Geometrical Realizations of Two- and Three-Dimensional Generalized *Tonnetze*

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Outline

(1) Two-Dimensional Generalized *Tonnetze*

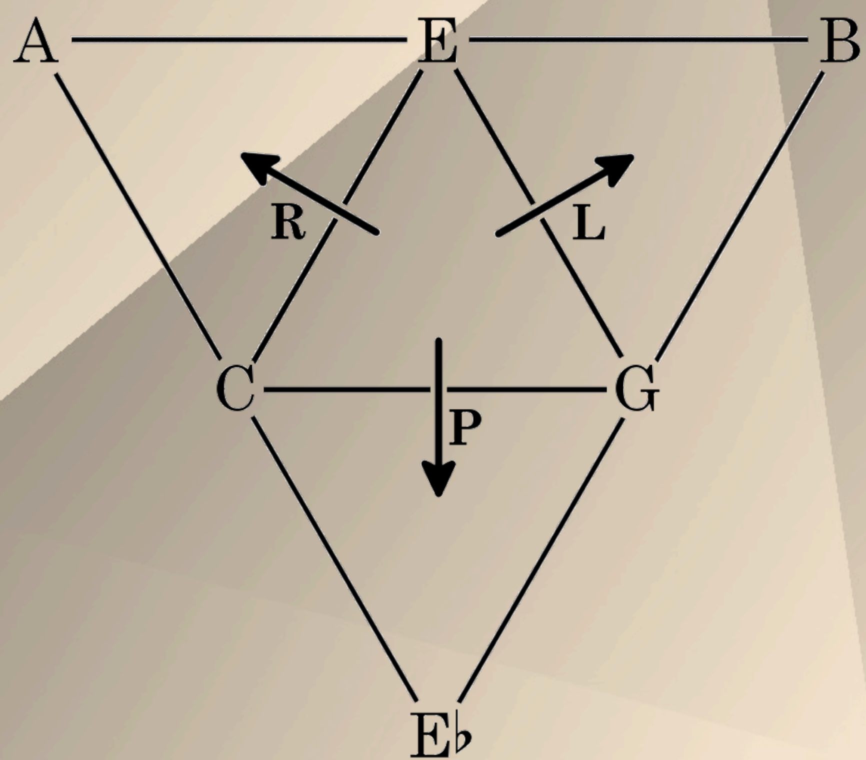
(2) Optimizing Spaces and Intervallic Duplications

(3) Examples of Music Analysis with Non-Triadic *Tonnetze*

(4) Three-Dimensional (tetrachordal) *Tonnetze*

(1) Two-Dimensional Generalized *Tonnetze*

Triadic *Tonnetz*



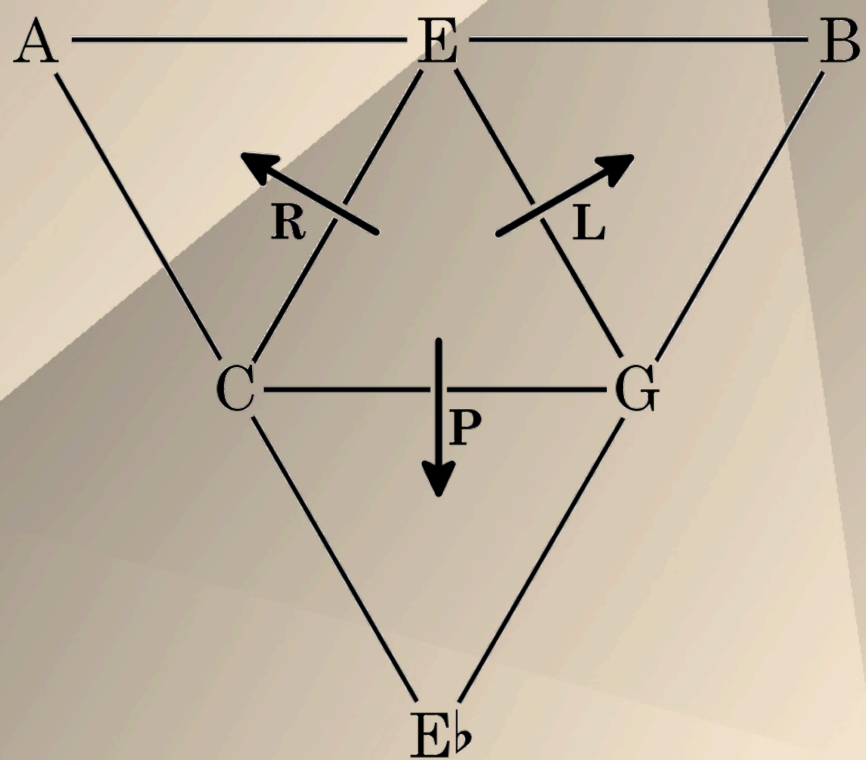
The triadic *Tonnetz* relates triads via

• Voice-leading efficiency and

• Common-tone retention

Tymoczko (*JMT* 2012) generalizes voice-leading properties of the triadic *Tonnetz*

