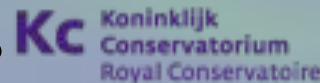


# Mathematical Approaches to Scale Degrees and Harmonic Functions in Analytical Dialogue

Thomas Noll,



Karst de Jong,



Jason Yust



and the Fundamental Bass  
Society for Music Theory, Oct 29, 2015

St. Louis, Missouri

# Outline

## I. The Structural Modes

- 1.) Concept lattice of scale theory
- 2.) Well-formedness and algebraic combinatorics on words
- 3.) Example: Handel, Minuetto, Concerto Grosso op. 3/4

## II. Triadic Voice-Leading Space

- 1.) Filtered-point symmetry and quantization in voice-leading space
- 2.) Tonal harmony in triadic voice-leading space (Handel Minuetto)

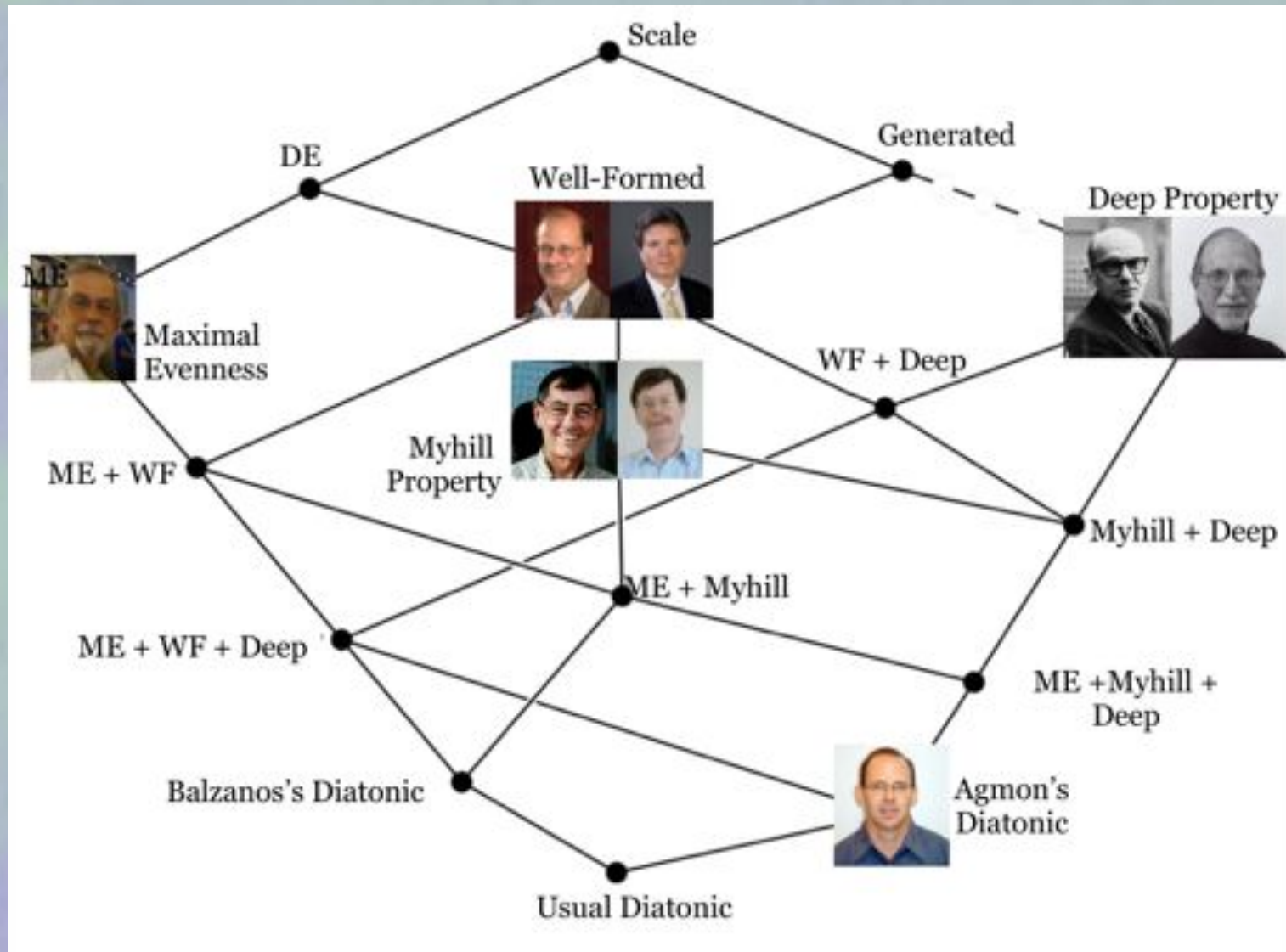
## III. Harmonic Function

- 1.) Example: Schubert, Piano Sonata in C minor (D.958), Trio
- 2.) Functions as vectors in voice-leading space

# Structural Modes



# Concept Lattice of Diatonic Scale Theory



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# The Structural Modes

## A Mathemusical Source of Inspiration:

### Aspects of Well-Formed Scales

Norman Carey; David Clampitt

*Music Theory Spectrum*, Vol. 11, No. 2. (Autumn, 1989), pp. 187-206.



A single structural principle accounts for pentatonic, diatonic, and chromatic scales. The same structure, that of the *well-formed scale*, also underlies the tonic-subdominant-dominant relationship, the 17-tone Arabic and 53-tone Chinese theoretical systems, and other pitch collections in non-Western music. This article shows that the concept of a well-formed scale can serve as a principled basis for tonal music.

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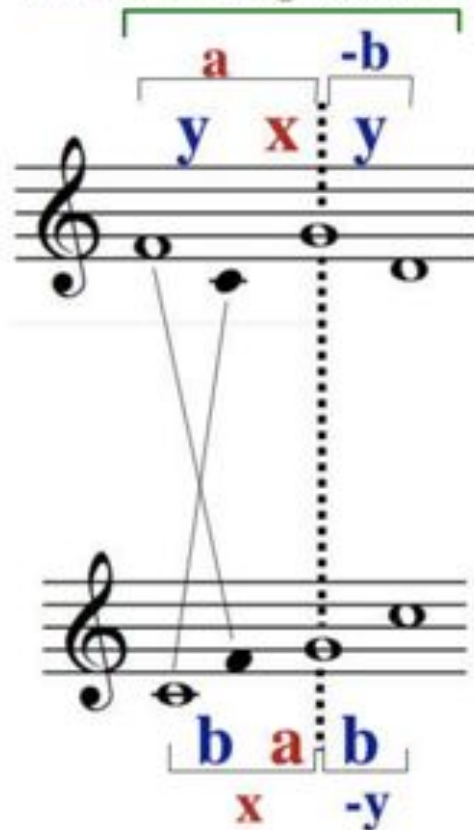
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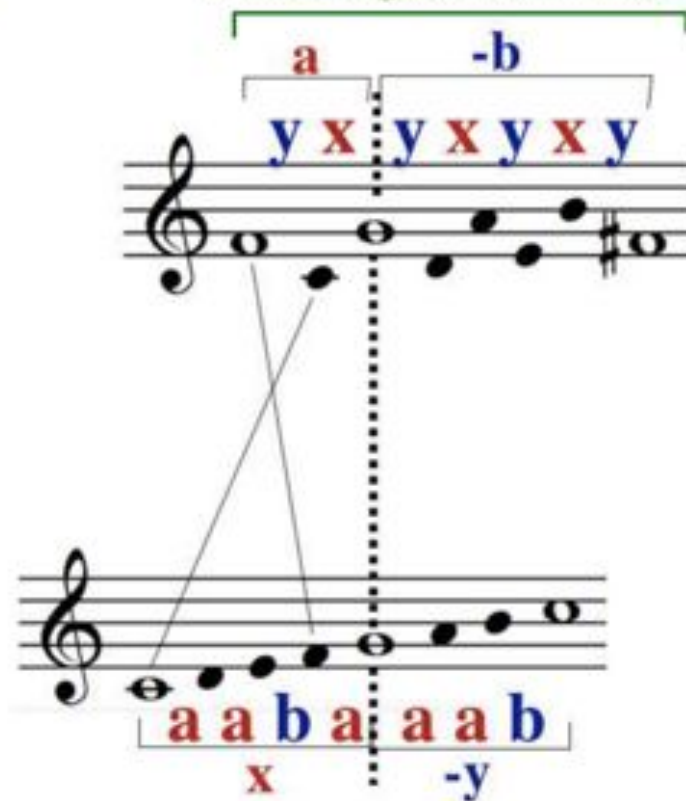
# The Structural Modes

Analogy between the Structural and the Diatonic Modes

Structural Augmented Prime



Usual Augmented Prime A1



# The Structural Modes

## Elementary Combinatorics and Arrow Notation

T S #S D T      T #T S D T      T S D #D T  
 ↓ ← ↓ ↓      ← ↓ ↓ ↓      ↓ ↓ ← ↓

**1st mode**

T S D (T)  
 S T D (#S)

**2nd mode**

T S D (T)  
 T D S (#T)

**3rd mode**

T S D (T)  
 D S T (#D)

The position of the minor third distinguishes the three common finalis modes

# Example: Handel, Concerto Grosso op. 3/4

The image displays a musical score for Handel's Concerto Grosso op. 3/4, consisting of three systems of music. Each system includes a treble and bass staff with various annotations below them. The annotations consist of letters (F, C, D, S, T, #S, #D, #T) and symbols (arrows, trills) indicating harmonic functions and performance techniques. The first system ends with a double bar line. The second system begins with a repeat sign. The third system ends with a 'Fine' marking.

