Schubert's Harmonic Language and the *Tonnetz* as a Continuous Geometry

Jason Yust, Boston University

Presentation to the
Society for Music Theory, October 31, 2013

Outline

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1. Periodicity in triads
2. Fourier balances
3. A continuous, cardinality-independent *Tonnetz*

II. Mediants in Schubert

Ex.: Menuetto from "Rosamunde" Quartet, D.804

III. *Tonnetz* regions vs. Weber regions

Ex.: Late B♭ major Piano Sonata, D.960

IV. Other compositional techniques

1. Modulation by scalar common tone
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   - Ex.: C major Quintet, Adagio
2. Transgression of modal boundaries
   - Ex.: C major Quintet, Adagio
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I. Discrete Fourier Transform

1. Periodicity in triads
2. Fourier balances
3. A continuous, cardinality-independent *Tonnetz*

Discrete Fourier Transform on Pcsets

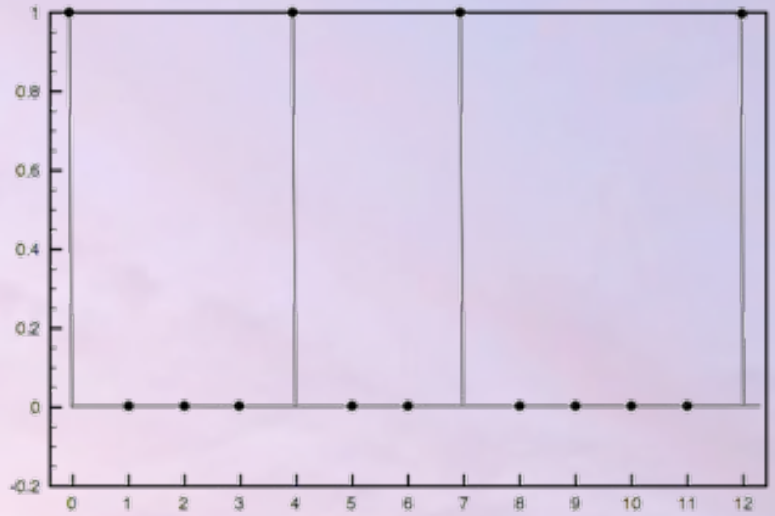
Lewin, David (1959). “Re: Intervallic Relations between Two Collections of Notes,” *JMT* 3.

Quinn, Ian (2006–2007). “General Equal-Tempered Harmony,” *Perspectives of New Music* 44/2–45/1.

Amiot, Emmanuel (2013). “The Torii of Phases.” *Proceedings of the International Conference for Mathematics and Computation in Music, Montreal, 2013* (Springer).

Discrete Fourier Transform: Periodicity

C major triad



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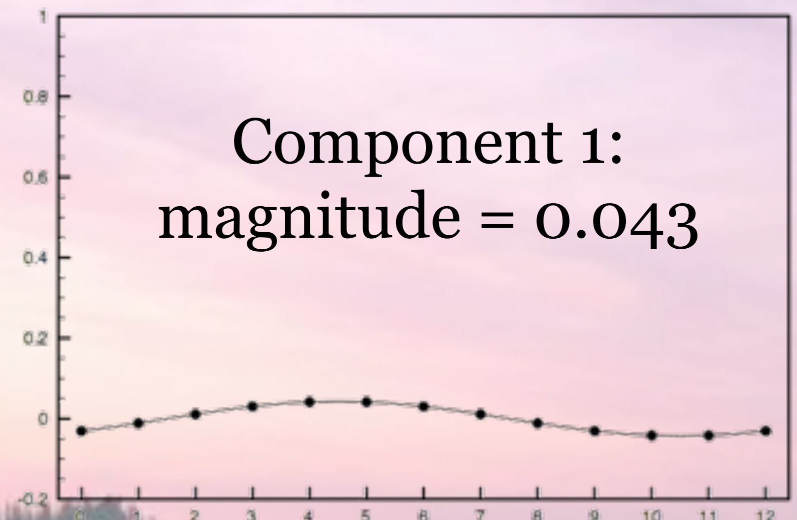

Component 1:
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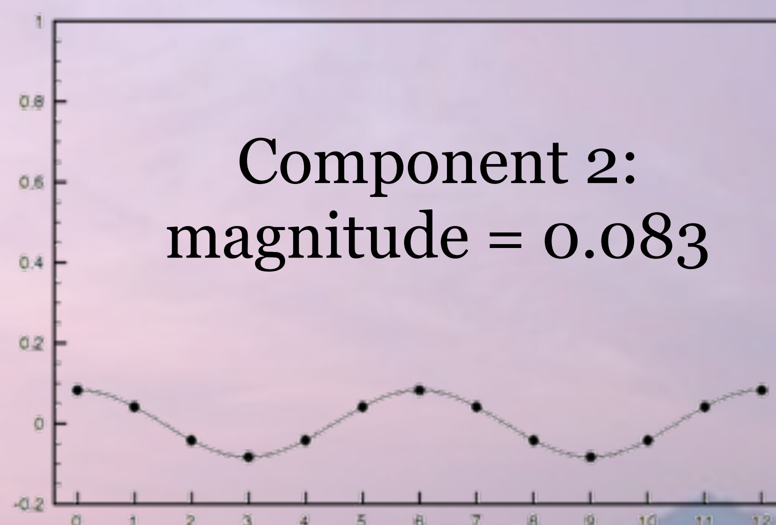

Component 2:
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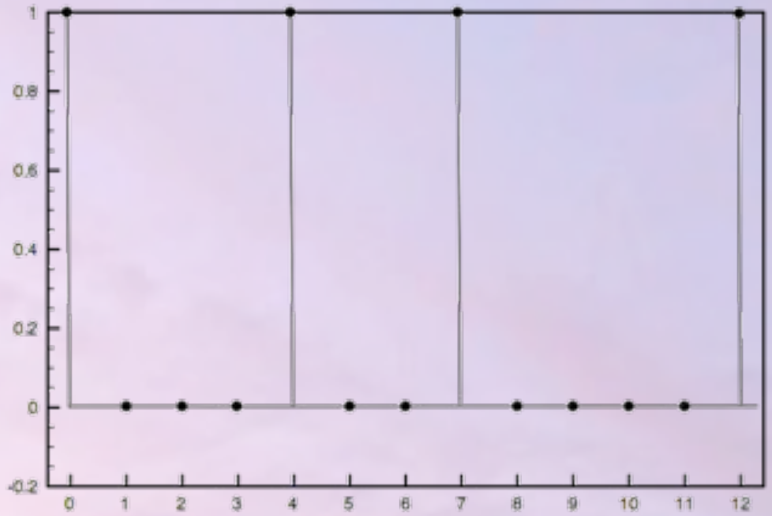
. . .

Discrete Fourier Transform: Periodicity

C major triad



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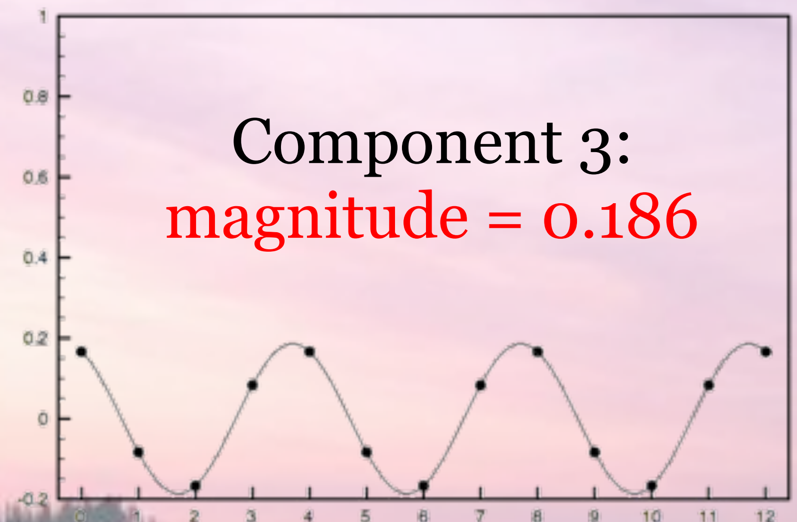

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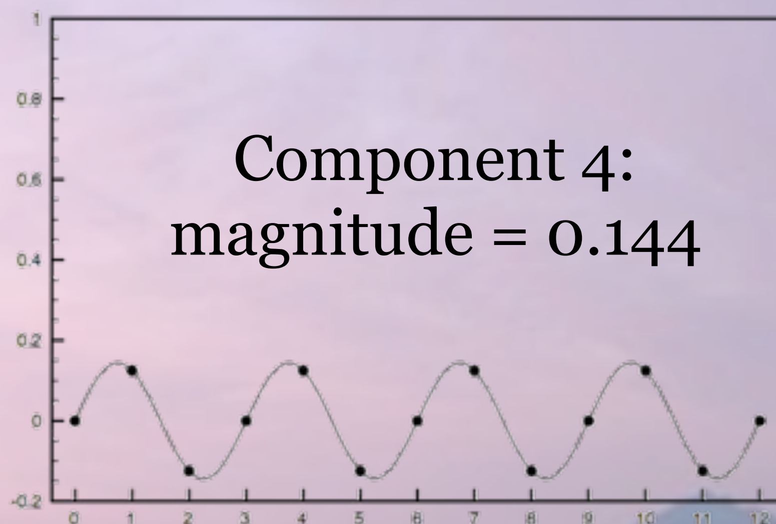

Component 4:
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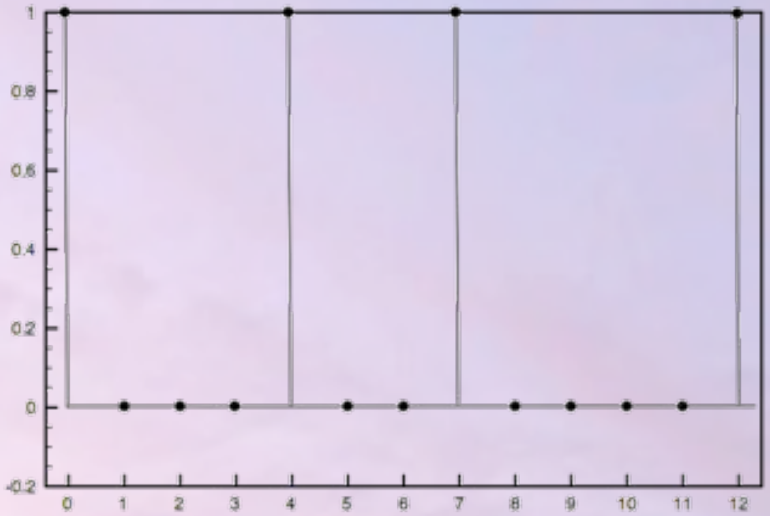
Discrete Fourier Transform: Periodicity

C major triad



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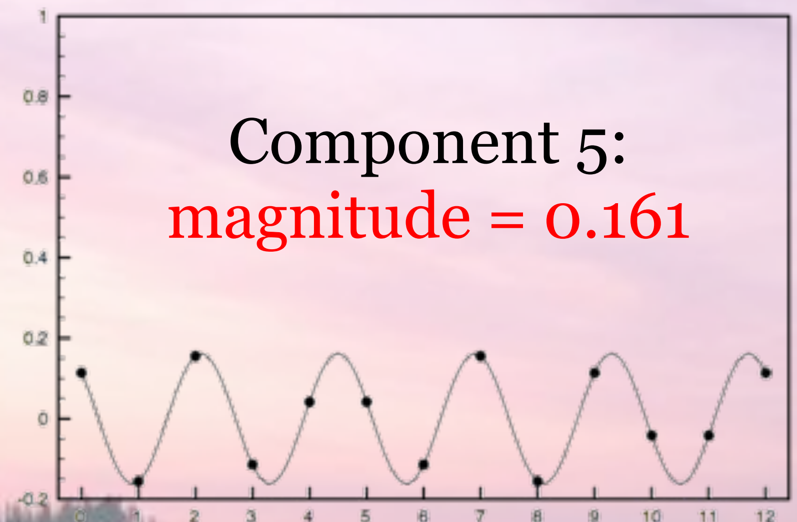

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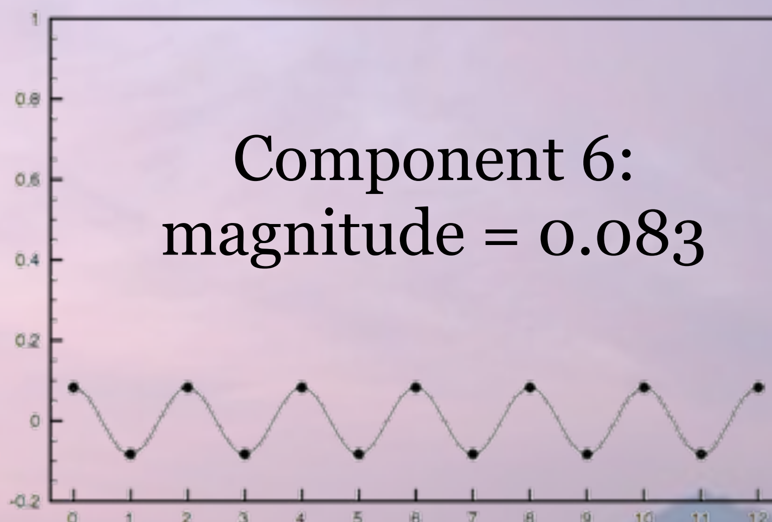

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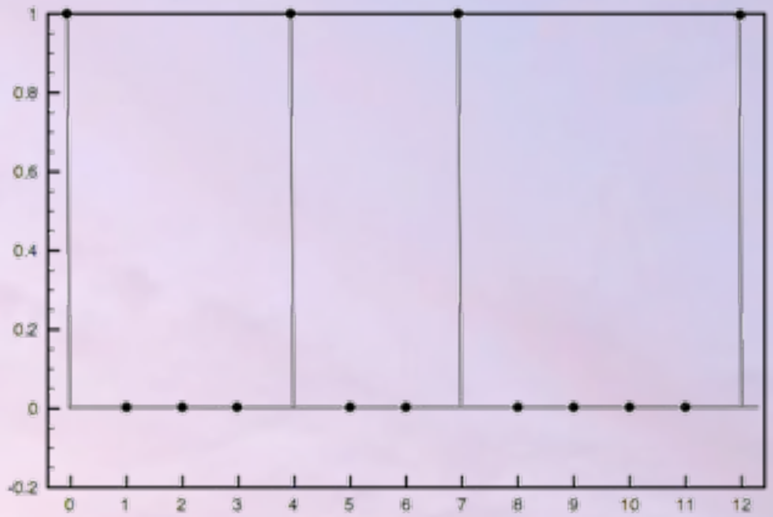
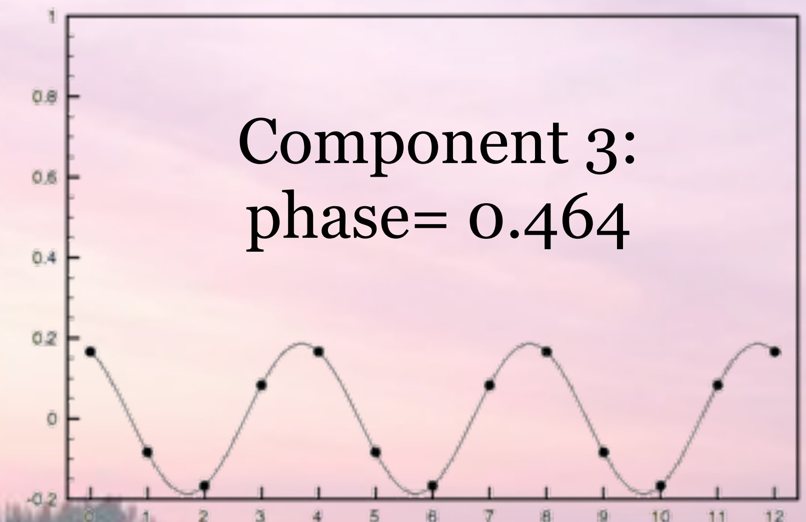
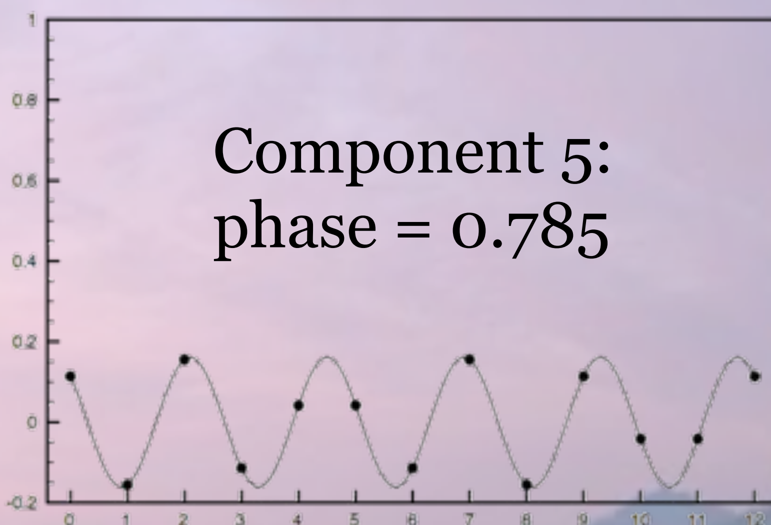
Discrete Fourier Transform: Periodicity

C major triad

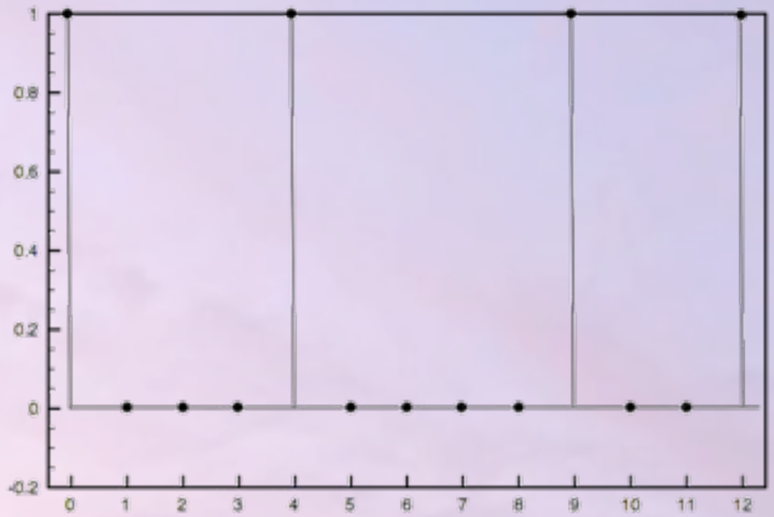


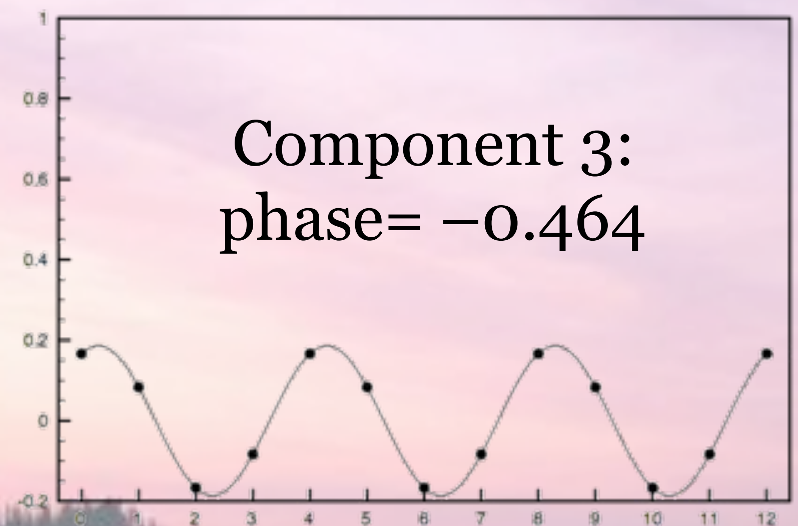


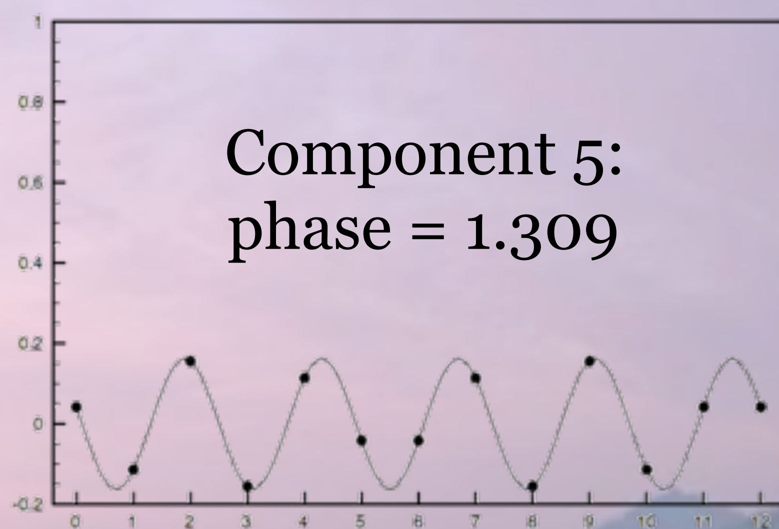


Discrete Fourier Transform: Periodicity

A minor triad

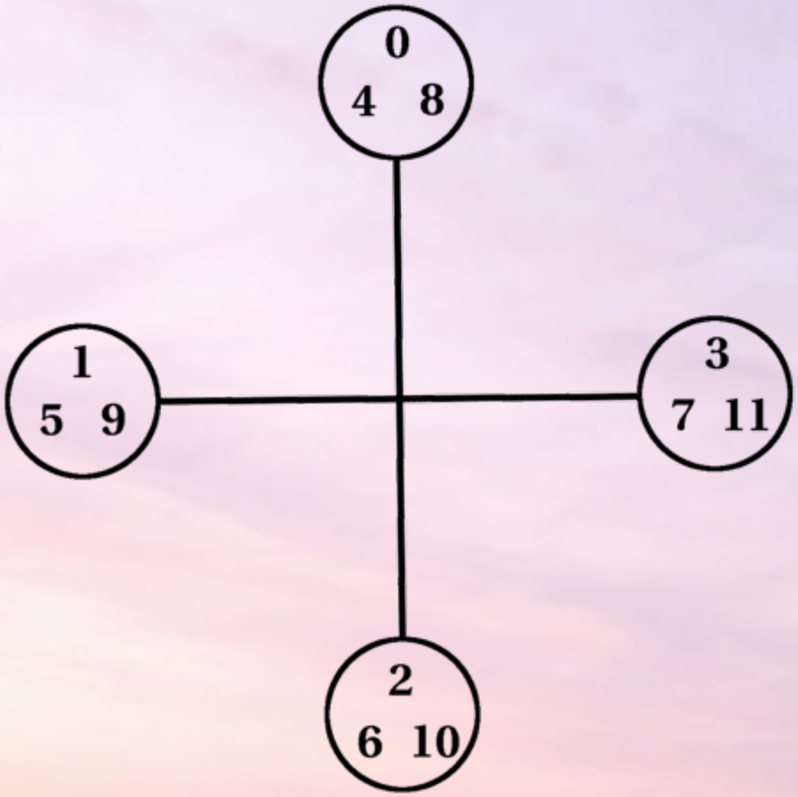




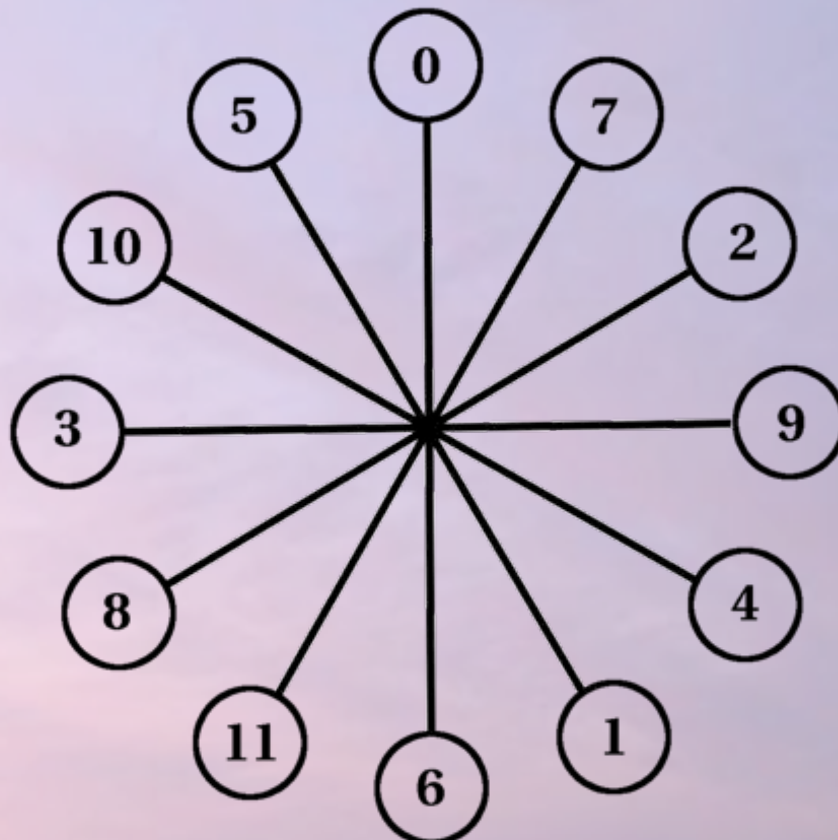


Discrete Fourier Transform: Balances

3-component balance



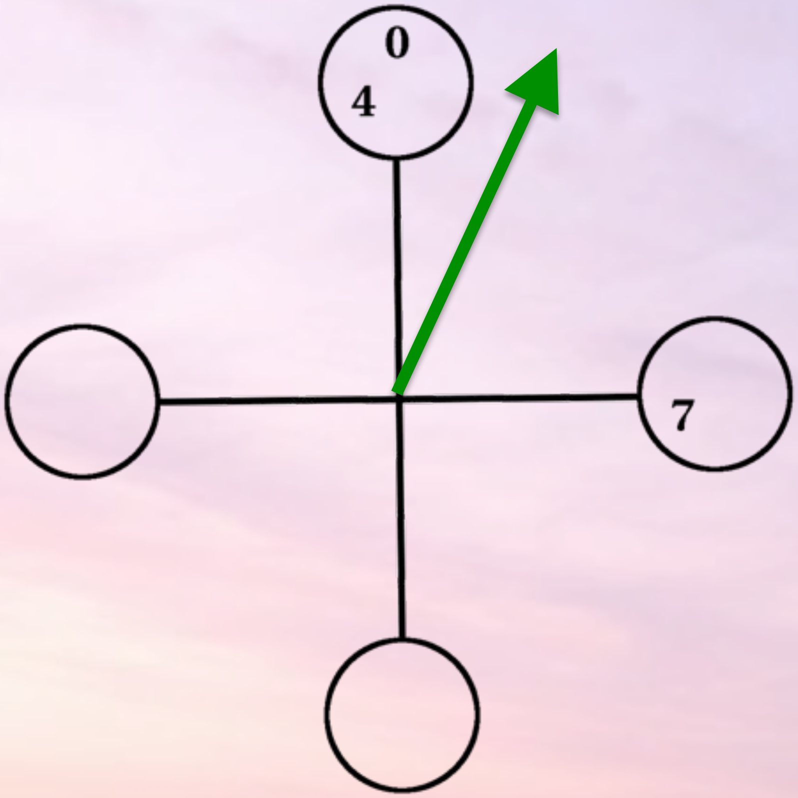
5-component balance



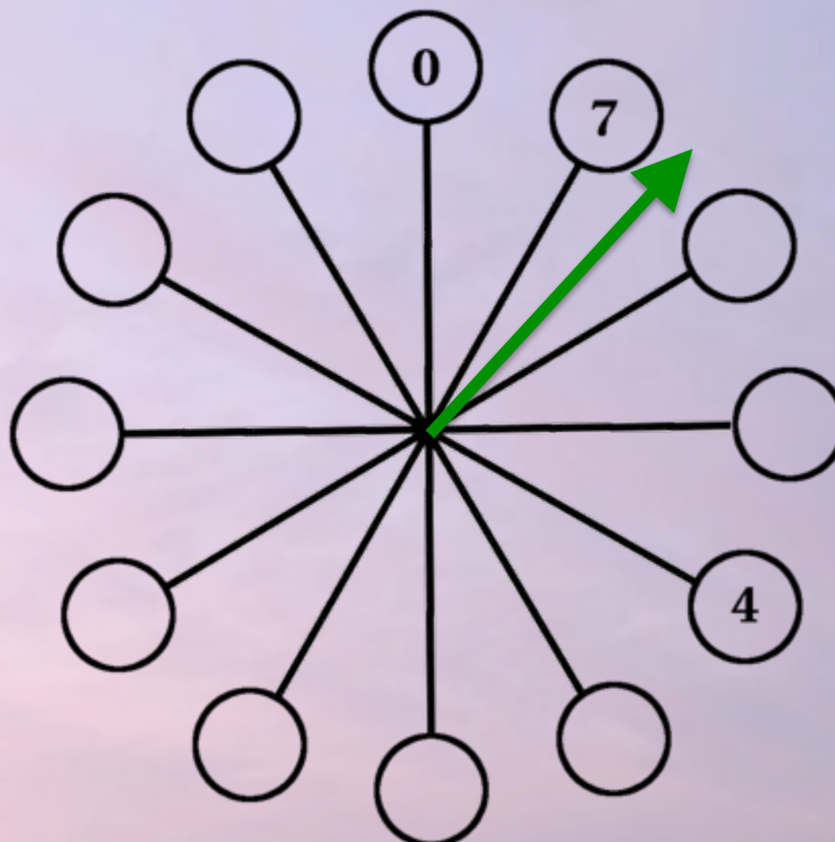
See Quinn "General Equal-Tempered Harmony," *Perspectives of New Music* 44/2–45/1 (2006–2007).

