections should be relatively balanced on the pc-circle, which means having low $|f_1|$s. However, low-$|f_1|$ collections may still seem uneven if they are imbalanced on a half-cycle, so even collections should also have low $|f_2|$s. To put it another way, even collections are those that least resemble a semitone, minimizing just those components that are high for ic1. Table 2 lists collections of 4–8 notes such that $|f_1|^2 + |f_2|^2 < 2$. These include all of the most familiar scalar collections. (Note that the seven- and eight-note collections are complements of the four- and five-note collections.)

Table 2: Set-classes of 4–8 notes with minimum $|f_1|^2+|f_2|^2$

	Collection	Set Class	Int.NF	Example	$\lvert f_1\rvert^2$	$\lvert f_2\rvert^2$	$\lvert f_3\rvert^2$	$\lvert f_4\rvert^2$	$\lvert f_5\rvert^2$	$\lvert f_6\rvert^2$
4	Diminished 7th	[0369]	(3333)	BDFA♭	0	0	0	16	0	0
	Dominant 7th	[0258]	(2334)	GBDF	0.27	1	2	7	3.73	4
	Minor 7th	[0358]	(3234)	DFAC	0.54	0	4	4	7.46	0
5	Pentatonic	[02479]	(22323)	CDEGA	0.07	1	1	1	13.93	1
	Dom.7th-add6	[01469]	(13233)	GBDEF	0.27	1	5	7	3.73	1
	Dominant 9th	[02469]	(22233)	GABDF	0.54	0	1	4	7.46	9
6	Whole tone	[02468t]	(222222)	CDEEF♯G♯B♭	0	0	0	0	0	36
	Hexatonic	[014589]	(131313)	CC♯EFG♯A	0	0	18	0	0	0
	Harm. minor subset	[013589]	(122313)	CDEFG♯A	0.27	1	8	3	3.73	4
	Acoustic subset	[013579]	(122223)	EFGABC♯	0.27	1	2	3	3.73	16
	Dorian/Lydian hex.	[013579]	(212223)	DEFGAB	0.80	1	0	3	11.20	4
	Guidonian hex.	[024579]	(221223)	CDEFGA	1.07	0	2	0	14.93	0
7	Diatonic	[013568t]	(1221222)	CDEFGAB	0.07	1	1	1	13.93	1
	Harmonic minor	[0134689]	(1212213)	ABCDEFG♯	0.27	1	5	7	3.73	1
	Acoustic	[013468t]	(1212222)	GABC♯DEF	0.54	0	1	4	7.46	9
8	Octatonic	[0134679t]	(12121212)	GA♭B♭BC♯DEF	0	0	0	16	0	0
	Harm. minor + $\sharp\hat{6}$	[0124578t]	(11212122)	ABCDEFF♯G♯	0.27	1	2	7	3.73	4
	Harm. minor + $\flat\hat{7}$	[0124579t]	(11212212)	ABCDEFGG♯	0.54	0	4	4	7.46	0

The scale-theoretic preference for even collections reflects a presupposition that no region of the pc-circle is privileged over any other. Hence, a high $|f_1|$ or $|f_2|$ collection on the surface of the music will be assumed to be a subset of a larger collection that reduces $|f_1|$ and $|f_2|$ by filling in gaps. Disregarding high values for f_1 and f_2 effectively equates smaller scalewise collections with larger, more well distributed collections that have similar values on f_3–f_6, by implicitly filling gaps and deemphasizing conflicting notes (i.e., potential non-harmonic tones).

The strength of $|f_5|$, or diatonicity (imbalance on the circle of fifths) for many of these collections is notable. All of the relatively even five- and seven-note collections have a high $|f_5|$, a strong diatonicity. Therefore position in the phase-space for f_5, the Ph$_5$-cycle or circle of fifths, should typically be a salient, musically significant factor for music based on these collections. Closely related collections, those that share a high degree of common pc-content or are related by efficient voice leading, will therefore tend to be close on the Ph$_5$ cycle. The exception to high $|f_5|$ values are the symmetrical collections, the whole-tone, hexatonic, and octatonic, where f_5 and f_1 are both zero. These collections are the principal prototypes for f_6, f_3, and f_4, and their role in scale theory will be discussed further in part 3.

2.2 Diatonicity in Satie's "Idylle"

Composers like Erik Satie and Claude Debussy founded a musical style in which prominent diatonicity is persistent, largely absent the triadic dimension that defines conventional tonality in coordination with diatonicity (as demonstrated by Amiot 2013 and Yust 2015c). Activity in the diatonic dimension in this music may substitute for the triadic motion of common-practice tonal harmonic progressions. In the following I will conduct two analyses along parallel tracks, one using a voice-leading network approach following Tymoczko (2004, 2011), the other using Ph_5-space. Comparison of these illustrates how motions in phase space reflect, and simplify, inferable voice leadings.

Satie's "Idylle" (*Avant dernières pensées* no. 1), Figure 10, is a miniature study in polytonality. Differences in diatonicity are significant both horizontally, between successive ideas in the right hand, and vertically, between the unchanging ostinato and the shifting diatonicity of the right hand.

Table 4 lists diatonicity values for the ostinato and a series of right-hand collections, and Figure 11 locates these collections around the Ph_5 cycle. Nicknames for the collections borrow from Satie's inscriptions in the score. The Ph_5 cycles in Figure 11 are labeled by the location of diatonic collections, indicated using Hook's (2011) pitch-center-neutral method of giving the number of sharps or flats. Hook's method works for any spelled seven-note collections (such as acoustic or harmonic minor collections).

Table 4: Diatonicity of collections in Satie's "Idylle"

	Pcset	**Right hand alone**	**Combined with LH**
		$(\lvert f_5\rvert^2,\ Ph_5)$	**$(\lvert f_5\rvert^2,\ Ph_5)$**
Ostinato	(ABCD)		$\langle\langle (6, 2.5)_5 \rangle\rangle$
Ruisseau (*a*)	(DEF♯A)	$\langle\langle (8.47, 3.67)_5 \rangle\rangle$	$\langle\langle (26.12, 3.14)_5 \rangle\rangle$
Ruisseau (*b*)	(EF♯G♯ABC♯)	$\langle\langle (14.93, 5.5)_5 \rangle\rangle$	$\langle\langle (20.93, 4.42)_5 \rangle\rangle$
Bois secs	(F♯G♯AA♯BC♯DEE♯)	$\langle\langle (3.73, 5.5)_5 \rangle\rangle$	$\langle\langle (9.73, 3.78)_5 \rangle\rangle$
Arbres (*a*)	(DFA)	$\langle\langle (3.73, 1.5)_5 \rangle\rangle$	$\langle\langle (17.93, 2.06)_5 \rangle\rangle$
Arbres (*b*)	(CE♭EFGA♭B♭)	$\langle\langle (8.46, 10.7)_5 \rangle\rangle$	$\langle\langle (8.46, 0.33)_5 \rangle\rangle$
Le Soleil	(F♯G♯ABC♯DE♯)	$\langle\langle (3.73, 5.5)_5 \rangle\rangle$	$\langle\langle (9.73, 3.78)_5 \rangle\rangle$
Froid dans le dos	(E♭FGA♭B♭CD)	$\langle\langle (13.93, 11)_5 \rangle\rangle$	$\langle\langle (15.20, 0.25)_5 \rangle\rangle$
La Lune	(EF♯GB^{2}D)	$\langle\langle (13.93, 4)_5 \rangle\rangle$	$\langle\langle (32.86, 3.41)_5 \rangle\rangle$
Last note	(C)	$\langle\langle (1, 0)_5 \rangle\rangle$	

The first idea, *Ruisseau*, moves progressively sharpward from the ostinato, while the second idea, *Arbres*, does the same on the flat side. This sets up a swinging pattern of sharp–flat alternation that continues in the second part of the piece.

An analysis of the piece using scale networks is more involved, because multiple possible scalar supersets must be considered for many of the collections. The ostinato, for instance, may be a subset of two possible diatonic scales, the 0-diatonic and 1♯-diatonic, but casting a wider net, we could also locate it as a subset of the 1♭-acoustic, 2♯-acoustic, or octatonic$_{2,3}$. The diatonicity value is directly in between the values for the 0- and 1♯-diatonics (or the 1♭- and 2♯-acoustics) at $Ph_5 = 2.5$.

I. Idylle


Modéré, je vous prie.
(Moderately, I beg you.)
Que vois-je?
(What do I see?)
Le Ruisseau est tout mouillé;
(The brook is all wet;)
La basse liée, n'est-ce pas?
(Basso legato, don't you think?)
et les Bois sont inflammables et secs comme des triques.
(and the wood dry and flammable as a switch.)
Mais mon cœur est
(But my heart is
tout petit.
very small.)
Les Arbres ressemblent à de grands peignes mal faits;
(The trees look like great misshapen combs;)

Figure 10: Satie, "Idylle," Avant Dernières Pensées *no. 1*

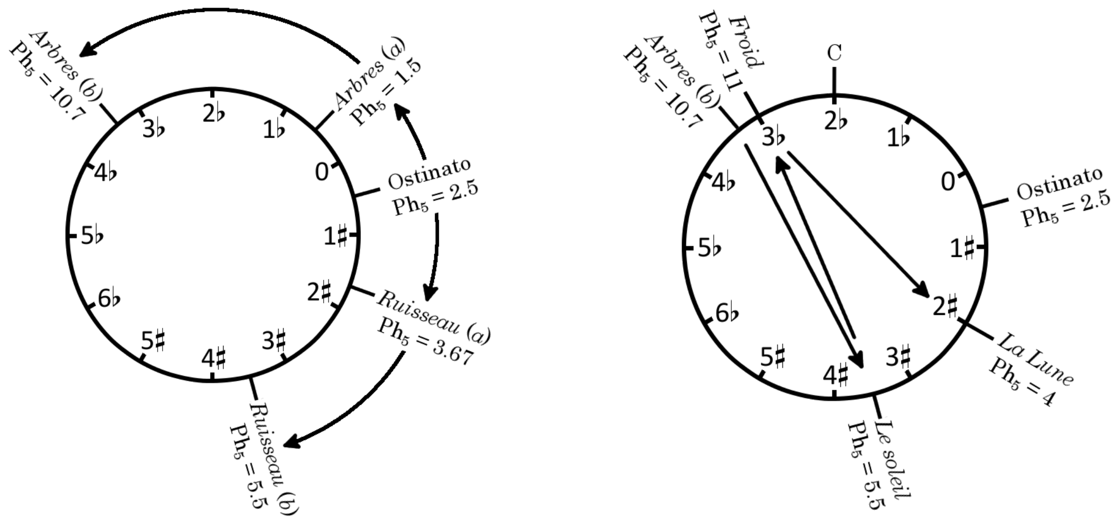

Figure 11: Collections from Satie's "Idylle" on the Ph_5-cycle.
Points on the cycle are labeled with the positions of diatonic collections.

Figure 12 diagrams a voice-leading based analysis of the first four ideas, which includes voice leadings between the right hand collections above and "vertical voice leadings" or polyscalar clashes, below, along with a listing of possible collections. Because of the subset problem, multiple collections are often inferable. The four or five possible collections for the ostinato are shown, then progressively narrowed down to give the minimal conflict with the right hand.