**System 1:**  
 Treble:  $F_1: T$  ↑  $D$  ↓  $T$   $C_1: S$  ↑  $T$  ↓ ↓ ↓  $S D T$   
 Bass:  $F_1: T$  ↓  $S$  ←  $\#S$  ↓  $D$  ←  $D_1: D$  ↓  $T$  ↓  $S D T$

**System 2:**  
 Treble:  $F_2: \#T$  ↓  $S$  ↓  $D$  ↓  $T$  ↓  $F_1: S$  ↑  $T$  ↓  $S D T$   
 Bass:  $F_1: S$  ↓ ←  $\#S \#D$  ↓  $\#T$  ↓  $S D T$

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# Triadic Voice-Leading Space

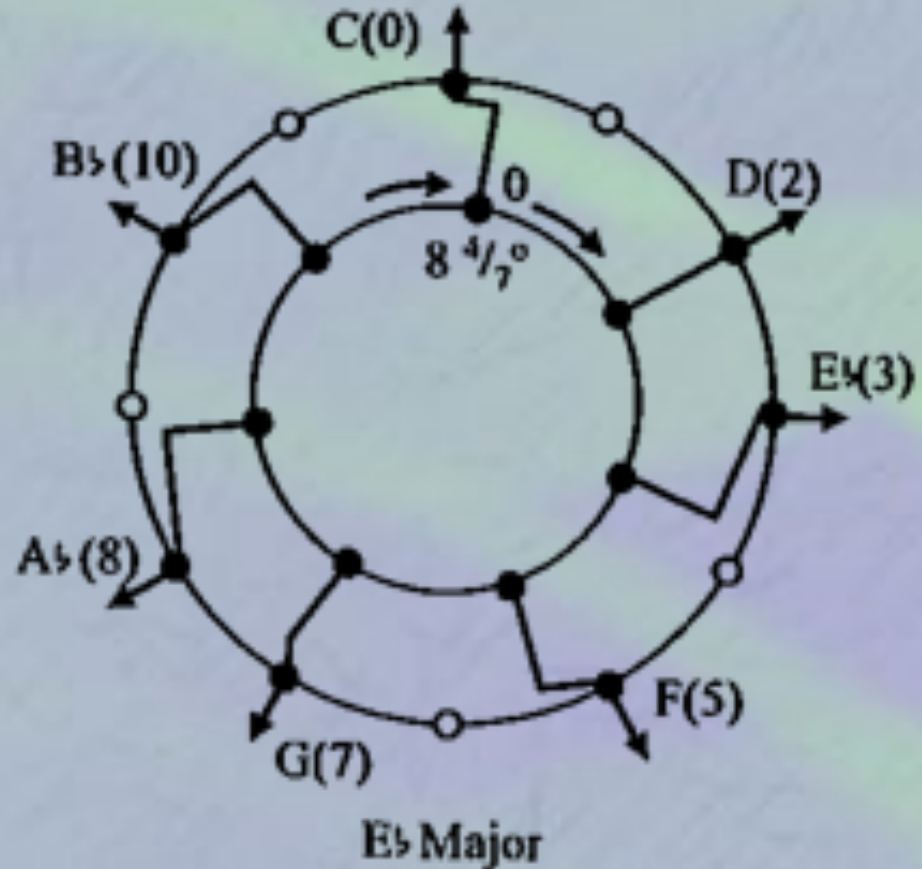


# Triadic Voice-Leading Space

## Douthett's Beacons and Filters

Because of the maximal evenness of the diatonic, a 7-equal scale rounded to the nearest points in 12-tET gives a diatonic scale.

As the 7-equal scale is continuously transposed, the diatonic scales traverse the circle of fifths.



From Douthett, "Filtered Point-Symmetry," 2008

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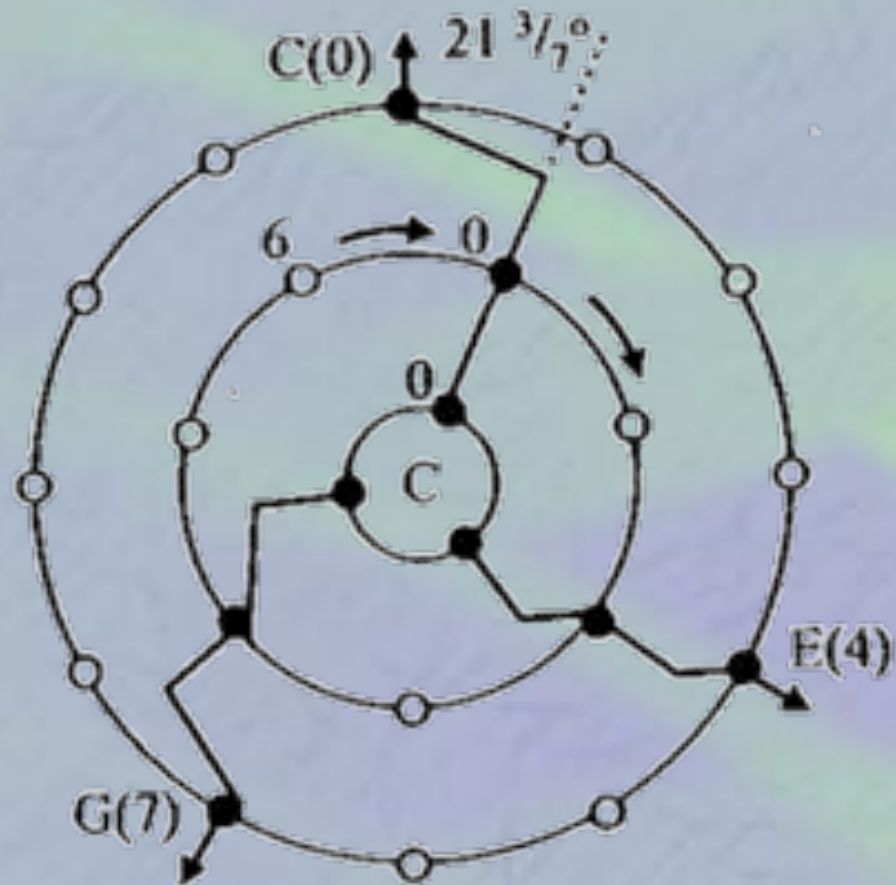
# Triadic Voice-Leading Space

## Douthett's Beacons and Filters

A *two-stage* quantization (rounding) process produces diatonic triads.

The first stage, rounding and even triad to 7-equal, produces a *7-equal* triad.

Rounding 7-equal triads to 12-tET gives a triad from the scale determined by the position of the 7-hole filter.



From Douthett, "Filtered Point-Symmetry," 2008

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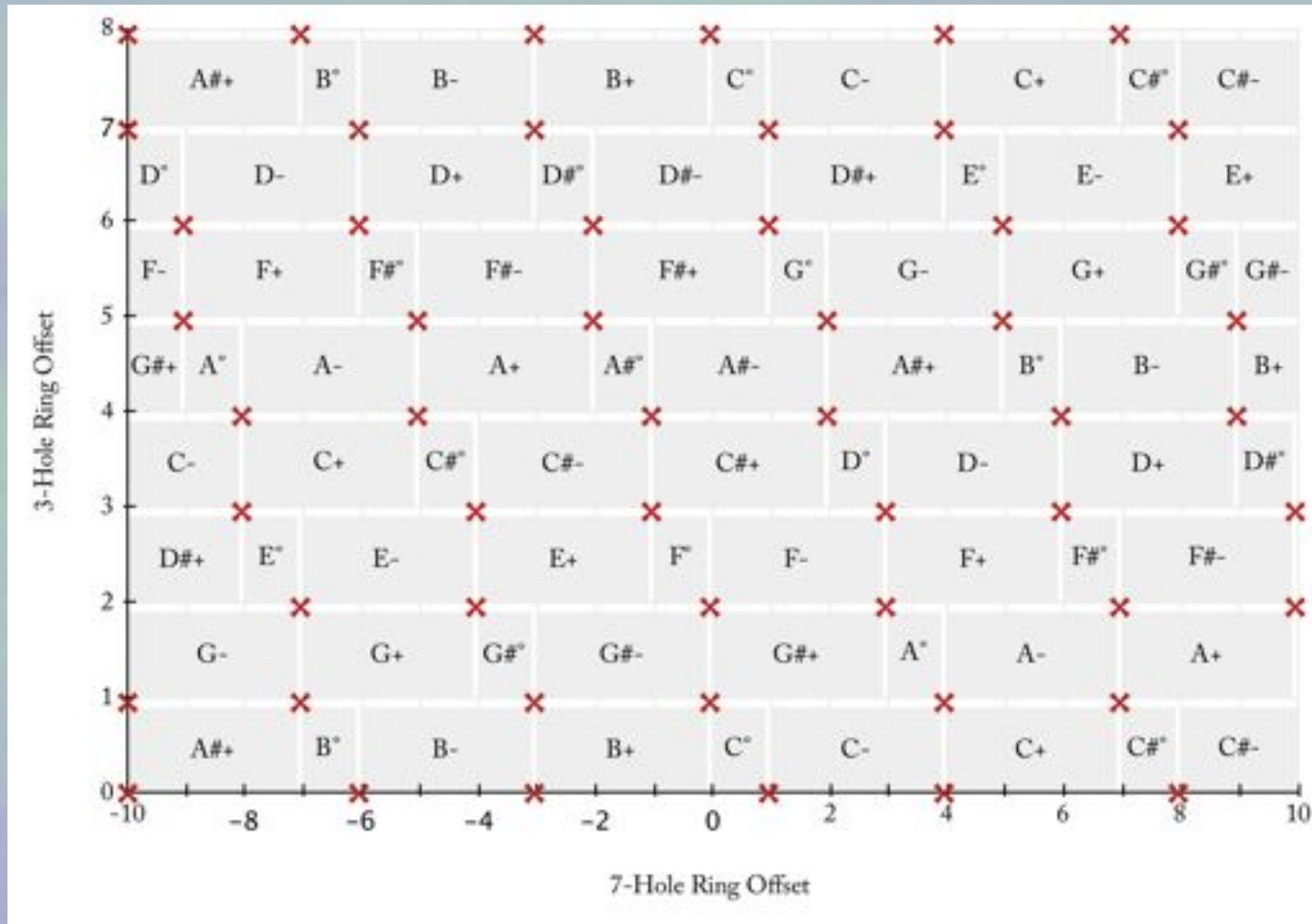
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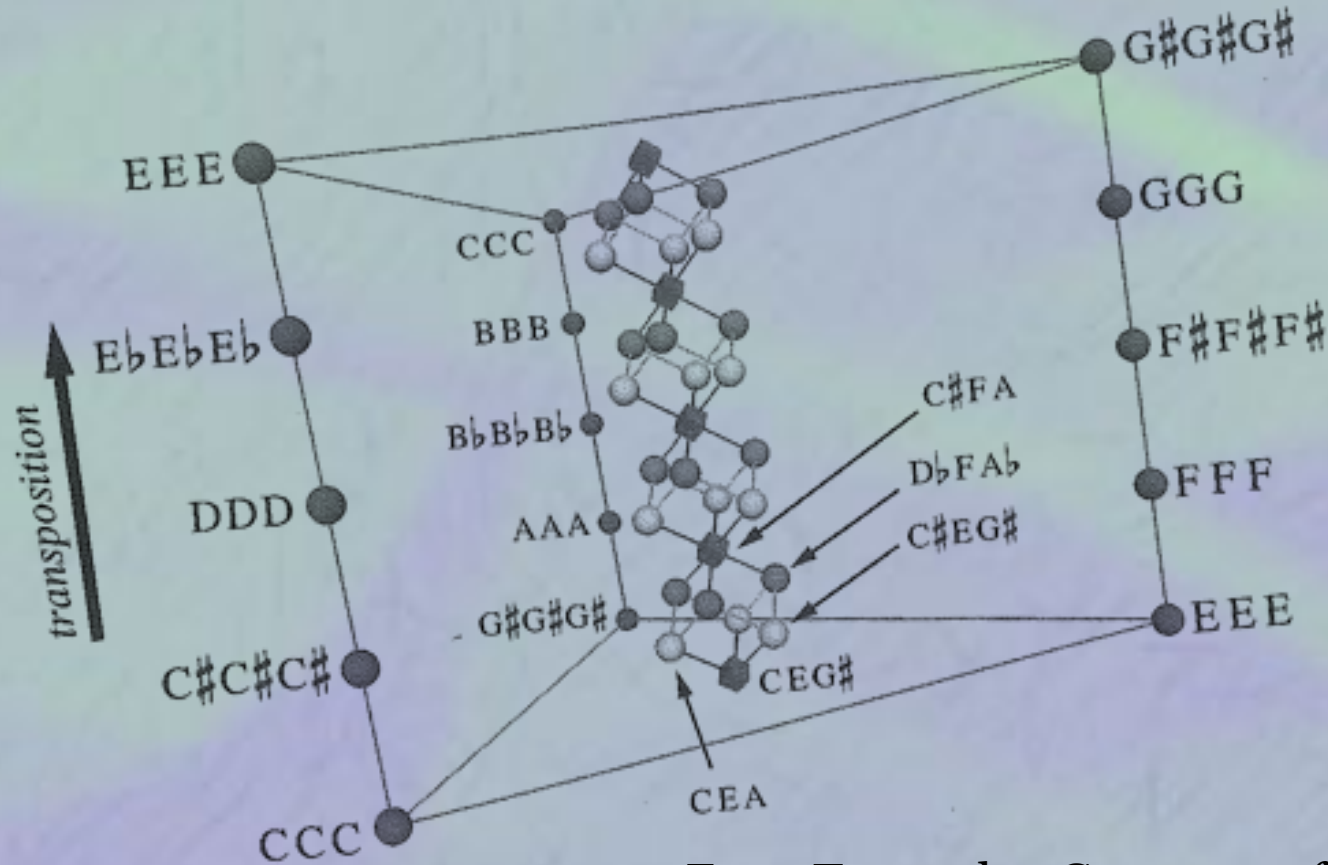
# Triadic Voice-Leading Space