Discrete Fourier Transform: Balances

3-component balance



5-component balance

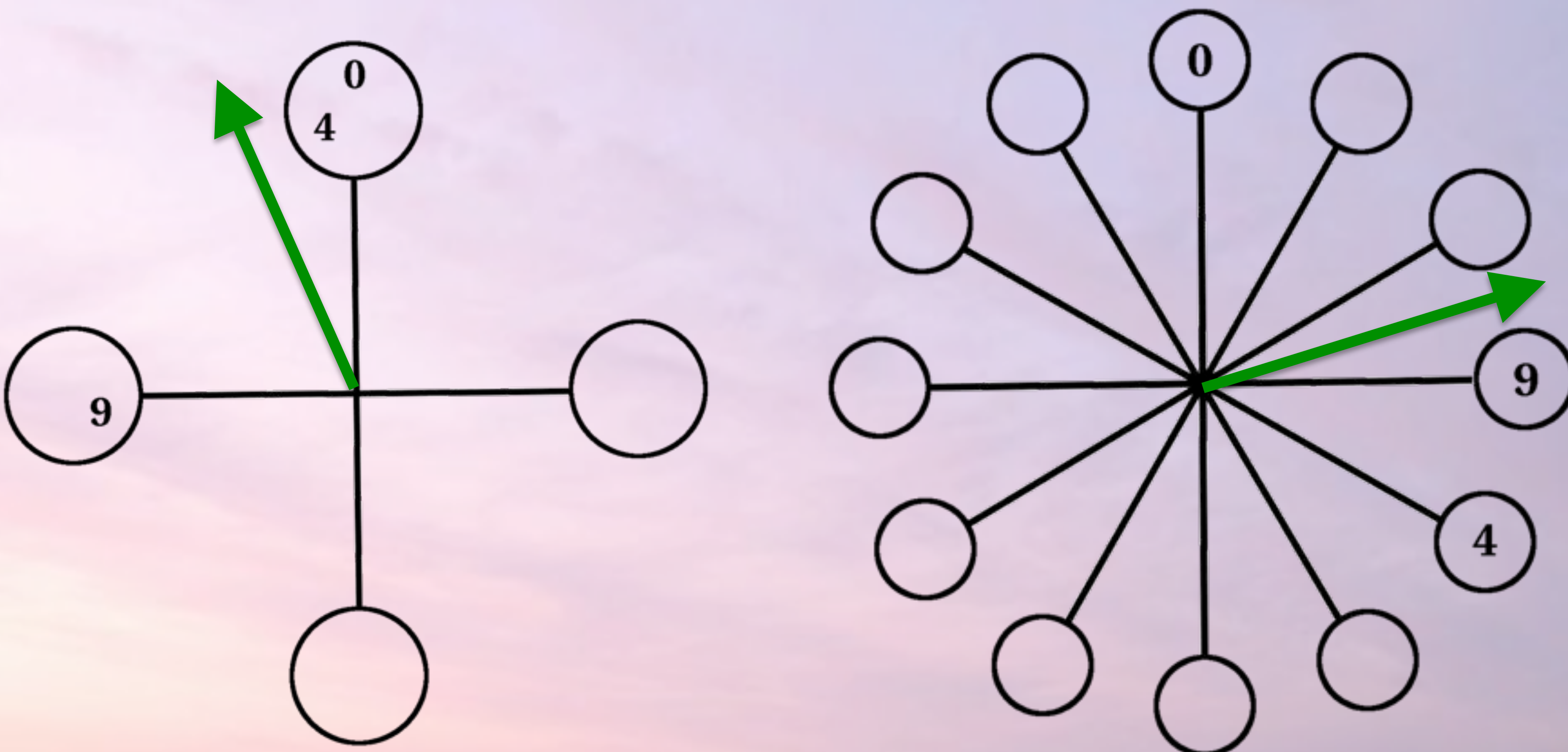


C major triad

Discrete Fourier Transform: Balances

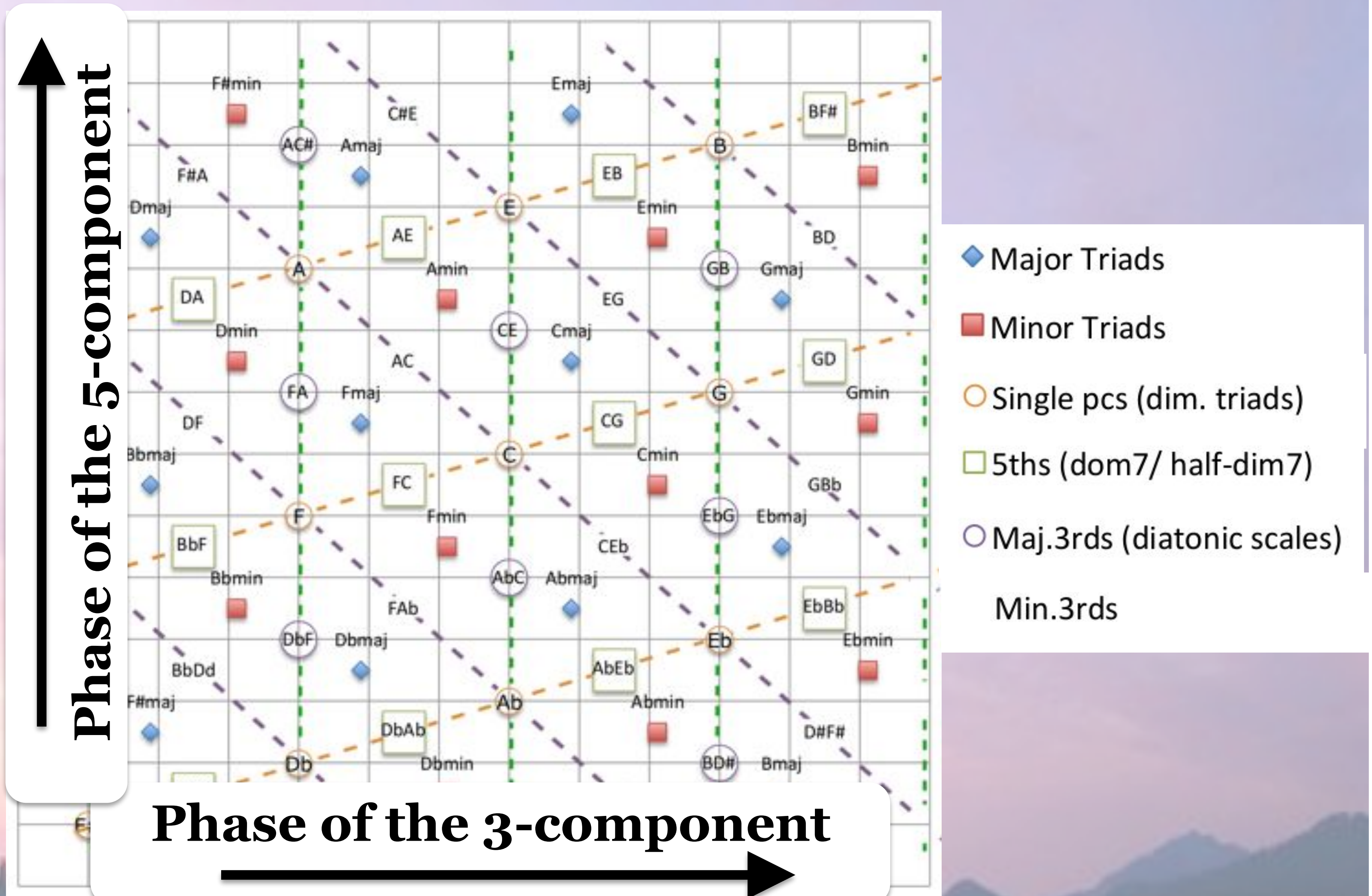
3-component balance

5-component balance

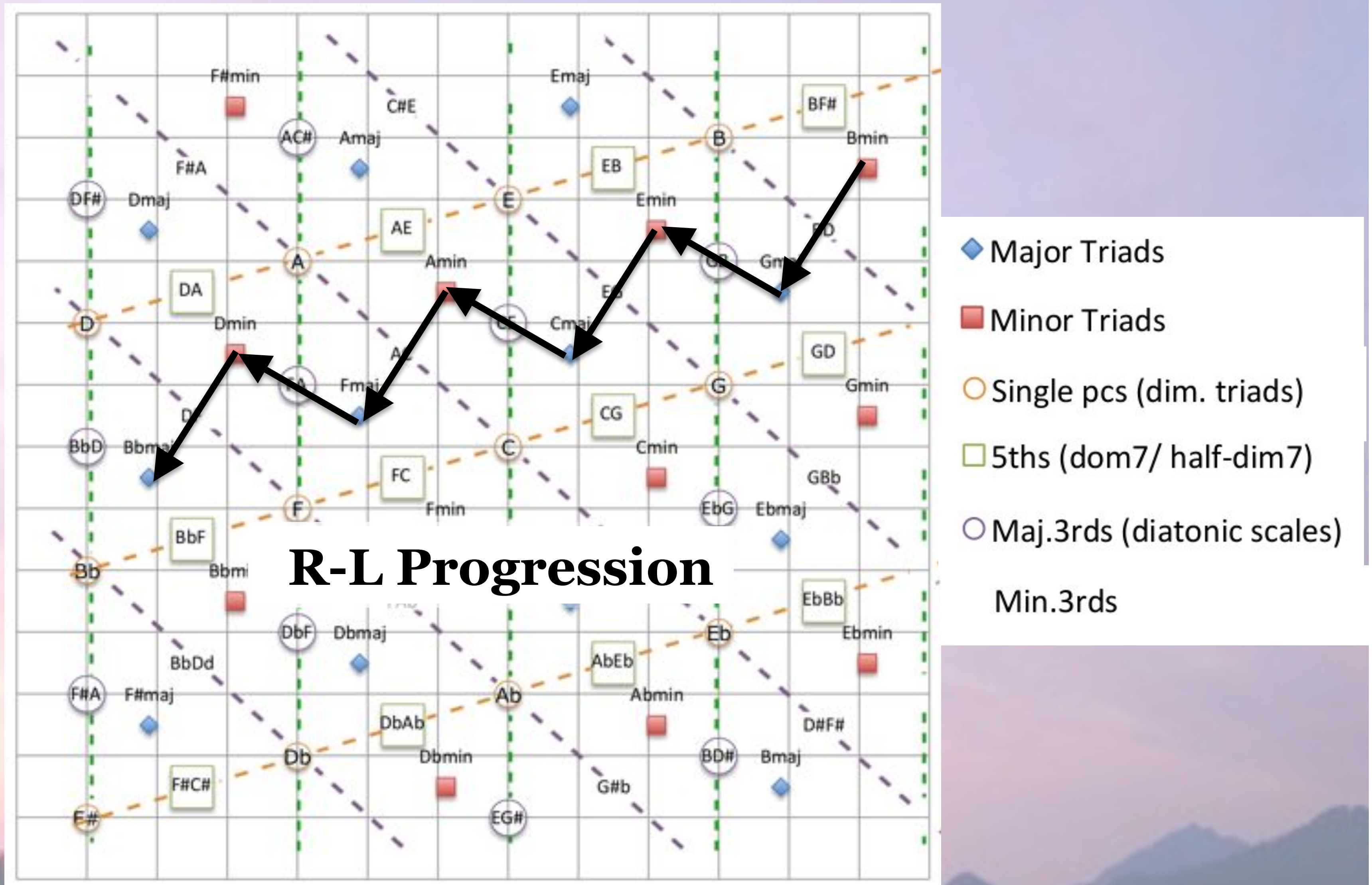


A minor triad

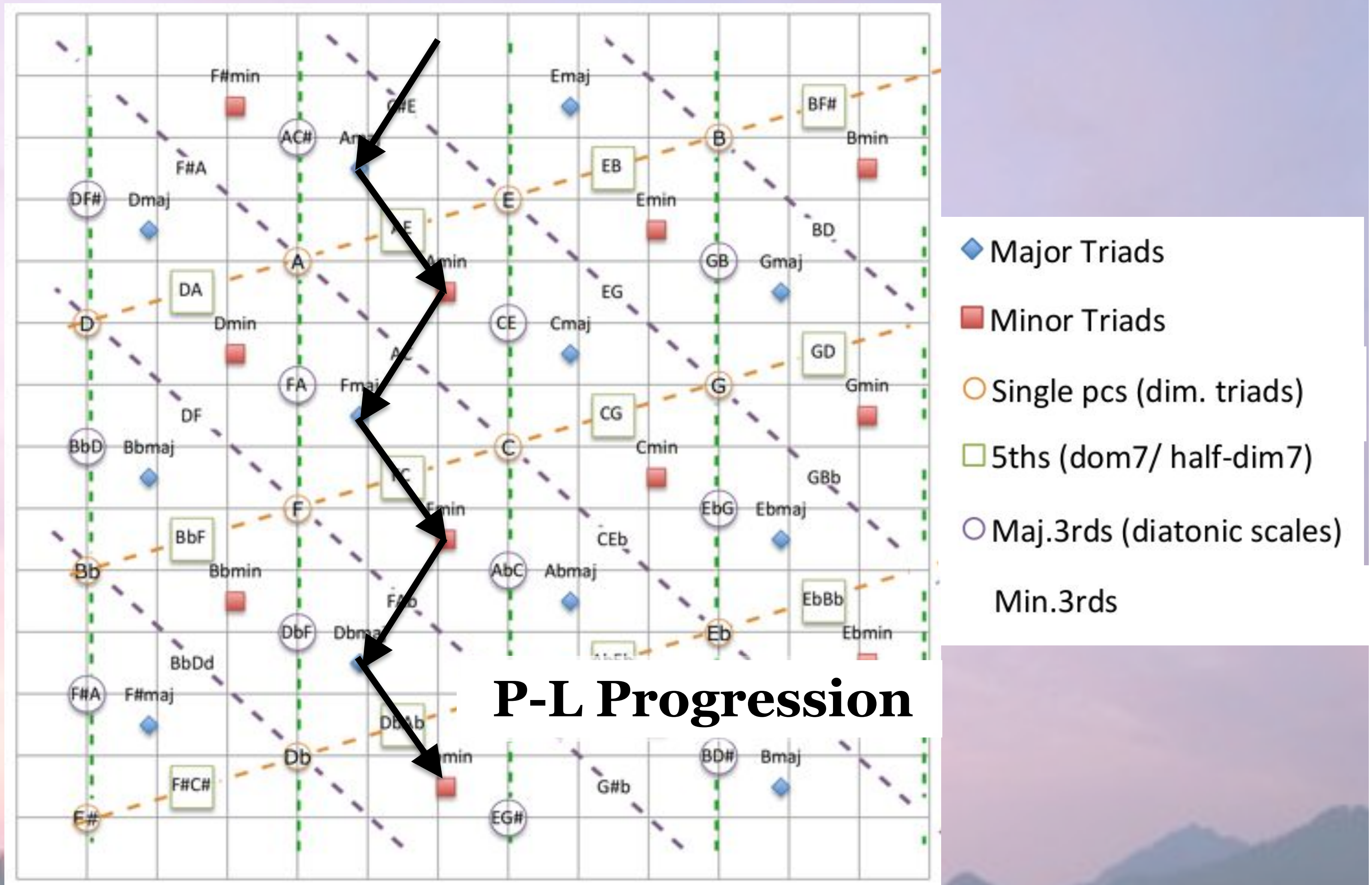
Discrete Fourier Transform: The *Tonnetz*



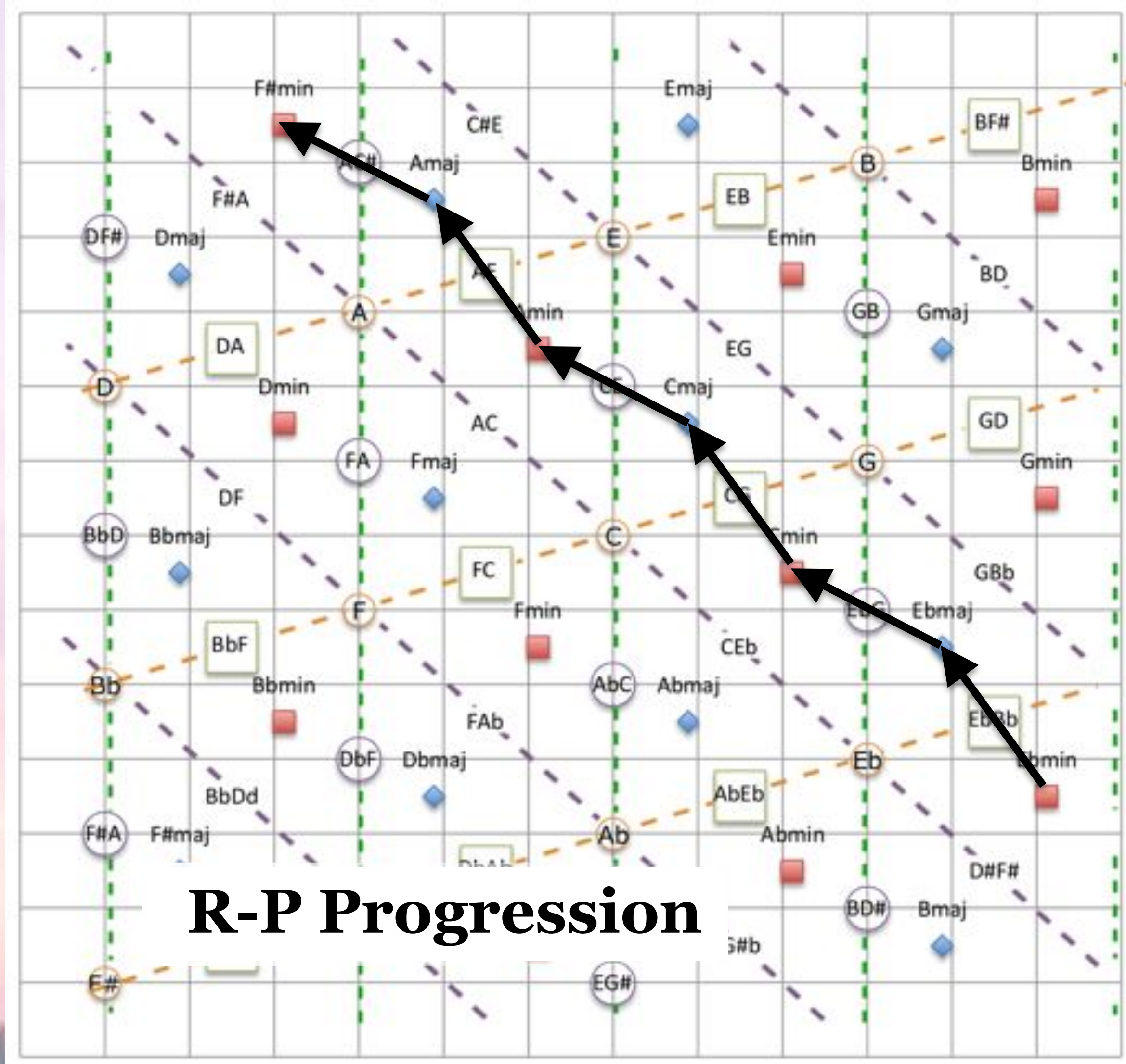
Discrete Fourier Transform: The *Tonnetz*



Discrete Fourier Transform: The *Tonnetz*



Discrete Fourier Transform: The *Tonnetz*



Major Triads
Minor Triads
Single pcs (dim. triads)
5ths (dom7/ half-dim7)
Maj.3rds (diatonic scales)
Min.3rds

R-P Progression

Mediants in Schubert

1. Example: Menuetto from the “Rosamunde” Quartet, D.804

String Quartet no. 13: Menuetto (mm. 1–11)

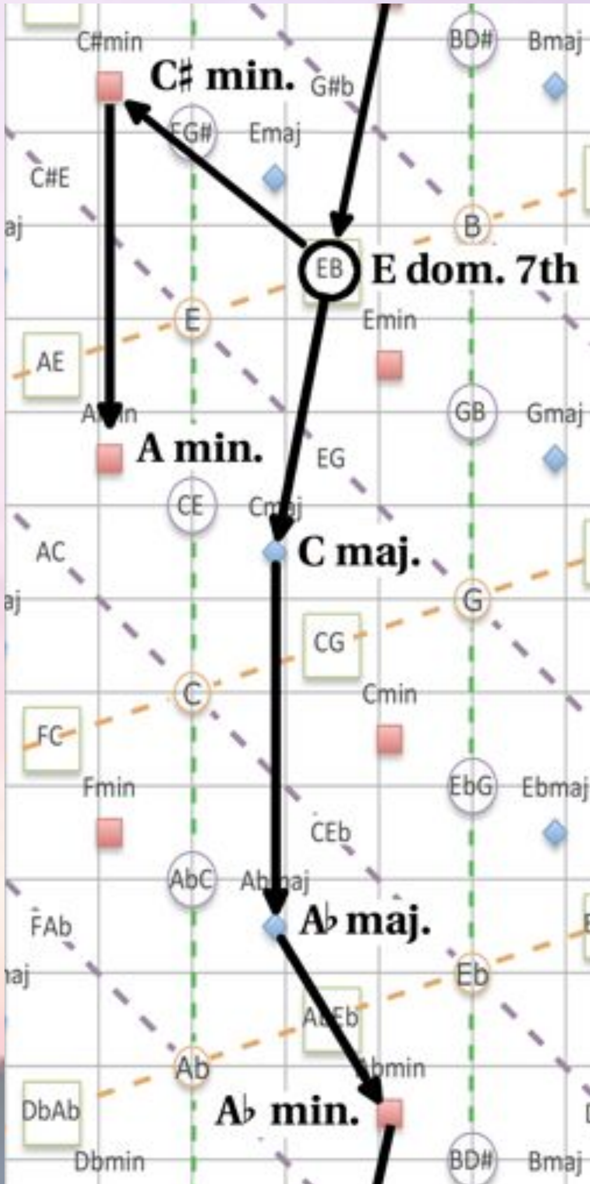
V of A min.

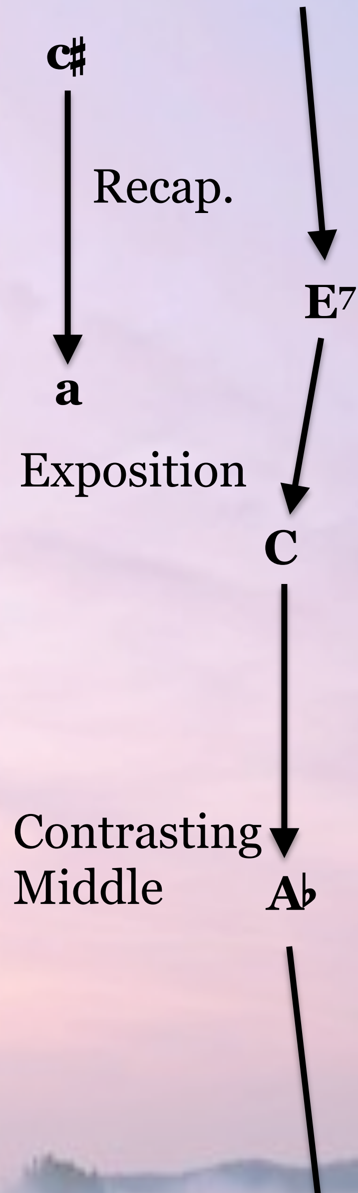
C maj.

See Sobaskie (2003) “Tonal Implication and the Gestural Dialectic in Schubert’s A minor Quartet,” in *Schubert the Progressive*, ed. Newbould (Ashgate).

String Quartet no. 13: Menuetto (mm. 28–46)

String Quartet no. 13: Menuetto (Tonal plan)

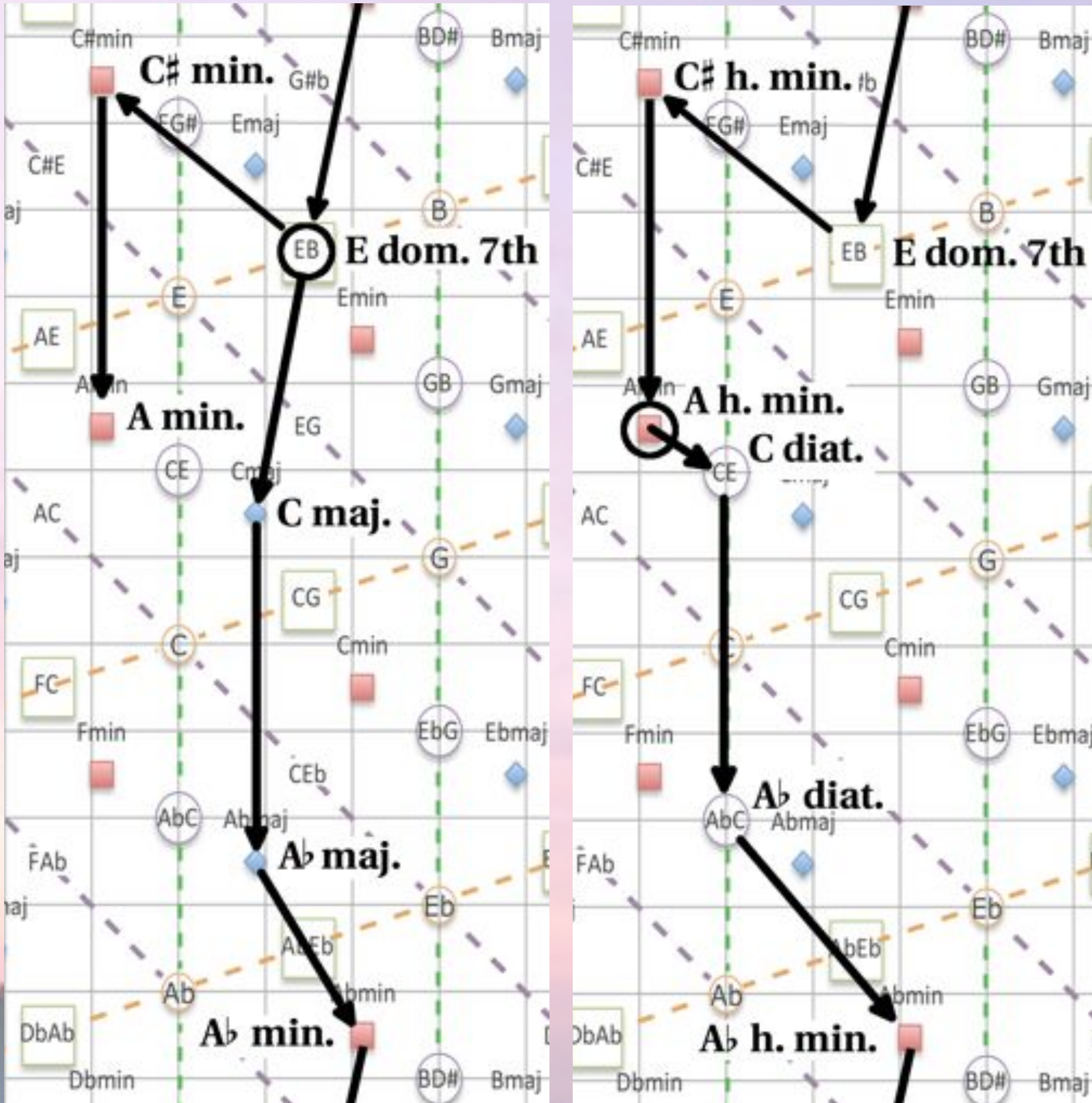




The tonal plan of the Menuetto as a pathway between *chords*.