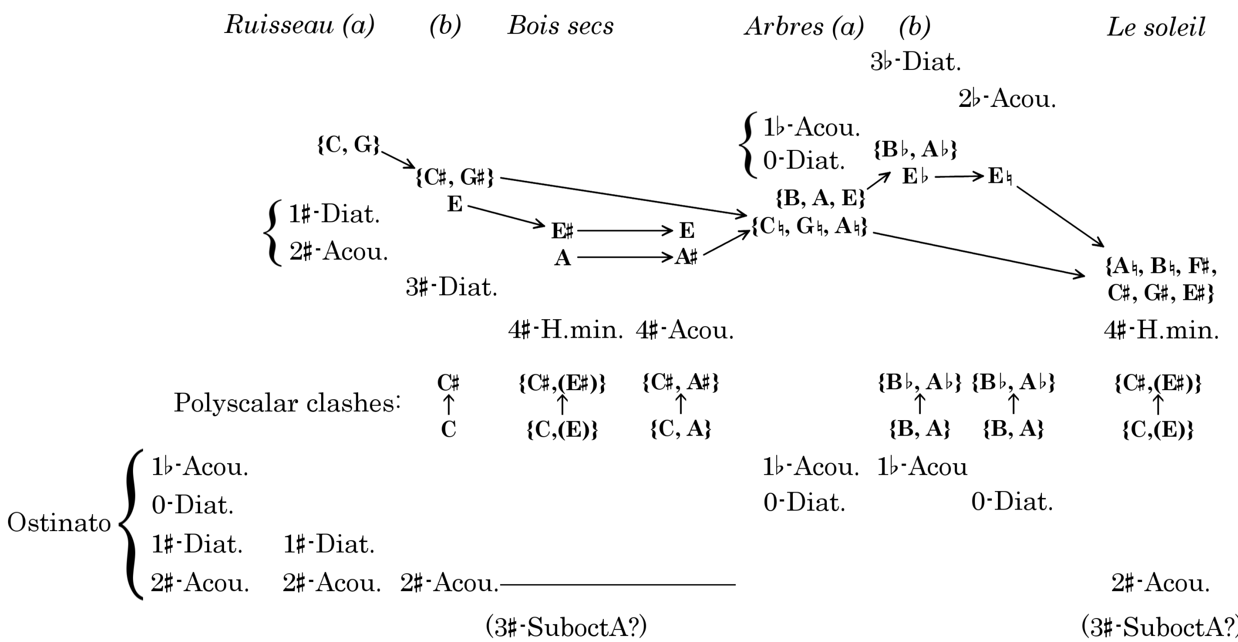

Figure 12: A description of Satie's "Idylle" using voice leadings between scalar collections.

The diatonicity values present essentially the same information as the voice-leading analysis in a somewhat generalized form. For example:

(1) Although *Ruisseau (a)* is not in direct conflict with the ostinato, it is a sharpward move in the sense that it rules out the more flatward possible supersets, 0-diatonic and 1♭-acoustic. This is shown by its higher Ph_5 value (3.67 > 2.5).

(2) *Ruisseau (b)* adds additional pitch classes, moving it further to the sharp side and making an explicit polyscalar conflict with the ostinato (C→C♯). The other new note, G♯, does not present an explicit clash, but the elimination of the 1♯-diatonic as the ostinato collection constitutes further sharpward movement. *Ruisseau (b)* is a subset of 3♯- and 4♯-diatonics, and is halfway between them in Ph_5-space (Ph_5 = 5.5).

(3) Satie's next gesture, *"Bois secs,"* is a little more complicated to describe in scale-theoretic terms, although an underlying language of familiar tonal scales is nonetheless evident. The idea begins with a scalewise ascent consistent with the previous 3♯-diatonic hexachord. The next part of the gesture requires scalar shifts in Figure 12 first to a 4♯-harmonic minor, then to a 4♯-acoustic, with F and B♭ understood as E♯ and A♯. The whole gesture is therefore still considerably sharp of the ostinato, creating sharpward polyscalar clashes. Hence its Ph_5 value (5.5) is the same as that of the previous gesture, *Ruisseau (b)*. The difference between these is rather in their degree of diatonicity: *Ruisseau (b)* is diatonically concentrated in a Guidonian hexachord, while *Bois secs* is spread out. This is reflected in the considerable decrease in overall diatonicity between them, from $|f_5|$ = 14.93 to 3.73.

The reduced diatonicity is also apparent in the voice-leading analysis. Although there is still only one literal polyscalar clash between the ostinato and the 4♯-harmonic minor (C→C♯), the 2♯-acoustic, the furthest sharp hypothetical superset for the ostinato, does not contain an E♯. It can only be included by expanding the range of possible scalar collections to include the seven-note octatonic subsets, or "suboctatonics," to use Hook's (2011) term. The ostinato (ABCD) plus the (E♯F♯G♯) of the right hand is Hook's 3♯-suboctA.

The respelling of E♯ and A♯ reflects their position with respect to the ostinato. The Ph_5 value of these pcs (10 and 11) is closer to 5.5 on the sharp side. Satie's spelling, on the other hand, is closer to what would be suggested by reference to the context created by the ostinato. Oriented from its Ph_5 value of 2.5 they are closer in the flatward direction, as F and B♭.

(4) As we saw in Table 2 in the previous section, the standard seven-note scalar collections are also the highest in diatonicity. The reasons for this can be explained by reasoning from voice-leading spaces (see Yust 2013a). For the most even, highest-diatonicity scales (diatonic and acoustic), the single-semitone voice leadings correspond directly to Ph_5 shifts of 1. As diatonicity gets slightly lower, as with the harmonic major/minor collections, the relationship of Ph_5 to voice leading starts to get more complex. A shift

away from diatonicity has a reduced effect on Ph$_5$, where a shift towards diatonicity has a correspondingly larger effect.

The 4♯-harmonic minor appears explicitly as *Le soleil*. This is a single-semitone voice leading from a 3♯-diatonic, E → E♯, but only 0.5 sharpward from it. The distance of E♯ from the diatonic center of the collection weakens its influence of the position of the collection, as shown in Figure 13. Therefore *Le soleil* occupies the same Ph$_5$ position as *Ruisseau (b)* (which is ambiguous between 3♯- and 4♯-diatonics) and *Bois secs*. However, like *Bois secs*, it has a much smaller $|f_5|^2$ (3.73 versus 14.93).

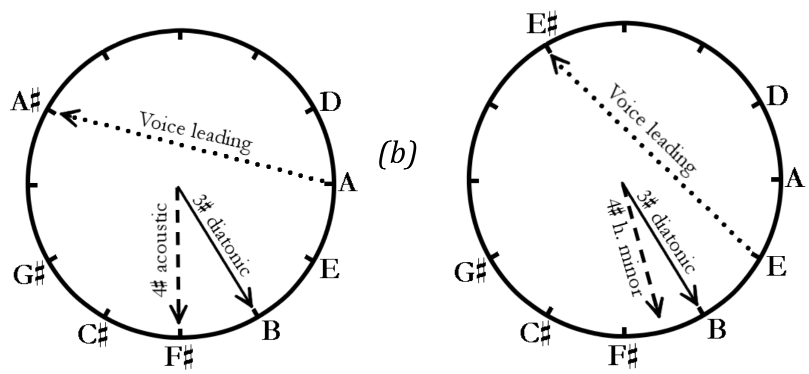

Figure 13: Effects of single-semitone voice-leadings on Ph$_5$

The evolution of Ph$_5$ over the course of the piece (in Figure 11) reveals Satie's overarching method: following the initial gradual sharpward move, the right hand collections swing like a pendulum from the sharp side to the flat side and back, always anchored off of the stable Ph$_5$ = 2.5 of the ostinato. The shifts mirror the constant back and forth of dichotomous imagery in Satie's inscriptions (wet–dry, small–large, warmth–cold), and at a slightly deeper level, the fickle narrator's vacillating mood. The swings are roughly aligned to make what would seem like the natural associations: dry, small, and warm with the sharp side and wet, large, and cold with the flat side. The last two inscriptions vaguely suggest that the narrator has tears in his or her eyes, and the description of the brook as "soaked through to the bones" ("trempé jusqu'aux os") recalls the wet, cold imagery (and also rhymes with "froid dans le dos," a chill down the spine). Satie then neatly provides a flat-side punctuation to match this text by isolating the furthest flatward note of the ostinato, C♮ (Ph$_5$ = 0), the note that has been central to polyscalar conflicts with all the sharp-side right hand gestures.

To some extent this compositional plan might be discerned, with some effort, from the voice-leading analysis in Figure 12. But there are two evident analytical advantages of explaining it through diatonicity. First, f_5 succinctly captures the basic features of interest, sharpward/flatward relationships (both horizontal in time and vertical), and overall degree of diatonicity. Second, it bypasses the tedious and tendentious step of inferring collections that are incomplete, ambiguous, or adulterated with extraneous pitch classes. The DFT works directly on pcsets and multisets of all cardinalities. For ambiguous subsets of multiple collections, like Satie's ostinato, it falls between possible scalar supersets. For supersets of multiple collections, like *Bois secs* or *Arbres (b)*, it negotiates between possible subsets. In both cases, it favors the more even subsets or supersets, like diatonic scales, over less even ones, like harmonic minors or more exotic suboctatonics, etc.

Another important aspect of the DFT is demonstrated by the last column of Table 4, which calculates f_5 for the polyscalar combinations. We have already seen that high degrees of polyscalar clash are reflected in big differences in Ph_5 values. These Ph_5 distances also determine the resulting $|f_5|$ of the combination. When pcsets are far apart their sum will have an attenuated $|f_5|$, while nearby collections reinforce one another's diatonicity, resulting in a high $|f_5|$.[5] Therefore, when the Ph_5 values of the right hand are close to the ostinato—i.e., a low degree of polyscalarity—as in *Ruisseau* (*a*), *Arbres* (*a*), and *La lune*, the combined $|f_5|^2$ in the last column of Table 4 is considerably higher than that of the individual parts. At the largest distance (*Arbres* (*b*)) the diatonicity of the combination is actually lower than that of the parts.

2.3 Diatonicity in Debussy's "Le vent dans la plaine"

In the preceding analysis we found that Ph_5-cycle plots may reflect voice leading between heptatonic scalar collections. Sharpward voice leadings move the collection clockwise and flatward ones move it counterclockwise. For the most even collections, a Ph_5 distance of 1 is equal to a one-semitone shift. Voice leadings that decrease the diatonicity of the collection (moving towards less even collections) lead to smaller Ph_5 distances. We also saw how diatonicity values bypass the explicit inference of supersets or subsets. The Ph_5 value of a subset of multiple collections averages those of the possible scalar supersets, weighted towards the more even ones. Non-scalar pitch-classes tend to be Ph_5 outliers and therefore have minimal influence over Ph_5 of the total collection.

Using diatonicity values therefore not only reproduces many of the conclusions we might derive from a scale-theoretic analysis, but also streamlines some of the *ad hoc* reasoning needed when the music does not present scales unambiguously in their entirety with no additional notes. This analytical consolidation is more than an expedient, though; it is essential to advancing the conclusions of scale theory, widening its applicability and integrating it with other theoretical perspectives. With a DFT-based methodology we can also isolate elements *within* a given scalar collection and characterize their relation to the whole, and representativeness of it.