In Plotkin's *configuration space*, each dimension tracks the position of one of the filters



# Triadic Voice-Leading Space

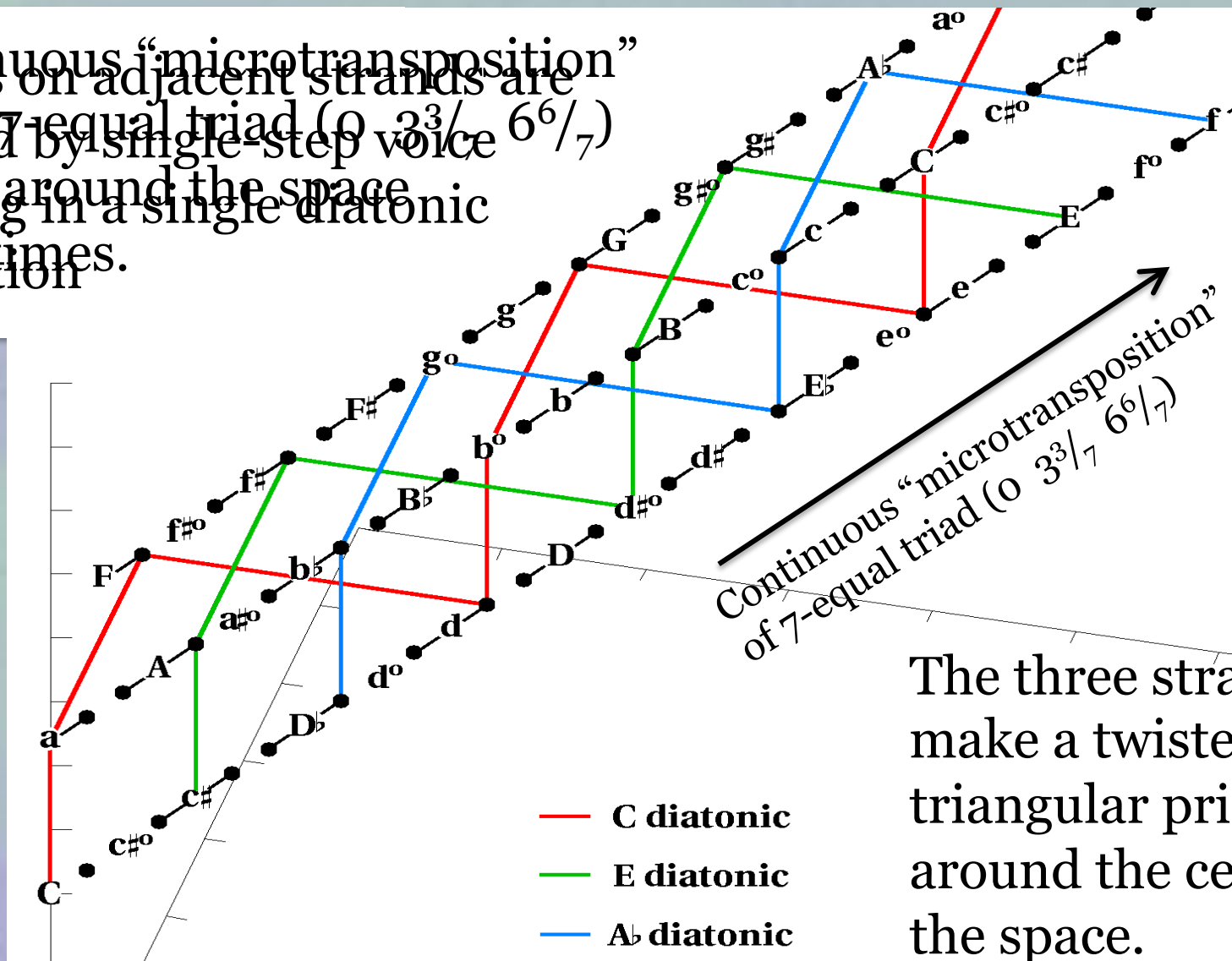
We can also understand the quantization as taking place in Callender, Quinn, and Tymoczko's *3-note chord space*.



From Tymoczko, *Geometry of Music*, 2011

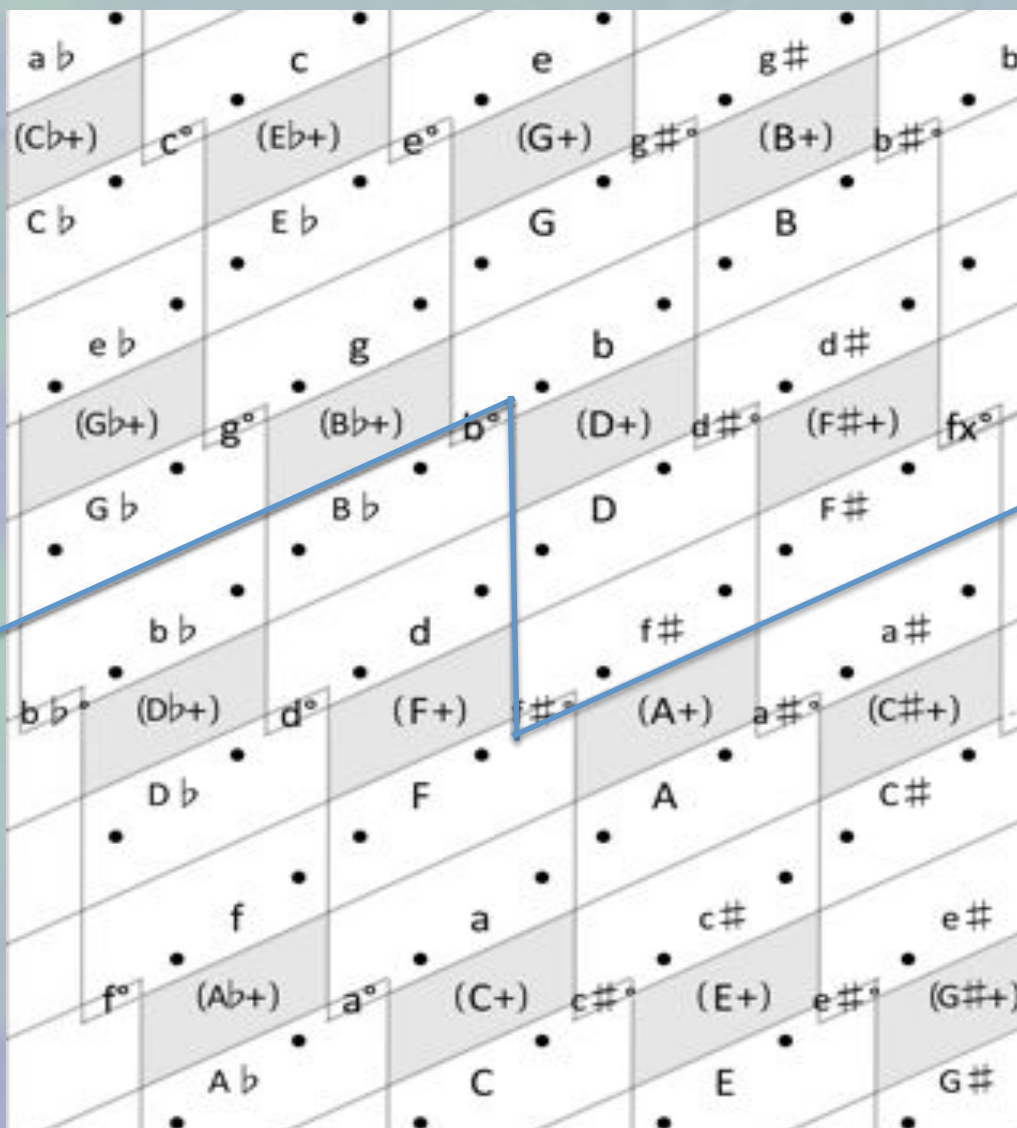
# Triadic Voice-Leading Space

Continuous "microtransposition" of the 7-equal triad ( $0\ 3^3/7\ 6^6/7$ ) wraps around the space leading in a single diatonic collection three times.