The dominant 7th on E has a distinct location from the E major triad.

String Quartet no. 13: Menuetto (Tonal plan)

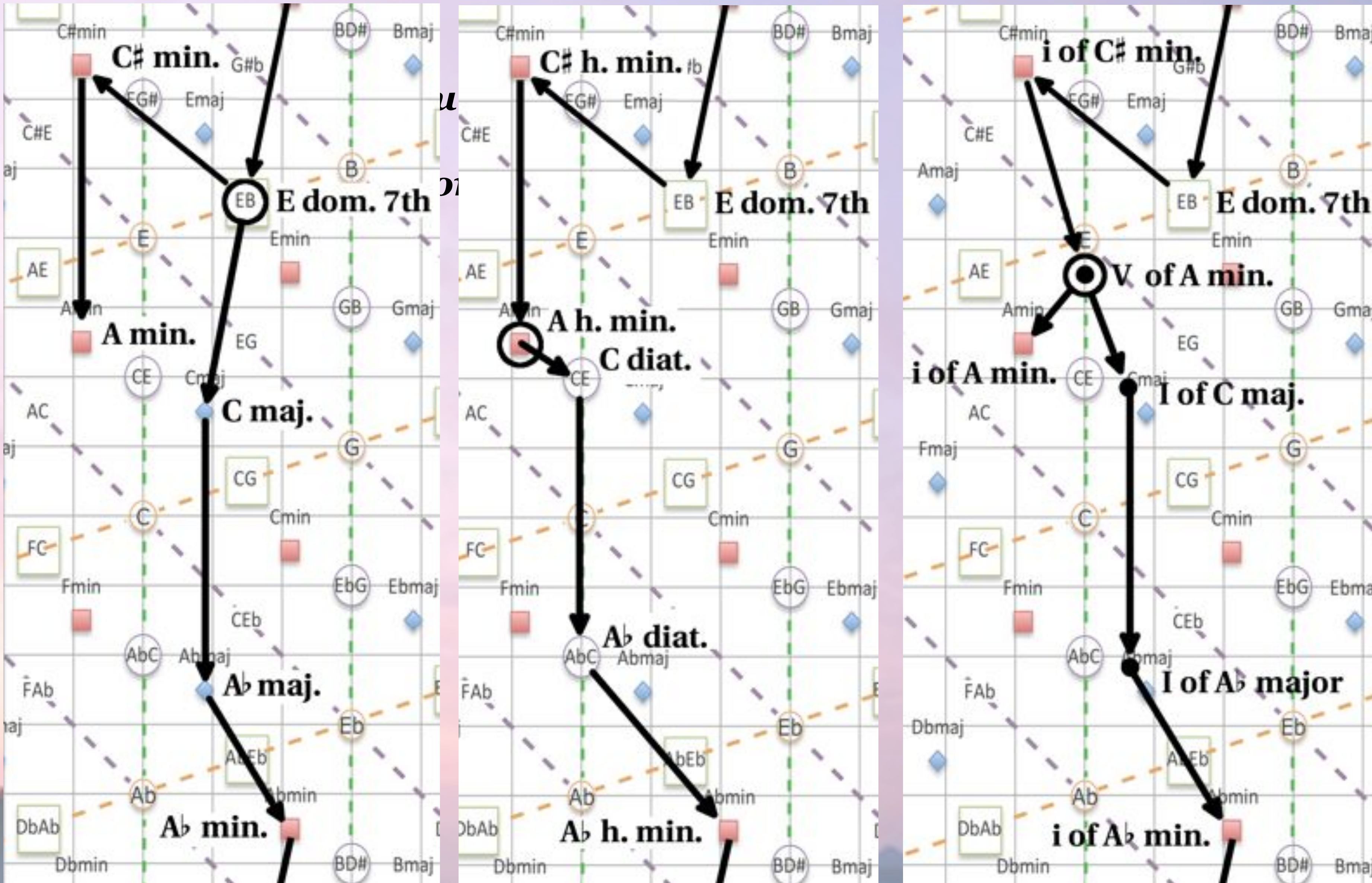


The tonal plan can also be charted as a pathway between *scales*.

A min. → **C maj.** Exposition

The exposition involves a much shorter path on scales.

String Quartet no. 13: Menuetto (Tonal plan)



String Quartet no. 13: Menuetto (Tonal plan)

This analysis shows each section of the piece moving in the *same direction*, a generalization of the *subdominant recapi-tulation principle.*

See Webster (1978–9). “Schubert’s Sonata Form and Brahms’s First Maturity,” *19th Century Music* 2(1)–3(1).

c♯: i

Recap.

a: V7

Exposition

C: I

Contrasting Middle

a♭: i

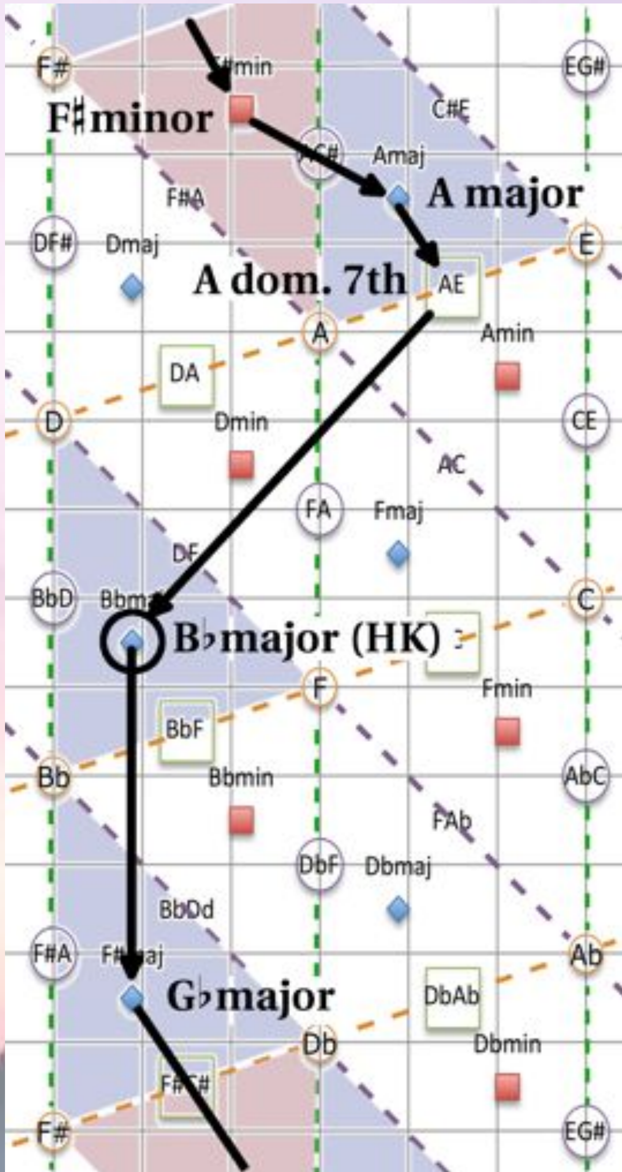
III. *Tonnetz* regions vs. Weber regions

Ex.: Schubert's Late B♭ major Piano Sonata, D.960

Schubert, D.960 Sonata (Recapitulation)

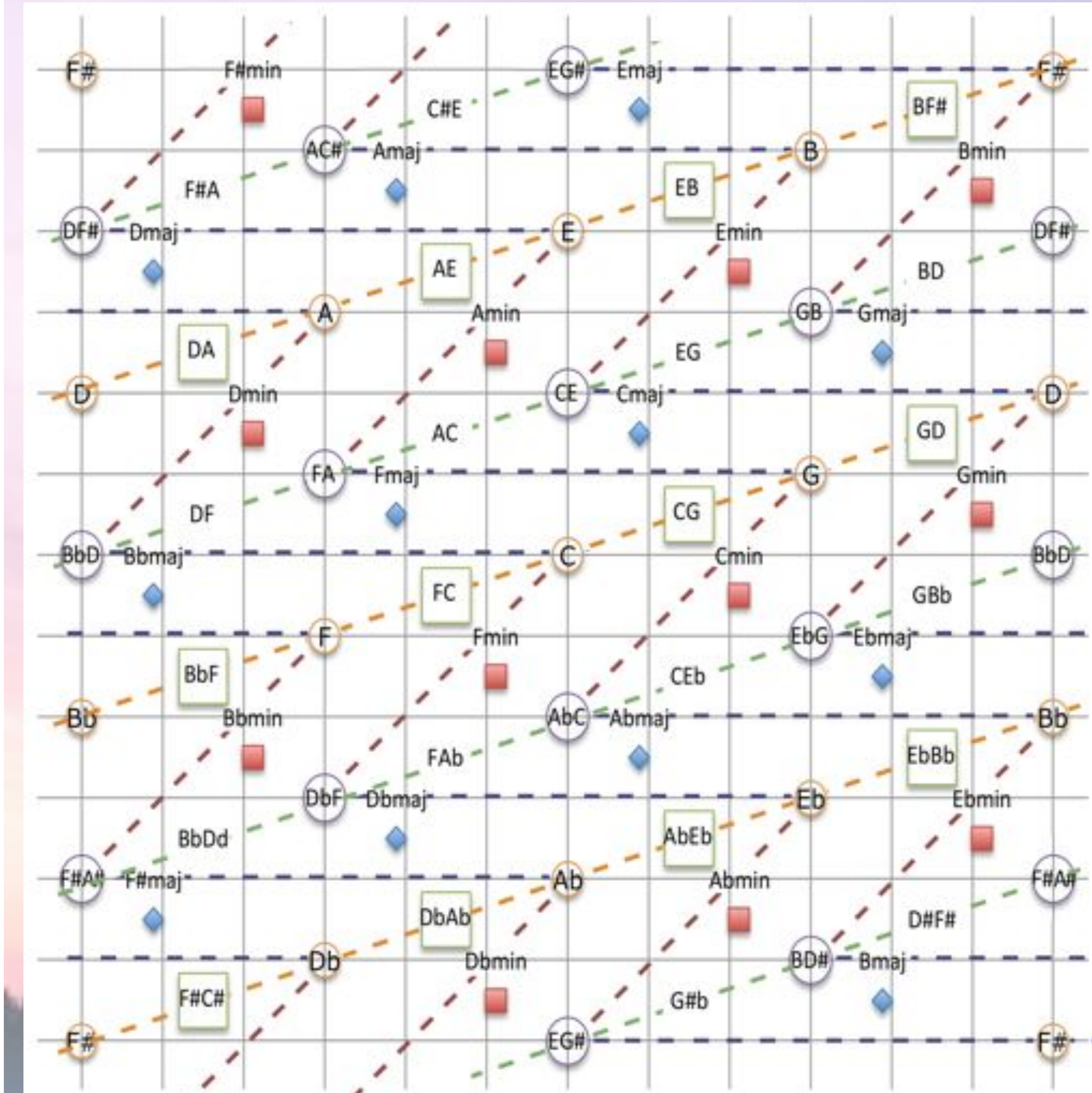
Schubert, D.960 Sonata (Recapitulation)

III. *Tonnetz* regions vs. Weber regions



In *Tonnetz* regions, the return to B♭ major passes through the same number of regions as the hexatonic pole B♭ maj. – F♯ min.

Weber Regions

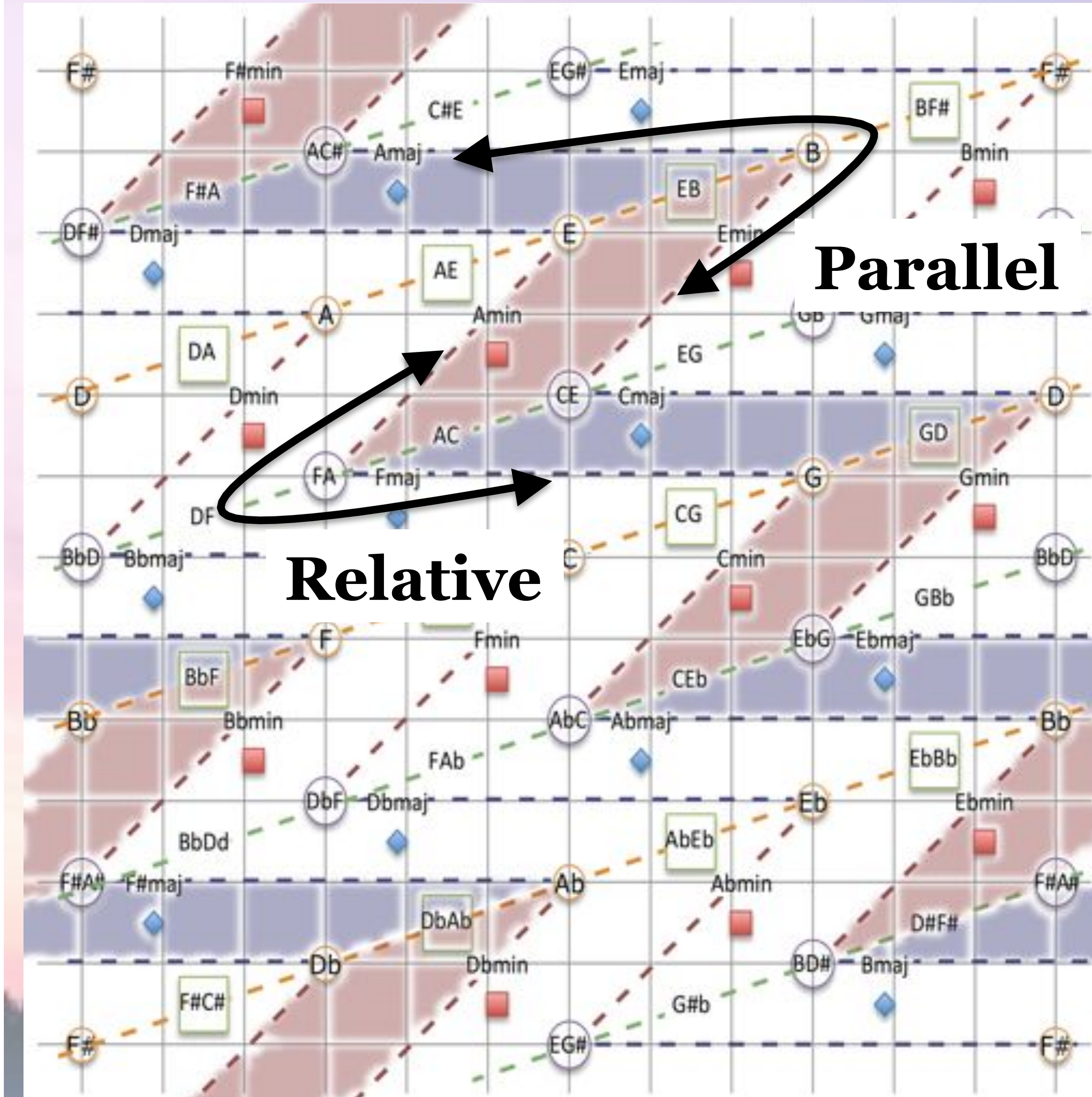


Regions in the space can be drawn differently.

These regions combine each triad with its *dominant* and its *characteristic scales*.

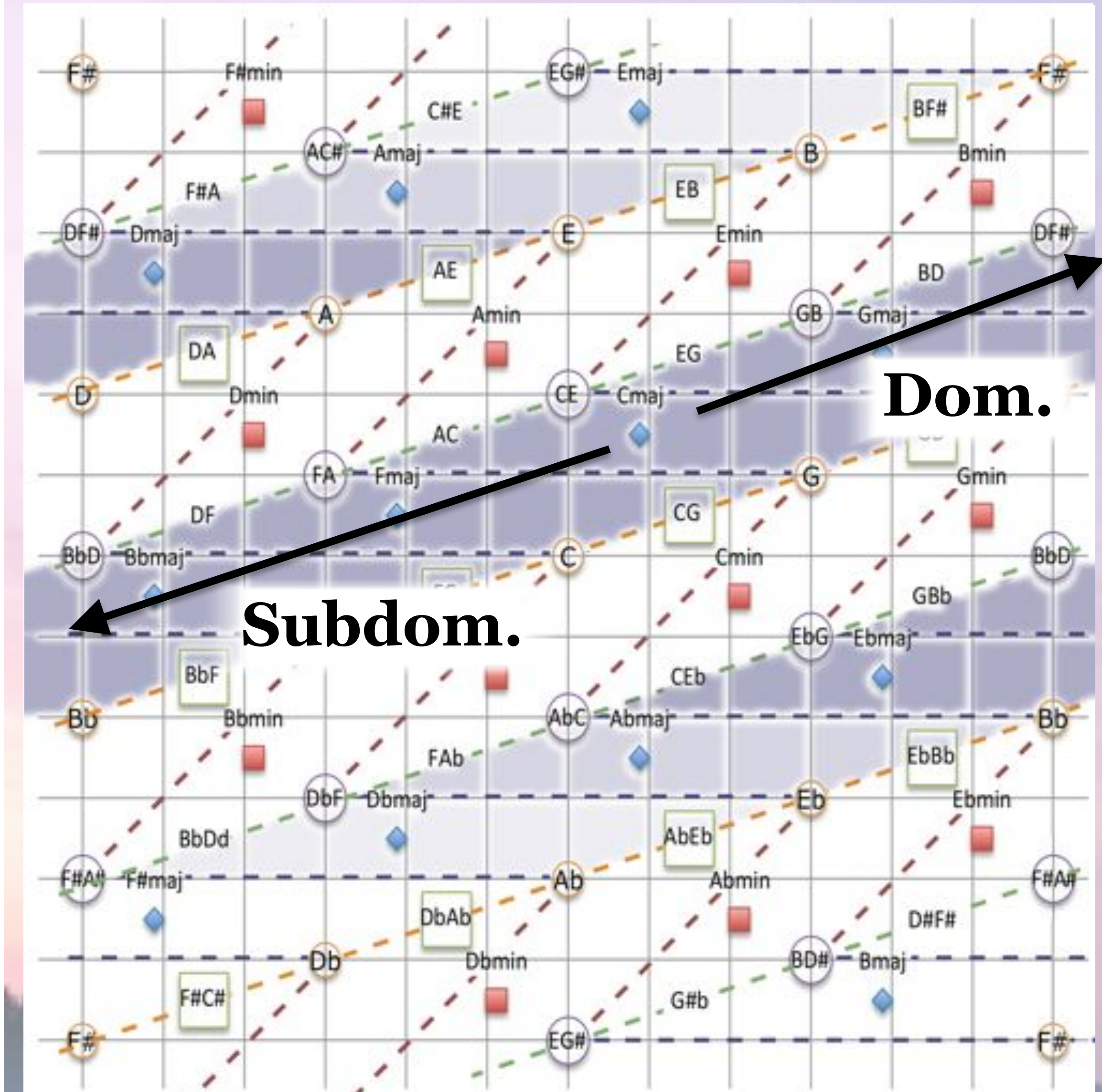
They reproduce the network characteristic of *Weber's chart of keys*.

Weber Regions



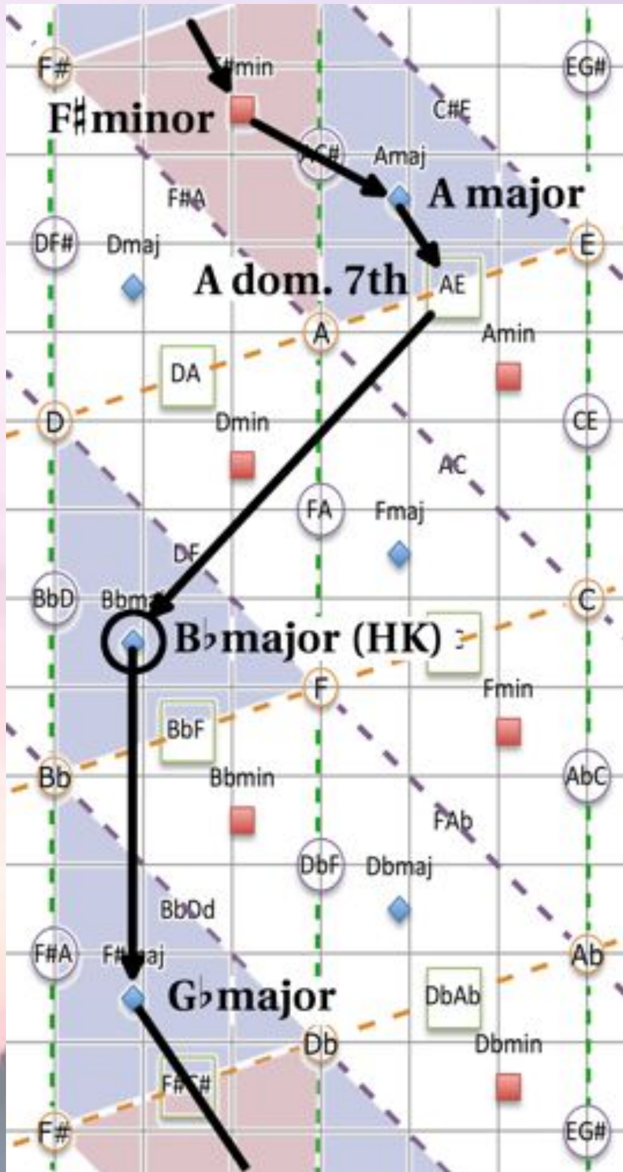
Parallel and **Relative** keys relationships zig-zag across the space.

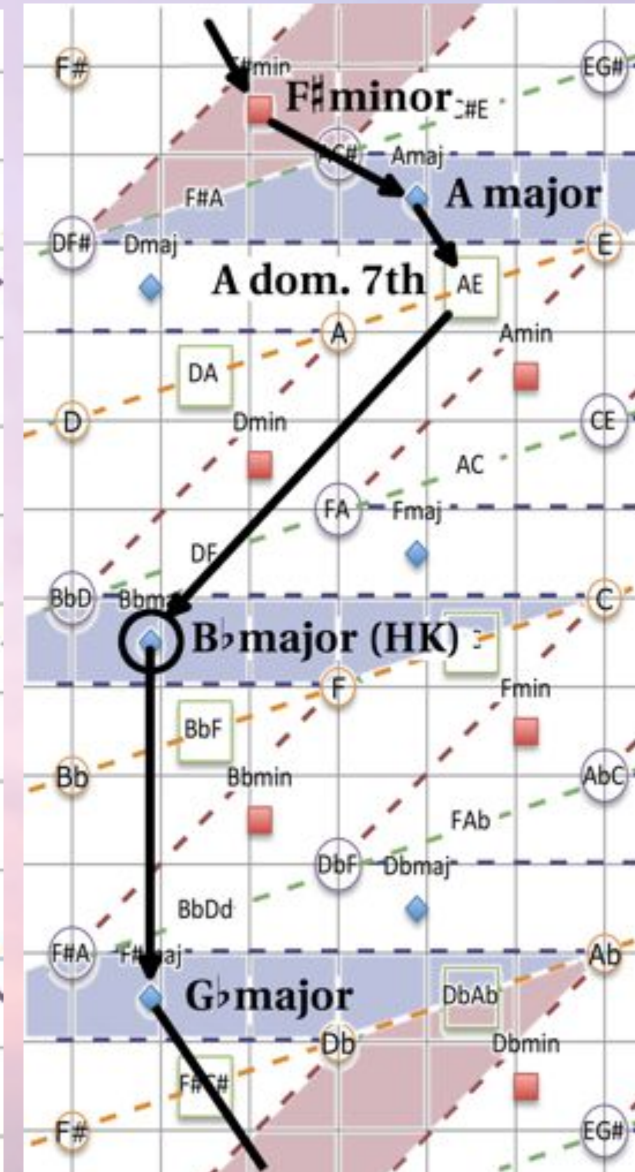
Weber Regions



The **circle of fifths** makes two (major / minor) bands that wrap around the space.

III. *Tonnetz* regions vs. Weber regions

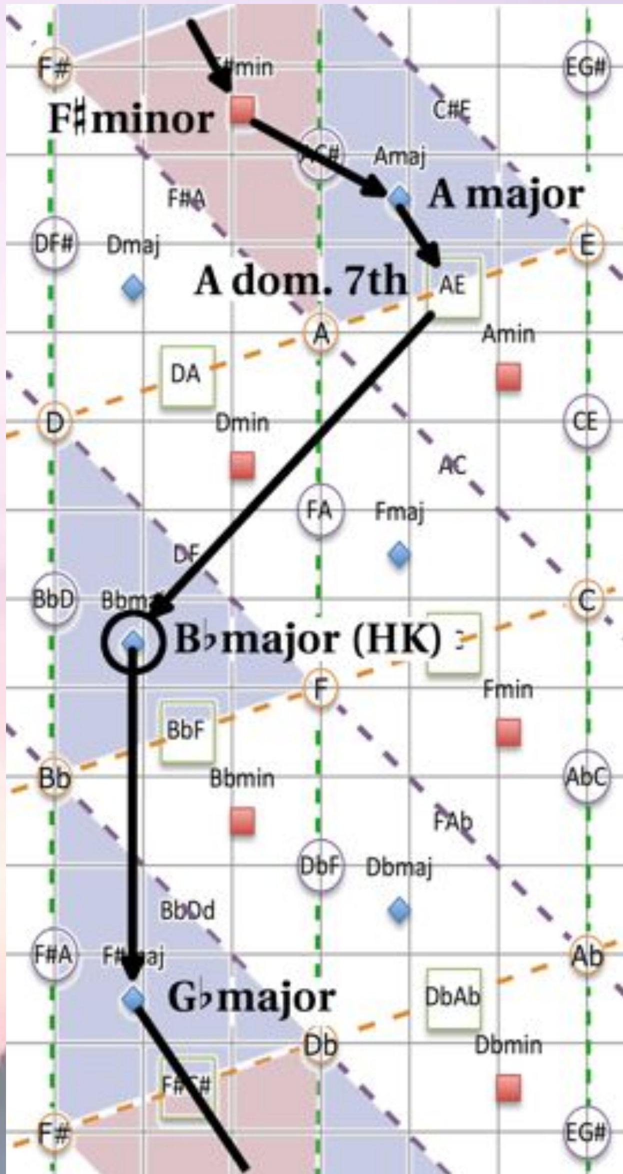


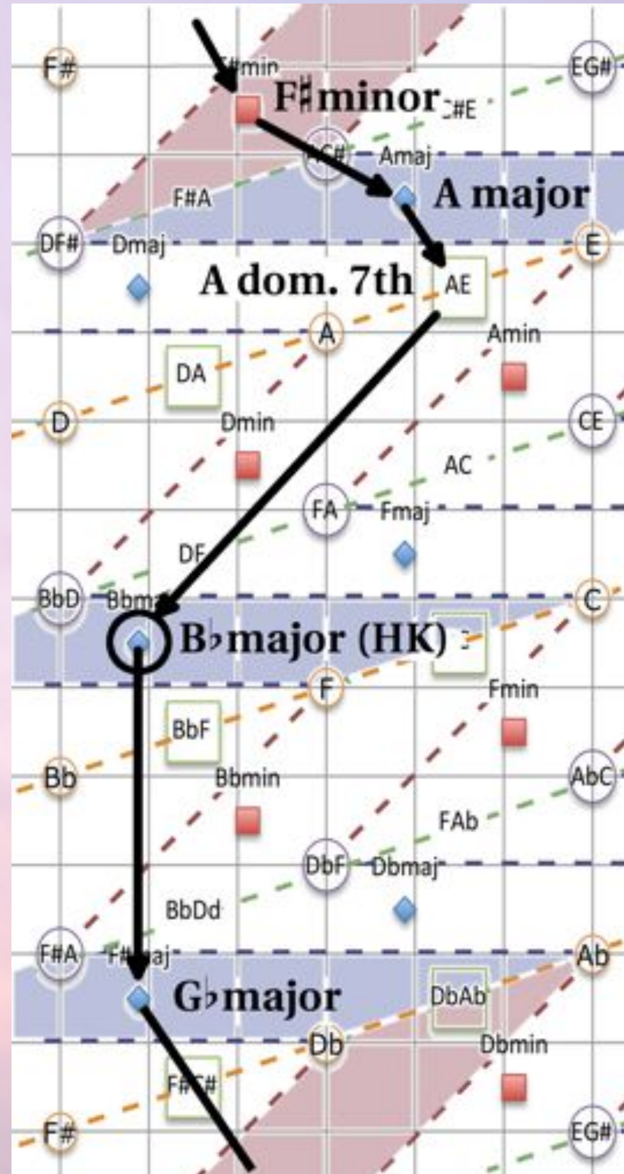


In Weber regions the **return to B♭ major** is relatively simple, crossing mostly through the D min. region (V^7–VI in d).