Among the most compelling analytical applications of scale theory are Dmitri Tymoczko's (2004; 2011, Ch. 9) analyses of Debussy using scale networks. Debussy's prelude "Le vent dans la plaine," was well chosen by Tymoczko (2004) as a piece that features important scalar collections (diatonic, acoustic, and whole-tone) and how they may be connected by maximally smooth voice leading. The first section, mm. 1–20, features clear use of scalar

[5] A formula for pcset addition, which demonstrates this principle mathematically, can be found in Yust 2015a.

collections, all of which are closely related (6♭-diatonic, 5♭-diatonic, and 7♭-acoustic). Figure 14 gives the beginning of the piece and the passage from measures 19–34. The section from measure 21 to 27 features clear use of whole-tone collections. All of the collections involved exhibit smooth voice leading except in two places: from measure 20 to 21, where Debussy drops his key signature and makes a sudden change of spelling from flats to naturals, and from 24 to 25, where a large shift results from a sequence by ascending semitone.

Figure 14(a): Debussy, "Le vent dans la plaine," mm. 1–4

The shift from m. 20 to m. 21 is somewhat tricky to manage from the scale-theoretic perspective, as is evident from Tymoczko who writes that measure 21 "involves a complex set of scalar affiliations" (253). The problem is that a complete scale is not present immediately after the large collectional shift, leaving some ambiguity as to what collection should be inferred. The set (DFGAB) is a subset of the 0-diatonic as well as the 1♯-acoustic. Tymoczko analyzes it both ways, with the 1♯-acoustic existing at a slightly larger-scale level, since a D♭ (= C♯) appears towards the end of the measure. He even shows a voice leading C→C♯ between these collections, even though the C is merely inferred on the assumption that the collection in the first part of the measure should be diatonic.

We can explain what happens more literally using Ph_5. The Ph_5 position of (DFGAB) is in fact the same as the 0-diatonic. This vindicates Tymoczko's intuition that the collection is D Dorian, not D melodic minor, until the D♭ appears literally, even though the 1♯-acoustic is closer to the preceding collection (the 7♭-acoustic, or 5♯-acoustic) on the Ph_5-cycle, meaning that scalar inertia would favor a 1♯-acoustic label for the entire measure. (That is, the D♭/C♯ might be presumed to carry over from the previous 7♭/5♯-acoustic collection.) See Figure 15(a).


19
21
pp
23
25
pp
27
f
p
pp
29
pp
f
p
f
p

Figure 14(b): Debussy, "Le vent dans la plaine," mm. 19–34

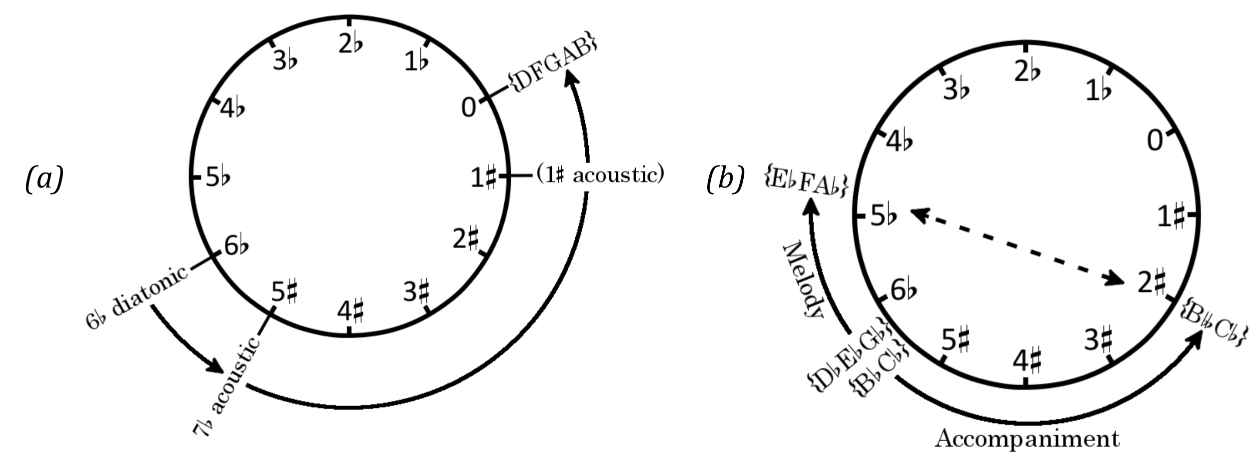

Figure 15: (a) Changes of scalar collection in mm. 19–21 in Ph_5-space, (b) Ph_5 evolution of the ostinato

The special relationship between phase distances and overall component magnitudes noted in the last section is useful for understanding what happens in the first 20 measures of "Le vent dans la plaine." Two elements close together in Ph_5 reinforce one another, to a high $|f_5|$, while ones far apart neutralize one another's diatonicity, leading to low $|f_5|$. The opening of the piece pointedly juxtaposes these two situations in the high-diatonicity subject and the low-diatonicity countersubject. The ostinato morphs over the course of the piece, moving towards a higher diatonicity when the B♭→B♭♭ voice leading changes it into a major second $|f_5|^2 = 3$.

The change from B♭-C♭ to B𝄫-C♭ has another effect, though, which is to pull the ostinato away from the diatonic center of the melody (from $Ph_5 = 7.5$ to 4), as shown in Figure 15(b). This is exacerbated when the melody is subsequently transposed by whole step, which moves it from $Ph_5 = 7.3$ to 9.3. The ostinato and melody at this point are separated by a phase distance of approximately 5, which corresponds to the interval of a semitone (i.e. it is the distance between T_1-related pcs or pcsets). Thus, while strengthening the diatonicity of the ostinato, Debussy replaces the semitonal conflict that occurred *within* the ostinato to a conflict *between* the ostinato and melody. The result is a weakening of diatonicity in the sum total of ostinato and melody. This weakening of overall diatonicity is important to the transition into the whole-tone materials that follow, which will be discussed further in the next section.

The process of increasing the diatonic strength of the ostinato culminates in measure 28 where the ostinato interval becomes the *fifth* of a G♭ major triad, $|f_5|^2 = 3.73$. Overall, then, this motive moves gradually from a soft-focus to a sharp diatonicity, and at the same time, as it comes into focus it also swings around more forcefully. This G♭ major triad occupies the same phase-space position ($Ph_5 = 7.5$) as the initial B♭-C♭ ostinato, constituting a return, in a sense, but one in which the protagonist has been radically transformed. The story is not completed, then, until after the ostinato resumes its initial dissolute state, which occurs suddenly in measure 34, but on a different semitone (G♯-A).

The passage starting in measure 28 features a different texture from what precedes it. Here Tymoczko's analysis stops short, because the music that follows is, as he says, "largely non-scalar." Indeed, at this point, the scale-theoretic framework becomes unmanageable, because it would involve both inferring collections from isolated triads, and, in other places, dealing with large collections of as many as nine pcs, none evidently treated as "non-scale-tones." This results in explanatory lacunae: for example, Debussy continues a motivic process over the scalar to non-scalar break, with the new idea of m. 23 (itself recalling the left hand of mm. 9–12) recast with harmonic thickening and contrary motion in mm. 29 and 32. While the scale-theoretic analysis gives important insight into Debussy's progressive development of the accompanimental motive and the main theme, the development of this parallel-fifths motive is oddly resistant even though it goes through a similar process.

For a diatonicity-based analysis, though, "non-scalarity" presents no barrier. Figure 16 shows how Debussy's collections move around the Ph_5-cycle in measures 22–28 and 28–34. In the earlier passage, Debussy stratifies the Ph_5 relationships so that smaller oscillations can occur at a more local level while the sequential transposition, an overall shift of –5, operates at a larger level. The collections in measures 22–24, before the sequential transposition, mimic the precise Ph_5 relationships of the D–E–G motive, major second (2) + perfect fifth (1) = descending minor third (3). In other words, Debussy "explodes" the harmonic content of his main motive here, turning it from a relationship between individual pcs to one between large collections. This begins a process of increasingly expansive and rapid motion in Ph_5-space.

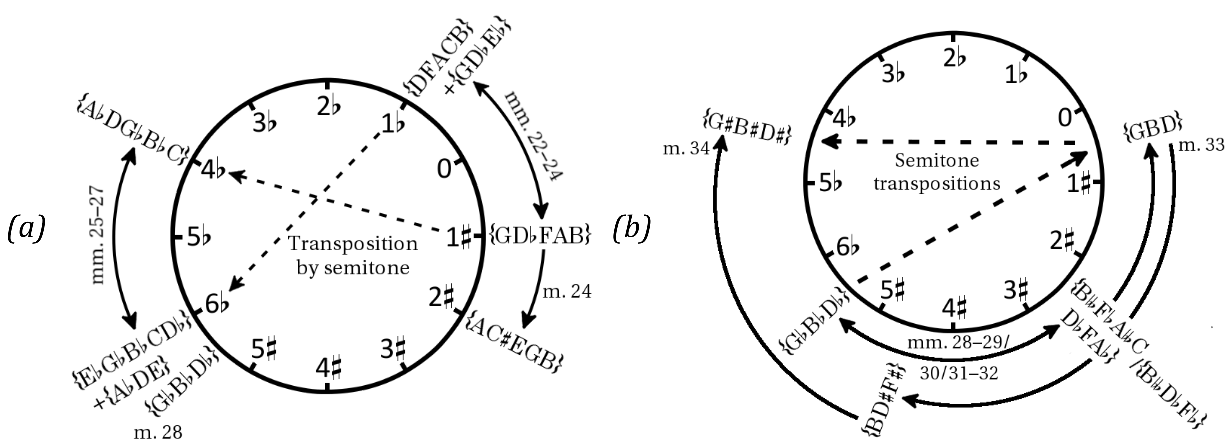

Figure 16: Ph_5 relationships between harmonic elements of mm. 22–28 (a), and mm. 28–34 (b)

The sequential transposition in measure 25 continues the process. Transposition by semitone invokes another motivic element of the piece, the accompaniment of the opening passage, which acts like a sort of countersubject to the subject, the melodic idea based on an (025) trichord. The accompaniment's interval of a semitone, a large Ph_5-distance of 5, is transformed from a melodic dyad into an interval of transposition. The high diatonicity of the subject ($|f_5|^2 = 5.73$) and low diatonicity of the countersubject ($|f_5|^2 = 0.27$) makes this procedure especially effective. The Ph_5-distant semitone transposition allows for two Ph_5-compact representatives of the (025) motive in mm. 22–27 to occupy distinct regions of diatonic space, as can be seen clearly in Figure 16(a).

The neatness of the motivic transformations in this earlier passage make room for another stage to Debussy's process of diatonic dissolution. This is realized in the next passage, which lurches more violently around the Ph_5 cycle, as is evident in Figure 16(b), so that now the semitonal relationships and those derived from the subject begin to blur together. The passage in mm. 28–29 continues to present the motive literally in the left hand, as in the preceding phrases (mm. 22–23, 25–26), and also, again like the preceding phrases, reflects the motivic Ph_5 relationships in its total harmonic content. Here, the Ph_5 spread of 3 appears in the relationship between the G♭ major triad and the contrasting harmonic content over the B𝄫–D♭ bass line in the latter part of measure 29. This relationship corresponds to a minor-third transposition, and therefore is exactly the same one between the G♭ and B𝄫 triads in measure 30. That is, measure 30 is an intensification (because of the higher diatonicity of the simple B♮ triad) of the Ph_5 motion in measure 29. Because of the complexity of the harmonic content of measure 29, the essential equivalence of the two measures would be hard to demonstrate without the DFT.