The three strands make a twisted triangular prism around the center of the space.

# Triadic Voice-Leading Space



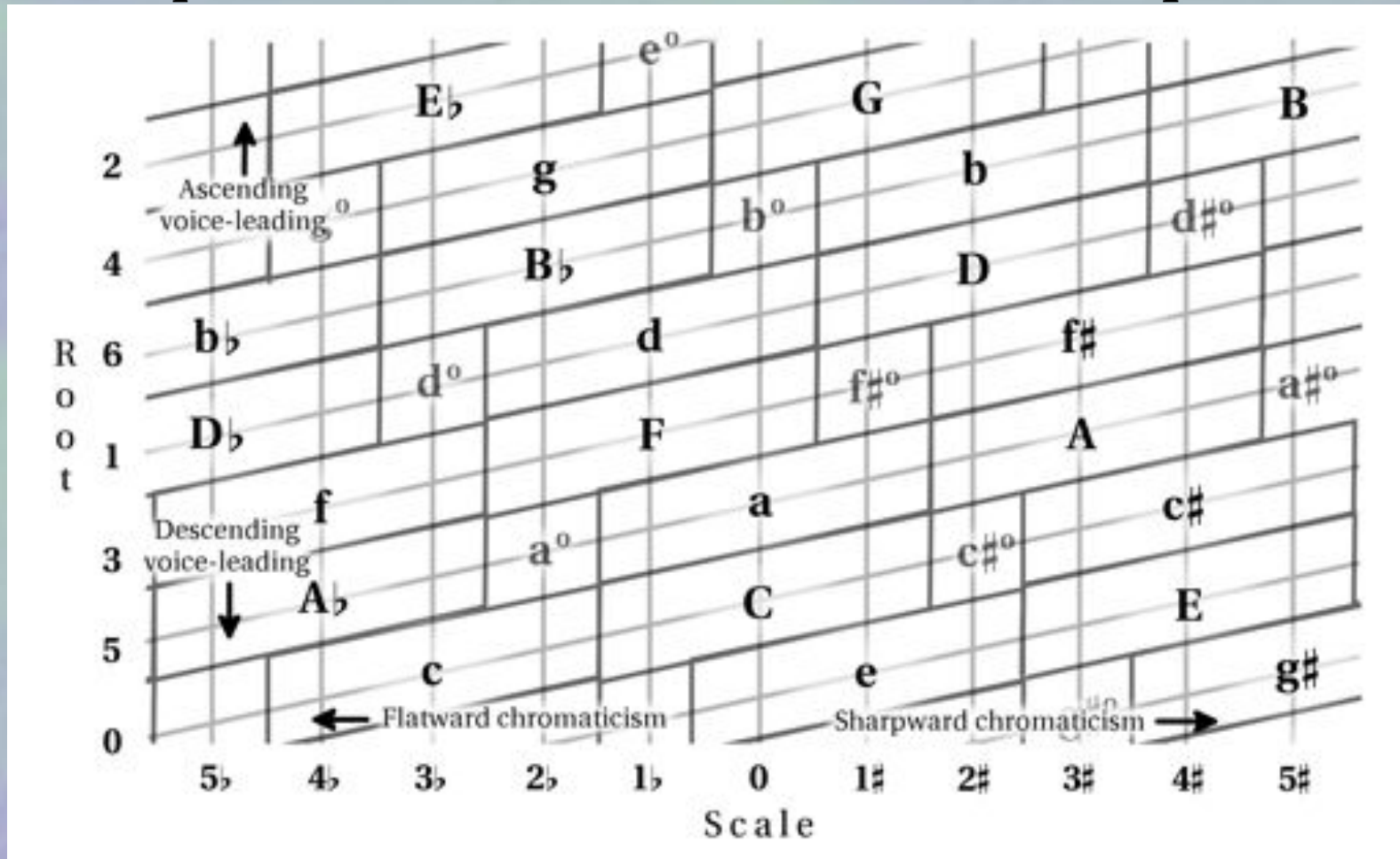
The surface of the prism makes a toroidal geometry with the kinks smoothed out

Each semitonal voice-leading (here F–F#) is a line that zig-zags across the space.

from Yust 2013b, “Space for Inflections,” *JMM* 7.3

# Triadic Voice-Leading Space

On this not-paired vertical axis the space of the augmented triads is the torus. There are twelve radiating axes, one for each scale.



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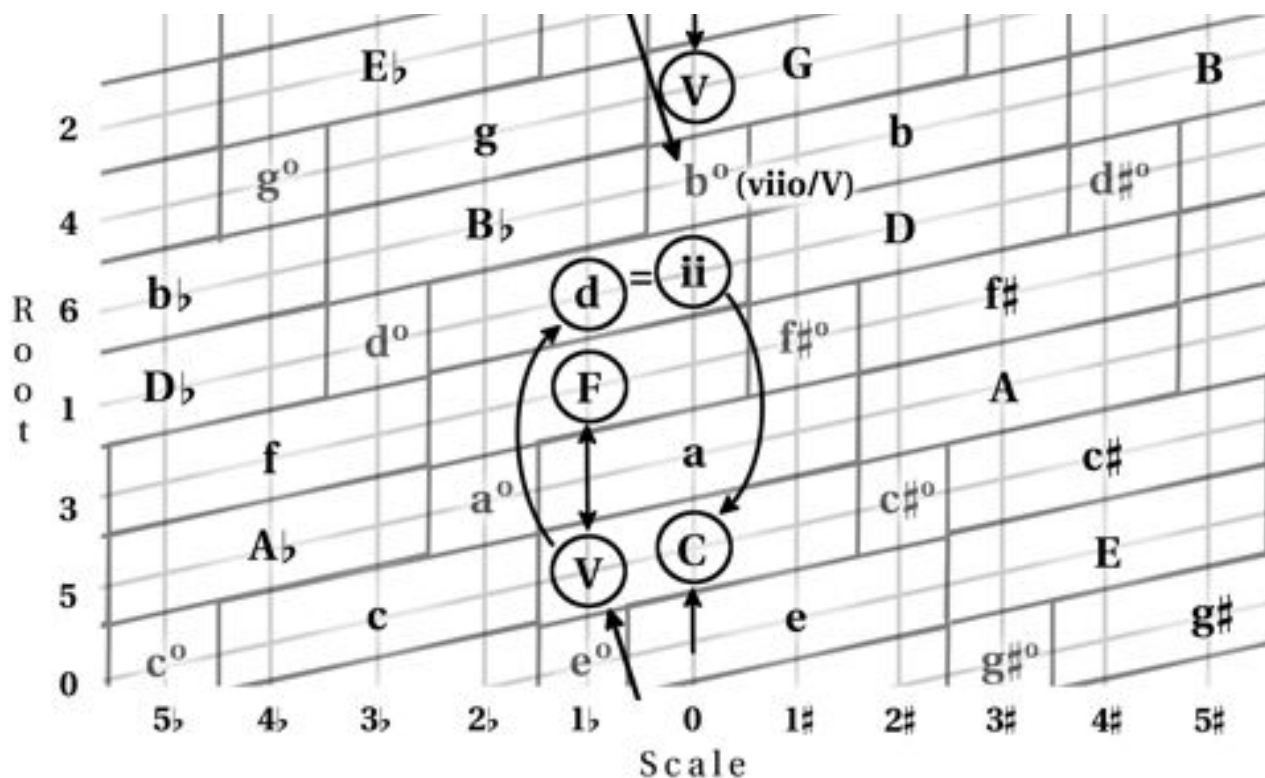
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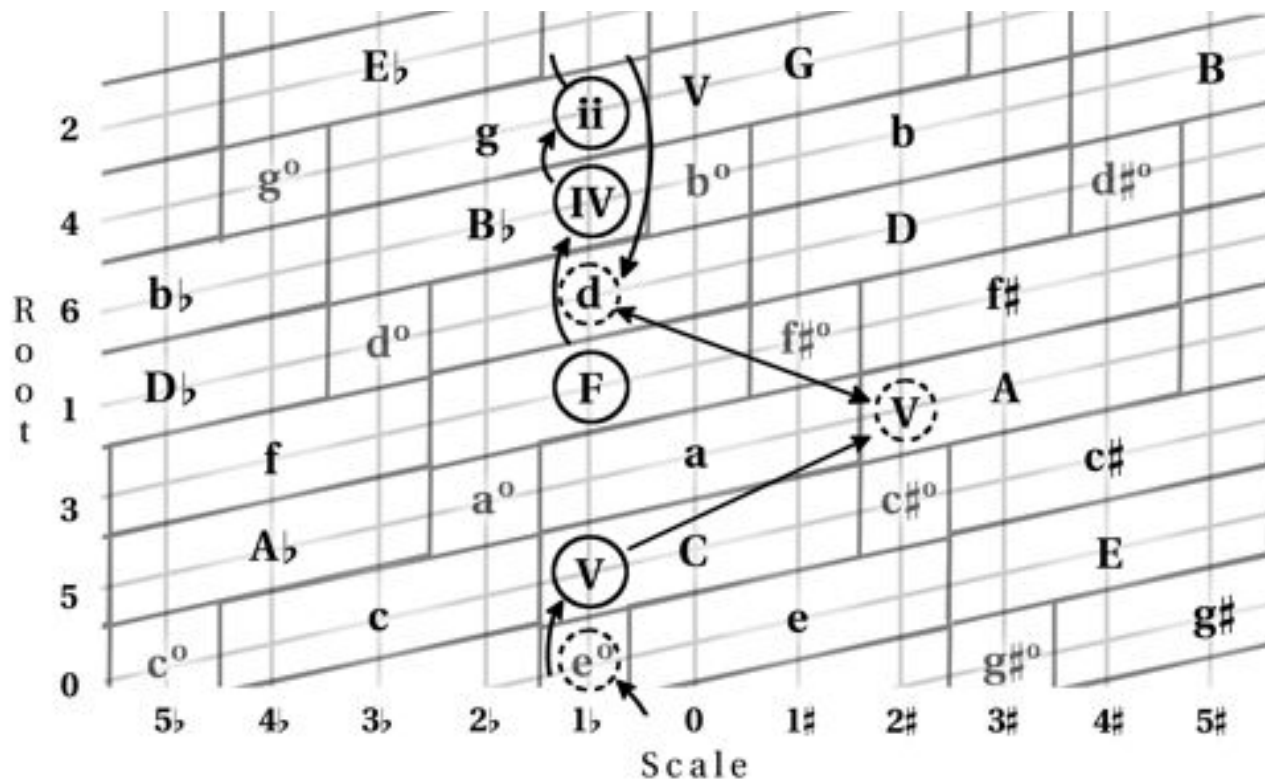


# Example: Handel Minuetto



Modulation to the dominant shifts one place to the right. The pivot chord is an “invisible” microtransposition.