The **hexatonic pole** (B♭ maj. – F♯. min.) however is complex, crossing **8 regions.**

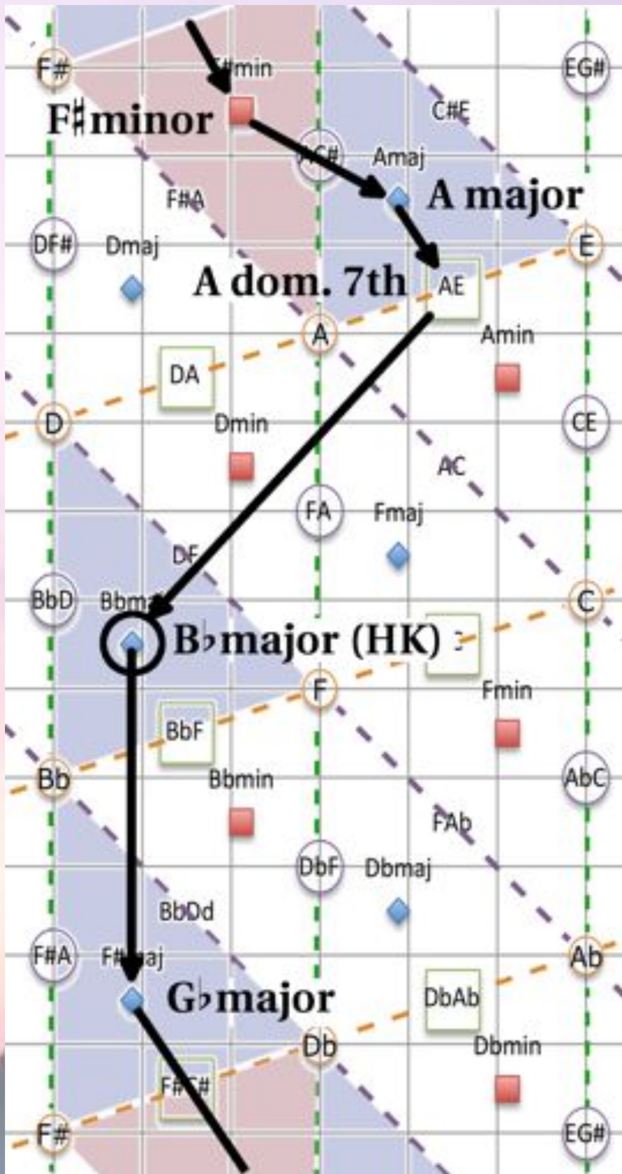
III. *Tonnetz* regions vs. Weber regions

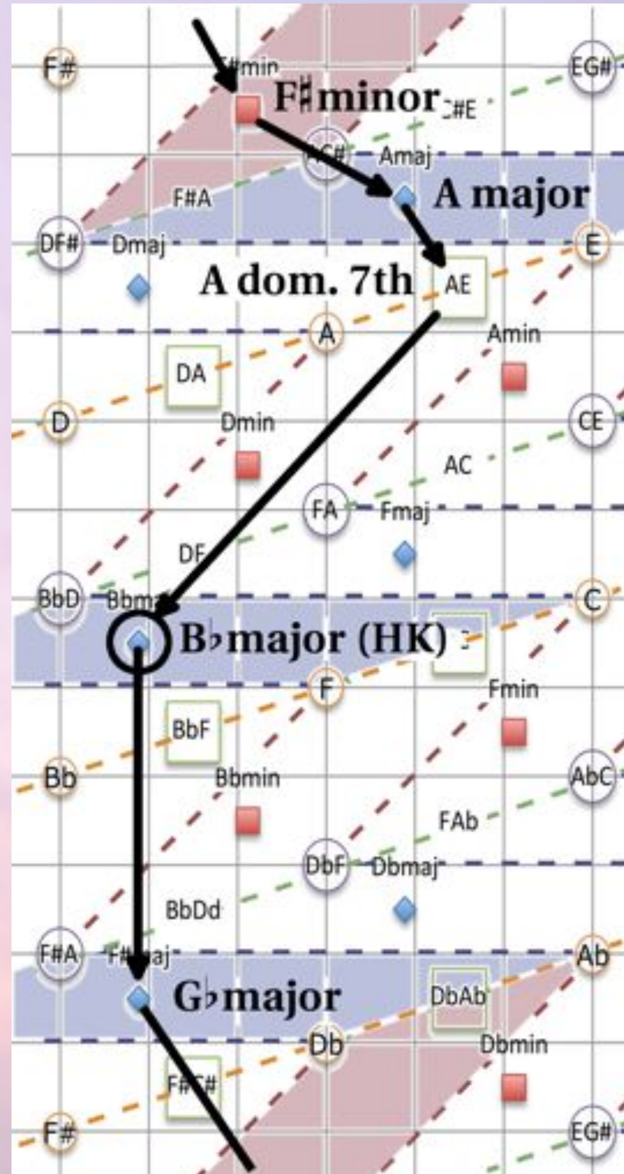




The **"lens of regions"** illuminates the **dispute between Richard Cohn, Peter Pesic, and Charles Fisk** over the status of the status of the hexatonic pole.

III. *Tonnetz* regions vs. Weber regions





See

Cohn (1999). “As Wonderful as Star Clusters . . . ” *19th Century Music* 22/3.

Pesic (1999). “Schubert’s Dream” *19th Century Music* 23/2.

Clark (2011). *Analyzing Schubert* (Cambridge Univ. Press).

Fisk (2001). *Returning Cycles* (UC Press).

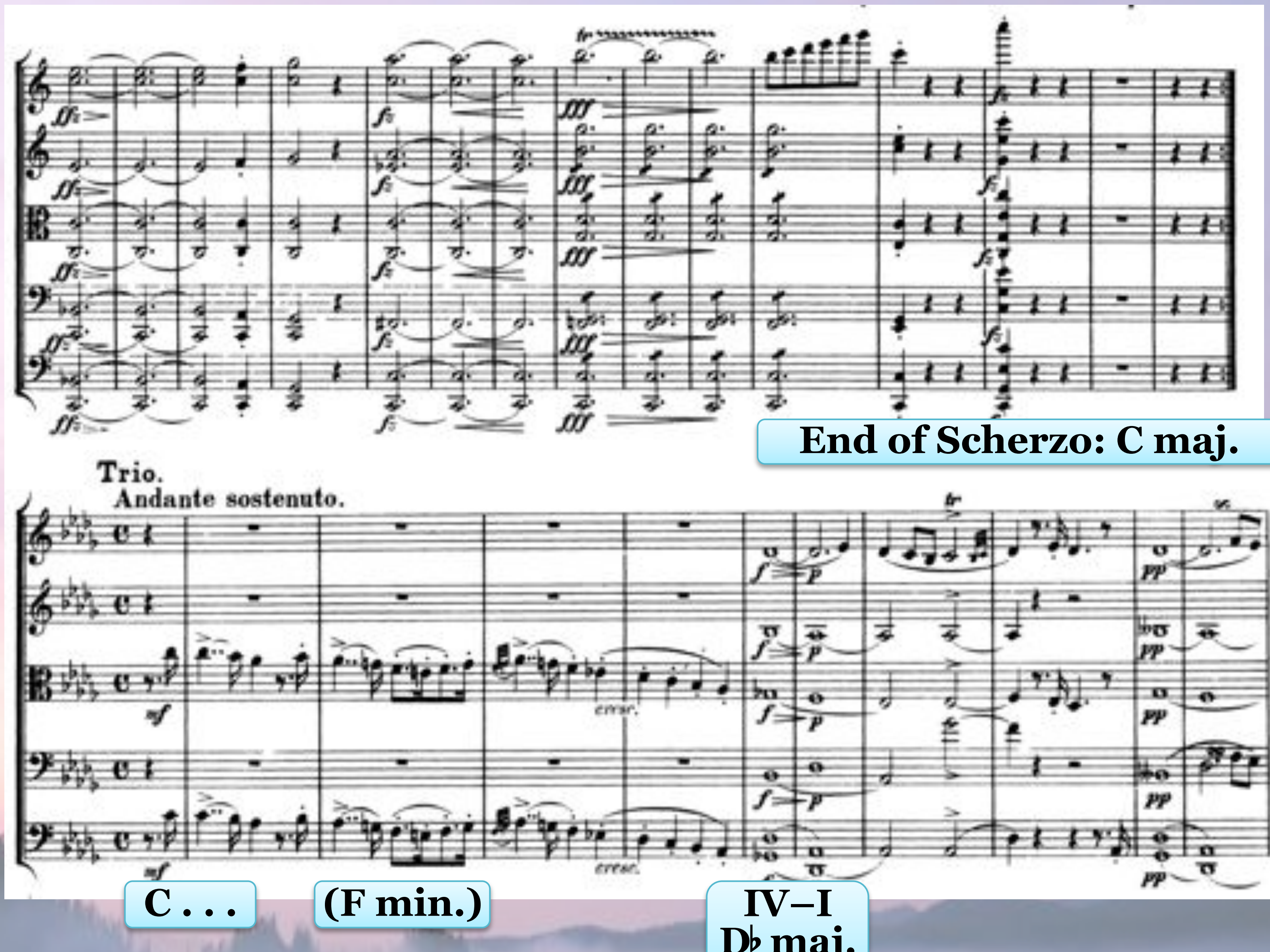
IV. Compositional Techniques

1. Modulation by scalar common tone

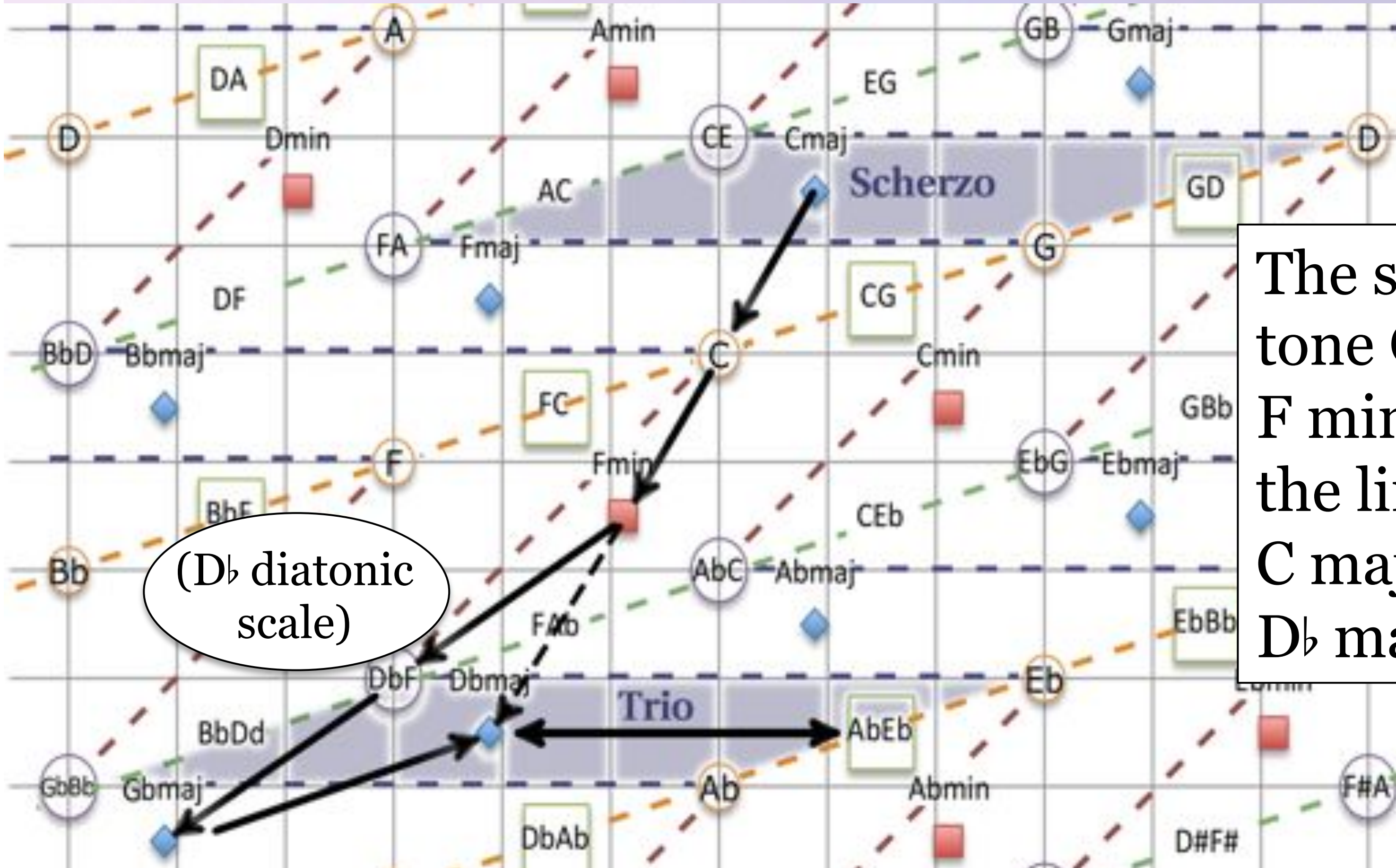
Ex.: C major Quintet, Scherzo–Trio

Ex.: C major Quintet, Adagio (MT–IT)

C major Quintet, Scherzo–Trio



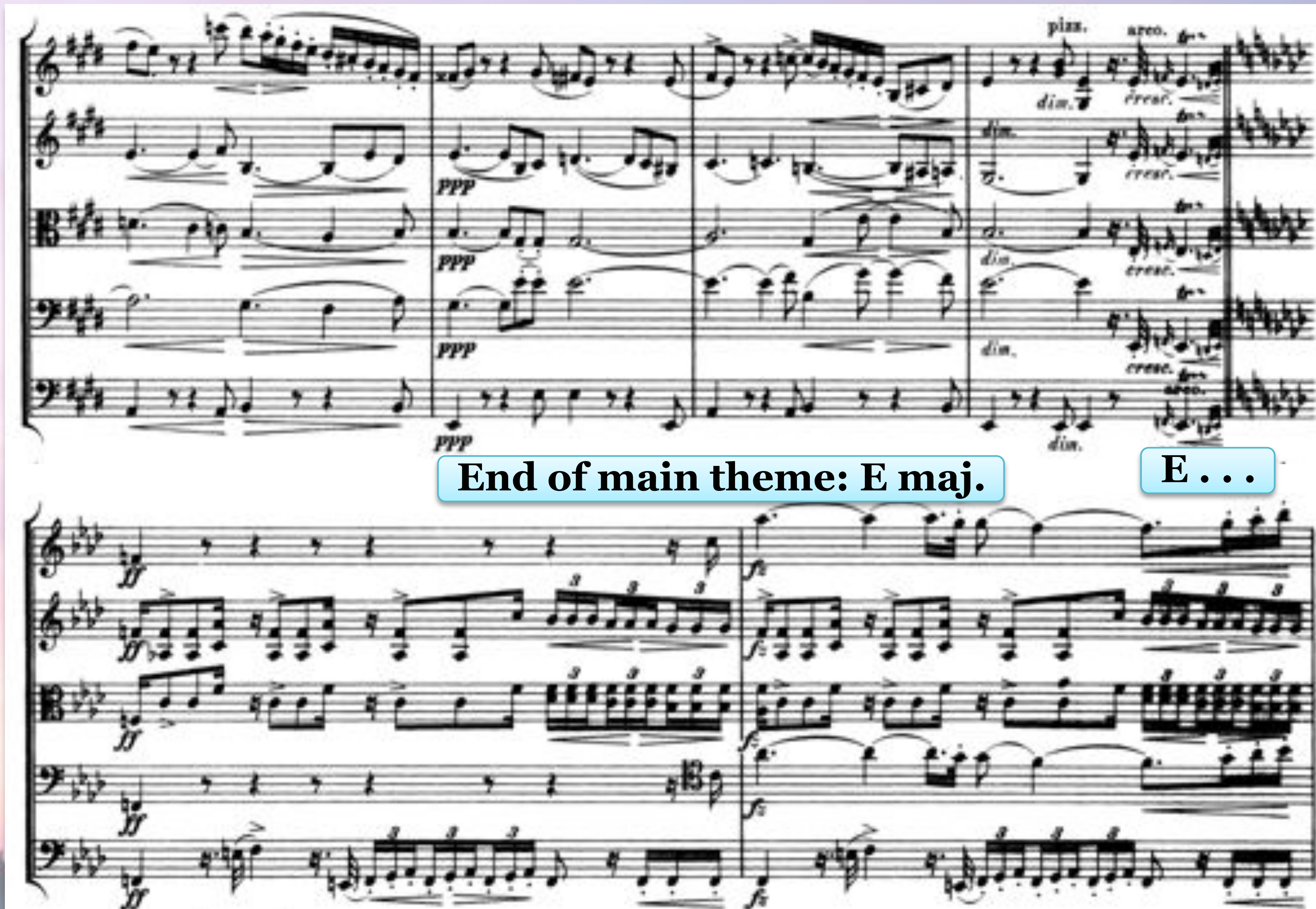
C major Quintet, Scherzo–Trio



The scalar common tone C and the F minor triad are on the line between the C major and D♭ major triads.

The route from F minor to G♭ major passes through the D♭ diatonic scale.

C major Quintet, Adagio

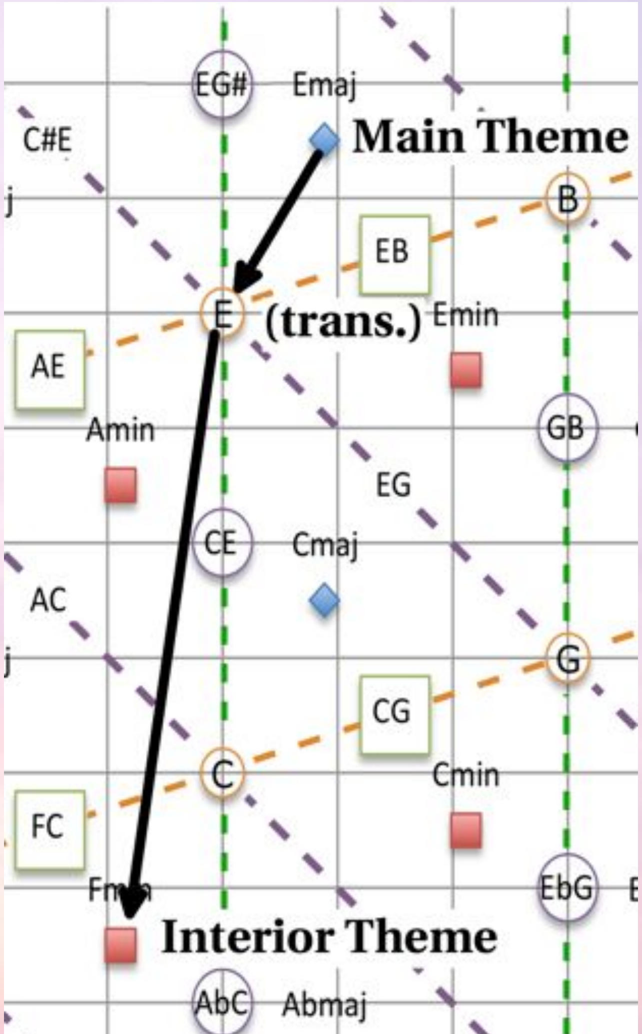


End of main theme: E maj.

E . . .

Interior theme: F min.

C major Quintet, Adagio



The scalar common tone **changes the enharmonic direction** of this modulation. (E major is otherwise closer to E♯ minor).

Even though E♯ minor and F minor are the **same point** in the space, the "slide" and the minor Neapolitan are **distinct paths**.

IV. Compositional Techniques

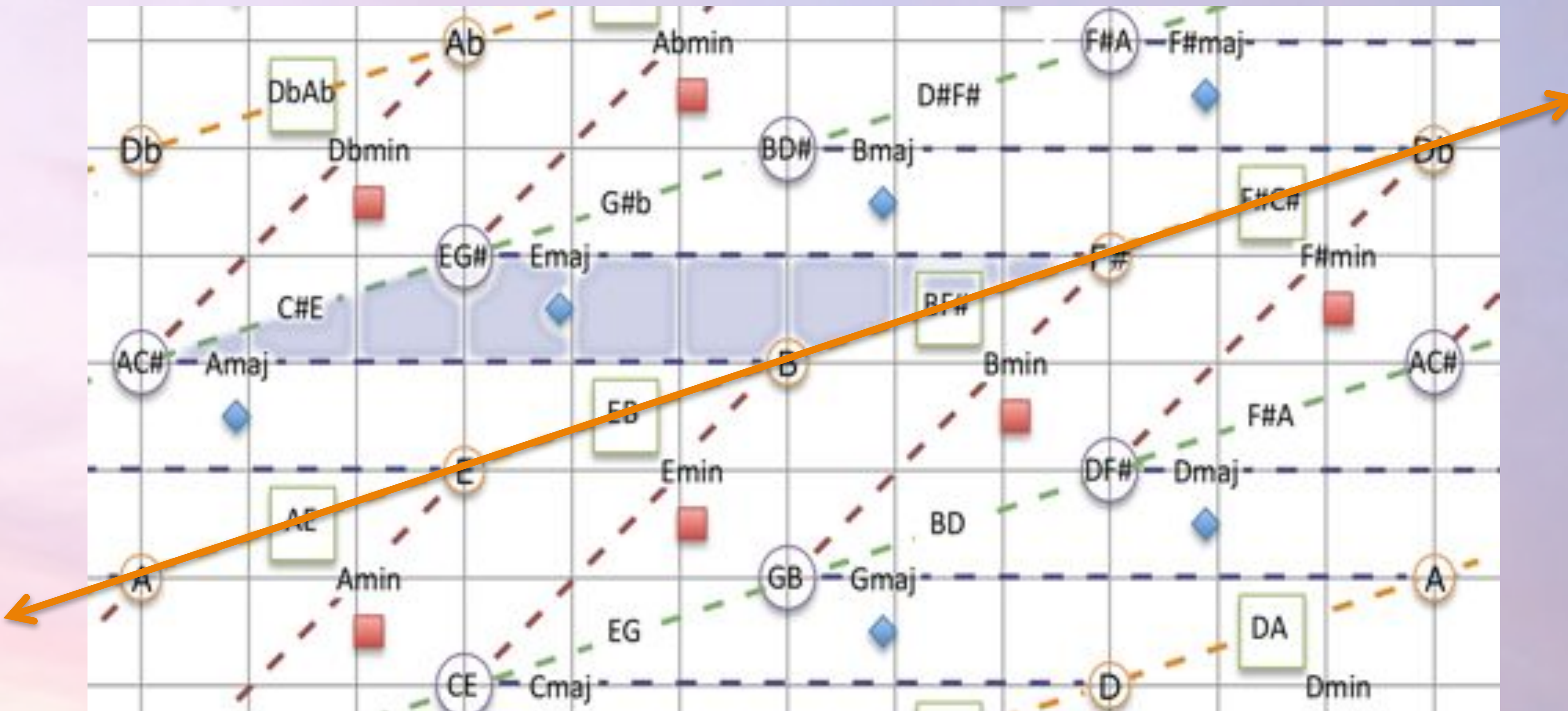
2. Transgression of modal boundaries

Ex.: C major Quintet, Adagio (MT)

Ex.: Late C minor Piano Sonata, Andante

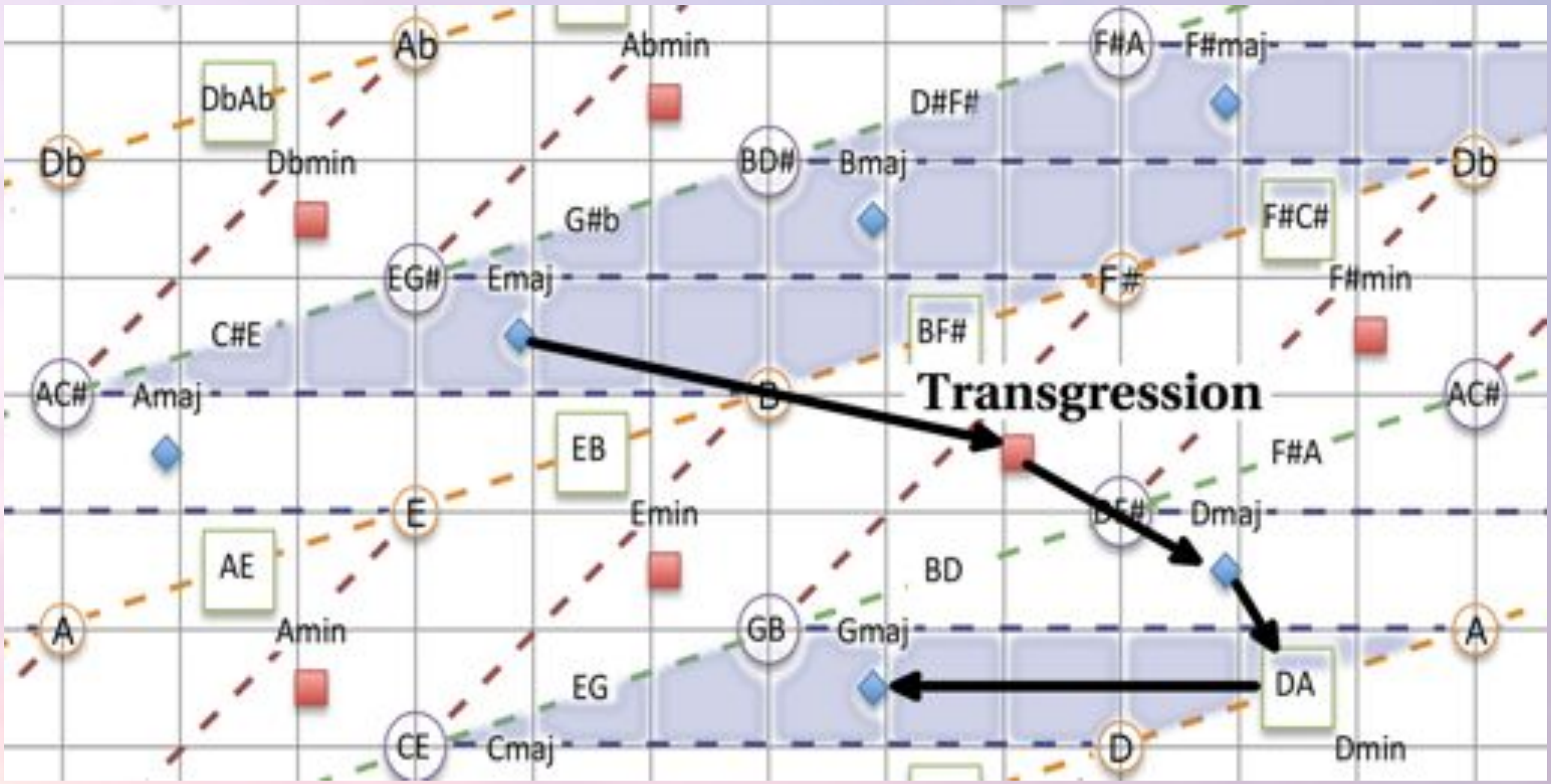
C major Quintet, Adagio

C major Quintet, Adagio



The parallel mode boundary (shared by Weber and *Tonnetz* regions) follows the circle of fifths on singletons (the orange lines).