The climax of the piece in measures 33–34 applies semitonal transposition (the sublimated countersubject) to the triadic outburst of measure 30. The semitonal progression from G♭ major to G major to G♯ major fills in an overall major second transposition of the minor-third related triads (B𝄫–G♭/B–G♯). This major second relation also appears on a yet grander scale in the transposition of the initial B♭–C♭ accompaniment down a whole step to G♯–A in measure 34. In other words, the diatonic agent (major second) and the anti-diatonic agent (semitone) once again swap roles. At this moment where the semitone once again becomes a melodic interval, the major second becomes an interval of transposition, relating entire passages on a formal level. This happens through what we might see as enharmonic reality impinging on an over-exuberant two-fold application of semitone transposition. A single semitone creates an impression of Ph_5 distance, but successive semitones cycle the space to create the uncannily *small* diatonic distance of a major second, effectively building small distances out of large ones.

The re-consolidation of the accompanimental motive, the countersubject, in measure 34 reverses the process charted in the exposition. In measure 34 the triads generated by the violent swings are consolidated into individual pitch classes. The pitch-classes of the resumed minor-second ostinato, G♯ and A, individually occupy approximately the same positions as the preceding G♭ major triad (Ph_5 = 7.5, approximated by Ph_5(G♯) = 8) and its semitone transposition to G major (Ph_5 = 2.5, approximated by Ph_5(A) = 3) on the downbeat of m. 33. The result is an ostinato (G♯-A) two paces away from the initial B♭-C♭ position. The process of thematic return is then completed by a retransition (mm. 34–43) that returns the motivic materials to their original transposition.

As this analysis demonstrates, in Debussy's compositional style, the manipulation of the diatonic dimension of harmony is a basic and highly significant factor. The use of various scalar collections, relatively even heptatonic collections with high diatonicity values and symmetrical collections with null diatonicity values, is a notable outcome of this special sensitivity to diatonicity. However, this significance of diatonicity is not limited to the use of scalar collections; individual pitch classes, dyads, and chords can also be assessed for their Ph_5 position and diatonic strength. Doing so makes more integrated and more comprehensive harmonic analysis of this music possible.

2.4 Diatonicity and Octatonicity in Stravinsky

Stravinsky begins his *Three Pieces for String Quartet* (1914) with a concise movement that, with its utter simplicity of construction, sparingly conveys the most essential features of his unmistakable style. The first violin repeats a small set of terse melodic motives in metrically irregular succession, never venturing beyond the narrow pitch set shown in Figure 17. The cello repeats an ostinato in pizzicato, also irregular, and also using a small set of pitches, and the viola plays a drone throughout. The second violin, meanwhile, intermittently interjects with a descending four-note line.

Figure 17: Summary of the three melodic parts in Stravinsky's Three Pieces for String Quartet, *first movement*

The pitch materials of the movement are therefore minimal. They are also striking, in that Stravinsky limits each instrument to a set of three or four pitches that are scalewise diatonic. Each instrument is a possible subset of at least two diatonic collections, yet the pitch content of no two instruments could be the subset of any single diatonic collection. The f_5 for each instrument explains why. As shown in table 5, f_5 is prominent for each instrument, because they are all diatonic subsets. However, they are distributed relatively evenly around the circle of fifths, as illustrated in Figure 18, with the melodic first violin somewhat more isolated than the others. Stravinsky's method is to construct a texture out of parts that are individually diatonic, but combine to make a markedly non-diatonic whole.

Table 5: DFT magnitudes of collections in Stravinsky's Three Pieces, *first movement*

	Pcset		$\lvert f_1\rvert^2$	$\lvert f_2\rvert^2$	$\lvert f_3\rvert^2$	$\lvert f_4\rvert^2$	$\lvert f_5\rvert^2$	$\lvert f_6\rvert^2$	
First violin	(GABC)	⟨⟨	5,	1,	2,	1,	5,	4	⟩⟩
Second violin	(C♯D♯EF♯)	⟨⟨	6,	0,	0,	4,	6,	0	⟩⟩
Cello	(CD♭E♭)	⟨⟨	5.73,	1,	1,	3,	2.27,	1	⟩⟩
Sum	(C^{2}C♯^{2}D♯^{2}EF♯GAB)	⟨⟨	8.46,	3,	5,	19,	1.54,	9	⟩⟩

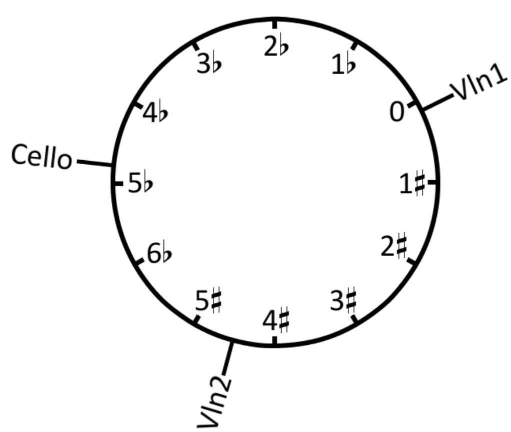

Figure 18: The three melodic instruments on the Ph_5 cycle

There is a tradition in Stravinsky analysis of explaining these non-diatonic elements of his harmonic language, especially in this early, so-called "Russian" period music, by invoking the octatonic scale (e.g., Van den Toorn 1983). The problem with this explanatory mode is that it easily dismissed as the arbitrarily imposition of a theoretical system by means of an overly loose interpretive methodology. The problem has led the idea of Stravinsky's octatonicism into a quagmire of unresolved contention, as evidenced by Tymoczko's (2002, 2003) dispute with Van den Toorn (2003), more recently rehearsed by Taruskin (2011) and Tymoczko (2011b). The latter exchange is particularly striking for Taruskin's retreat into radical historicism, claiming in essence that the method of octatonic analysis is justified regardless of whether the music provides any clear evidence of octatonicism.

The movement from *Three Pieces for String Quartet* demonstrates the problems that have led to the calcification of this debate. The evidence deducible though scale theory or pcset theory does not conclusively show any influence of the octatonic on Stravinsky's distribution of pitch materials. Two out of the three instruments have a three- or four-note octatonic subset, a statistic that is consistent with a random selection of scalewise diatonic subsets. While two of the instruments that do have octatonic subsets (second violin and cello) do belong to a common octatonic, they add up to only a five-note subset, something that could also be easily attributed to chance, especially when factoring in the avoidance of common diatonic supersets. The total pc content of the three instruments might be characterized as an octatonic with one wrong note (B instead of B♭), but a skeptic might then ask why not, say, an A acoustic scale with an added note (C)? Thus, Van den Toorn's (1983, 152) claim that this is "a piece in which octatonic relations . . . interpenetrate with the diatonic (A G F♯ E D C) hexachord" is readily susceptible to Straus's (1984, 132) and Tymoczko's (2002, 80) skeptical point that any harmony could be "explained" as the result of diatonic-octatonic interpenetration.

Nonetheless, there is a defensible argument to be made for octatonicism in the piece that does not rely upon *ad hoc* or circular reasoning. The DFT components of the three sets (Table 5) evince three general features, in order of prominence: (1) A consistently high $|f_1|$, showing that Stravinsky restricts each instrument to narrow pc ranges, (2) A high $|f_5|$, especially in the violins, showing the use of diatonic subsets, and (3) a high $|f_4|$ in the accompanying instruments, showing the use of octatonic subsets. When the pc content of the three instruments is added together, f_4 stands out as overwhelmingly dominating the total harmonic content. This is because the individual collections are distributed relatively evenly around the circle of fifths (as already noted) and also around the pc circle (Ph_1). However, they are concentrated in a narrow Ph_4 region, as can be seen in the $\mathrm{Ph}_{4,5}$-space of Figure 19. This region is not centered on a specific octatonic collection, but is somewhat to the right of $\mathrm{Oct}_{0,1}$, towards $\mathrm{Oct}_{2,3}$.

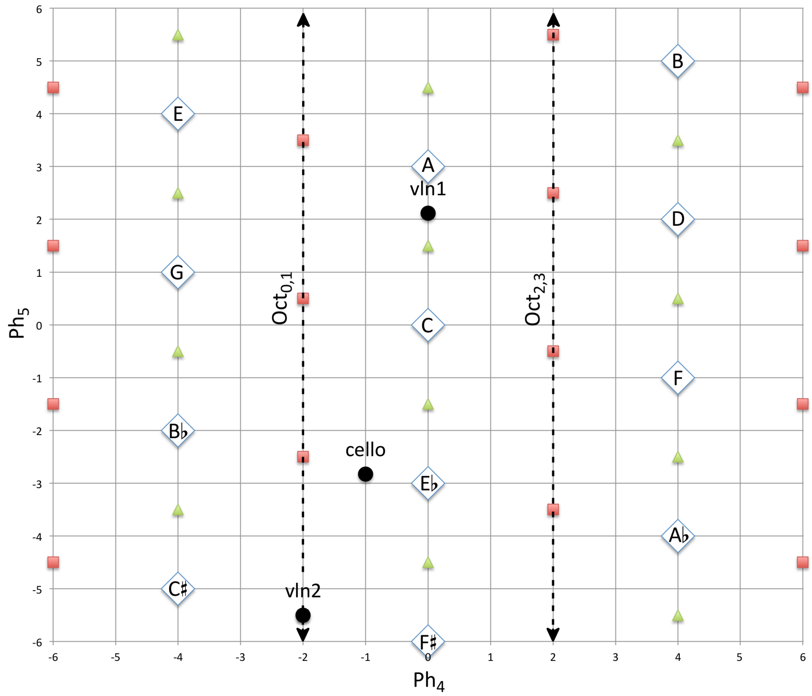

Figure 19: The collections used by Stravinsky in Ph$_{4–5}$-space. = *5ths,* = *minor 3rds.*

The reason that the DFT can lead us to this result is that prominent use of the f_4 and the Ph_4 dimension of phase space is a much more general definition of "octatonicism" than the scale-theoretic one. It is also more precise, leaving no ambiguity as to, e.g., what collections are privileged as scales and why. The fact that the DFT components define an algebraic *basis* for harmonic content is crucial here: f_4 is independent of all other DFT components, so we can be sure there is no overlap or redundancy between our explanation of what Stravinsky is doing with the diatonicity of his collections (emphasizing it within collections but nullifying it between them) and their octatonicity (which is emphasized within some collections but even more so in their combination). At the same time, the relevance of these features is clear: it is rather unlikely to have a single component dominate as strongly as f_4 does in the combined pc content here. On the other hand, octatonicity is only one out of six of the possible elemental harmonic characters. Only an examination of more music could conclusively show that this harmonic character has a special status for Stravinsky. Furthermore, even assuming that this may be shown, it is hard to make too strong a historical claim about why Stravinsky favors octatonicity. He might have discovered this sound in a particular work of Debussy, Ravel, or even Wagner or Strauss as easily as Scriabin or Rimsky-Korsakov, or he might have hit upon the sound independently, since it is only one out of six possible such elemental harmonic qualities, and among those six are ones with well-established associations such as chromaticism (f_1), diatonicity (f_5), and whole-tone balance (f_6). In particular, the idea that a property of such generality could have functioned as a calling card for Russianness is quite tenuous.