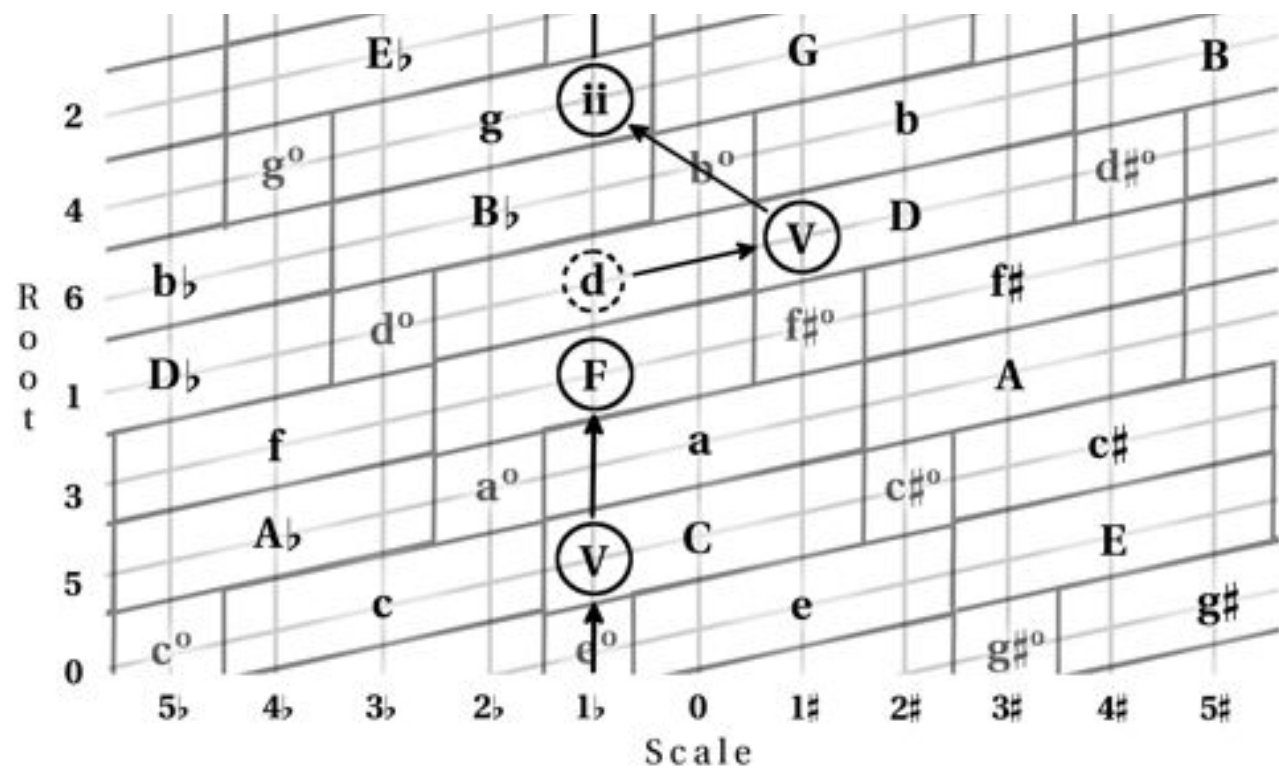
# Example: Handel Minuetto



The dominant of a minor key comes from three places to the right (from its parallel major).

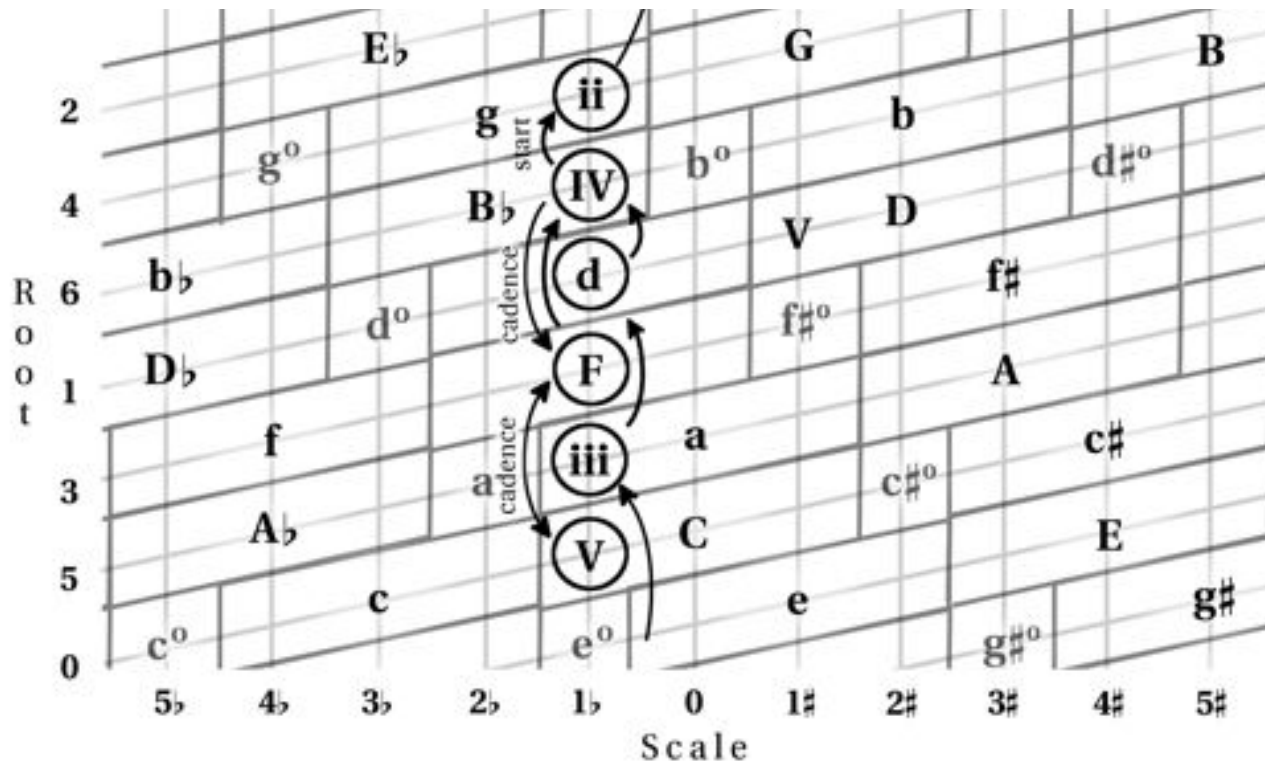
The sequence tours the space while the cadence circles the tonic.

# Example: Handel Minuetto



The sequence tours the space while the cadence circles the tonic.

# Example: Handel Minuetto



The sequence tours the space while the cadence circles the tonic.

# Harmonic Function



# Harmonic Function

## Structural Modes and their Scale Degrees

The 3 degrees of the structural modes can be associated with the 3 tonal functions T, S, D (in the spirit of the motivating remark from Carey and Clampitt 1989)

## Functional Modes and Functions

The embedded modes comply with Dahlhaus's (1990) attempt to integrate elements of Riemann's function theory with Sechter's scale degree theory

## Functional Regions

As will be shown by Jason, functionally indifferent chords are neighbors in the chordal/voice-leading spaces. This allows identification of functional regions.

## Functional Vectors

The symmetric placement of the regions allows a geometrical interpretation of the prototypical(?) harmonic progressions

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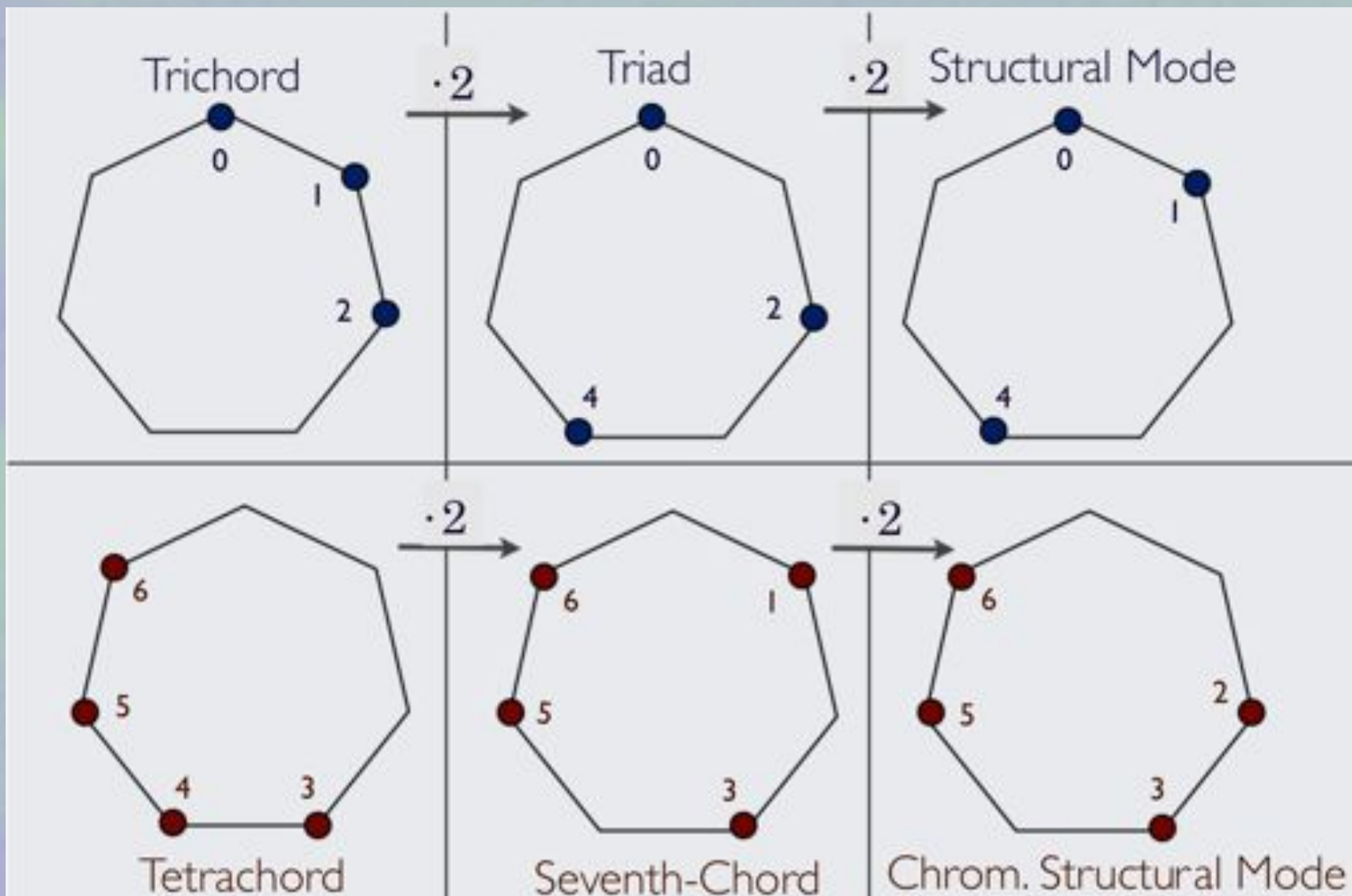
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# Conceptual Crossrelations and Transformations