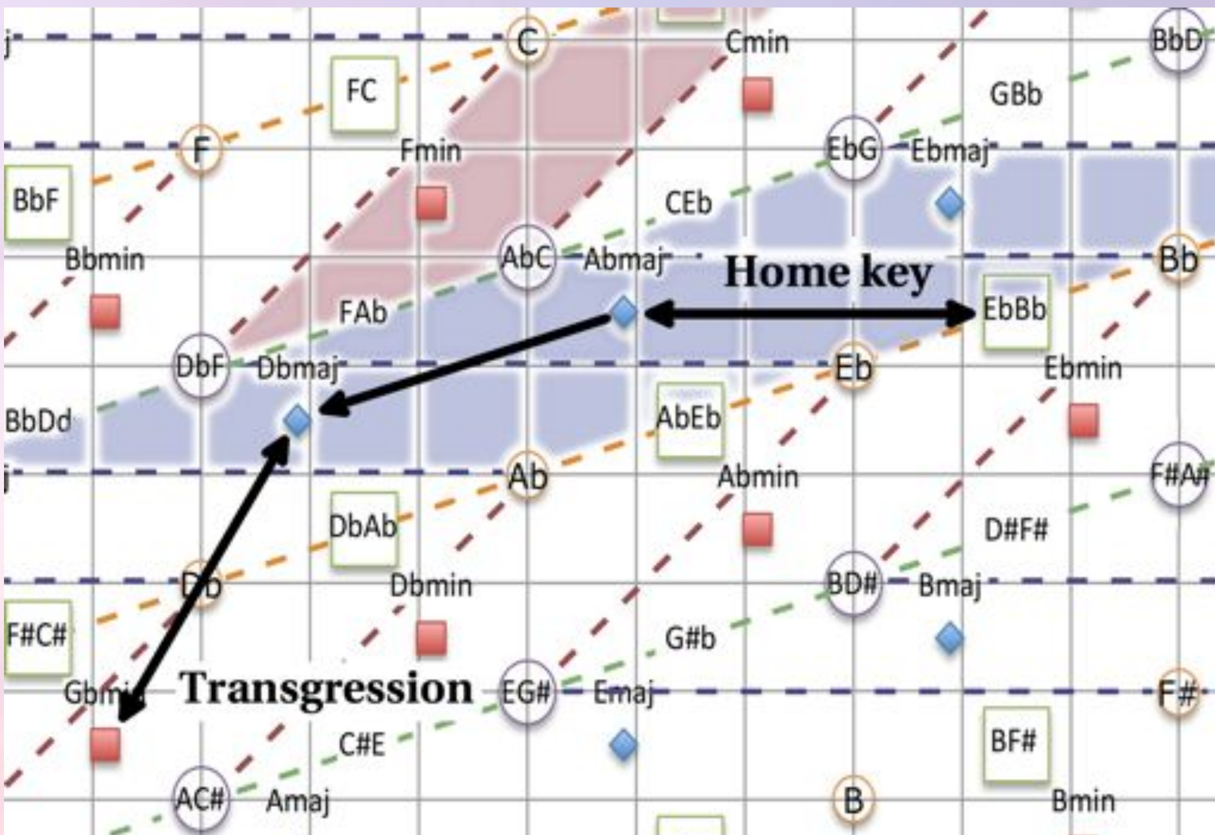
C major Quintet, Adagio



The transgression sets up the flatward "catastrophe" modulation to F minor that happens soon after.

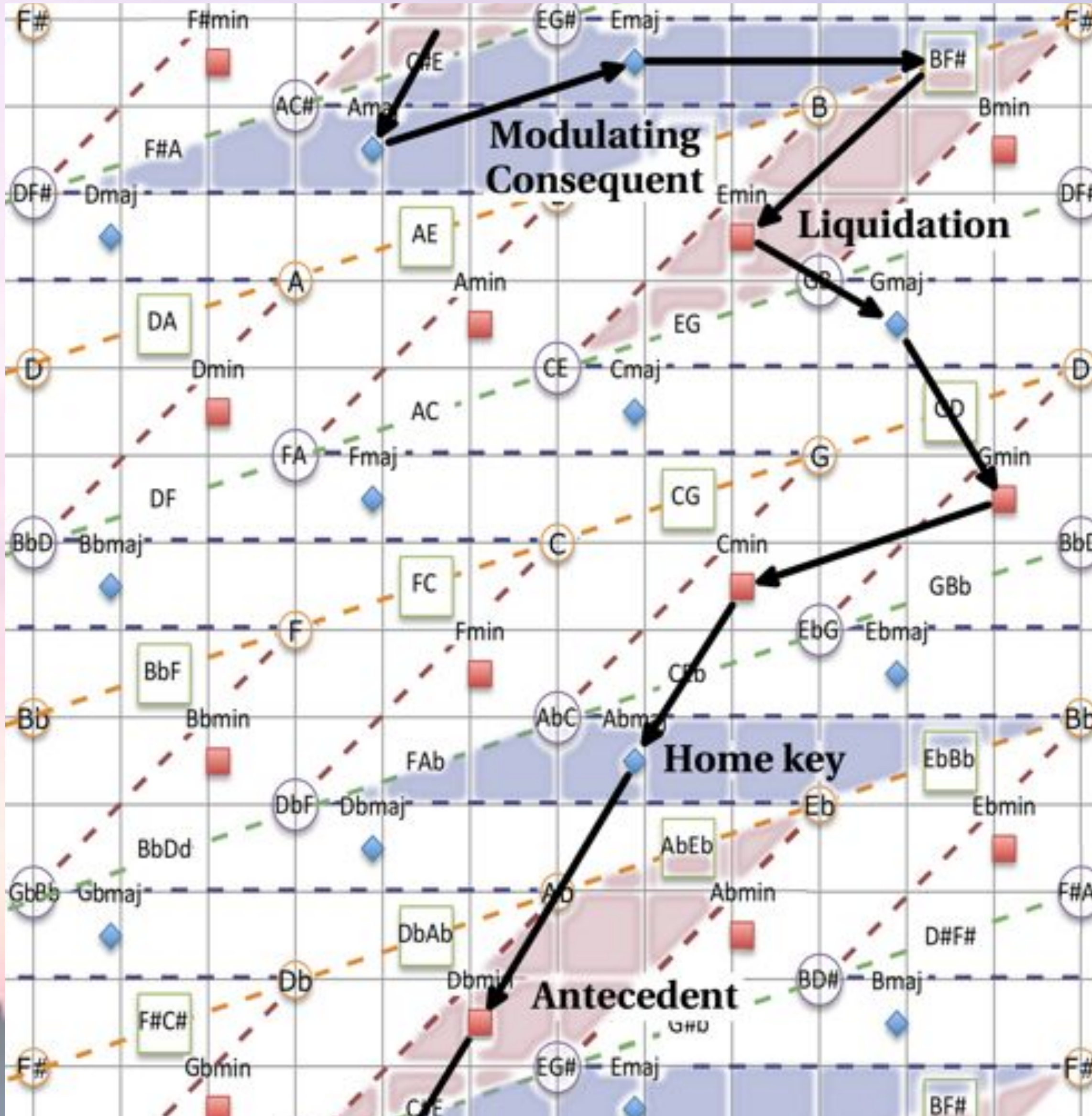
Late C minor Piano Sonata, Adagio

Late C minor Piano Sonata, Adagio



The G♭ minor chord breaks the parallel modal boundary of the main theme key of A♭ major.

Late C minor Piano Sonata, Adagio



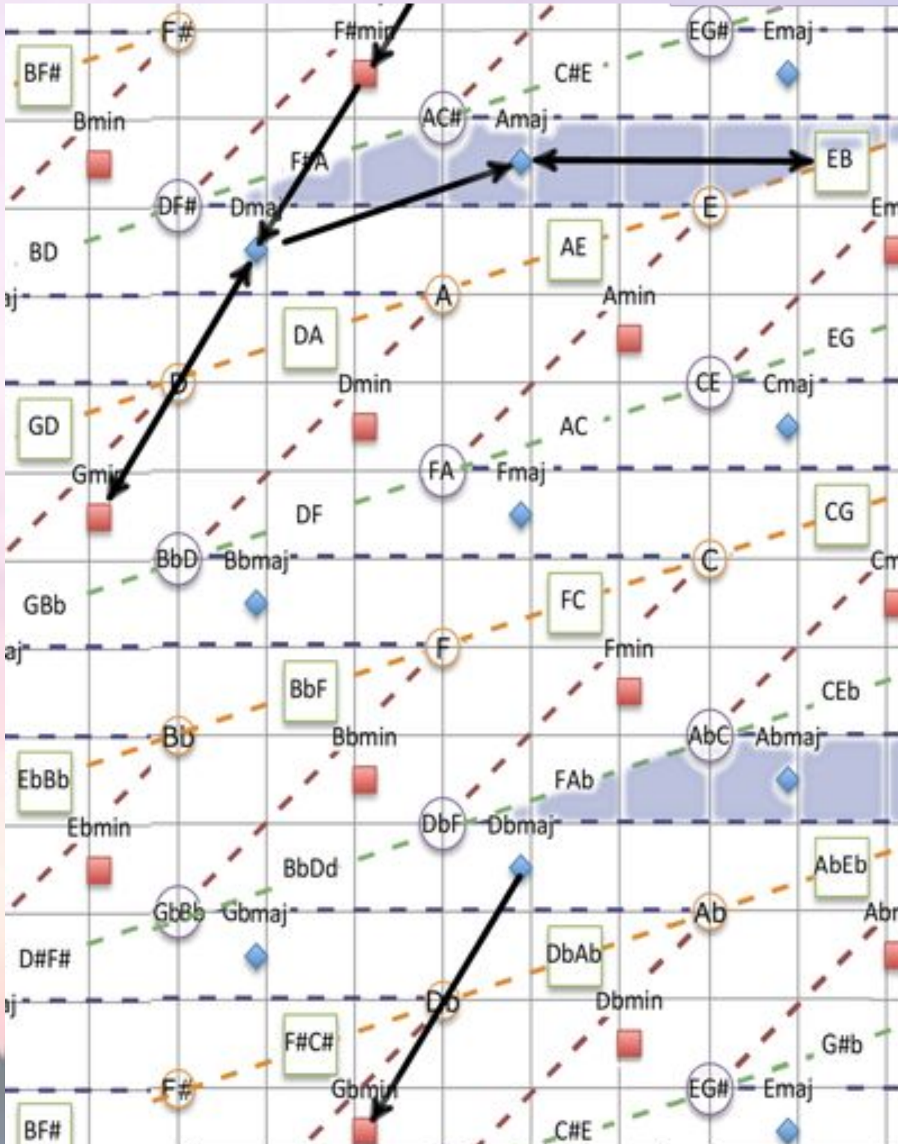
The first episode executes **a flatward enharmonic tour**, motivated by the modal transgression of the main theme

Late C minor Piano Sonata, Adagio

A♭ major . . . transgression . . . transposed → A major (PAC)

Schubert later confirms the significance of the transgression by letting it derail the theme itself from its tonal course. (The passage here is from the second refrain.)

Late C minor Piano Sonata, Adagio



Schubert later confirms the significance of the transgression by letting it derail the theme itself from its tonal course. (The passage here is from the second refrain.)

Schubert's Harmonic Language and the *Tonnetz* as a Continuous Geometry

Jason Yust, Boston University

Presentation to the
Society for Music Theory, October 31, 2013

Bibliography

Amiot, Emmanuel. (2013). "The Torii of Phases." *Proceedings of the International Conference for Mathematics and Computation in Music, Montreal, 2013,* ed. J. Yust, J. Wild, and J.A. Burgoyne (Heidelberg: Springer).

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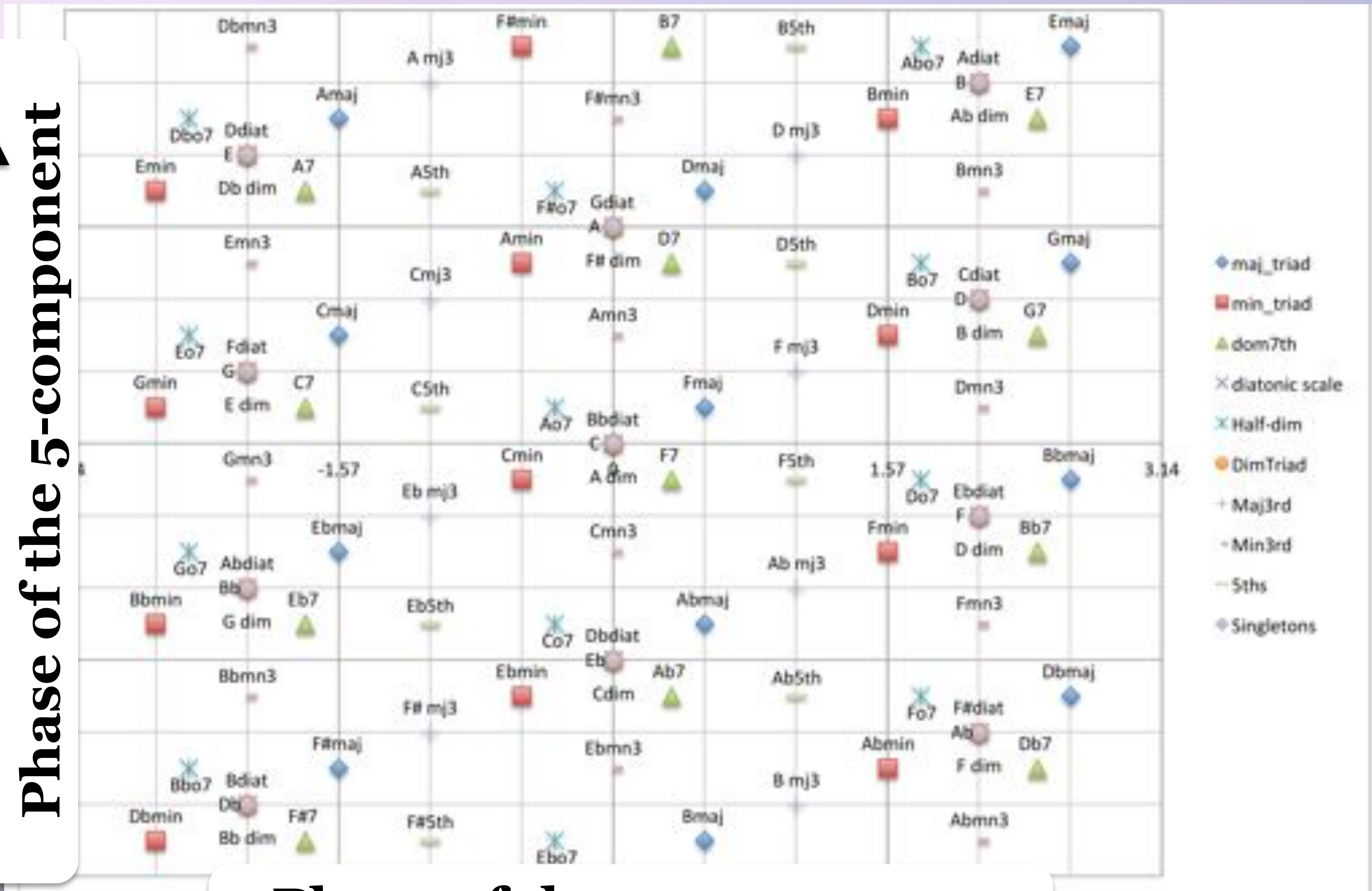
Appendices:

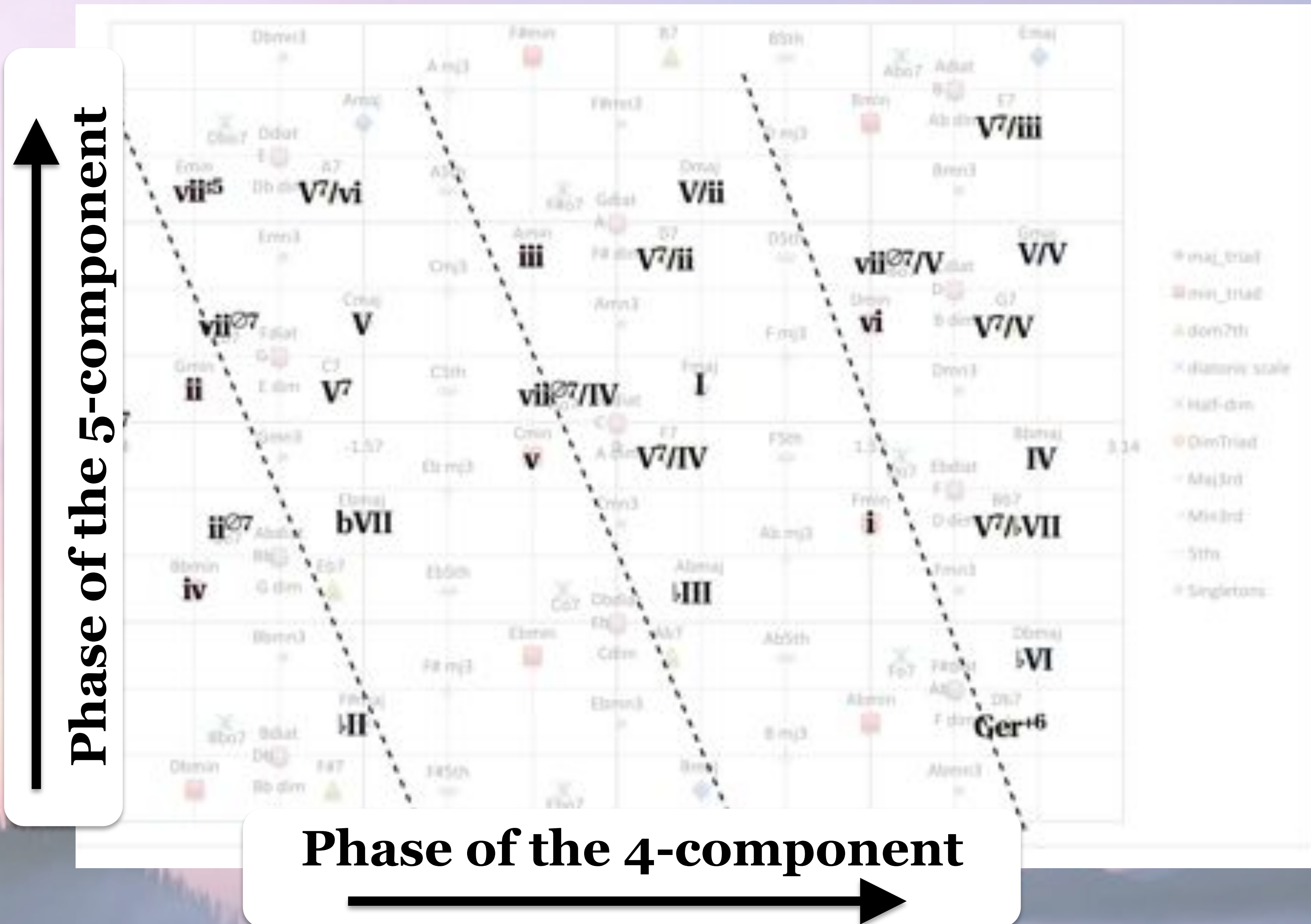
A1: Functional categories in c5/c4 space



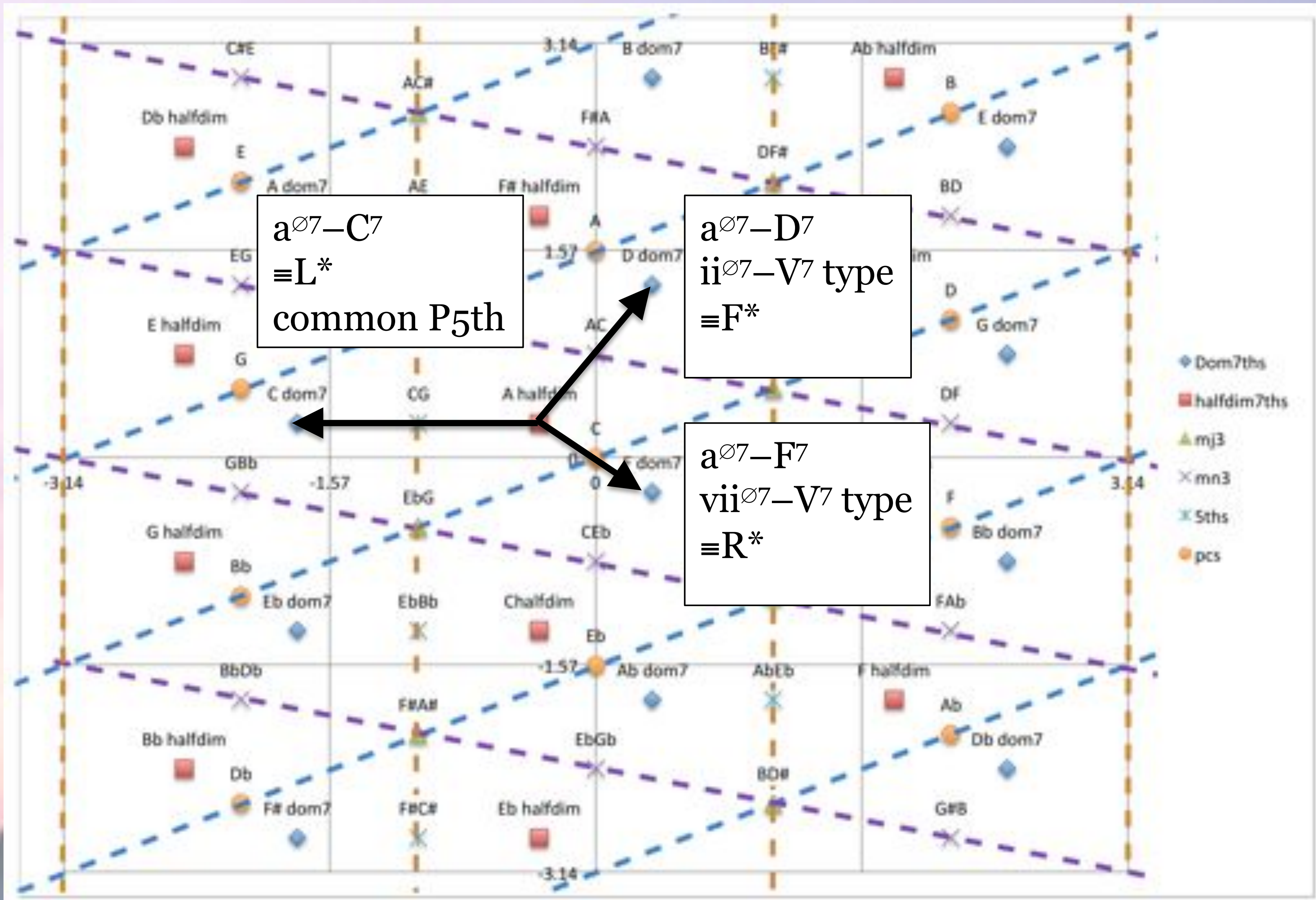
Phase of the 5-component

Phase of the 4-component

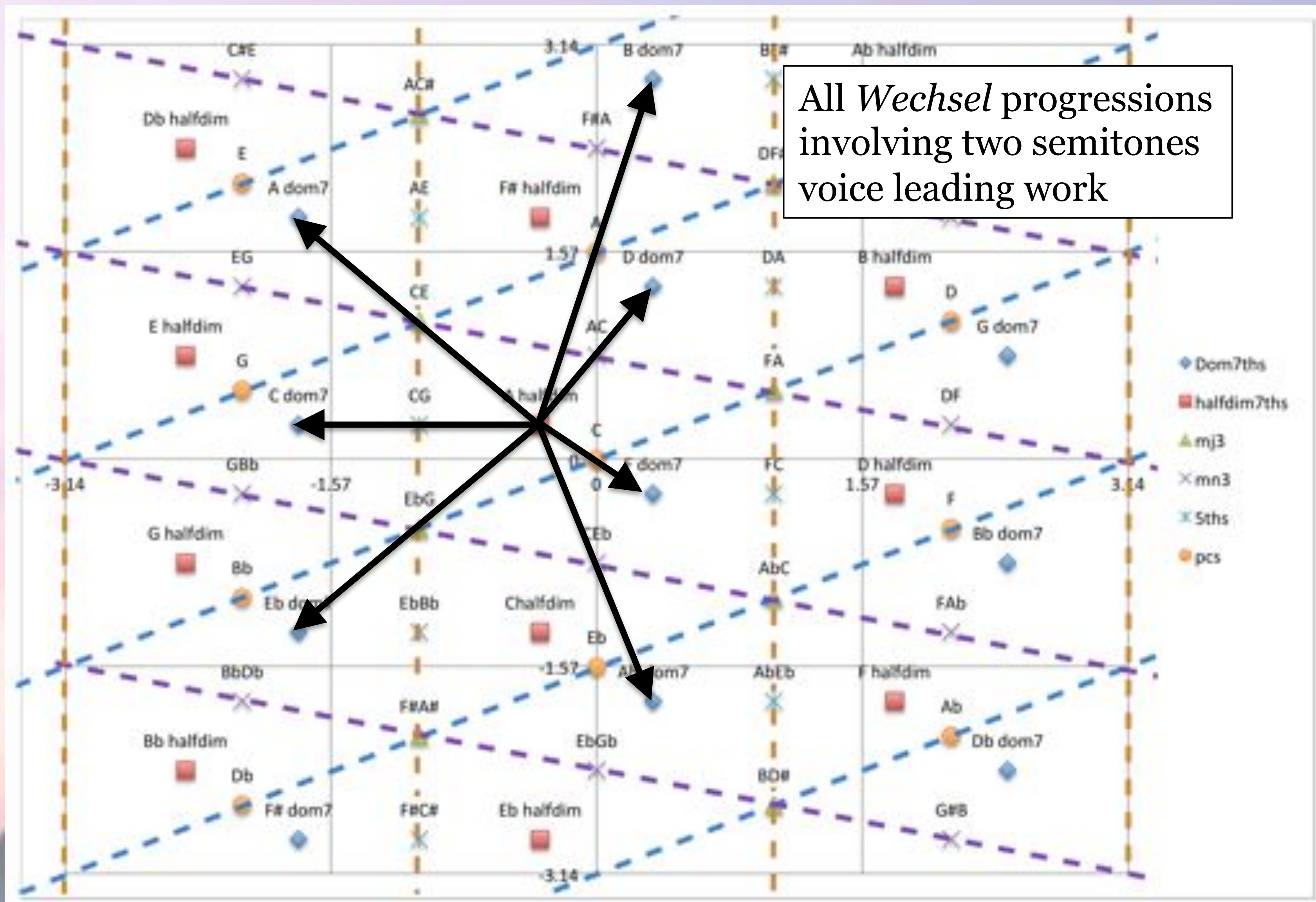
A1: Functional categories in c5/c4 space