3. Common-tone Relations and Scale-Network Wormholes

3.1. Common Tones and Cross-Correlation

In his Debussy analyses, Tymoczko (2004, 2011a) emphasizes inter-cardinality relationships between scales that are evident in Debussy's music, particularly the interaction of whole-tone scales (cardinality 6) with acoustic and diatonic scales (cardinality 7). Although Tymoczko folds these relationships into a larger theoretical perspective centered on the concept of voice leading, these intercardinality relationships are not geometric voice leading in the sense of Callender, Quinn, and Tymoczko 2008 and Tymoczko 2011a. The former article demonstrates that the metrics of these geometries cannot be preserved under intercardinality equivalences. As I point out in Yust 2015c, intercardinality voice-leading relationships like Callender's (1998) split and fuse operations are better handled by the DFT.

The links in Tymoczko's networks relating scales of different cardinalities are best understood as *maximal intersections*. Maximal intersection is a common-tone property, and as such it is well-reflected by phase-space distances. This may be demonstrated using one of Lewin's (2001, 2007) original applications of the DFT, the *interval function*, which is the *cross-correlation* of two pc-vectors.

As a demonstration, Figure 20 calculates the zeroeth entry of the cross-correlation between the ostinato from Satie's "Idylle," (ABCD), with the first melodic idea ("*Ruisseau* (*a*)"). Other entries of the cross-correlation are calculated by rotating (transposing) the second vector backwards by one place (semitone) at a time. The zeroeth entry of the cross-correlation gives the number of common tones between two pcsets. A well-known result from Fourier analysis, the convolution theorem, states that cross-correlation is equivalent to multiplying Fourier magnitudes and subtracting the phases. This implies that the number of common tones between two pcsets may be calculated from the DFT as follows:

$$\frac{1}{12}\sum_{n=0}^{11}|f_n(A)||f_n(B)|\cos(\varphi_n(A)-\varphi_n(B))$$

So, if we understand number of common tones as a measure of harmonic closeness, then it is equivalent to distance in phase space weighted towards the dimensions of high-magnitude components. When the pcsets involved tend to be consistently weighted towards two (or some small number of) specific components, distance in the phase space for these components (using circular averages, which is the role of the cosine functions) gives a good approximation to the common-tone function. We might also want to deliberately ignore common tones or lack of common tones attributable to certain components. For example, using the scale-theoretic assumptions of evenness explained in section 2.1 and ignoring f_1 and f_2, we capture the common-tone closeness of possible scalar supersets of two harmonies.

$$
\begin{array}{rcccccccccccc}
 & \text{C} & & \text{D} & & & & & & & \text{A} & & \text{B} \\
 & (1, & 0, & 1, & 0, & 0, & 0, & 0, & 0, & 0, & 1, & 0, & 1) \\
 & & & \text{D} & & \text{E} & & \text{F}\sharp & & & \text{A} & & \\
\times & (0, & 0, & 1, & 0, & 1, & 0, & 1, & 0, & 0, & 1, & 0, & 0)
\end{array}
$$

$$= 0 + 0 + 1 + 0 + 0 + 0 + 0 + 0 + 0 + 1 + 0 + 0 = 2$$

Figure 20: Calculating the zeroeth component of a cross-correlation between (ABCD) and (DEF♯A)

Table 6 illustrates this with pcsets from Satie's "Idylle." The first calculation shows why the ostinato and *Ruisseau* (*a*) have two common tones. The first consideration (f_0) is cardinality: larger cardinalities generally lead to more common tones. Two four-note sets have on average 1.33 common tones. Second, both sets are imbalanced on the pc-circle, as indicated by $|f_1|$, and in roughly opposite directions (a Ph$_1$ difference of 5.2). This reduces the common tone total by about 0.47—not a huge amount because the right-hand set is still fairly well spread out. The more important factor is the diatonicity of the two collections and their closeness on Ph$_5$, which increases the total by 0.97. Components 2, 3 and 6 are irrelevant because they have zero magnitude in the ostinato, and component 4 makes only a small contribution to the total. Compare this to the second example in Table 6, which compares the ostinato to *Arbres* (*b*). Between a four-note and seven-note set, we should expect more common tones—2.33 on average. But the diatonicity of the collections works against common tones here because of the large Ph$_5$ difference of 3.83. Also the Ph$_1$ difference indicates that *Arbres* (*b*) has a gap in the vicinity of the ostinato's tetrachord, further reducing the common-tone total.

Table 6: Calculating common tones with the DFT. The last line multiplies the two above it and divides by twelve.

	$\|f_0\|$, Ph$_0$	$\|f_1\|$, Ph$_1$	$\|f_2\|$, Ph$_2$	$\|f_3\|$, Ph$_3$	$\|f_4\|$, Ph$_4$	$\|f_5\|$, Ph$_5$	$\|f_6\|$, Ph$_6$	$\|f_7\|$, Ph$_7$	$\|f_8\|$, Ph$_8$	$\|f_9\|$, Ph$_9$	$\|f_{10}\|$, Ph$_{10}$	$\|f_{11}\|$, Ph$_{11}$
Ostinato	4, 0	2.4, 0.5	0, —	0, —	2, 2	2.4, 2.5	0, —	2.4, 9.5	2, 10	0, —	0, —	2.4, 11.5
Ruisseau (*a*)	4, 0	1.2, 7.2	1, 6	1.4, 7.5	1, 0	2.9, 3.7	2, 0	2.9, 8.3	1, 0	1.4, 4.5	1, 6	1.2, 4.8
$\|f_n(A)\|\|f_n(B)\|$	16	3.04	0	0	1	5.83	0	5.83	1	0	0	3.04
$\cos(\varphi_n(A) - \varphi_n(B))$	1	–0.93			0.5	0.82		0.82	0.5			–0.93
common tones	1.33	– 0.24	+ 0	+ 0	+ 0.08	+ 0.49	+ 0	+ 0.49	+ 0.08	+ 0	+ 0	– 0.24
												= 2

	$\|f_0\|$, Ph$_0$	$\|f_1\|$, Ph$_1$	$\|f_2\|$, Ph$_2$	$\|f_3\|$, Ph$_3$	$\|f_4\|$, Ph$_4$	$\|f_5\|$, Ph$_5$	$\|f_6\|$, Ph$_6$	$\|f_7\|$, Ph$_7$	$\|f_8\|$, Ph$_8$	$\|f_9\|$, Ph$_9$	$\|f_{10}\|$, Ph$_{10}$	$\|f_{11}\|$, Ph$_{11}$
Ostinato	4, 0	2.4, 0.5	0, —	0, —	2, 2	2.4, 2.5	0, —	2.4, 9.5	2, 10	0, —	0, —	2.4, 11.5
Arbres (*b*)	7, 0	1.2, 6.2	1, 4	2.2, 0.9	1, 8	2.9, 10.7	1, 0	2.9, 1.3	1, 4	2.2, 11.1	1, 8	1.2, 5.8
$\|f_n(A)\|\|f_n(B)\|$	28	3.04	0	0	2	7.13	0	7.13	2	0	0	3.04
$\cos(\varphi_n(A) - \varphi_n(B))$	1	–0.93			0.5	0.82		0.82	0.5			–0.93
common tones	2.33	– 0.25	+ 0	+ 0	– 0.17	– 0.25	+ 0	– 0.25	– 0.17	+ 0	+ 0	– 0.25
												= 1

3.2 Debussy's Whole-Tone Wormhole

An interesting feature of intercardinality relationships in Tymoczko's (2004) scale networks is the "wormhole" phenomenon, where the use of a different-cardinality intermediary brings otherwise distant collections closer together. This can be understood by the role that DFT magnitudes play in determining common-tone relations, as explained in the previous section.

Debussy's whole-tone scale transitions are one instance of this phenomenon. Consider the 6♭-diatonic and 0-diatonic (white-note) that he uses in "Le vent dans la plaine." As the network in Figure 21 shows, when restricted to 7-note collections it takes six single-semitone voice leadings to get from one to the other. But using WT_1 as an intermediary, only *four* moves are needed, two single-semitone voice leadings and two intercardinality maximal intersections.

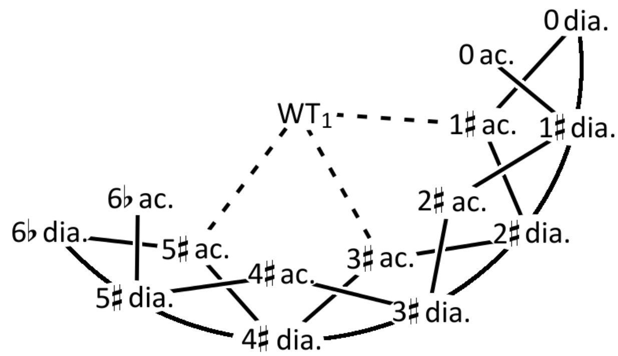

Figure 21: A voice-leading network for connecting 6♭-diatonic and 0-diatonic scales

The wormhole phenomenon can be understood by invoking another Fourier component besides f_5. The simple additive voice-leading relationships between 7-note collections all occur around the circle-of-fifths, or in the one-dimensional Ph_5-space. As long as we are restricted to collections in which f_5 dominates (e.g., diatonic, acoustic, and harmonic minor/major scales), this simple one-dimensional space adequately reflects common-tone relations.[6] However, the T-symmetric scales, whole-tone, octatonic, and hexatonic, all have $f_5 = 0$, so Ph_5 is totally irrelevant to common-tone relations involving these scale types. In fact, these three scale-types are all *perfect representatives* of the 3rd, 4th, and 6th DFT components (hexatonic, octatonic, and whole-tone, respectively) meaning that they have zero magnitude on all other components. Therefore the common tones function from any collection to one of these is determined entirely by Ph_3, Ph_4, or Ph_6 distance (and cardinality).

The harmonic process in the first part of "Le vent dans la plaine," then, can be understood through a combination of f_5 and f_6. Ph_6 is in a sense degenerate, in that it can only possibly take on two values, 0 and 6. The phase space for Ph_5 and Ph_6 may therefore be depicted as two Ph_5-cycles, one for $Ph_6 = 0$ and one for $Ph_6 = 6$, as shown in Figure 22. The positions in Figure 22 are labeled with two set types relevant to the Debussy analysis: acoustic scales and dominant ninths. Dominant ninths are the complements of acoustic scales, a fact that is

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[6] This is only true, however, if the diatonic is a persistent intermediary. The acoustic scale is actually dominated more by f_6 than f_5, so common tone relations directly from one acoustic scale to another may defy the one-dimensional Ph_5 logic. For example C acoustic and F♯ acoustic are opposite on Ph_5, but share four common tones. Similarly, harmonic minor/major collections' have stronger f_4 and f_3 than f_5.

useful since this means that the DFT magnitudes of both set classes are the same. Dominant ninths are also diatonic subsets, and each is in the same position as its diatonic superset in $Ph_{5,6}$-space. For example, G^9, which describes the pc content of the first part of measure 21, is in the same position as C diatonic. The dominant ninths are better representatives of $Ph_{5,6}$ positions because diatonic collections are imbalanced, with a large $|f_5|$ ($|f_5|^2 = 13.9$) and a small $|f_6|$ ($|f_6|^2 = 1$), while dominant ninths and acoustic scales are relatively balanced between these components ($|f_5|^2 = 7.5$, $|f_6|^2 = 9$).