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# Embedded Structural Modes

LT-Dominant			Neapolitan		
M2 P4 P4	P4 M2 P4	P4 A4 m2	m2 A4 P4	P4 M2 P4	P4 P4 M2
M2 P4 P4	P4 M2 P4	P4 P4 M2	m2 P4 A4	P4 m2 A4	P4 P4 M2
M2 P4 P4	P4 M2 P4	P4 P4 M2	M2 P4 P4	A4 m2 P4	A4 P4 m2
M2 P4 P4	P4 M2 P4	P4 P4 M2			



# Embedded Structural Modes

The image displays three diagrams illustrating embedded structural modes, each represented as a heptagon with vertices labeled with notes. The C<sub>2</sub> mode is highlighted in blue in each diagram.

- C<sub>2</sub> in C-Dorian:** The heptagon has vertices C, G, D, A, Ab, C. The C<sub>2</sub> mode is highlighted in blue, showing a path from C to G to D to A to Ab to C.
- C<sub>2</sub> in C-Aeolian:** The heptagon has vertices C, G, D, A, Ab, C. The C<sub>2</sub> mode is highlighted in blue, showing a path from C to G to D to A to Ab to C.
- C<sub>2</sub> in C-Phrygian:** The heptagon has vertices C, G, D<sup>b</sup>, A, Ab, C. The C<sub>2</sub> mode is highlighted in blue, showing a path from C to G to D<sup>b</sup> to A to Ab to C.

Below each diagram is a musical score in bass clef, showing the mode's structure with notes and chords.

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# Example: Trio of Schubert's C minor Sonata, D.958

**Trio.**

The image shows a musical score for the Trio of Schubert's C minor Sonata, D.958. The score is in 3/4 time and C minor. It consists of four systems of music, each with a treble and bass staff. The first system is marked '1' and 'p'. The second system has a first and second ending. The third system is marked 'pp'. The fourth system is marked 'ritard.' and 'fp'. Below the score, various harmonic functions are annotated with letters and symbols: A<sub>b2</sub>: T, #T, S, D, F<sub>1</sub>:D, #T, S, D, T, E<sub>b1</sub>: T, D, T, G<sub>b2</sub>: #T, S, D, T, A<sub>b2</sub>: D, (S), D, T. Arrows indicate the direction of the harmonic function changes.

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# Example: Chopin Mazurka in F# minor

M. M.  $\text{♩} = 132.$   
 N<sup>o</sup>. 1.

The image displays three systems of musical notation for Chopin's Mazurka in F# minor. Each system consists of a piano score with treble and bass staves, followed by a line of harmonic analysis. The first system includes dynamics like *p*, *cres.*, and *deces.*, and performance instructions like *Ped.* and *legato.*. The second system includes *rubato.* and *cres.*. The third system includes *p* and *ritenuto.*. The harmonic analysis below the first system is: F#<sub>2</sub>: D, T, A<sub>2</sub>: D, T, C#<sub>2</sub>: S, D, T. The harmonic analysis below the second system is: B<sub>2</sub>: S, D, T, A<sub>2</sub>: S, D, T, G: S, D, F#<sub>2</sub>: S, D, T.

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# A voice-leading based concept of function

## Some principles

- Functions are types of *relationship* rather than types of *chords*. Therefore a function is a kind of direction in the space rather than a region.
- Functional relationships involve strong (not weak) voice leading
- It is not necessary for functions to exhaustively classify all chords, or to be symmetrical.

# A voice-leading based concept of function

## Some principles

Dominant: • Includes lower neighbor to  $\hat{1}$   
• Diatonic or sharpward

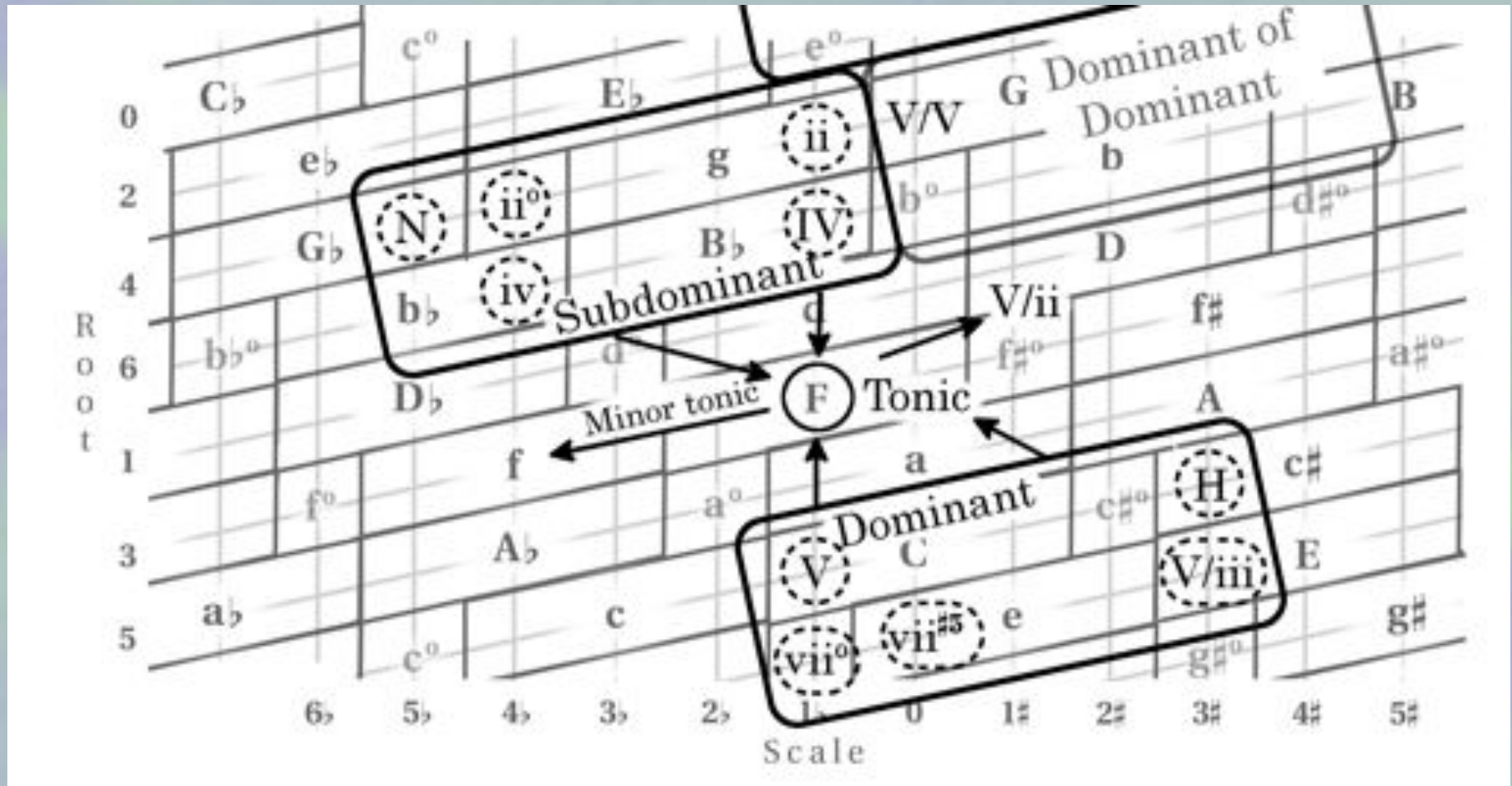
Subdominant: • Includes upper neighbor to  $\hat{5}$   
• Diatonic or flatward

Hence:

Dominant: A path down and rightward from tonic

Subdominant: A path up and leftward from tonic

# A voice-leading based concept of function



Harmonies southwest or northeast of tonic are unclassified, unless we add secondary relations (dominant of V, subdominant of IV)