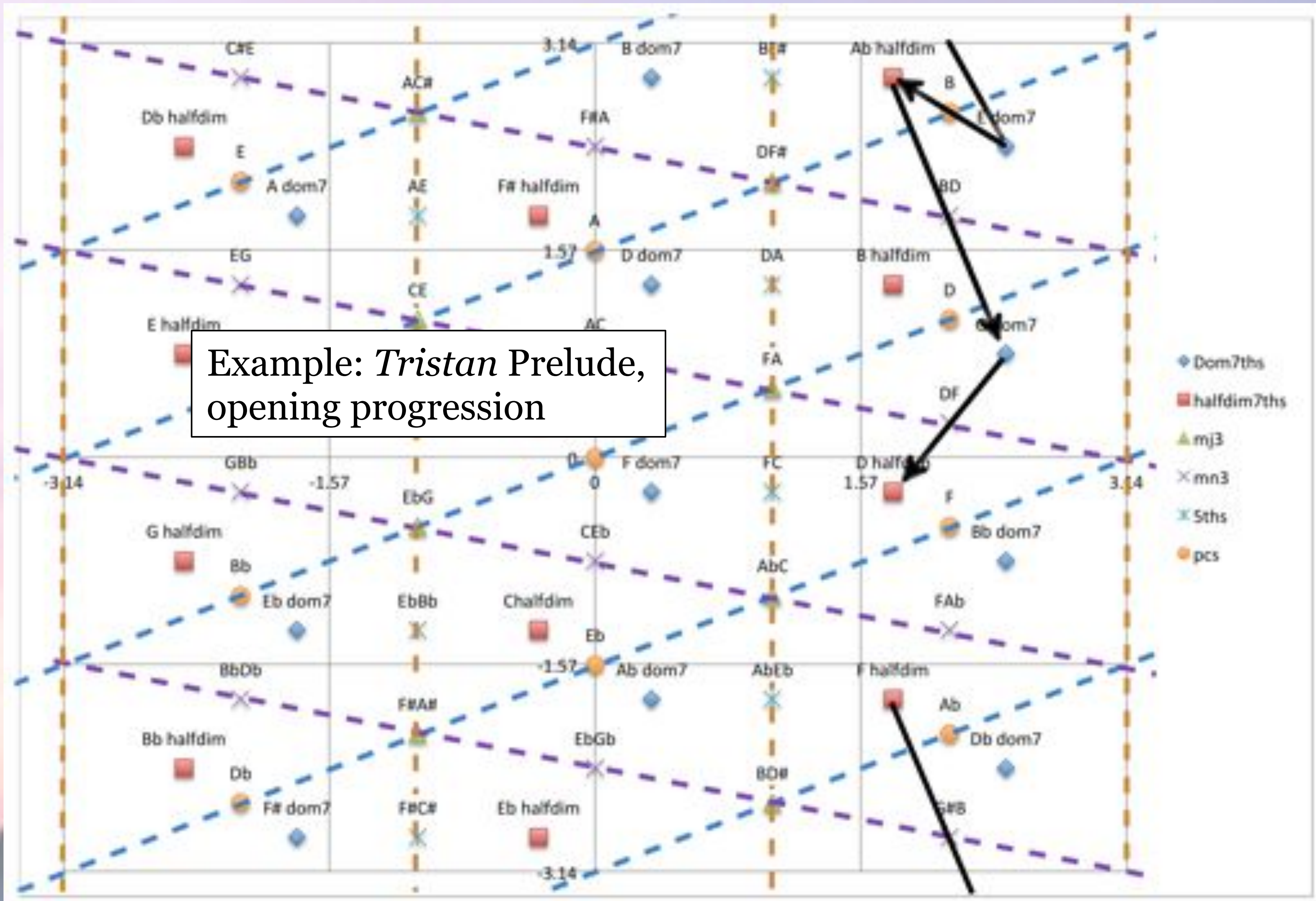
A2: *Tonnetz* of seventh chords in c4/c5 space



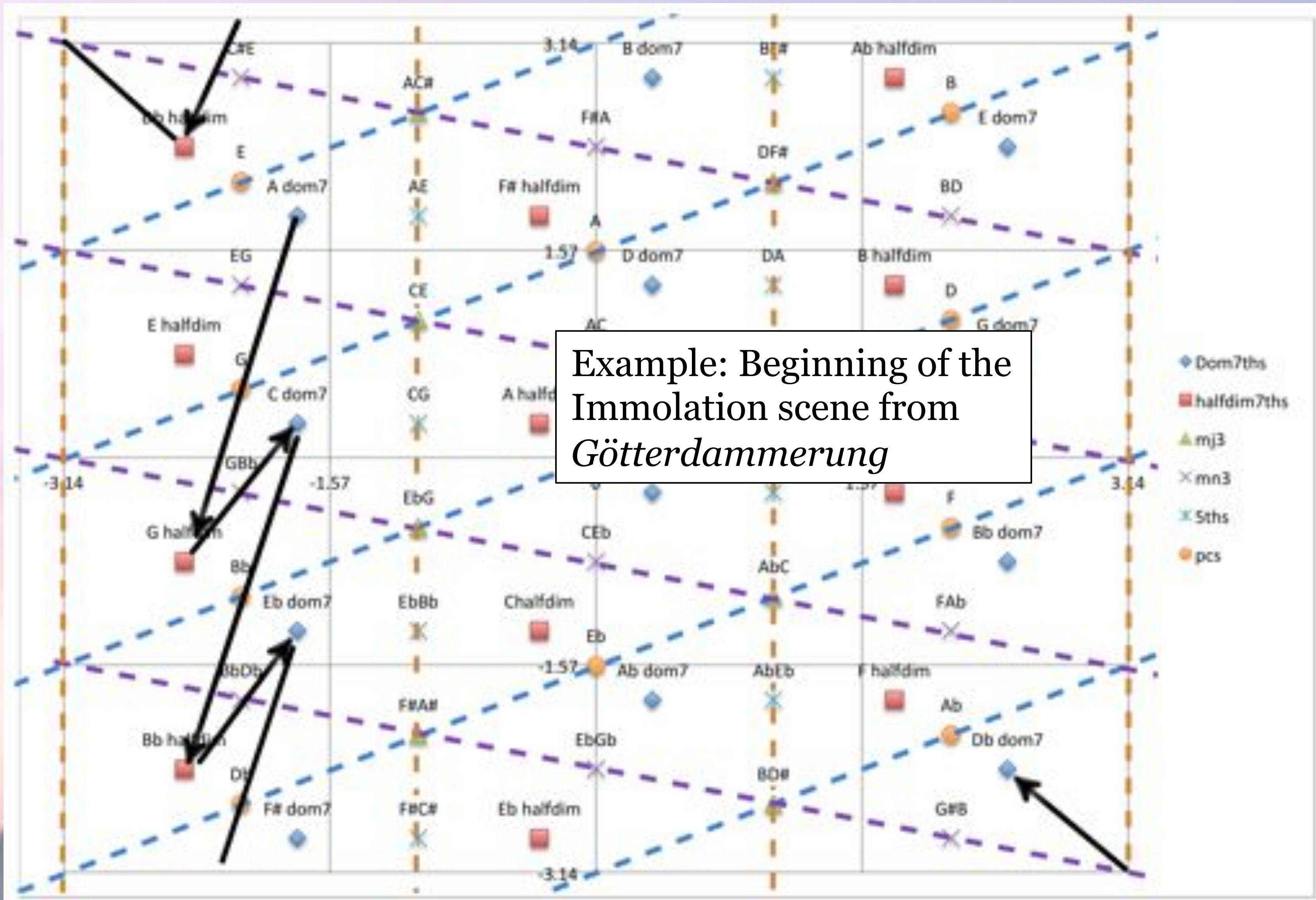
A2: *Tonnetz* of seventh chords in c4/c5 space



A2: *Tonnetz* of seventh chords in c4/c5 space

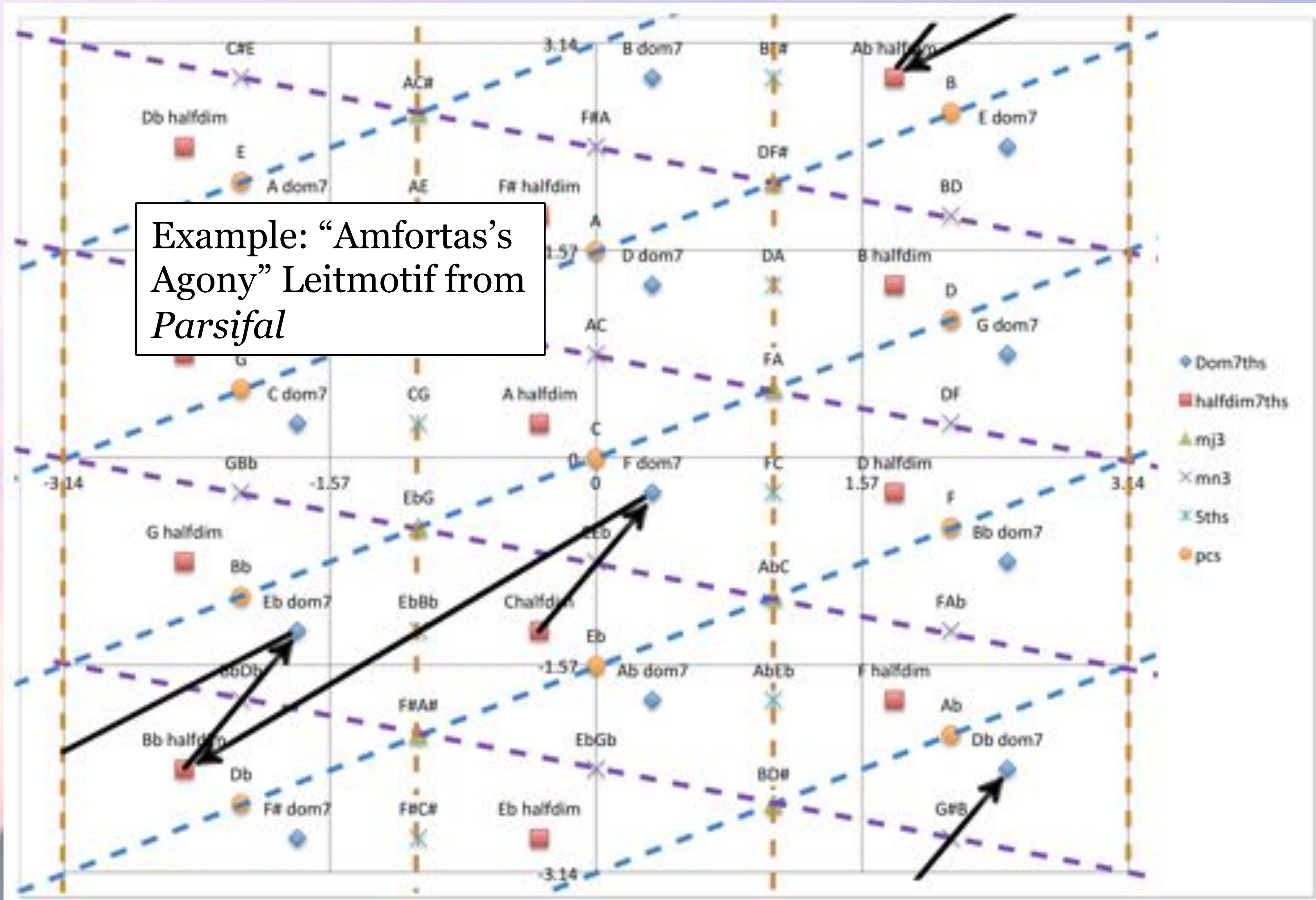


A2: *Tonnetz* of seventh chords in c4/c5 space



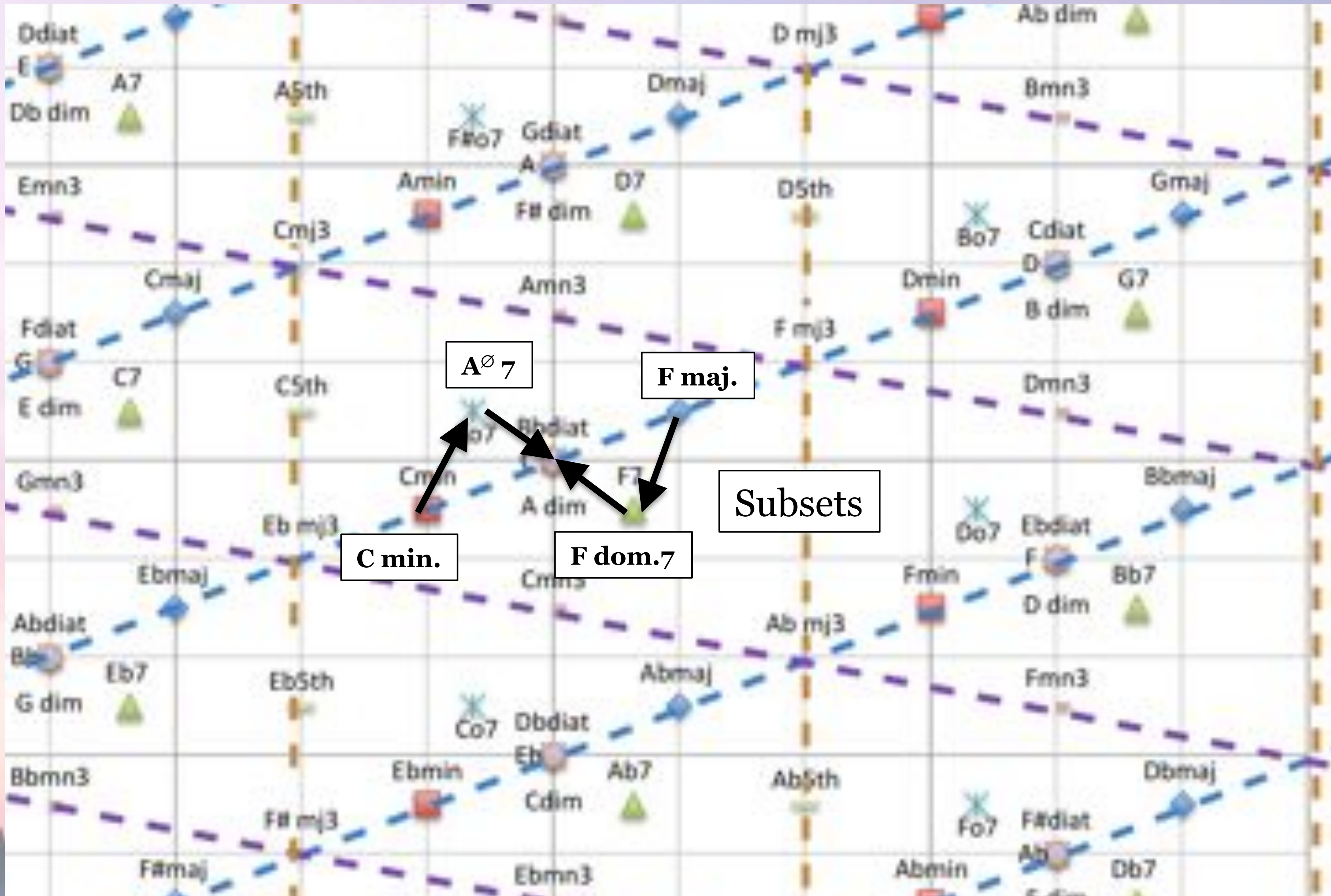
Example: Beginning of the Immolation scene from *Götterdammerung*

A2: *Tonnetz* of seventh chords in c4/c5 space

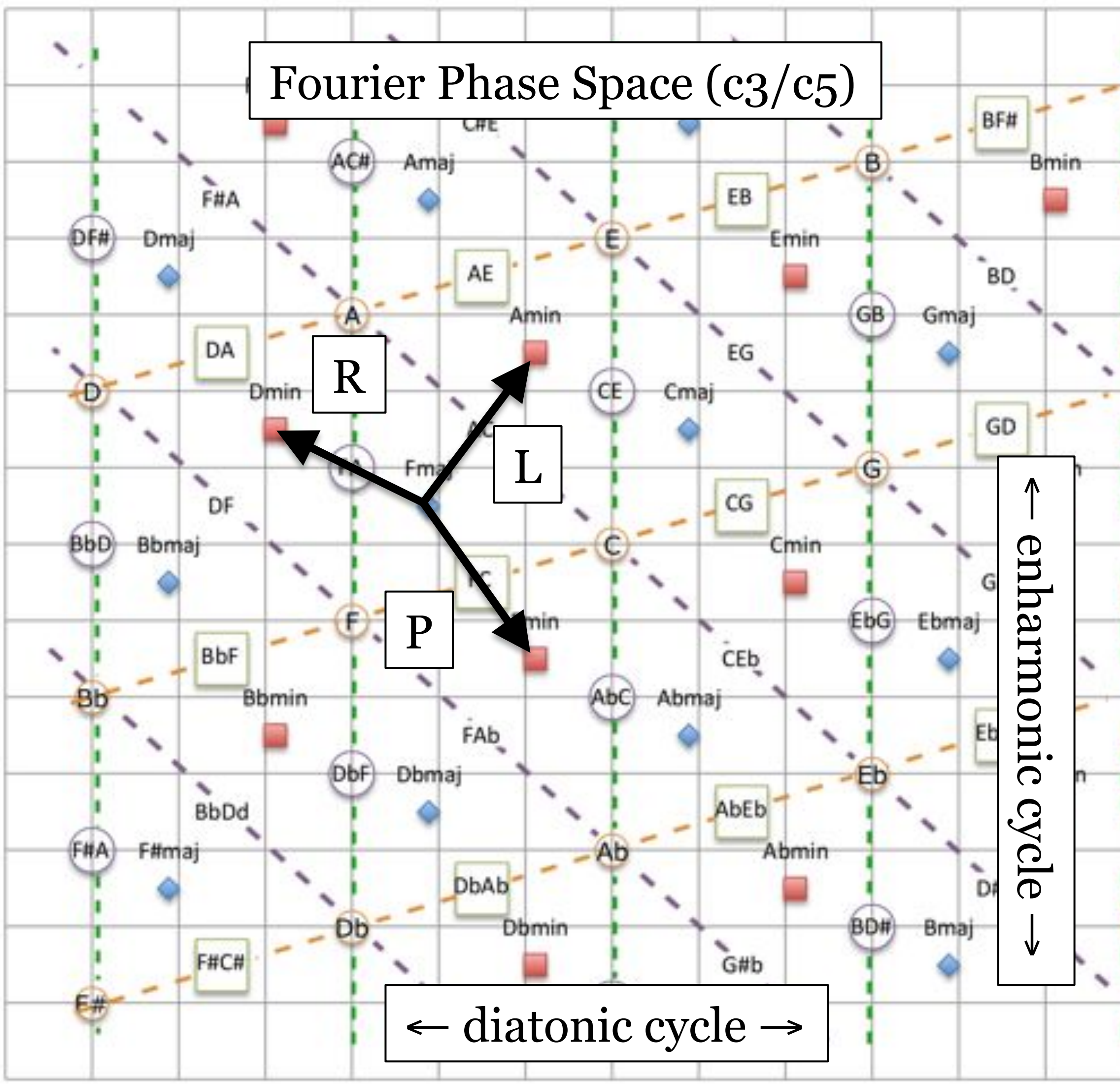


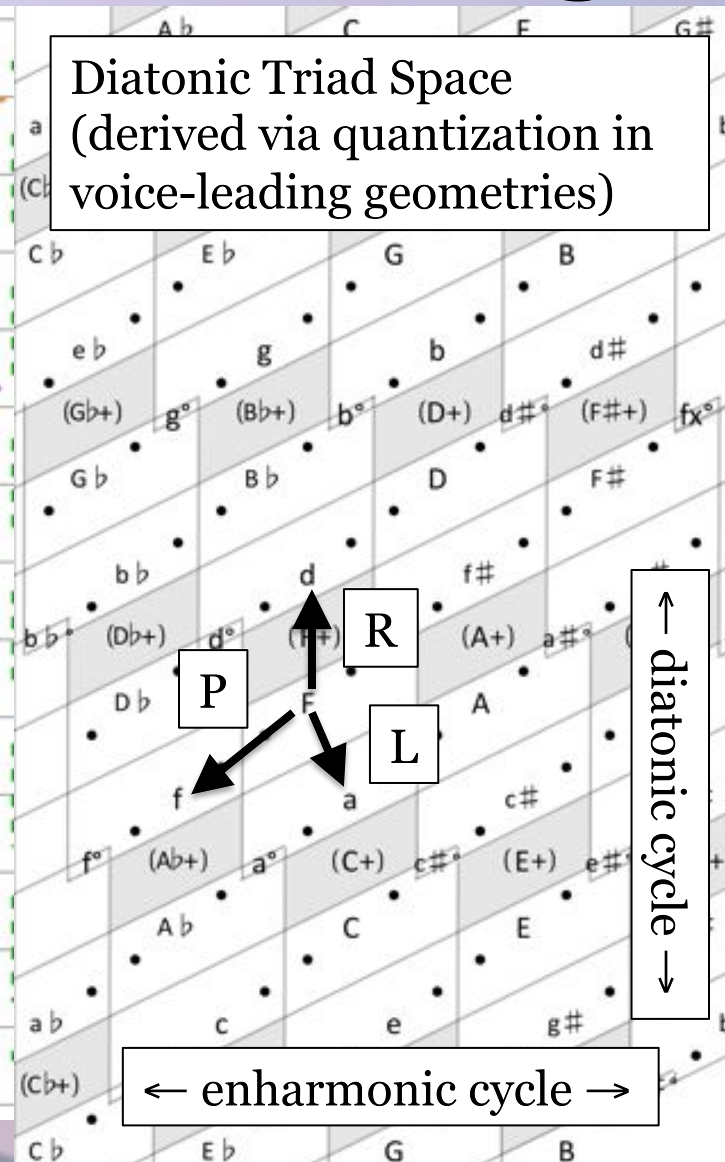
Example: "Amfortas's Agony" Leitmotif from *Parsifal*

A2: *Tonnetz* of seventh chords in c4/c5 space



A3: Phase space vs. voice leading





A3: Phase space vs. voice leading

The pitch-class quantities model

Voice leading model

A3: Phase space vs. voice leading

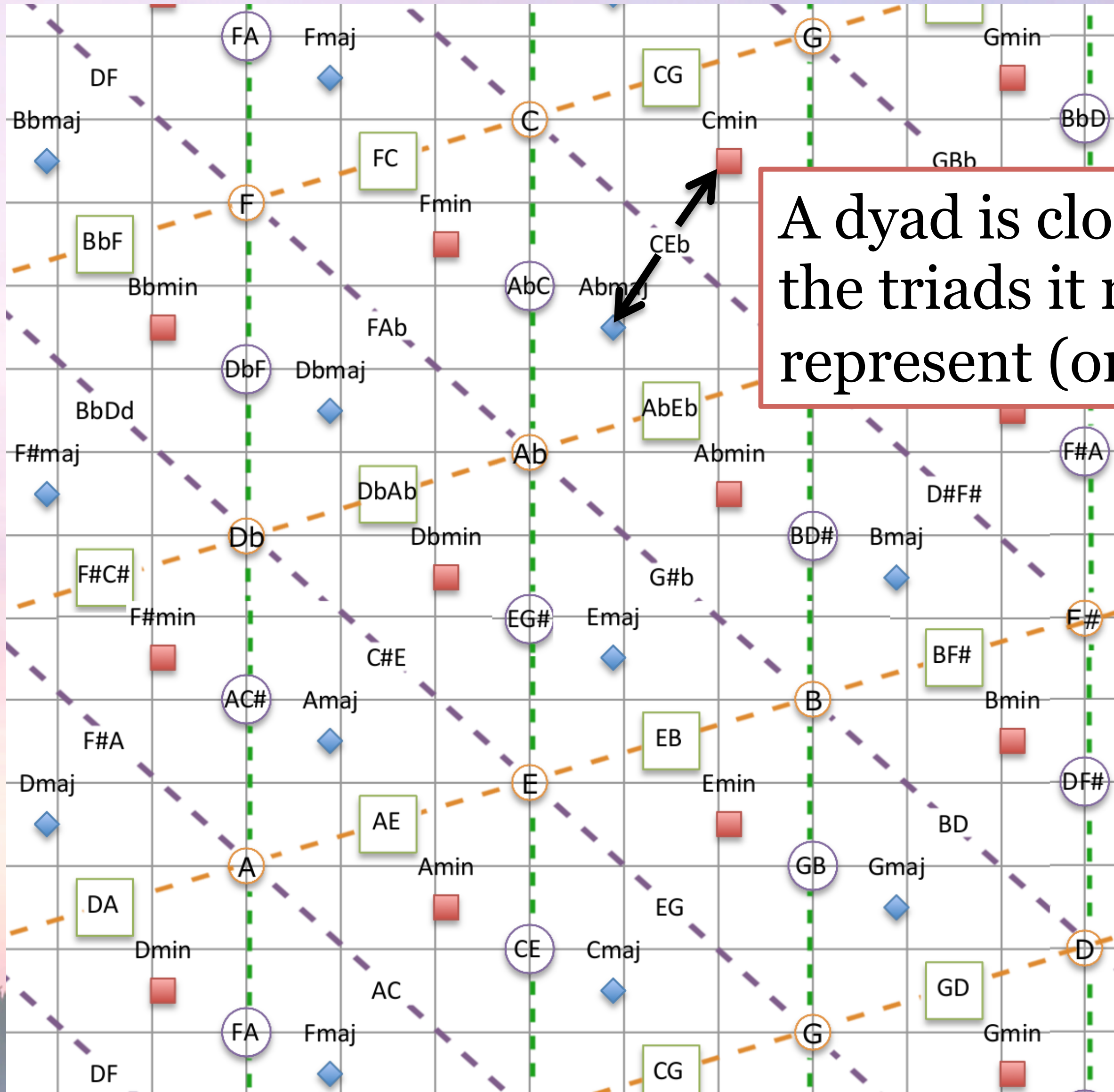
The phase spaces can be interpreted as *kinds of voice-leading spaces!*

Unlike voice-leading geometries from Callender, Quinn, & Tymoczko 2008 they are **robust with respect to omissions and doublings**. This is possible because an additional assumption is made: chords are **assumed to be nearly even.**

Component 3 shows voice-leading direction between the nearest even 3-note chords.

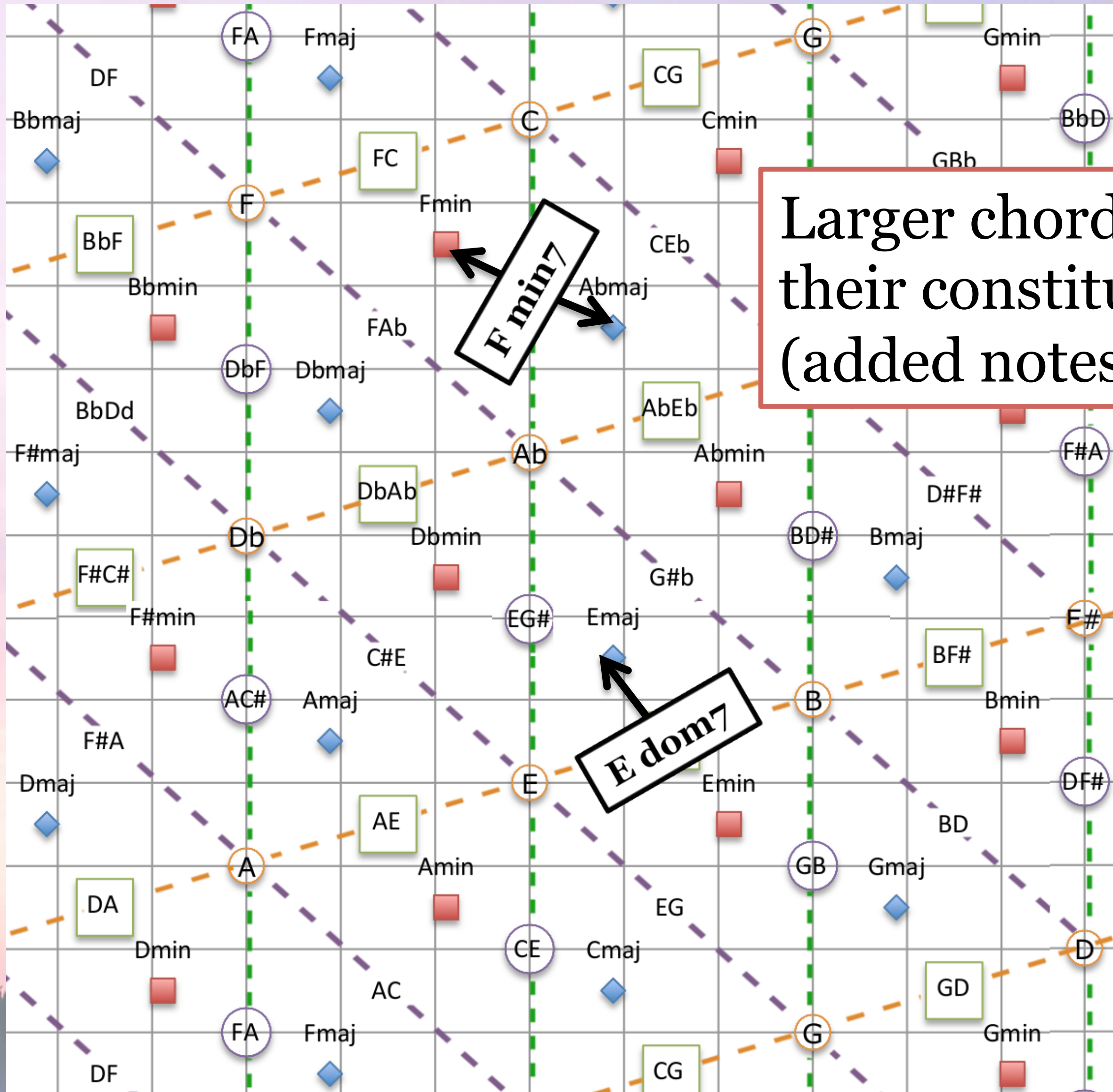
Component 5 shows voice-leading directions between implied diatonic scales.