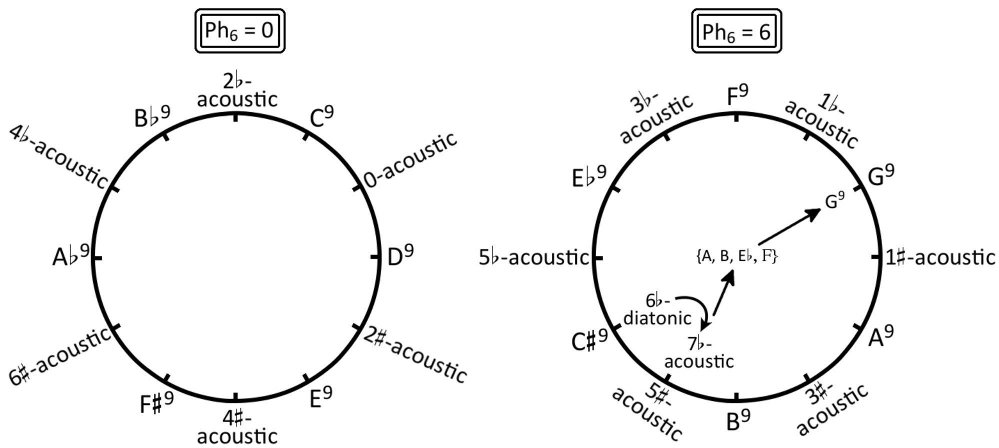

Figure 22: The harmonic process of mm. 13–22 of "Le vent dans la plaine" in $Ph_{5,6}$-space

The important shift from the 6♭-diatonic to the 7♭-acoustic therefore serves two functions: it is a small move in Ph_5, but more importantly, it elevates f_6 in importance at the expense of f_5. Debussy throws further weight towards f_6 towards the end of m. 20 by isolating just four notes of this acoustic scale, (ABE♭F). This set has $|f_5|^2 = 0$ and $|f_6|^2 = 16$. At the beginning of m. 20, the G^9 collection also has a high $|f_6|$, with the same phase, so the transition into m. 20 is actually quite smooth, as evident in the high number of common tones over the barline (A, B, and F, all of the notes sustained through the last beat of m. 19). The drastic shift in Ph_5 is hidden, so to speak, by the reduction of $|f_5|$ to zero within the progression where that shift takes place.

3.3 An Octatonic Wormhole in Shostakovich's String Quartet no. 11

Similar kinds of scale-network wormholes can be created through f_4, suggesting an octatonic intermediary. Figure 23 shows an example from Shostakovich's Eleventh String Quartet. The movement is an "Elegie" written in remembrance of the Beethoven Quartet's second violinist. (Lesser 2011, 200–205) The viola and cello intone a somber elegiac subject in bare octaves, evoking a chorus singing in unison, while the violins alternate with more soloistic laments. In the passage shown in the example, the lower strings, leaping up to F♯, escape for the first time the low C♯, which has been the *terra firma* from which the small melodic arcs of the subject have arisen and to which they have up to this point inevitably withdrawn. The lower parts then drop out, also a first for the movement, leaving the first violin to play alone. The F♯ major triad is also striking as a sudden departure from the prevailing sonority of the movement.[7] All of this conjures a crossing over of some spiritual boundary, perhaps from the realm of the living into that of the dead, to commune with the beloved lost soul.

Figure 23: Meas. 27–44 of Shostakovich's String Quartet no. 11

In the passage preceding the crossing-over, starting from measure 30, the first violin's melody seems to be based on a B natural minor, or 2♯-diatonic, collection. However, it is not a complete collection: there is no A♮, meaning that it is also a subset of B (3♯) harmonic minor, and the melody focuses primary on the C♯ diminished pentachord, (C♯DEF♯G), in measures 33–6. The second violin rises initially from a repeated G♯, playing a variation on the elegiac subject a perfect fifth above. The G♯ is left behind, however, in measures 34–6, where the second violin circles around a chromatic trichord (A, B♭, B). Of all the possible collections inferred for measures 33–36, the F♯ major triad belongs only to the B harmonic minor. After the crossing-over (mm. 39–43), the solo violin plays a mostly scalewise line in an unambiguous 0-harmonic minor. The 0-harmonic minor does not relate directly to any of the preceding scales by voice leading and maximal intersection, as can be seen in Figure 24. Restricted to heptatonics, there are at least two hypothetical intermediaries between the 3♯-harmonic minor and the 0-harmonic minor. They are more directly related through an

[7] McCreless (2009, 24) mentions multiple striking uses of F♯ major triads from earlier quartets of Shostakovich recollected by this moment.

inferred octatonic, however. This collection, $Oct_{1,2}$, also happens to be the octatonic superset of the central (0134) of the elegiac subject, (C♯DEF), and appears explicitly at the end of the piece (mm. 64–69, lacking only B♮). Shostakovich's orthography reflects the idea that this shift occurs in a non-diatonic dimension: he spells the B♭ of the 0 harmonic minor as an A♯, showing its status as a common tone from the 3♯-harmonic minor. (The extra voice leading of the path restricted to 7-note scales is required to avoid this orthographic anomaly, first moving A♯→A♮, then restoring the same pitch class with B→B♭.)

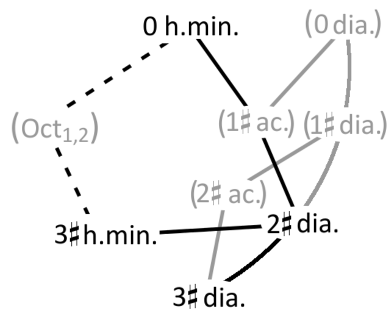

Figure 24: A scale network showing the octatonic wormhole

We can clarify and solidify these observations using f_4 and f_5. Figure 25 plots some significant collections in $Ph_{4,5}$-space. First, we note that the F♯ major triad and D harmonic minor are f_5 antipodes, with the same magnitude $|f_5|^2 = 3.73$, and opposite phases. Yet they are relatively close in Ph_4, which explains why they have a substantial number of common tones (2) despite being diatonically opposed.[8] The music leading up to the crossing over makes the shift from diatonicity to octatonicity that sets up this relationship. The total pc content of measures 30–33, as shown in Table 6, is dominated by f_5, supporting the idea of a mixture of 2♯- and 3♯-diatonics. The move in the second violin away from G♯ and adding B♭, however, shifts the balance to where f_4 and f_5 have equal magnitudes. The values of these components and the position in $Ph_{4,5}$-space are precisely the same as for the first violin's C♯ diminished pentachord by itself (which is a subset of both large collections). The positions of G♯ and B♭ in $Ph_{4,5}$-space show how the evolution of the second violin part accomplishes this change. While G♯ is a similar distance from the C♯ diminished pentachord in both dimensions, the A♯ lines up perfectly with the pentachord in f_4, and is directly opposite in f_5. The F♯ major chord in measures 38–39 isolates this A♯ from its chromatic neighbors in the second violin, weighting f_4 even more strongly, as shown in the last line of Table 6. This collection is similar to the 3♯-harmonic minor.

[8] This also has to do with their similar Ph_3 values, but I focus on f_4 here because it is more significant in the preceding music.

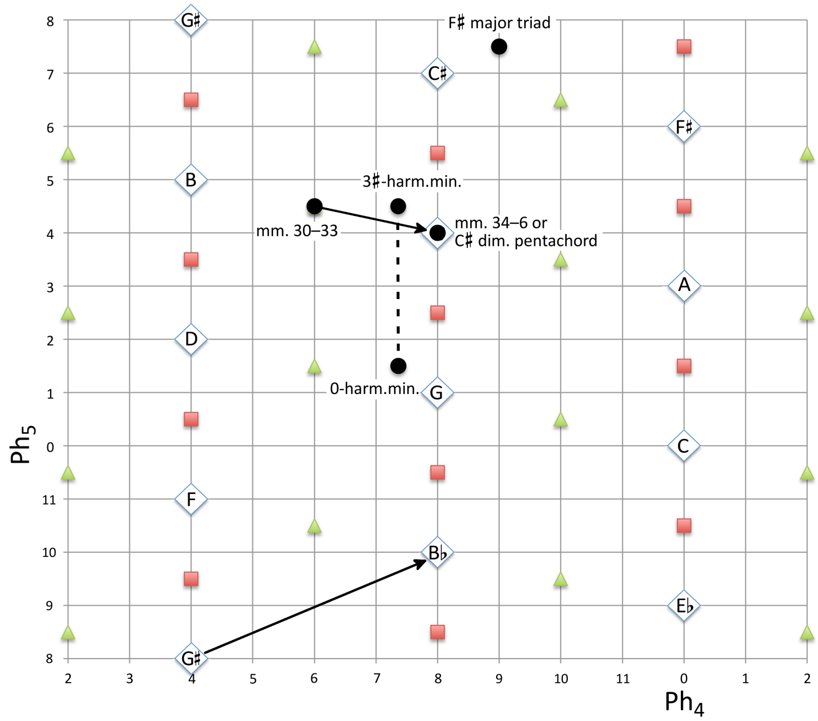

Figure 25: Collections from Shostakovich String Quartet no. 11, mm. 30–43, in Ph$_{4,5}$-space. ■ *= 5ths,* ▲ *= minor 3rds.*

Table 6: DFT magnitudes of collections in Shostakovich's String Quartet no. 11, mm. 30–43

	Pcset	$\vert f_1\vert^2$ $\vert f_2\vert^2$ $\vert f_3\vert^2$ $\vert f_4\vert^2$ $\vert f_5\vert^2$ $\vert f_6\vert^2$
Meas. 30–33	(C♯DEF♯GG♯AB)	⟨⟨ 0.8, 3, 0, **1,** **11.2,** 0 ⟩⟩
Vln1, meas. 32–39	(C♯DEF♯G)	⟨⟨ 4, 4, 1, **4,** **4,** 1 ⟩⟩
Meas. 34–39	(C♯DEF♯G♯AB♭B)	⟨⟨ 0.54, 0, 4, **4,** **4,** 0 ⟩⟩
Harmonic minor	(C♯DEF♯GA♯B) or (DEFGAB♭C♯)	⟨⟨ 0.27, 1, 5, **7,** **3.73,** 1 ⟩⟩
Vln1, meas. 32–39 + A♯	(C♯DEF♯GA♯)	⟨⟨ 1, 1, 4, **9,** **1,** 4, ⟩⟩

The harmonic minor/major is also dominated by f_4 (and to a lesser extent by f_3), so in a direct relationship between harmonic minors, the value of f_4 is a primary determinant of distance, not f_5. This explains the octatonic wormhole of Figure 24. In the DFT-based explanation, however, it is not the intercession of an imaginary octatonic *scale* that brings the 3♯- and 0-harmonic minors together. Rather, it is the more general property of octatonicity, of which the scale is a prototype, but not the sole exemplar. Other high-f_4 set classes include (0134) and the diminished pentachord (01356), which are more directly evident as a feature of Shostakovich's harmonic language than the octatonic scale *per se*.

3.4 Diatonic-Octatonic Interaction in Ravel's *Duo*

In classical forms, positions in tonal space articulate the major stations of the formal process. Drawing upon the interpretation of tonal regions as areas in $Ph_{3,5}$-space in Yust 2015c, we may note that the concentration of harmony in specific locations (high overall $|f_3|$ and $|f_5|$) is indicative of certain formal functions (main theme and secondary theme) while more diffusion through this space typifies such functions as transitions and development sections.