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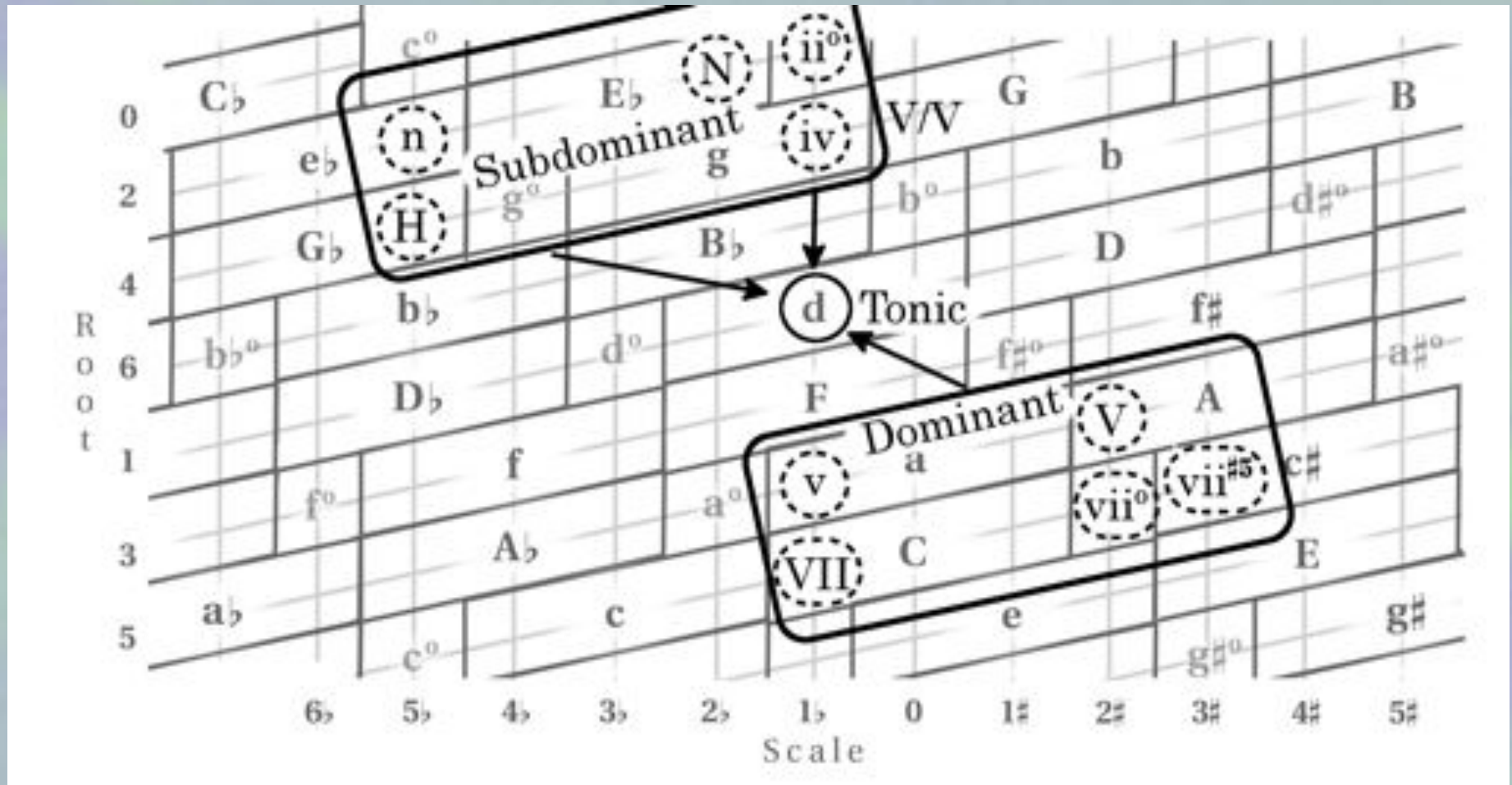
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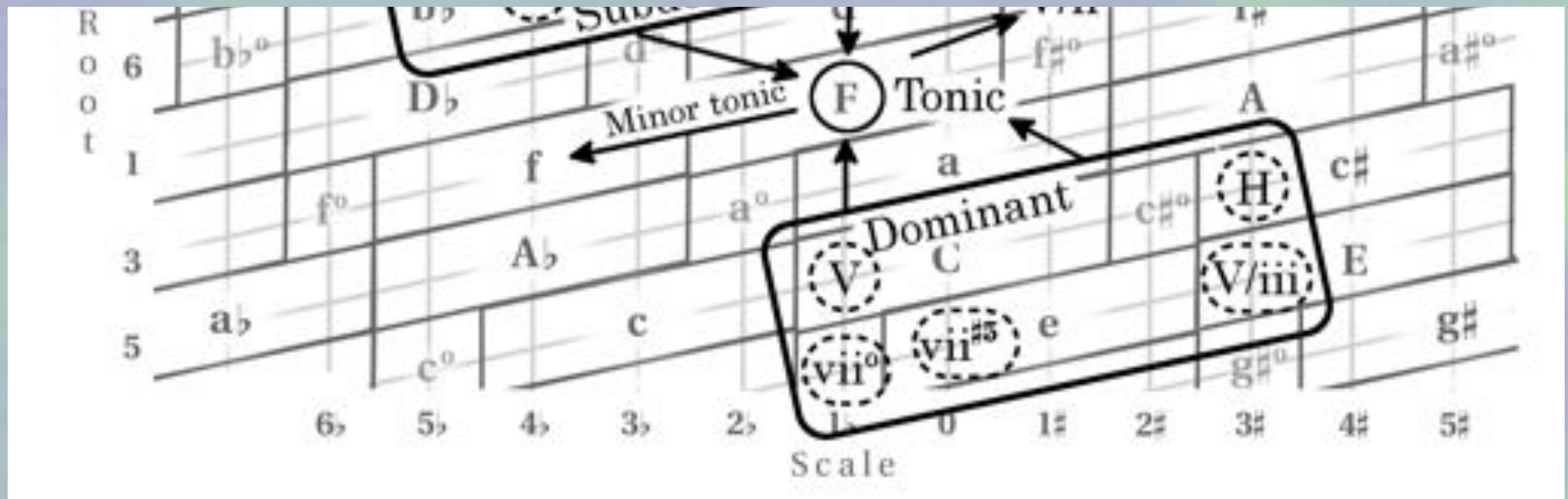
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# A voice-leading based concept of function



# A voice-leading based concept of function

Surprising predictions: Dominant function in major includes the minor triad on the leading tone, V/iii, and the hexatonic pole (spelled as  $\#v^{\#5}$ )





# A voice-leading based concept of function

Example:  $vii^{\#5}$  as a dominant

The image displays two systems of musical notation for Schubert's "Auf dem Flusse" (Winterreise no. 7). The first system shows the vocal line and piano accompaniment. The vocal line has the lyrics: "Der du so lu - stig rauschtest, du hel - ler, wil - der Fluss, wie still bist du ge - wor - den, gibst". The piano accompaniment features a *ppp* dynamic marking. The second system continues the vocal line with the lyrics: "kei - nen Schel - do - gruss! Mit har - ter, star - rer Rin - de hast". The piano accompaniment features a *pp* dynamic marking. Between the two systems, the text "e: V# ... #vii#5" is written, indicating a harmonic analysis of the progression.

Schubert: "Auf dem Flusse" *Winterreise* no. 7

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# A voice-leading based concept of function

Example:  $vii^{\sharp 5}$  as a dominant (Lydian cadence: Machaut)

LA MESSE DE NOSTRE DAME  
I. Kyrie

25

e - ley - son.

e - ley - son.

e - ley - son.

VII

- ley - son.

d: i  $\sharp vii^{\sharp 5}$  i

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# A voice-leading based concept of function

Example:  $VII^{\sharp 5}$  (V/III) as a dominant

d: V = B $\flat$ : VII $\sharp 5$

Schubert, Piano Sonata in B-flat Major, D.960, mm. 251–8  
Recapitulation (retransition to A' of main theme)

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# A voice-leading based concept of function

Example:  $\#v^{\#5}$  (H) as a dominant

hal - ten will ich den Schmerz, ob auch zer - spal - ten mir ist das Herz, se - lig doch hal - ten will ich den

$A_b$ : I . . .  $\#v^{\#5}$  (= H) - - - - I

Schmerz, ob auch zer - spal - ten mir ist das Herz, se - lig doch hal - ten will ich den Schmerz,

The image shows two systems of musical notation. The top system includes a vocal line with German lyrics and a piano accompaniment. The piano part features chords and arpeggios, with dynamic markings like *fff* and *p*. The bottom system continues the vocal and piano parts. A harmonic analysis is overlaid on the piano part, identifying a specific chord as  $\#v^{\#5}$  (= H) and showing its resolution to the tonic I.