A3: Phase space vs. voice leading



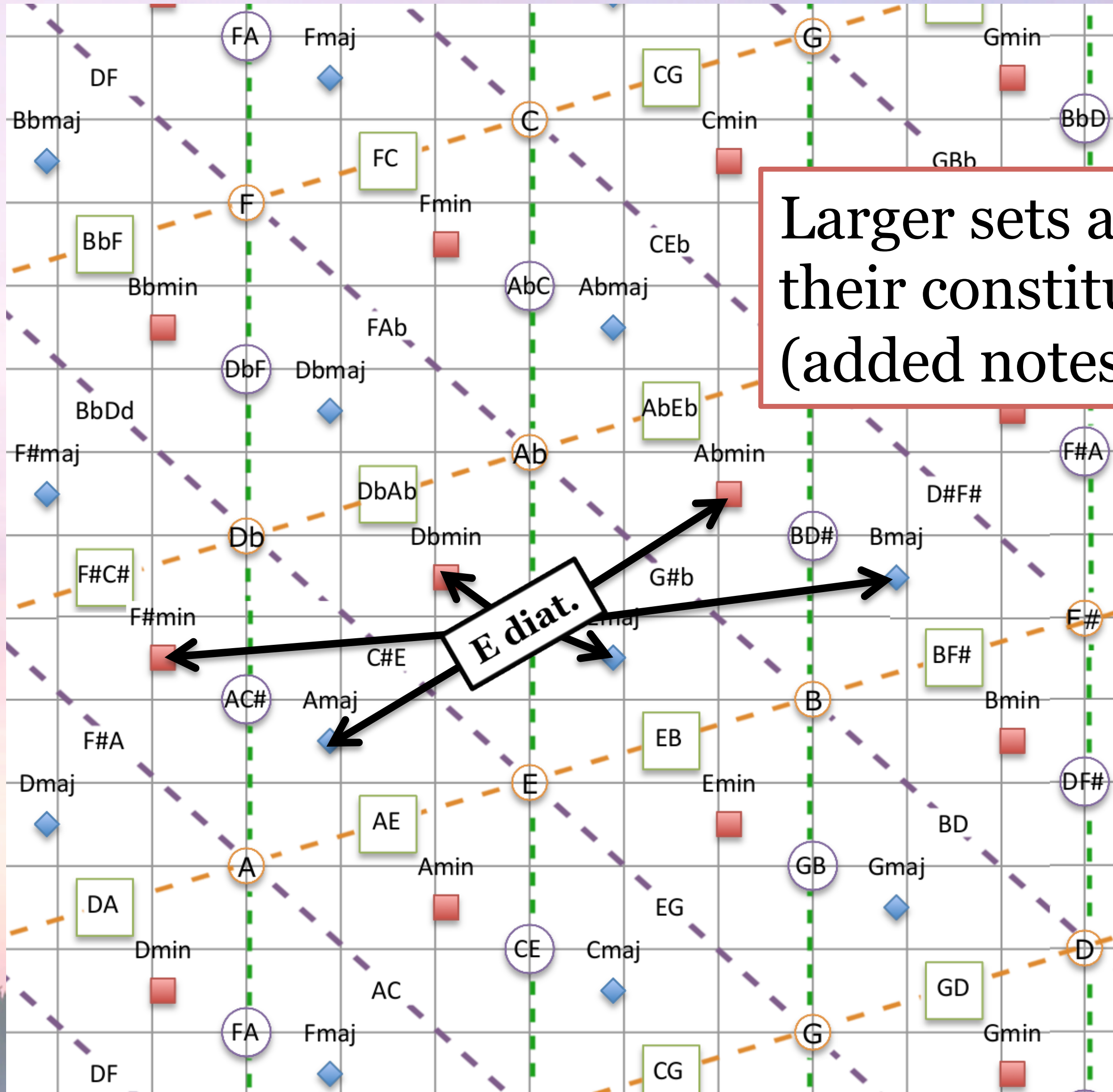
A dyad is close to the triads it might represent (omission)

A3: Phase space vs. voice leading



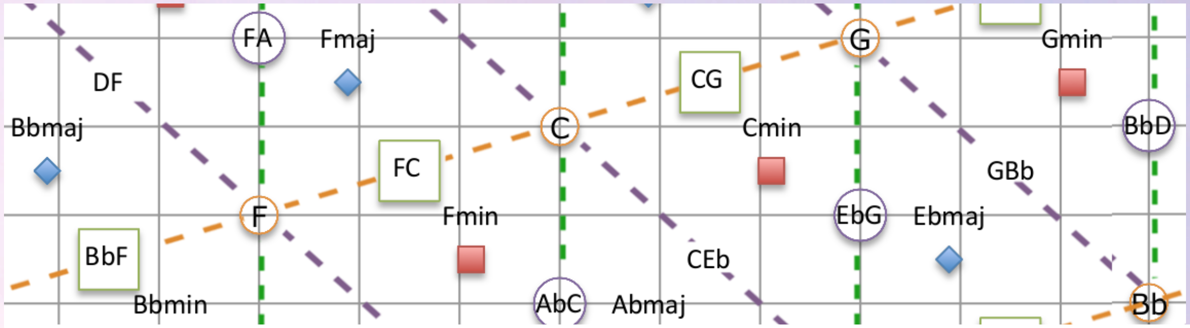
Larger chords are close to their constituent triads (added notes)

A3: Phase space vs. voice leading

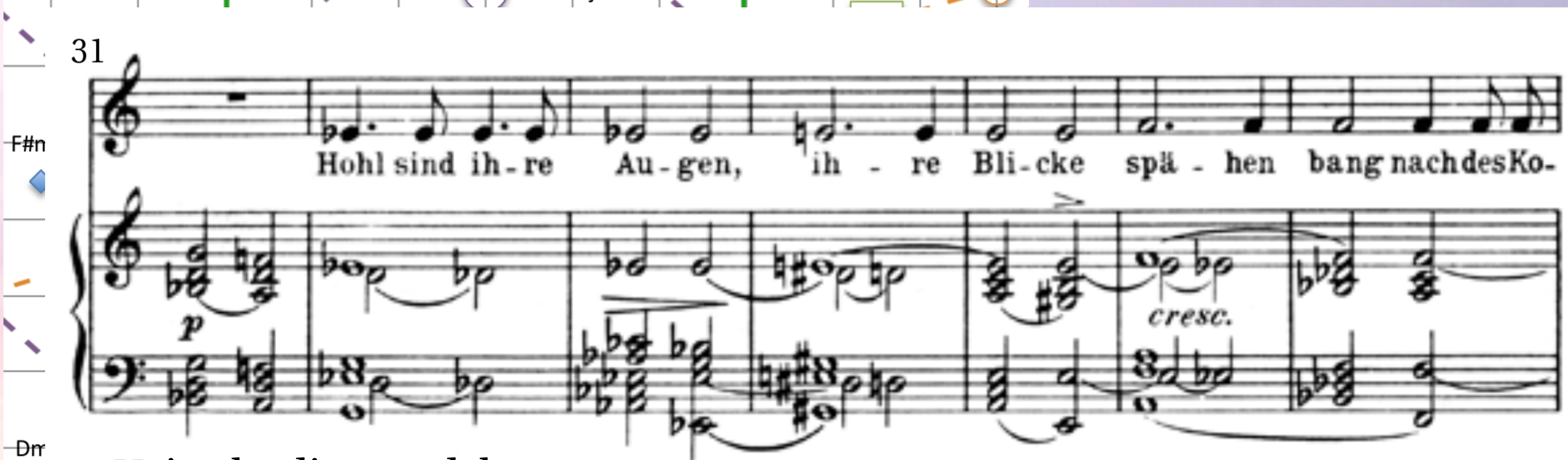


Larger sets are close to their constituent triads (added notes)

A3: Phase space vs. voice leading

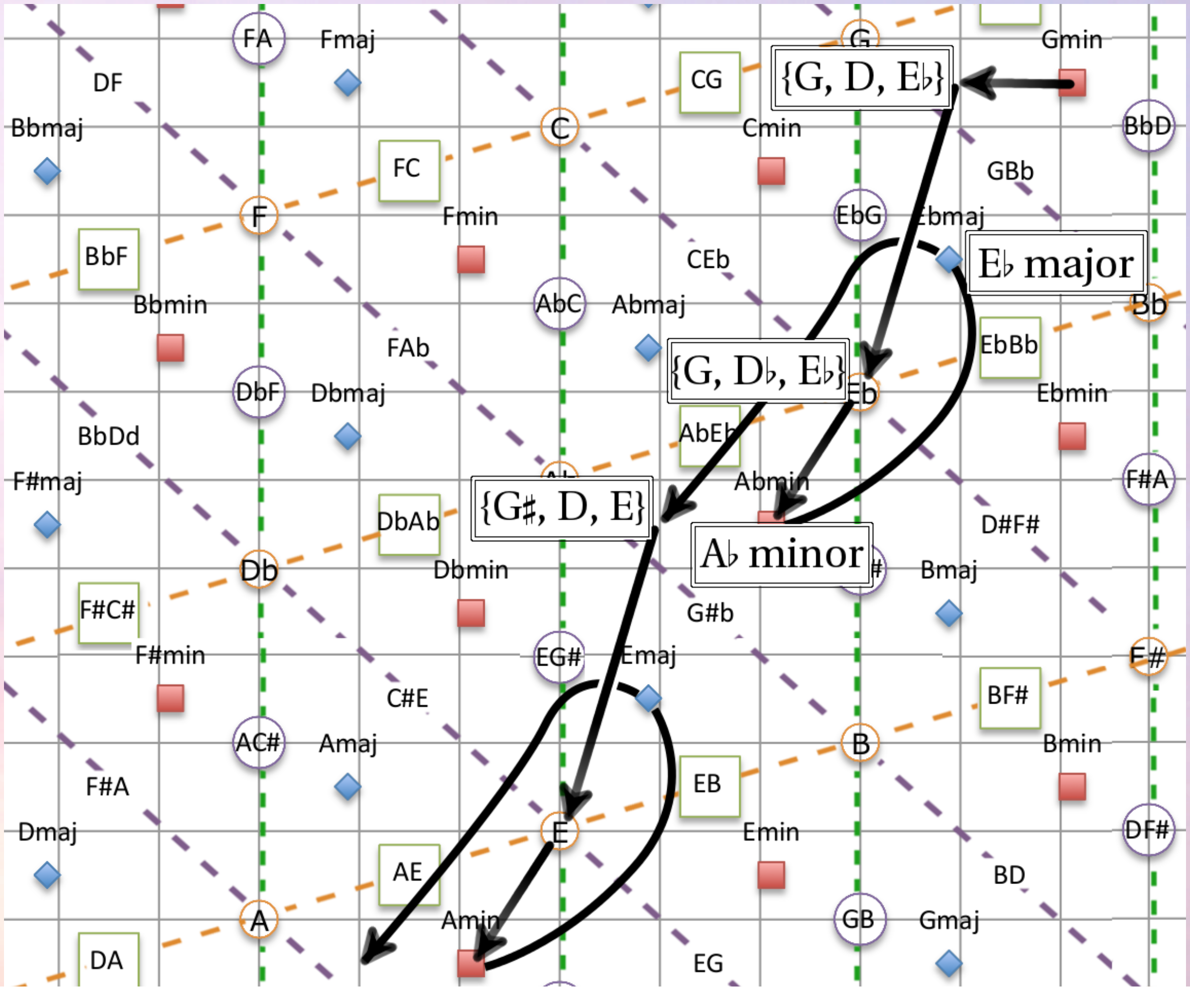


Example: Sequence from "Gruppe aus dem Tartarus" (D. 583)



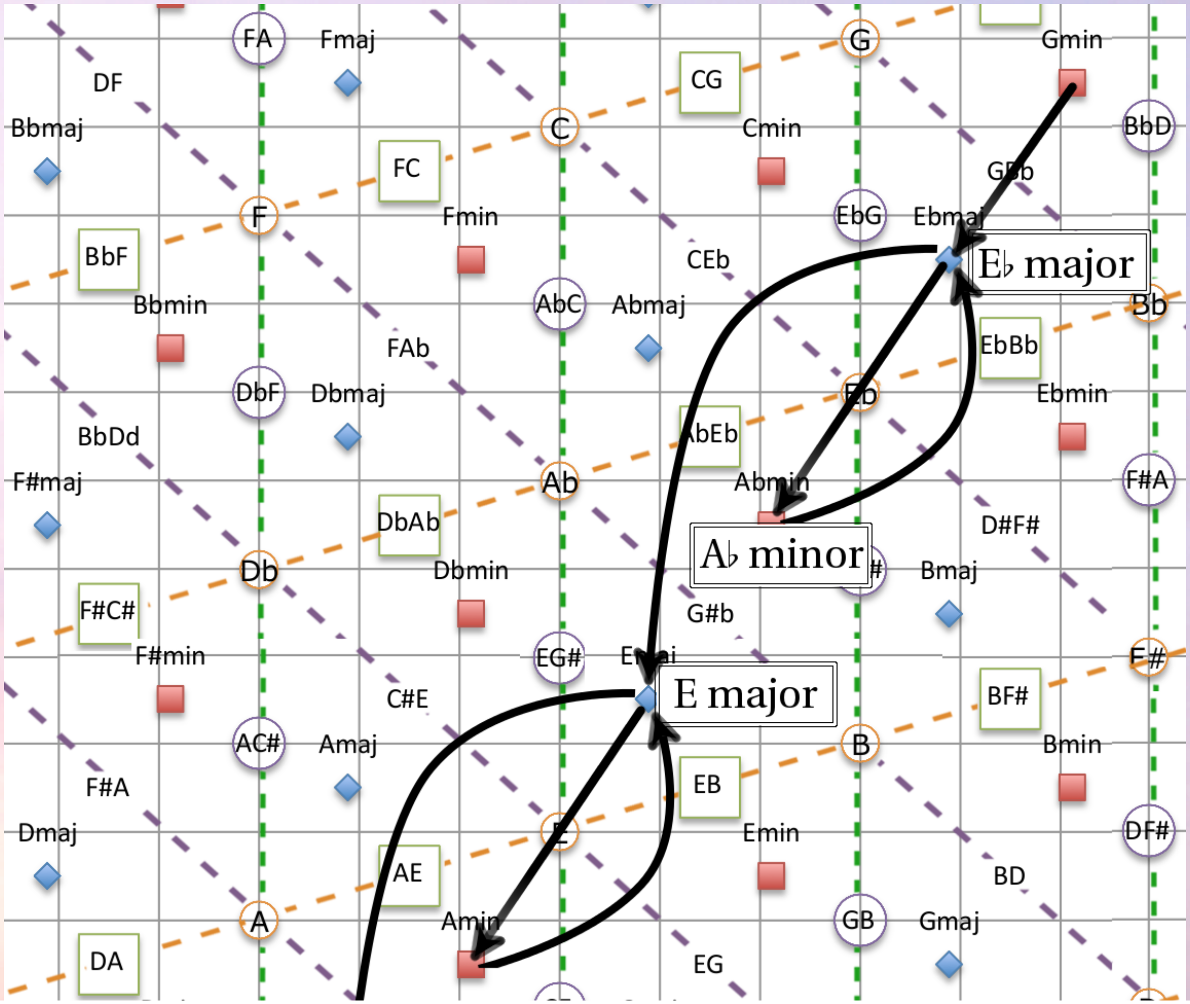
Voice-leading model:

A3: Phase space vs. voice leading



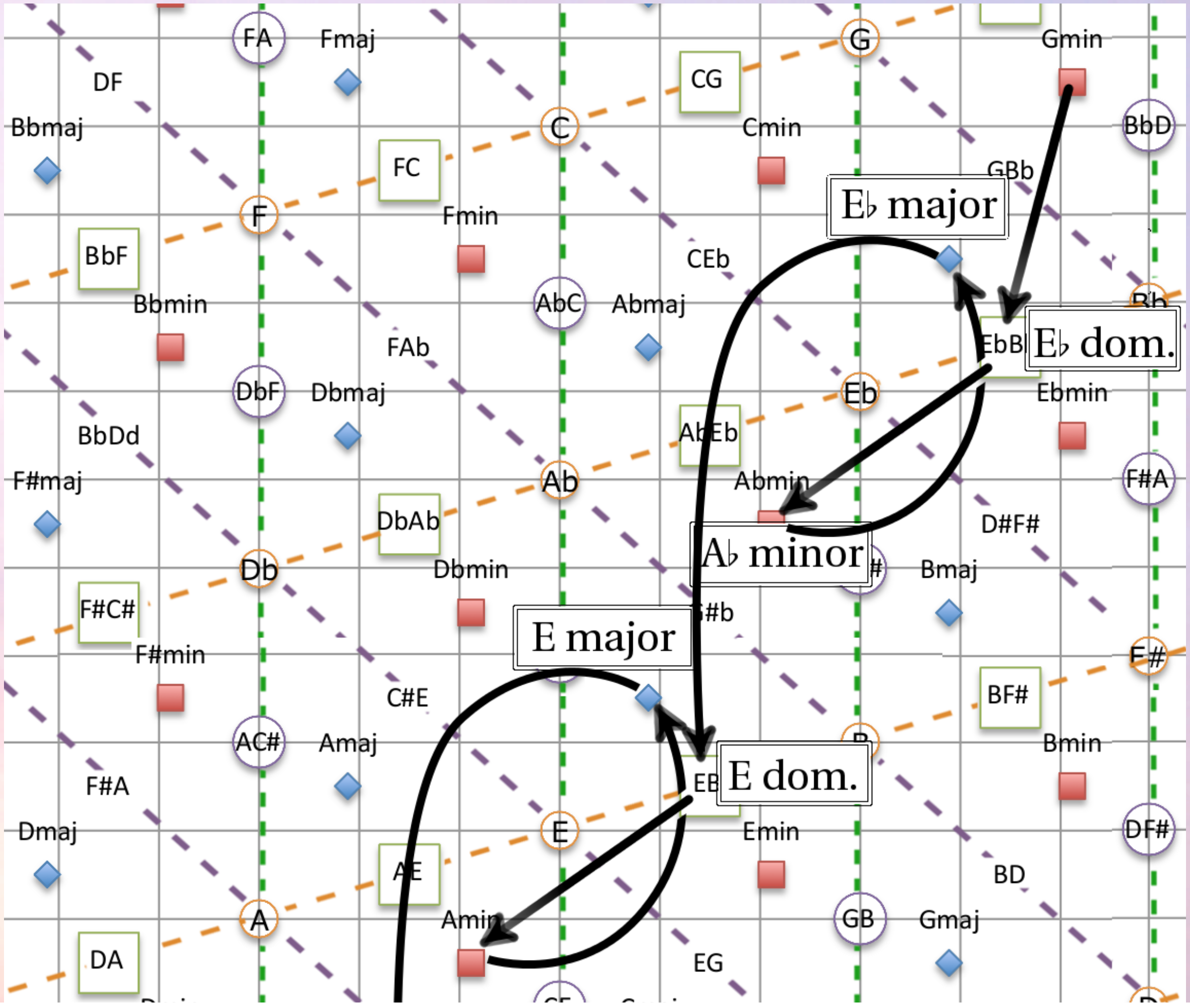
Example: Sequence from "Gruppe aus dem Tartarus" (D. 583)

A3: Phase space vs. voice leading



A similar, nearby, sequence on triads

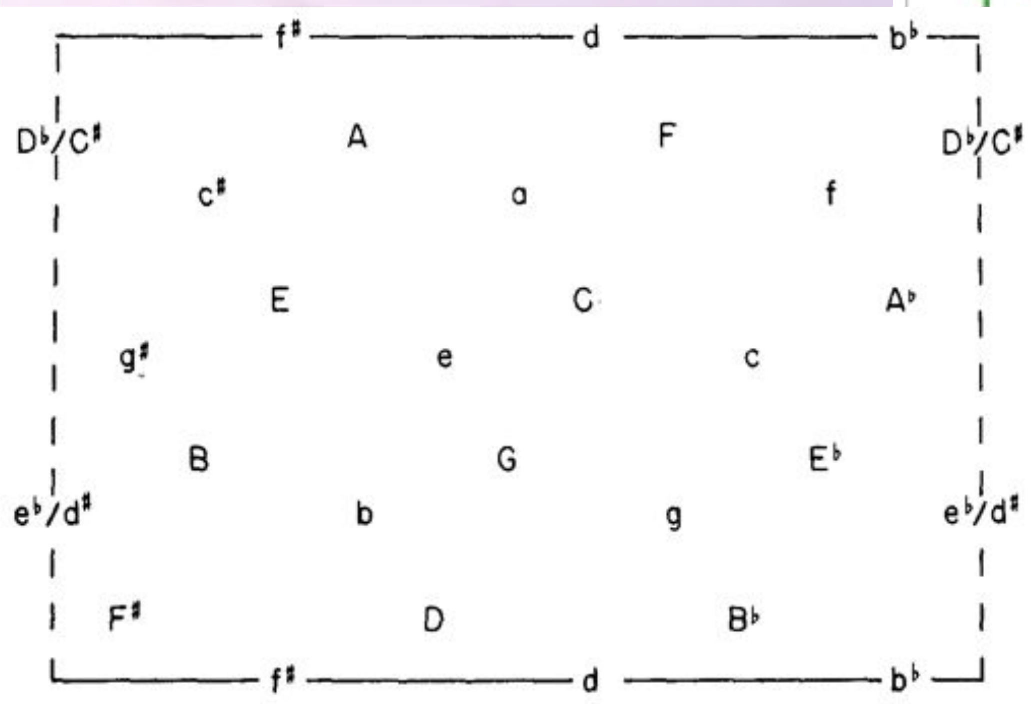
A3: Phase space vs. voice leading



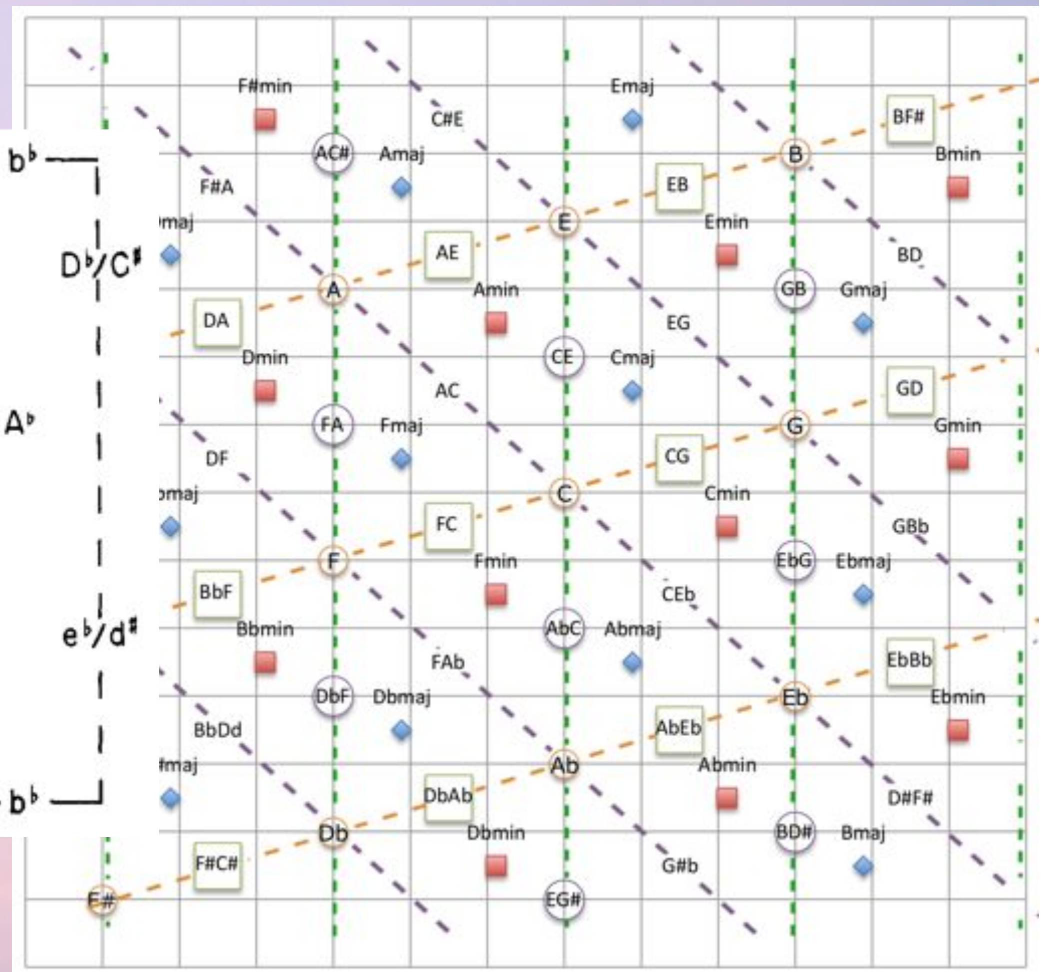
A nearby sequence on triads and seventh chords.