Ravel often writes chamber works in a version of sonata form (Howat 2000, Aziz 2015). Given Ravel's non-traditional harmonic language, however, the use of such conventional forms poses a compositional problem: how are the usual usages of harmony to articulate form to be modified to fit this harmonic language? Kaminsky (2011) observes that Ravel was deeply concerned with form and shows that a successful formal plan depends for Ravel on the effective interface of harmonic structure and formal design. As is evident in Heinzelmann's (2011) analysis of Ravel's String Quartet, one harmonic feature that may interact with the form is the play between diatonic collection and symmetrical (octatonic and whole tone) collections, an important aspect of Ravel's harmonic language.

Ravel's *Duo for Violin and Cello* (first movement of the *Sonata for Violin and Cello*) was a watershed in his career and especially notable for its rigorously contrapuntal conception and use of polyscalar stratification. The exposition (Fig. 26) effectively establishes the principal dimensions of harmonic activity for the piece, the diatonic (f_5) and the octatonic (f_4). Ravel's main theme is a modal (Dorian) diatonic melody, using a subset of the 1♯diatonic (CDE F♯GA). This choice from among the possible diatonic hexachords (in particular, the use F♯ rather than F♮) gives the theme a presence of f_4 in addition to the prominent f_5 (Table 7). The accompanying idea has the converse property—it has a relatively large f_5 for a 5-note subset of the octatonic.[9] The distinctness of the two collections is therefore evident (one a diatonic, the other an octatonic representative) while at the same time they can play on one another's turf. And Ravel chooses precisely those transpositions that are perfectly aligned in Ph_5 and close in Ph_4.

Table 7: DFT components 3–6 of collections in the exposition of Ravel's Duo

	Pcset	$(\lvert f_3\rvert^2, Ph_3)$ $(\lvert f_4\rvert^2, Ph_4)$ $(\lvert f_5\rvert^2, Ph_5)$ $(\lvert f_6\rvert^2, Ph_6)$
Main theme melody	(CDEF♯GA)	$\langle\langle\ (0, –)_3\ \ (3, 11)_4\ \ (11.2, 2.5)_5\ \ (4, 0)_6\ \rangle\rangle$
Main theme acc.	(ACC♯EG)	$\langle\langle\ (5, 11.1)_3\ \ (7, 9.4)_4\ \ (3.73, 2.5)_5\ \ (1, 6)_6\ \rangle\rangle$
Accomp., mm. 29–37	(GG♯B♭BDF)	$\langle\langle\ (2, 4.5)_3\ \ (12, 5)_4\ \ (2, 11.5)_5\ \ (0, –)_6\ \rangle\rangle$
Melody, mm. 30–37	(CDEFG)	$\langle\langle\ (1, 0)_3\ \ (1, 6)_4\ \ (10.5, 1.1)_5\ \ (1, 0)_6\ \rangle\rangle$
4♯-acoustic	(EF♯G♯A♯BC♯D)	$\langle\langle\ (1, 6)_3\ \ (4, 6)_4\ \ (7.46, 6)_5\ \ (9, 0)_6\ \rangle\rangle$

[9] The largest possible being set class (02358), a dominant seventh add-6 or half-diminished minor ninth chord. The set Ravel uses is the octatonic complement of a minor triad, and as such has the same magnitude f_5 (and similarly for all components other than f_4)

Figure 26: Ravel, Duo, *mm. 6–46*

The exposition begins by stating this diatonic subject – octatonic countersubject pair, then transposing it up by fourth and exchanging the roles of the instruments. This transpositional process casts the diatonic dimension as the agent of stable continuity. As is evident from the phase-space plot in Figure 27, the transposed melodies are close in Ph_5, but distant in Ph_4.

As the exposition proceeds, Ravel hints at a continuation of this transpositional process: the next version of the countersubject in the violin, measures 29–37, is a variation on the melody transposed up another fourth, but one of the Gs replaced by a G♯. This further intensifies the octatonicity and weakens the diatonicity of the melody (Table 7). The new melody of these measures is new, a distinct shape from the main theme melody, but its pc content does carry forward the established trajectory, evident in the phase-space plot of Figure 27. At the same time, the new melody maintains a strong diatonicity while shedding the trace of octatonicity evident in the main theme's hexachord.

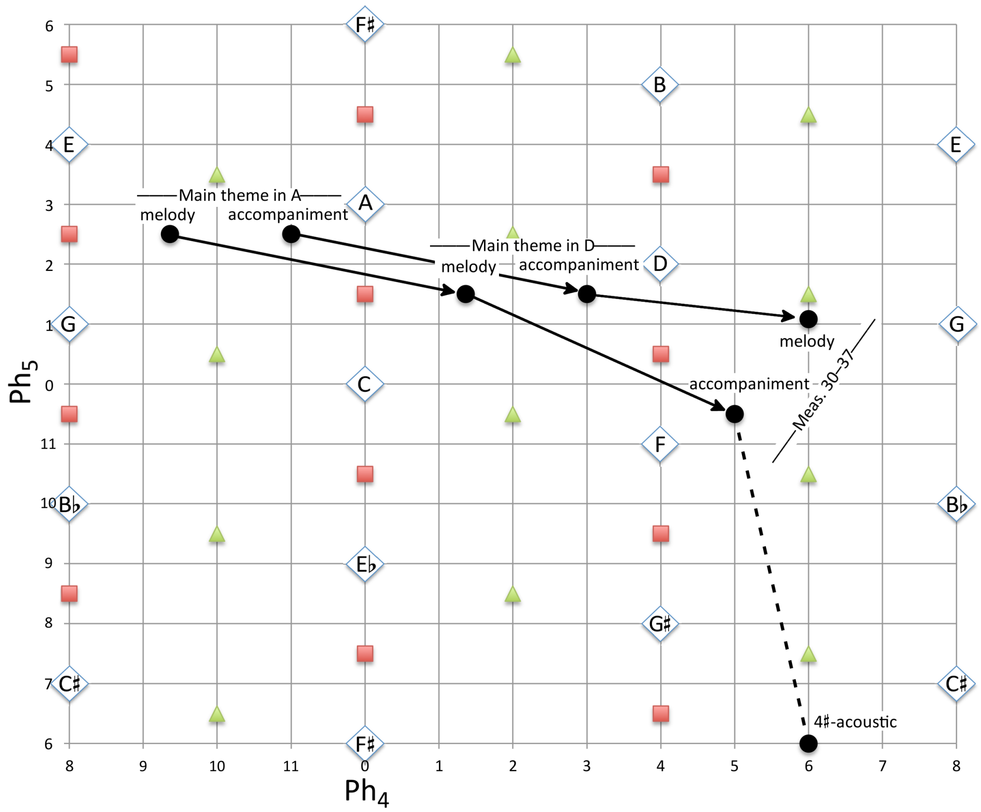

Figure 27: The exposition of the Duo *in* $Ph_{4,5}$*-space.* ■ *= 5ths,* ▲ *= minor 3rds.*

This analysis reinforces a point made by Antokoletz (2011, 218–20) about the formal significance of movement between contrapuntal "fusion" and "polarity" in the piece. In his set-theoretic analysis, he finds that the ambiguity of the octatonic quality of the initial accompaniment (because of the incompleteness of its parent octatonic scale) allows for "fusion" with the diatonic melody. The more robustly octatonic countersubject of measures 29–37, on the other hand, moves towards a stronger polarity. Using the DFT, the point can be made without invoking any ambiguity, because there is no need to reference an incomplete source scale. The countersubject mixes diatonic and octatonic qualities in precise measure, and the sense of fusion has specifically to do with its closeness to the diatonic melody in phase space. Furthermore, we can see that as Ravel's transitional process gets underway, the two collections remain close in phase space as they diverge in harmonic quality, the melody becoming more exclusively diatonic while the accompaniment becomes more exclusively octatonic.

This divergence is important to what happens next. The violin takes over the new transition theme in measures 39–46, transposed by tritone, the interval of f_5 *negation*, in stark contrast to the Ph_5 consistency of all the preceding music. The cello initially plays the same theme in a close stretto (mm. 39–42), but then changes to an accompanimental role with the neighbor figure E–D played on harmonics. The pcs of this accompaniment combine with the melody to give a governing 4♯-acoustic collection for the entire passage. Like the main theme collections, the acoustic is representative of both f_5 and f_4 qualities (Table 7), and in phase space (Figure 27) it is close to the preceding music in Ph_4 while it is distant in Ph_5. In other words, Ravel switches to using octatonicity to thread a disruption of diatonicity. The reorientation is prepared by the divergence between the parts, so that the octatonic continuity is now localized to the countermelody, whose harmonic quality now governs the coherence of the transition in contrast to the diatonic continuity that governed the main theme.

While the main theme and transition are characterized by polyscalar tensions and differentiated by swapping the roles of f_4 and f_5 as agents of fusion and polarity, the second theme (mm. 69–104) is distinguished by its thoroughgoing consonance and unity of harmonic quality. The theme is consistently dominated by diatonicity in both parts, never deviating beyond one flat or one sharp. Ravel replaces the harmonic tensions with rhythmic ones, by writing the two parts metrically out of phase by an eighth-note. The last part of the theme before the development (mm. 93–104), shown in Figure 28, is restricted to a Guidonian hexachord, the set class of maximal diatonicity ($|f_5|^2 = 14.9$). The octatonicity of the earlier material is wholly absent; the Guidonian hexachord has a zero-valued f_4.

The extreme diatonicity of the second theme provides Ravel with his method of announcing the main formal division of the piece, the beginning of the development (m. 105). The element of polarity, linear and vertical, suspended in the second theme, reappears forcefully in the development, shown in Figure 28. This first part of the development divides into four phrases, phrase 1 (mm. 105–109), phrase 2 (mm. 112–18, *En animant*), phrase 3 (mm. 122–27), and phrase 4 (mm. 127–35, *Assez vif*).

When the familiar octatonic countersubject appears in the violin in phrase 1 the cello has a new melody that recalls the rhythm and contour of the second theme's countermelody while sharply contrasting in its lack of clear diatonic focus. Ignoring the cello's low double stops, which serve as a reinforcement for the accompanimental melody, the notes used in the melody are perfectly chosen to negate the diatonicity of the accompaniment. As Figure 29 and Table 8 show, the two collections are f_5-antipodes, meaning they have a zero-diatonicity sum. At the same time, they reinforce one another strongly in the non-diatonic f_3 and f_4 dimensions.