Schubert, "Fülle der Liebe" D.840

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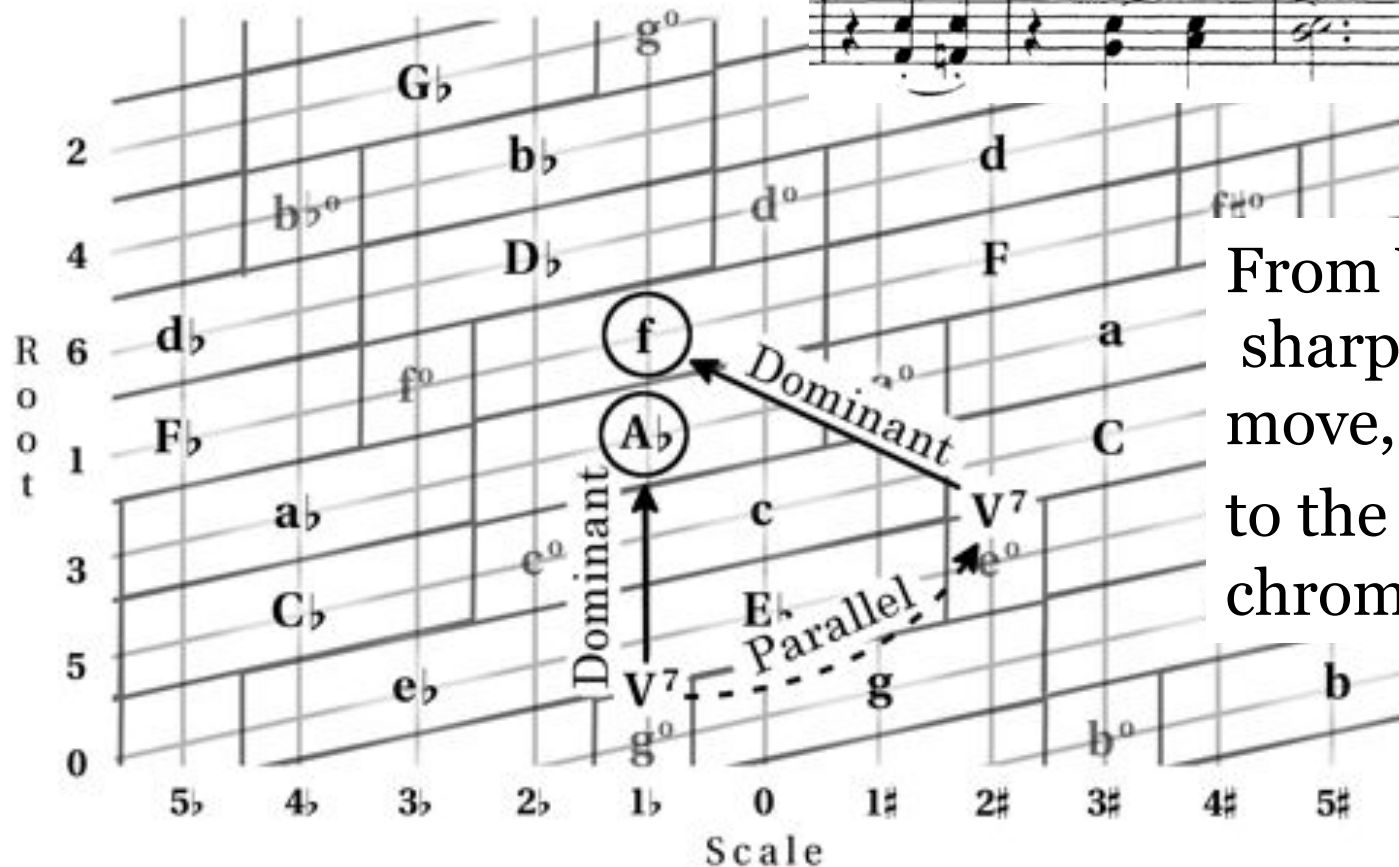
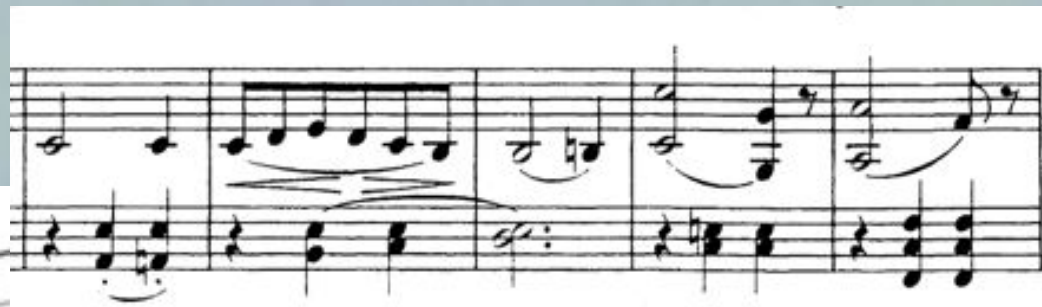
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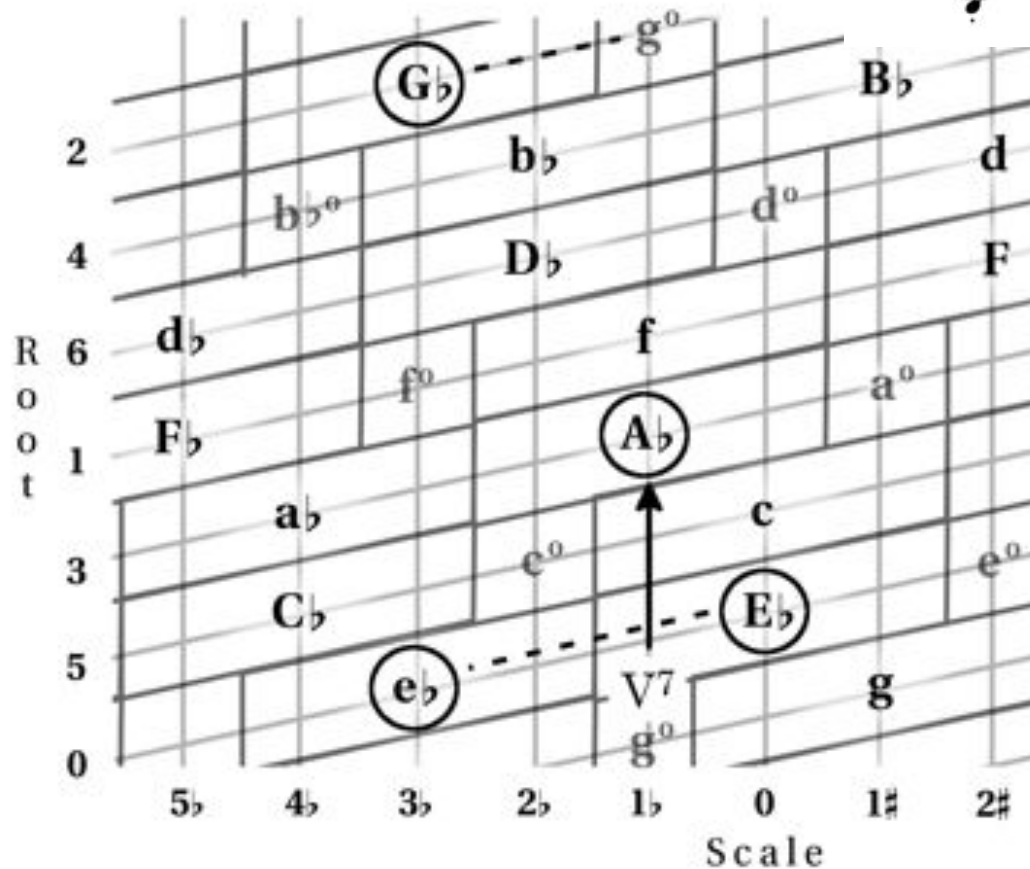
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# Voice-leading functions in D.958 Trio



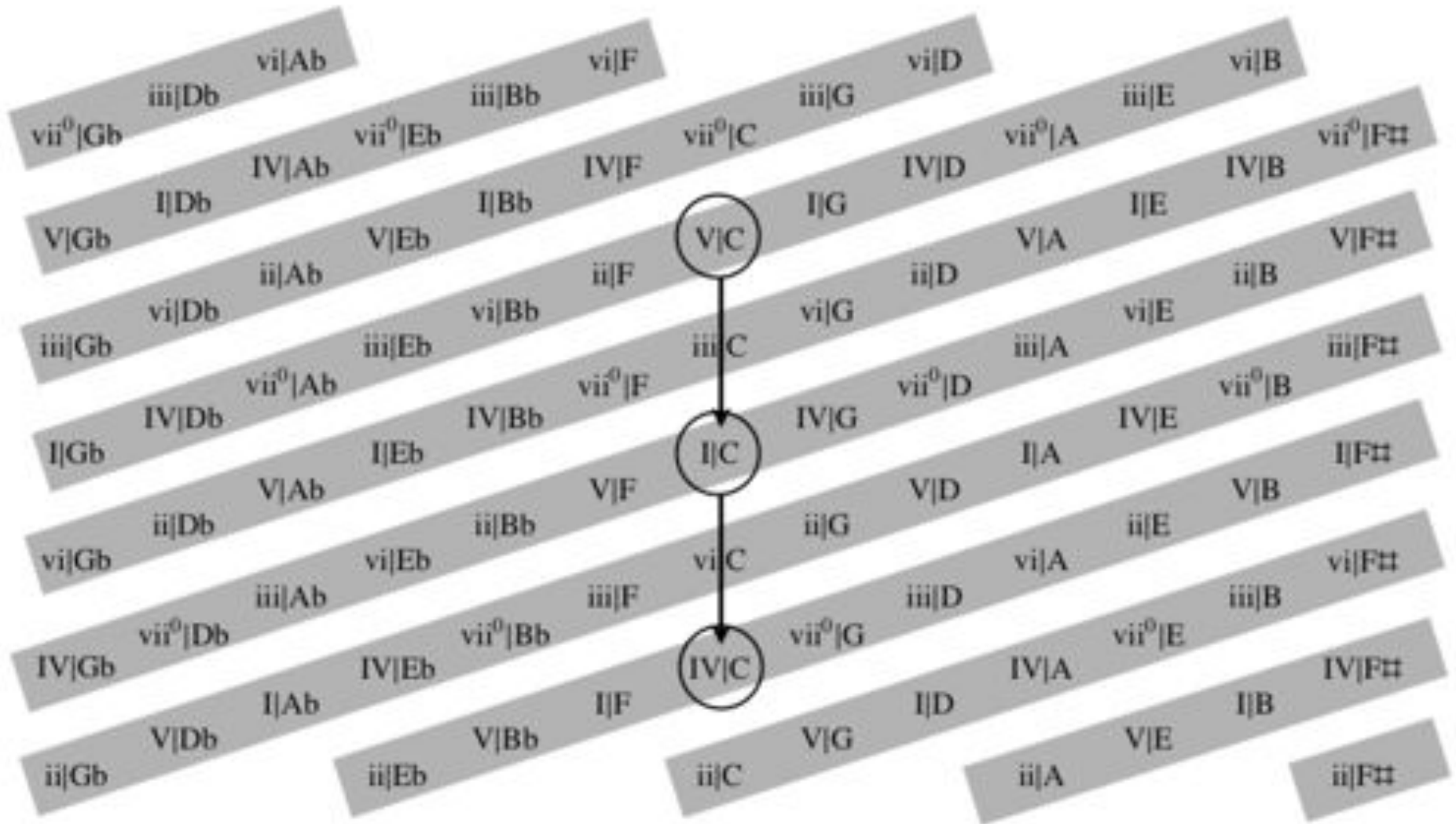
From  $V^7$  to  $V^7/vi$  is a sharpward parallel move, corresponding to the  $E_b \rightarrow E$  chromatic semitone.

# Voice-leading functions in D.958 Trio



The  $\flat VII \rightarrow V^7$  retransition reverses the parallel move of the mode change with a  $G\flat \rightarrow G$  chromatic semitone. Although the parallel move is functionally neutral, The  $G\flat$  and  $E\flat$  chords are *not* functionally equivalent relative to an  $A\flat$  tonic.

# Structural modes as ME divisions of triad space



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# Example: Handel, Concerto Grosso op. 3/4

4 (a). Minuetto Alternativo

Violino I ed Oboe I

Violino II ed Oboe II

Viola

Fagotti

Continuo

17

Fine

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# Example: Trio of Schubert's C minor Sonata, D.958

The image shows a page of musical notation for the Trio of Schubert's C minor Sonata, D.958. The score is written for piano and bass clef staves. It begins with the word "Trio." and a first ending bracket. The music features various dynamics including *p*, *pp*, *ritard.*, and *sp*. The tempo marking *tempo* appears at the start of the fifth system. The notation includes notes, rests, slurs, and articulation marks.

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