A4: Key profiles and phase space

MDS of Key Profiles from Krumhansl and Kessler 1982

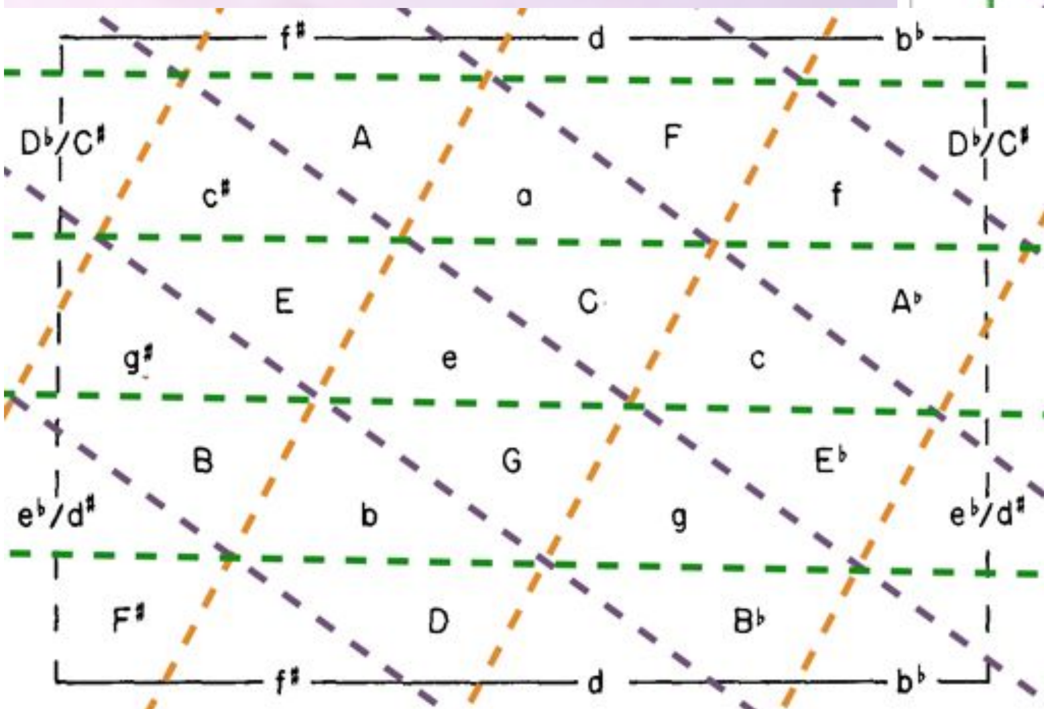


Harmonic Phase Space

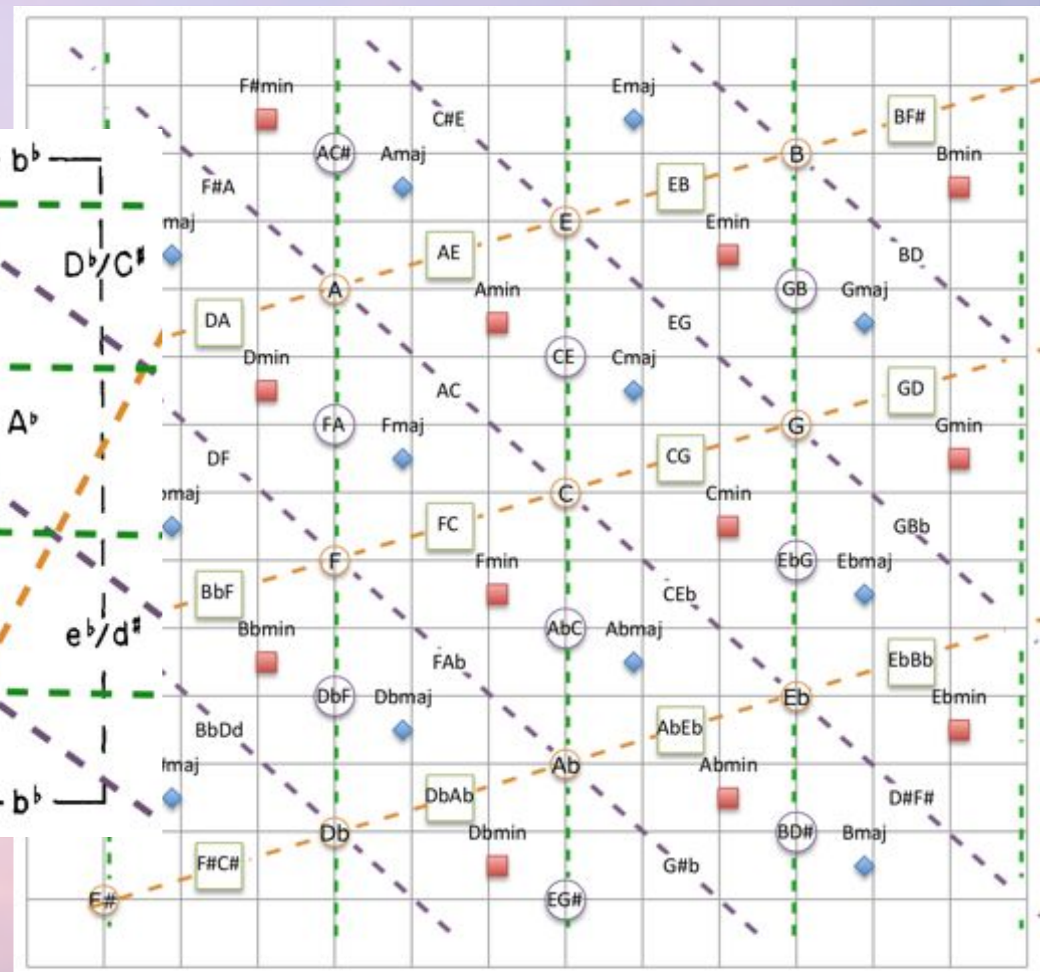


A4: Key profiles and phase space

MDS of Key Profiles from Krumhansl and Kessler 1982

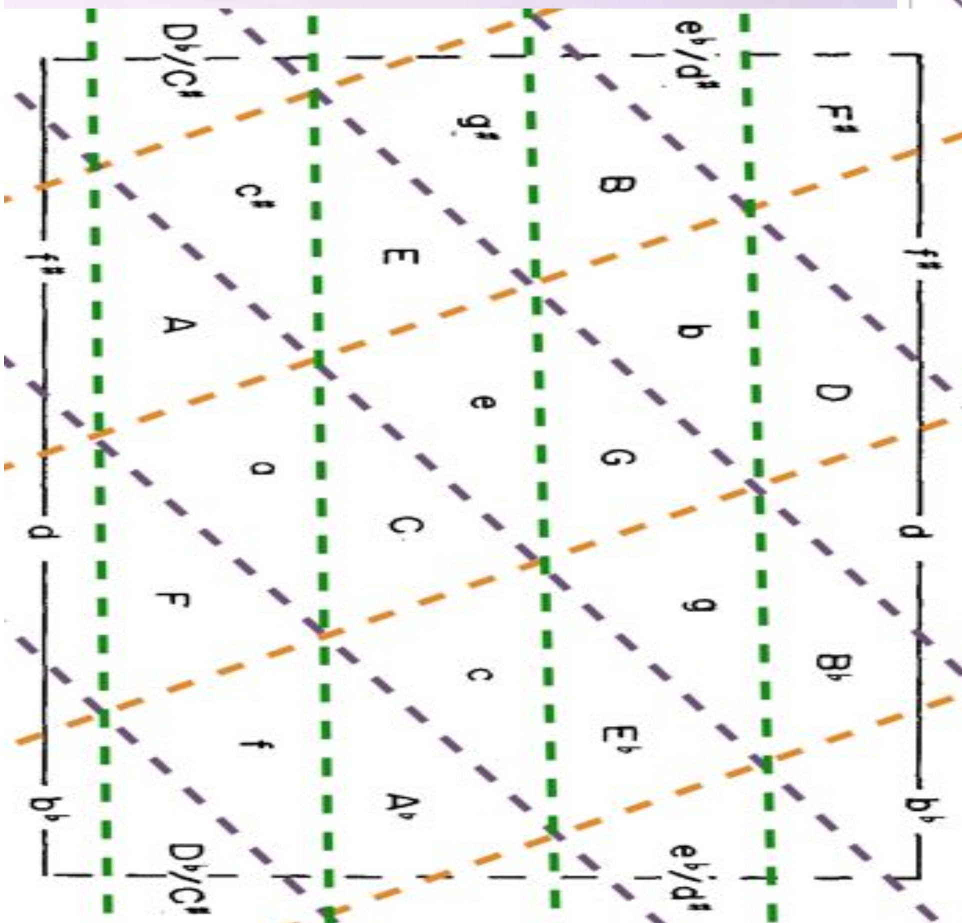


Harmonic Phase Space

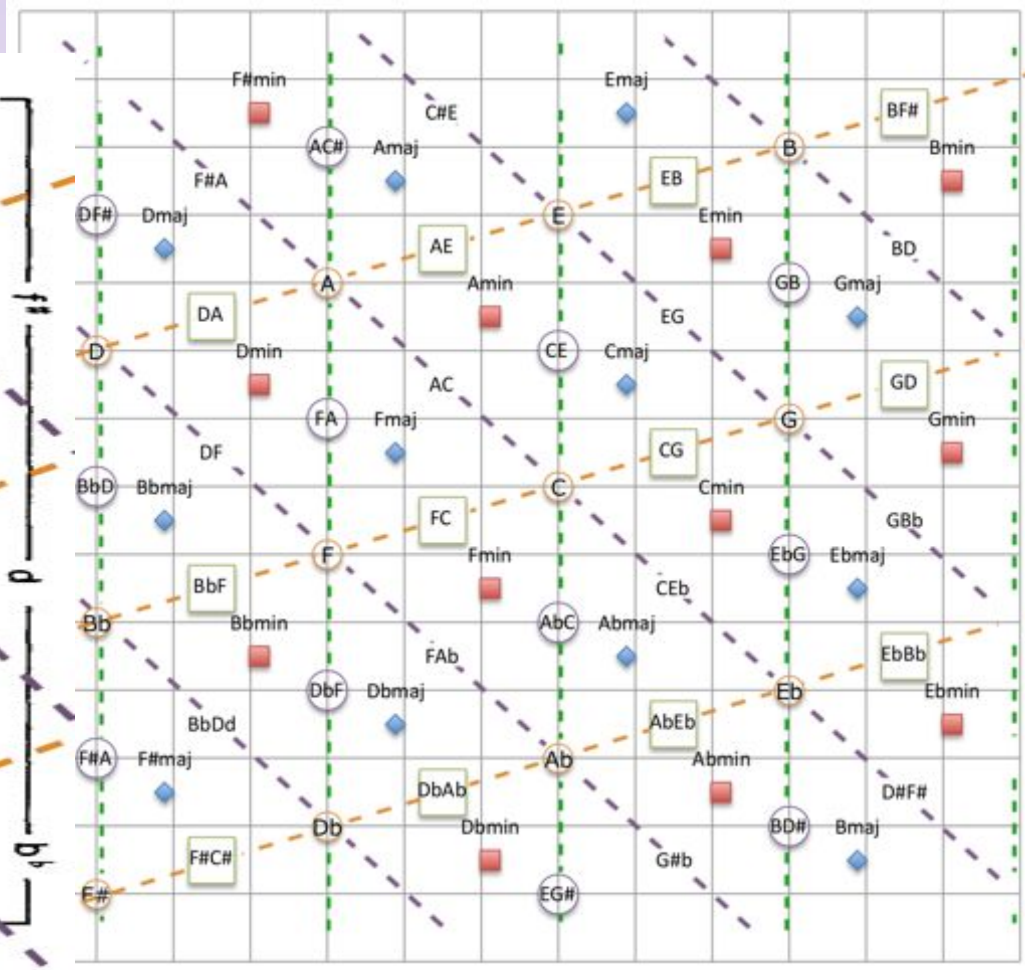


A4: Key profiles and phase space

MDS of Key Profiles from Krumhansl and Kessler 1982



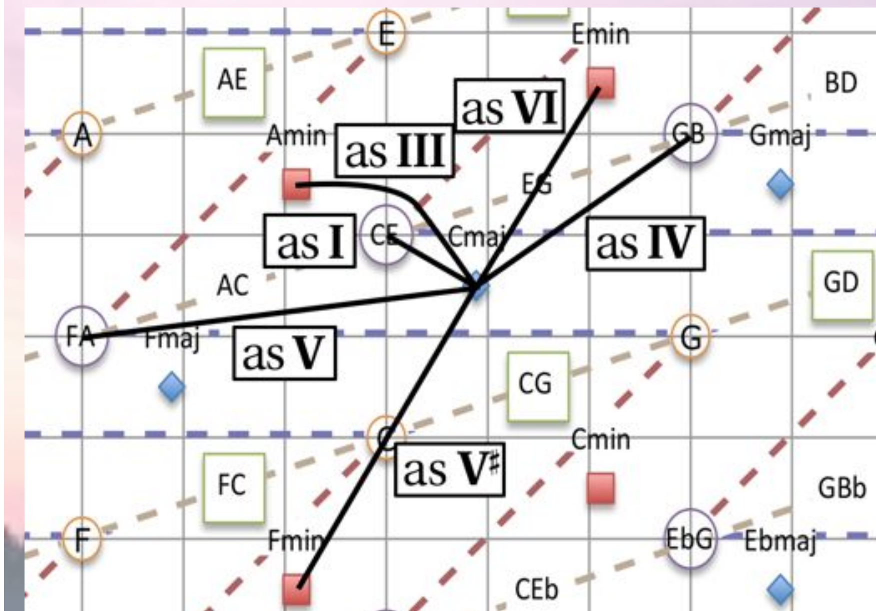
Harmonic Phase Space



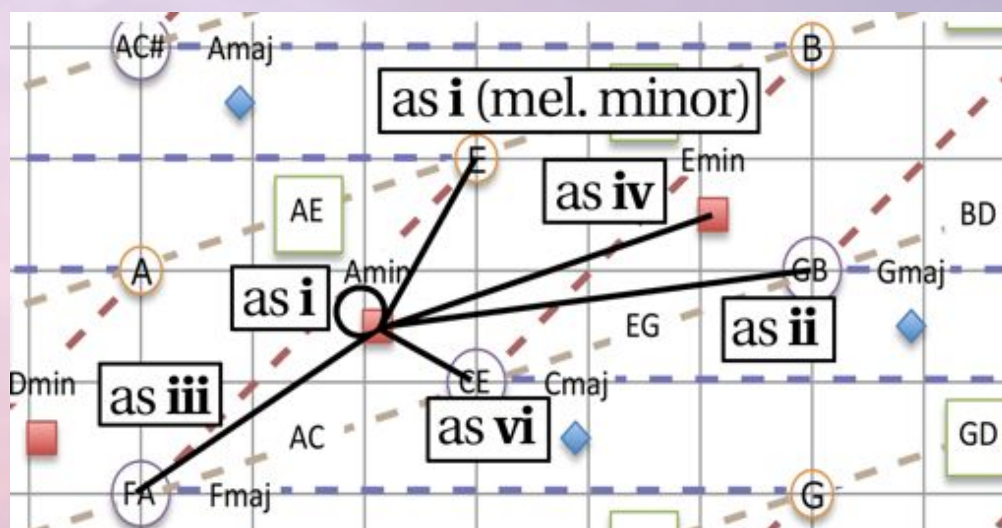
A5: Chord–scale relationships

Major and minor chords are generally closest to the scales of the keys in which those chords have some basic function.

Major chord inclusions:



Minor chord inclusions:



Notes on slides

I. Discrete Fourier Transform on Pcsets

(4) This work is based on the idea of treating a pcset as a signal and taking its Fourier transform to isolate significant intervallic properties. The idea was first proposed by Lewin and further developed by Ian Quinn as a measure of set class similarity. Both Lewin and Quinn focus on *magnitudes* of the resulting Fourier components, which contain the transposition- and inversion-independent interval information. The current paper and Amiot's are interested instead in the *phases* of the components, which contain the residue of transposition-dependent information. The phases therefore can show relationships between instances of the same set class, not just between different set classes.

(5) The Fourier transform described in Lewin (1959) and Quinn (2006–7) treats a pcset as a signal by interpreting each pc as a spike in pc space as shown in the first graph. The Fourier transform decomposes this into 12 sinusoidal components, the first two of which are shown below.

Components 1 and 2 have very small magnitudes for consonant triads. The nth component can be interpreted as an equal division of the octave into n parts. The phase of the component is how it is transposed to best match the give pcset. Component 0 indicates the cardinality of the pcset, and Components 7–11 duplicate the information given by components 1–5.

(6) Component 3 is the largest magnitude component for major and minor triads. (Note that component 4 is also of substantial magnitude.)

(7) Component 5 is the second largest for consonant triads.

(8) We will consider just the largest components, numbers 3 and 5. Every transposition or inversion of a major or minor triad has a different phase on one or both of these components. Component 4 will be revisited in appendices A1 and A2.

This slide shows how a change from C major to A minor affects the phase of these two components.

(9) Quinn's Fourier balances are a convenient way to conceptualize the meaning of the phase components. The 3^{rd} component shows which augmented triad the chord is closest to. The 5^{th} component gives a position around the circle of fifths. Note that tritones cancel out on both of these balances (meaning that adding a note a tritone away from another note has the same effect as removing that note). On the other hand, major seconds have the same canceling effect on the 3^{rd} component, but are actually a reinforcement on the 5^{th} component.

(10) Example: A minor is close to C major. (Small change in phases).

(13) We interpret the phases of components 3 and 5 as axes in a continuous space on chords. Each dimension is cyclical, so that the graph is actually a flattened torus.

This slide begins by plotting major and minor triads in this space, and then adds other consonant intervals and singletons. The resulting layout reflects the *Tonnetz*, which can be shown by drawing triangular regions around each consonant triad with a singleton at each vertex and a consonant interval in the center of each side. (One could also draw the dual Tonnetz, by connecting each triad to its nearest three neighbors. The consonant intervals would again be in the center of each edge, but the single pcs would be at the center of hexagonal regions.)

Note that many set classes share positions in the space. As indicated in the caption, diminished triads coincide with the position of their thirds (as singletons), dominant and half-diminished sevenths coincide with their constituent fifths, and diatonic scales coincide with their $\hat{1}$–$\hat{3}$ major third. In many cases, the cancelation of tritones explains these coincidences. Not shown are harmonic minor scales, which coincide with their tonic minor triads, and melodic minor scales, which coincide with their ^5 as singleton (or $\hat{1}$ of the acoustic scale).

All of these sets would be disambiguated if other components, such as the 4th component, were included.

II. Mediants in Schubert: Example: Menuetto from the "Rosamunde" Quartet, D.804

(18) This *Menuetto* nicely illustrates Schubert's use of mediant relationships. Note how, at the beginning, the key is A minor but the main harmony is the dominant, E major.

(19) Although the contrasting middle section sets up A minor, the beginning of the recapitulation is recomposed in C♯ minor.

(20) A traditional Tonnetz shows relationships between major and minor triads, but cannot easily assimilate other chords such as dominant sevenths. Any chord can be plotted in Fourier phase space, however, so that the path of Schubert's *Menuetto* can be shown using the E dominant seventh as a representative of the main theme.

(The circle denotes the starting point.)

Notice that Schubert's tonal plan involves an *enharmonic tour*, cycling the space. In mathematical terminology, the enharmonic tour is a kind of *homotopy equivalence class*.

(21) Another possible approach is to equate each tonal region with its characteristic scale. Note that the enharmonic tour is preserved.

(22) One could argue that neither of the first two methods presents a complete picture, because both chord and scale are significant. The third plot treats each tonal area as

a combination of chord and scale by adding together the pcsets of the chord and scale. This is possible because pc-multisets can be plotted in the same way as ordinary pcsets.

(23) The third graph shows that Schubert's unusual recapitulation can be seen as a mediant-based version of the subdominant recapitulation principle.

III. *Tonnetz* Regions vs. Weber Regions. Ex.: Schubert's Late B♭ major Piano Sonata, D.960

(25) Richard Cohn discusses the hexatonic pole relationship between B♭ major and F♯ minor in the exposition of this movement (between the main theme and beginning of the second theme). In the recapitulation (shown here), the same relationship appears within the main theme.

(26) Schubert returns to Bb major by reinterpreting the relative of F♯ minor (A major) as a dominant and resolving it deceptively (reinterpreting the resulting B♭ major chord as tonic).

(27) This is another enharmonic tour.

The regions, which are drawn here to reflect the *Tonnetz*, influence our perception of distance in the space.

(28) This different way of drawing regions better reflects traditional tonal relationships. These regions combine each triad with its dominant seventh (which borders two parallel regions). The diatonic scales are on vertices of the parallelograms, coincident with the major third that falls between the tonic chords of that diatonic's major and natural minor modes. (Harmonic minor scales are within the minor regions, coinciding with their tonic triads. The same is true of harmonic majors. Melodic minors are on a node between the regions of their tonic, subdominant, and their parallels).

The resulting regions reflect Gottfried Weber's chart of key relationships (i.e., by taking the dual network). (The same chart is used by Schoenberg and others.)

The zig-zag pattern reflects a fundamental difference between traditional major and minor keys: major keys are strongly localized along the c5 (circle of fifths) axis. Minor keys require more of a spread along the circle of fifths, and therefore are more localized along the c3 (circle of major thirds) axis.

(31) The different regions give different impressions of distance.

(32) The different analytical perspectives can be thought of as different overlays onto the same set of musical relationships, which are well-represented by the two different divisions of the space into tonal regions.

IV: Compositional Techniques (1): Modulation by scalar common tone, Exx.: C major Quintet, Scherzo–Trio and C major Quintet, Adagio (MT–IT)

(35) We have already seen how the possibility of plotting larger cardinality chords and multisets expands the possible analytical applications of a *Tonnetz* conceived as a continuous geometry using Fourier decompositions. The same can be said for *smaller* cardinality sets, including singletons.

In the String Quintet, the C major Scherzo transitions into the D♭ major Trio by isolating a common tone between the two scales (N.B., not between their tonic chords!).

Notice that between the isolation of the common tone and the arrival at D♭ major there is a fleeting tonicization of F minor, then a full D♭ diatonic scale down to G♭, where the D♭ major area begins on its subdominant chord.

(36) The first chord to appear in the D♭ major tonality is actually the subdominant, G♭ major, which is a tritone away from the previous tonic chord. This chord is maximally distant in all four directions from C major, so the mediation of the common tone and the fleeting F minor tonicization is significant in moving us through the space in an unambiguously directed fashion.

(37) Another dramatic example occurs in the second movement of the same work, in the transition between a languid main theme in E major and a turbulent F minor interior theme.

(38) The paths are distinct in the sense of homotomy.

IV: Compositional Techniques (2): Transgression of modal boundaries, Exx.: C major Quintet, Adagio (MT) and Late C minor Piano Sonata, Andante

(40) In the Adagio Schubert skates across the E major–E minor boundary inconspicuously and irresistibly, without ever strongly suggesting a minor key on the surface. Although this phrase returns to E major, the consequences of the transgression are realized when the F minor interior theme later appears. (This modulation was shown in the previous example.)

(42) By “sets up” I mean a kind of “action at a distance” which might be compared to Edward Cone’s (1982) “promissory notes.” Cone’s promissory notes in Schubert are unresolved chords, suggesting a harmonic area without fully realizing it. Here it is a harmonic invocation of the parallel minor, which doesn’t fully realize the implications of mode change. Cone’s idea of a promissory note is curious in that it seems to ask us to imagine a long-range resolution, even though that resolution is clearly not structural. However, the idea makes sense when considered spatially: it is “priming” a region of the space, by moving to it briefly without fully committing to

it. Such a priming might help create a feeling of inevitability to a later exploration of that region.

(43) The excerpt from the Adagio of D.958 shows the modal transgression in the theme (a minor subdominant of the subdominant) and the first part of the subsequent episode, which takes a precipitously flatward enharmonic course.

(45) Here are the consequences of the modal transgression, an enharmonic tour in the first episode. The first part of this tour appears in the musical example two slides back (A♭ major to D♭ minor and then to A major). The subsequent liquidation, or developmental passage, which completes the tour, does not.

(46) The episode following this passage begins the same way as the first episode, but in D minor rather than D♭ minor.

(47) The illustration in phase space shows that the minor-key V-i progression (or mode mixture I-iv) goes in the same direction as the "L" progression. Cohn (2011) discusses the common sequential progression that results from combining these, which he calls the N-L pattern. Phase space reveals an interesting feature of this pattern, its unidirectionality—i.e., if you connect a major triad to its Neapolitan, the minor subdominant of the first triad falls almost directly on this line. Amiot (2013) points out some interesting consequences of this serendipitous mathematical fact.

The third and final appearance of the main theme in the adagio extends the N-L sequence by another step.

Appendix A1: Functional categories in c4/c5 space

(53) We can also plot chords using the fourth Fourier component instead of the third.

(54) The phase of the fourth Fourier component segregates chords according to harmonic function in a given tonal context. In this example, F major is chosen as tonic, and regions are drawn with dashed lines. In the middle, around the tonic, are other tonic functioning and tonic substitute chords, as well as applied chords to predominants. To the right of the tonic chords, and wrapping around to the far left, are predominant chords, and to the left of the tonic chords are dominant functioning chords. (Note the inclusion of vii♯5 and ♭VII in the dominant-functioning region, validating a claim made by Damschroder (2010,*Gamut*) for these as substitute dominants in his work on Schubert.)

Appendix A2: A *Tonnetz* on dominant and half-diminished sevenths

(55) The c4/c5 plot distinguishes dominant and half-diminished seventh chords, which are on top of one another in the c3/c5 plots (where dominant and half-diminished sevenths are in the same position as their constituent perfect fifth). This means we can use the c4/c5 plot to draw a seventh-chord *Tonnetz* analogous to the triadic *Tonnetz*. This way of generalizing the *Tonnetz* to four-note chords follows a different

logic than other proposals (e.g., those of Childs, Gollin, and Tymoczko). The three adjacencies of this *Tonnetz* are minimal voice-leadings, but not the only minimal voice leadings (there are many—the ones here are distinguished by c5 proximity, meaning they occur in the same or closely related keys). The three axial directions lead to sequences by minor third (vertical), major second (/), and perfect fifth (\).

(56) Relative voice-leading parsimony is very common for dominant and half-diminished sevenths, so it is not in itself a sufficient criterion for something like a *Tonnetz*. (The minimal voice-leading distance is two semitones, but there are more *Wechsels* with two semitone voice leadings than there are with larger ones.) Nonetheless, the seven two-semitone *Wechsels* are also the smallest distances in c4/c5 space.

(57) Here are examples of some well-known Wagner progressions involving dominant and half-diminished sevenths (discussed by Lewin (1996) and Douthett and Steinbach (1998)) . The progressions tend to involve one larger motion (augmented-sixth like, creating a large circle-of-fifths distance) and one simpler one.

(58) The Immolation progression is a reflection of the Tristan progression on the *Tonnetz*, even though they are not, e.g., inversionally related.

(59) The Amfortas progression also looks similar, a rotation on the *Tonnetz*.

(60) Position of triads in the space. They fall on the border between the dominant and half-diminished sevenths that share three common tones ("R*"-related), closest to the seventh chord that contains them (the triad given by removing the seventh of a dominant chord or the root of a half-diminished one).

Appendix A3: Phase space and voice leading

(61) A similar *Tonnetz*-like geometry can also be derived from voice-leading considerations. The figure on the right is from an article forthcoming in *Journal of Mathematics and Music* 7.3, but the principle upon which it's based (voice-leading quantization) can also be found an article in the current issue of *Journal of Mathematics and Music* (7.2) as well as a forthcoming article in *Music Theory Spectrum*. Both are continuous geometries, each is a torus with an enharmonic cycle and a perpendicular "diatonic" cycle, and the layout is essentially the same if we focus on *triads only*. The underlying principles are different however, which changes the interpretation of the space between the triads. In the voice-leading geometry, the space traversed, e.g., in an R progression includes all the chords that occur as one pitch slides continuously to the other (hence the gray augmented-triad regions). In the Fourier phase space, this traversal consists of continuous changes in the quantities of the pcs present. For example, an R progression from F maj → D min can be traversed by continuously fading out C while fading in the D (or doing these two things one after the other). This explains why the midpoint of every progression

is the position of set consisting just of the common tones for that progression. For instance, halfway between F maj. and D min. is the major third F-A.

An important advantage of the Fourier approach is that it is not restricted to one cardinality at a time. In voice leading spaces, it is not possible represent the relationship between, e.g., a triad and a seventh chord without specifying a doubling for the triad.

Each of these systems has its virtues, but it is important to recognize that one is not more or less abstract than the other. Both systems, when applied to music, envision a harmonic progression as a kind of imaginary continuo (to use Rothstein's term). Voice leading geometry sees the imaginary continuo as consisting of a *fixed number of pitch classes* and continuous motion as a kind of glissando in one or more of these voices. Fourier phase space instead views the imaginary continuo as a kind of sound board with twelve sliders, one for each pitch class. Continuous motion in the space involves gradually "fading in" some pcs while fading out others. Both abstractions involve mapping real music onto a highly circumscribed controlled universe, which is supposed to represent the essence of the harmonic progression.

(62) Here's a picture. The Fourier analysis is based on a model where the pitch classes are fixed but their presence or absence can be adjusted along a continuum. The voice leading model treats the number of voices as fixed, and their pitch classes as continuously variable. (However, note also that the discrete Fourier transform is usually considered a sampling of a continuum, so it is robust with respect to variable intonation, etc., and other sampling rates are possible in principle).

(63–66) Although relationships between dyads, seventh chords, and other non-triadic pcsets and multisets is not directly a voice-leading relationship, it is indirectly related to voice-leading. Specifically, these sets are arranged horizontally on the basis of ascending, descending, or balanced voice-leading on their perfectly-even three-note image. (The reason it is "three-note" is that this dimension is associated with component 3). That image is determined by assuming that the pcset represents a relatively even three note chord in which some notes may be omitted, others may be added or doubled. So, roughly speaking, if one moves from left to right between consonant dyads in the space, that indicates a descending voice leading between the triads of which those dyads might be incomplete representatives. Etc. Note that (somewhat confusingly) a progression that ascends by fifths and goes rightward in the space is actually associated with *descending*, not ascending, voice leading.

(67–70) This is the example that convinced me that the DFT procedure may actually *better* reflect musical intuitions about voice leading than voice-leading spaces. Even though this sequence is literally a progression on three-note chords (two of which are seventh chords with omitted fifth), it is impossible to plot it in three-note chord space without implying certain large voice-leading moves that do not accurately reflect the music. That's because at each repetition of the sequence, the suspension is introduced by splitting one voice into the two that make the suspension, and allowing one of the other voices to disappear. The DFT procedure does this essentially automatically, because it assumes that the underlying three-note voice leading is on chords that are relatively even. A chord like {G, D, Eb} is therefore close to G minor and Eb major (which it would not be in a voice-leading space). This means that other progressions that we would tend to assert as models for this sequence (as the triadic and seventh-chord/triad progressions on slides 69–70) trace a similar path in the space.

Appendix A4: Key profiles and phase space

(71) One familiar application of the quantity-of-pc way of thinking is the kind of statistical distribution used in key profiles (e.g. Krumhansl-Kessler 1982, Krumhansl 1990, Temperley 2007). On the left is Krumhansl and Kessler's multi-dimensional scaling solution for key relatedness based on the correlations of the experimentally derived major and minor key profiles. In these key profiles, "quantities of pc" are average goodness-of-fit ratings for the given pc in the given context. They also have been interpreted as *statistical likelihood* of a note appearing in a given tonal context, or as the distribution of pcs in an actual musical passage or experimental stimulus (Temperley 2007, Temperley and Marvin 2008).

(72) Krumhansl and Kessler's key-correlation space is essentially the same as the c4/c5 harmonic phase space, up to an arbitrary 90-degree rotation and reflection, and a normalization of the parameters.