Figure 28: The beginning of the development of the Duo

Table 8: DFT components 3–6 of collections in the development of Ravel's Duo

	Pcset	$(\lvert f_3\rvert^2, \mathrm{Ph}_3)$	$(\lvert f_4\rvert^2, \mathrm{Ph}_4)$	$(\lvert f_5\rvert^2, \mathrm{Ph}_5)$	$(\lvert f_6\rvert^2, \mathrm{Ph}_6)$
Melody, mm. 93–104, ST	(CDEFGA)	⟨⟨ $(2, 10.5)_3$,	$(0, –)_4$,	$(14.93, 1.5)_5$,	$(0, –)_6$ ⟩⟩
Accomp., phrase 1	(CE♭EGB♭)	⟨⟨ $(5, 2.1)_3$,	$(7, 9.4)_4$,	$(3.73, 11.5)_5$,	$(1, 0)_6$ ⟩⟩
Melody, phrase 1	(EF♯B♭B)	⟨⟨ $(2, 4.5)_3$,	$(1, 8)_4$,	$(3.73, 5.5)_5$,	$(4, 0)_6$ ⟩⟩
Accomp., phrase 2, odd meas.	(BDF♯)	⟨⟨ $(5, 5.1)_3$,	$(3, 3)_4$,	$(3.73, 4.5)_5$,	$(1, 6)_6$ ⟩⟩
Accomp., phrase 2, even meas.	(A♯C♯E♯)	⟨⟨ $(5, 8.1)_3$,	$(3, 7)_4$,	$(3.73, 9.5)_5$,	$(1, 6)_6$ ⟩⟩
Accomp., phrase 2, sum	(A♯BC♯D♯E♯F♯)	⟨⟨ $(1, 6)_3$,	$(3, 1)_4$,	$(5.73, 4.7)_5$,	$(1, 6)_6$ ⟩⟩
Melody, phrase 2, odd meas.	(F♯A)	⟨⟨ $(2, 7.5)_3$,	$(4, 0)_4$,	$(2, 4.5)_5$,	$(0, –)_6$ ⟩⟩
Melody, phrase 2, even meas.	(BC♯DE)	⟨⟨ $(0, –)_3$,	$(4, 6)_4$,	$(6, 4.5)_5$,	$(0, –)_6$ ⟩⟩
Melody, phrase 2, sum	(ABC♯DEF♯)	⟨⟨ $(2, 7.5)_3$,	$(0, –)_4$,	$(14.93, 4.5)_5$,	$(0, –)_6$ ⟩⟩
Accomp., phrase 3	(GB♭BDF)	⟨⟨ $(5, 5.1)_3$,	$(7, 5.4)_4$,	$(3.73, 0.5)_5$,	$(1, 6)_6$ ⟩⟩
Melody, phrase 3	(EF♯G♯ABC♯)	⟨⟨ $(2, 10.5)_3$,	$(0, –)_4$,	$(14.93, 5.5)_5$,	$(0, –)_6$ ⟩⟩
Accomp., phrase 4	(ABEF♯)	⟨⟨ $(0, –)_3$,	$(1, 0)_4$,	$(11.20, 4.5)_5$,	$(0, –)_6$ ⟩⟩
Melody, phrase 4	(EF♯G♯ABC)	⟨⟨ $(4, 0)_3$,	$(3, 1)_4$,	$(5.73, 4.71)_5$,	$(4, 0)_6$ ⟩⟩

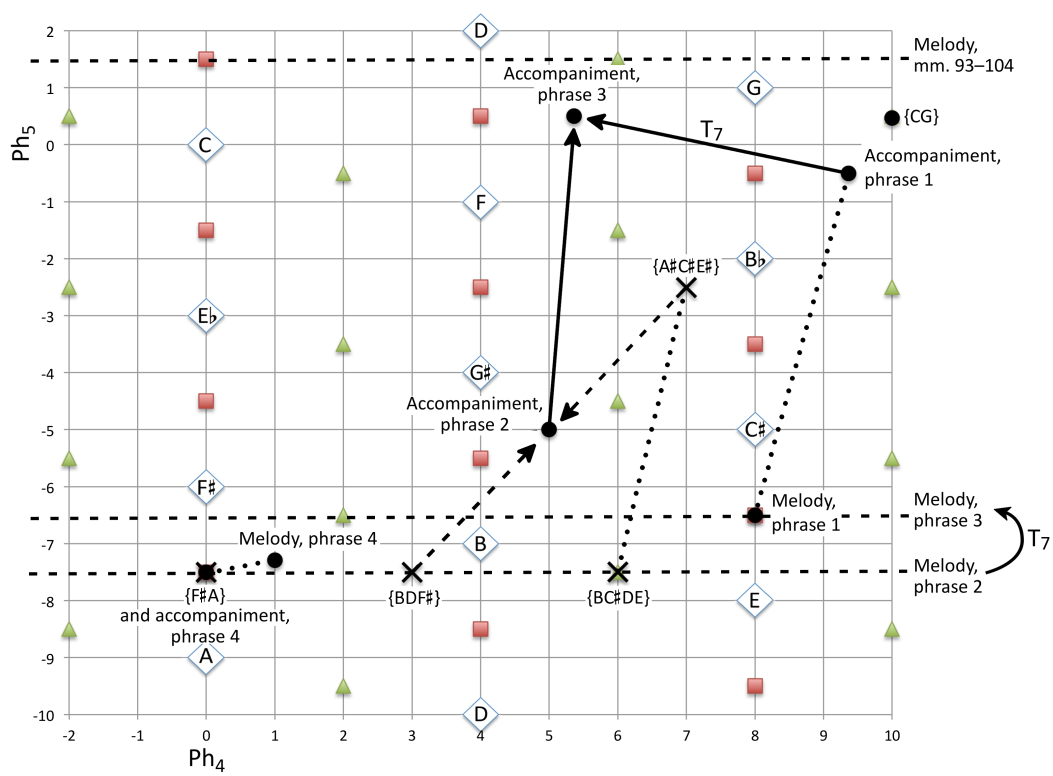

Figure 29: Collections from the development in $Ph_{4,5}$-space. Dots indicate collections for an entire phrase in one part, while crosses indicate partitions of those collections. Dashed horizontal lines give the Ph_5 values of sets with undefined Ph_4. Dotted lines connect collections that appear simultaneously. Dashed arrows show the combination of two subsets of a larger collection.

Figure 29 plots a number of collections from the first part of the development in $Ph_{4,5}$-space. Some melodic ideas are based on Guidonian hexachords, which have undefined Ph_4 and are therefore shown as dashed horizontal lines. From this, one feature of this section is evident: a bifurcation of diatonicity between the melodic ideas between the second theme (centered on $Ph_5 = 1.5$) and the melodic ideas of the four phrases of this section of the development, which are consistently in the range of $Ph_5 = 4.5$–5.5. After measure 135, the second theme material returns in roughly its original Ph_5 position. These two diatonic zones are exemplified by the antipodal f_5s of the melody and countermelody in phrase 1.

Ravel uses the kind of T_5/T_7 transpositions featured in the main theme in this section, but in a more concealed fashion. There is a clear T_7 transposition of the countersubject from phrase 1 to phrase 3 that is obscured by the intervening material in phrase 2. As Figure 29 shows, the two appearances of the countersubject are in a distinct Ph_5 area from the other material of the section, in the zone associated with the preceding second subject. Another T_7 occurs between the melodies in phrases 2 and 3, which is obscured by the fact that it begins as a sequence up a whole step, not a fifth. Ravel modifies the melodic shape, avoiding the D♯ that would appear in a real T_2 transposition and adding an A.

The bifurcated diatonicity of the passage thus plays three roles: it generates polyscalar tension in phrases 1 and 3 alternating with passages that are more diatonically consonant. In addition, these alternating sections produce large swings in the diatonic content of the accompaniment, meaning that there is a linear diatonic tension specific to the accompanying parts. Finally, another larger-scale diatonic tension, specific to the melodic parts, occurs between the flanking second theme material and these first four phrases of the development.

The role of octatonicity in the section is less obvious because of the many Guidonian hexachords with $f_4 = 0$, but a closer looks shows important activity in this dimension. First, we can see that in both places where Ravel uses the original countersubject (phrases 1 and 3) the diatonic polarities are mitigated by octatonic links. We already noted the octatonic bond between counterpointing parts in phrase 1. The countersubject in phrase 3 (which accompanies a $f_4 = 0$ melody) instead links linearly with the preceding accompaniment, which has approximately the same Ph_4 (see the arrow in Fig. 29). These two associations create two octatonic zones, which, because they are separated by a T_7 relationship, recall the octatonic relationships created by T_7 transposition in the main theme and transition. It is especially interesting, then, that Ravel varies the thematic idea of this section in phrase 4 to include a C♮ instead of a C♯ (which would appear if the melody of phrase 2 were precisely transposed). The result is a diatonic hexachord that, like the main theme melody and unlike the Guidonian hexachord, has a significant non-zero f_4. The Ph_4 value of this melody (which agrees with the weaker f_4 of its accompanying melody—see Table 8) then occupies the next T_7-related octatonic zone, replicating the Ph_4 progression of the main theme and transition in retrograde.

These phrase-by-phrase relationships are complemented by more local activity. The new accompaniment that appears at the *En animant* consists of alternating arpeggiations of B minor and A♯ minor triads. The T_1 relationship of these triads puts them in opposing zones in both diatonic and octatonic dimensions. The B minor triad has exactly the same Ph_5 as the melody, making the A♯ minor triad the diatonic alien element. In other words, much like the phrase-to-phrase activity, there is a measure-to-measure oscillation between diatonic consonance and dissonance. The pc-content of the melody of this section also divides up measure-by-measure into {F♯A}, which coincides with the B minor triads in the accompaniment, and {BC♯DE}, which tends to coincide with the A♯ minor triads. This division separates this octatonically neutral hexachord into more octatonically concentrated subsets ($|f_4| = 4$). Furthermore, the octatonic subset that coincides with the A♯ minor triad also links to it in the octatonic dimension (despite the large Ph_5 difference: see Fig. 29), recalling the similar phrase-level octatonic link in phrase 1.

On the whole, then, Ravel uses his development to mix the different kinds of linear and contrapuntal fusion and bifurcation in the diatonic dimension that have been explored in previous material, and continues to weave octatonic threads to bring together diatonically disparate material, and animate diatonically consistent material with coordinated octatonic drift. The section is distinguished, however, by exchanging the exposition's smooth evolution of such relationships with rapid, restless, phrase-by-phrase and measure-by-measure changes of orientation and the embedding of different processes of fluctuating linear and vertical diatonic polarity within one another on multiple time scales.

Summary

This article demonstrates the concept of *harmonic quality* and the use of *phase space* plots in the analysis of twentieth century music, focusing on how these reproduce and extend claims made from methodologies that rely on referential collections. An important advantage of the DFT-derived concepts of harmonic quality and phase space is that they circumnavigate the dubious ontological status of imaginary referential collections and the question of where to draw the line between the elite class of privileged scalar collections and the rest of the pcset universe. In this sense they accomplish what might be seen as one of the original core goals of Forte's set theory, the development of a theoretical framework that embraces scale-based music along with more radically atonal repertoire.

Prototypes of harmonic qualities include many of collections that have played an important role in our understanding of twentieth-century composers. For composers like Satie, Debussy, Ravel, Stravinsky, and Shostakovich, the manipulation of *diatonicity*, the fifth Fourier component, is a salient feature of their compositional styles. For some composers, whole tone quality and/or *octatonicity*, the sixth and fourth Fourier components respectively, play a significant role, regardless of whether the whole tone and octatonic actually appear as explicit collections. This fact validates the sometimes seemingly undermotivated use of, e.g., octatonic collections as a conceptual filter for music of Stravinsky, Ravel, and others. Often, the role of these other qualities is to create an alternate form of coherence to link diatonically distant harmonic material. The result in part three of the paper, derived from the convolution theorem of Fourier theory, shows how the DFT can subdivide common-tone relations between collections into parts attributable to individual harmonic qualities. According to this result, distances in phase space weighted towards the most prevalent harmonic qualities determine the number of common tones between collections. Debussy, Ravel, and Shostakovich sometimes de-emphasize diatonicity in favor of other harmonic qualities to invoke alternate tonal topographies.

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