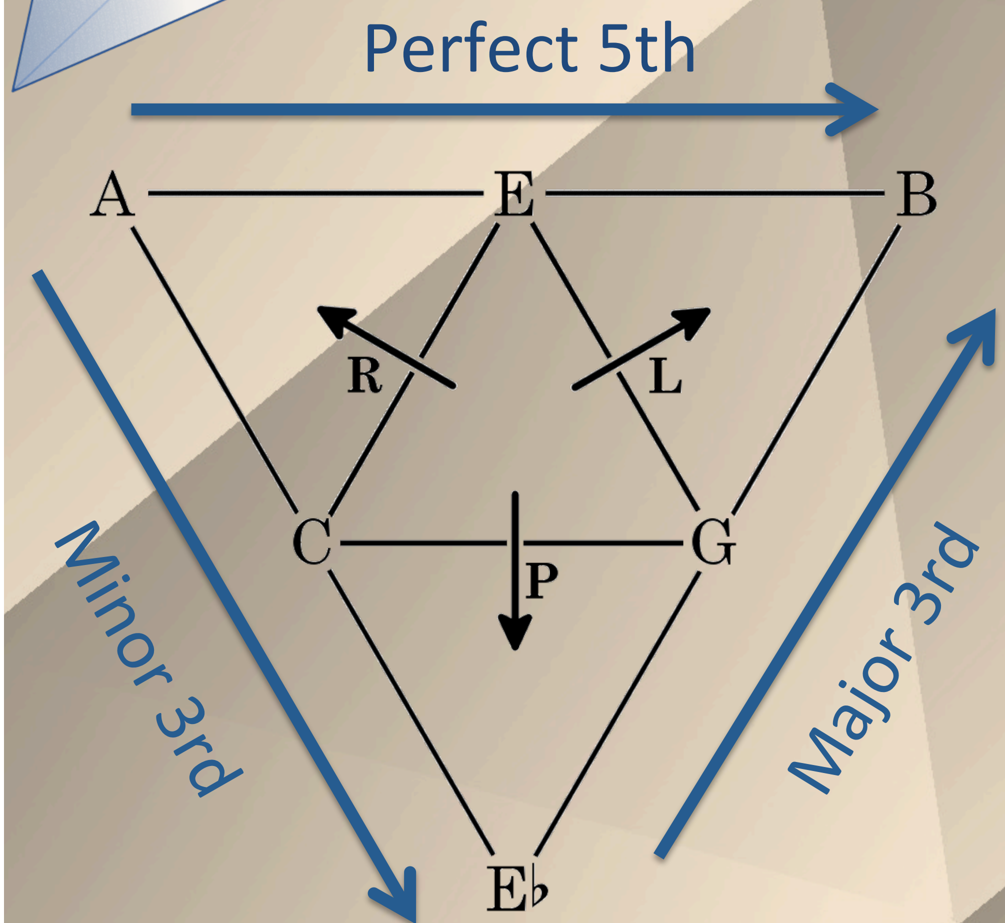
Triadic *Tonnetz*



Cohn (*JMT* 1997) generalizes the common-tone aspect of the triadic *Tonnetz.*

Previous approaches (Cantazaro *JMM* 2011, Bigo et al. *MCM* 2013, Bigo *CMJ* 2015) have treated it as a *network* or *topology* rather than a *geometry*

Triadic *Tonnetz*



Intervallic Axes
of the triadic *Tonnetz*

General 2-dimensional *Tonnetz*

Geometric stipulations:

- Each pc is represented by a single distinct point
- Finite number of pcs (universe $\mathbf{Z}_u$)
- Transposability: All transpositions (= translations) of $\mathbf{Z}_u$ correspond to rigid geometric transformations (e.g., translations or rotations).

General 2-dimensional *Tonnetz*

Definition of a *Tonnetz:*

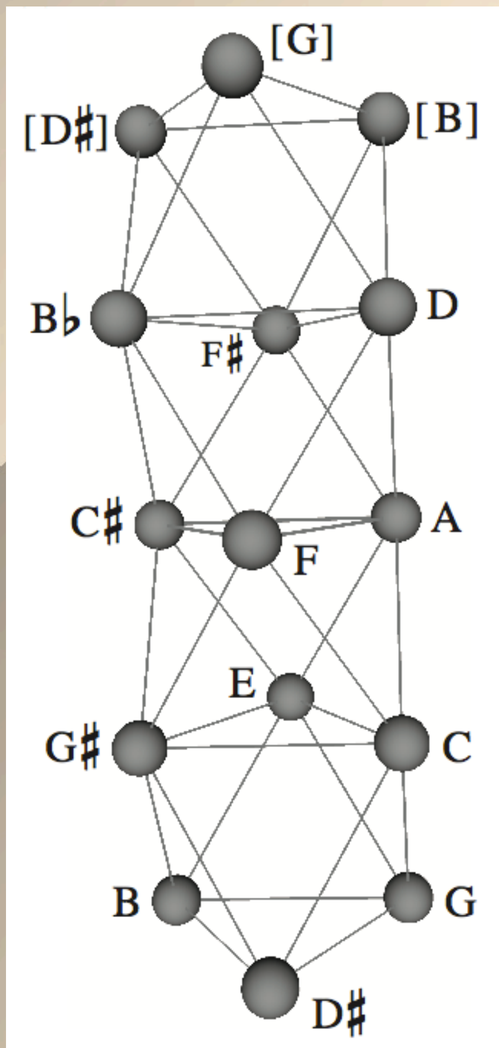
- Simplicial decomposition of the space
- Vertices of the simplicial regions include all and only the pc points.
- Transposablility

General 2-dimensional *Tonnetz*

All transpositions of $\mathbf{Z}_u$ are cyclic, therefore the transposability conditions lead naturally to toroidal geometries (by representing transpositions as geometric translations).

Other topologies (e.g., non-orientable) can be made to work in special cases by embedding in higher-dimensional spaces or by folding the torus to equate distinct axes.

Example: Tymoczko's triadic *Tonnetz*



- As a two-dimensional space (surface of the figure) it is toroidal. It satisfies transposability through translations.

- Embedded in a three-dimensional space it satisfies transposability through rotations and screw rotations.

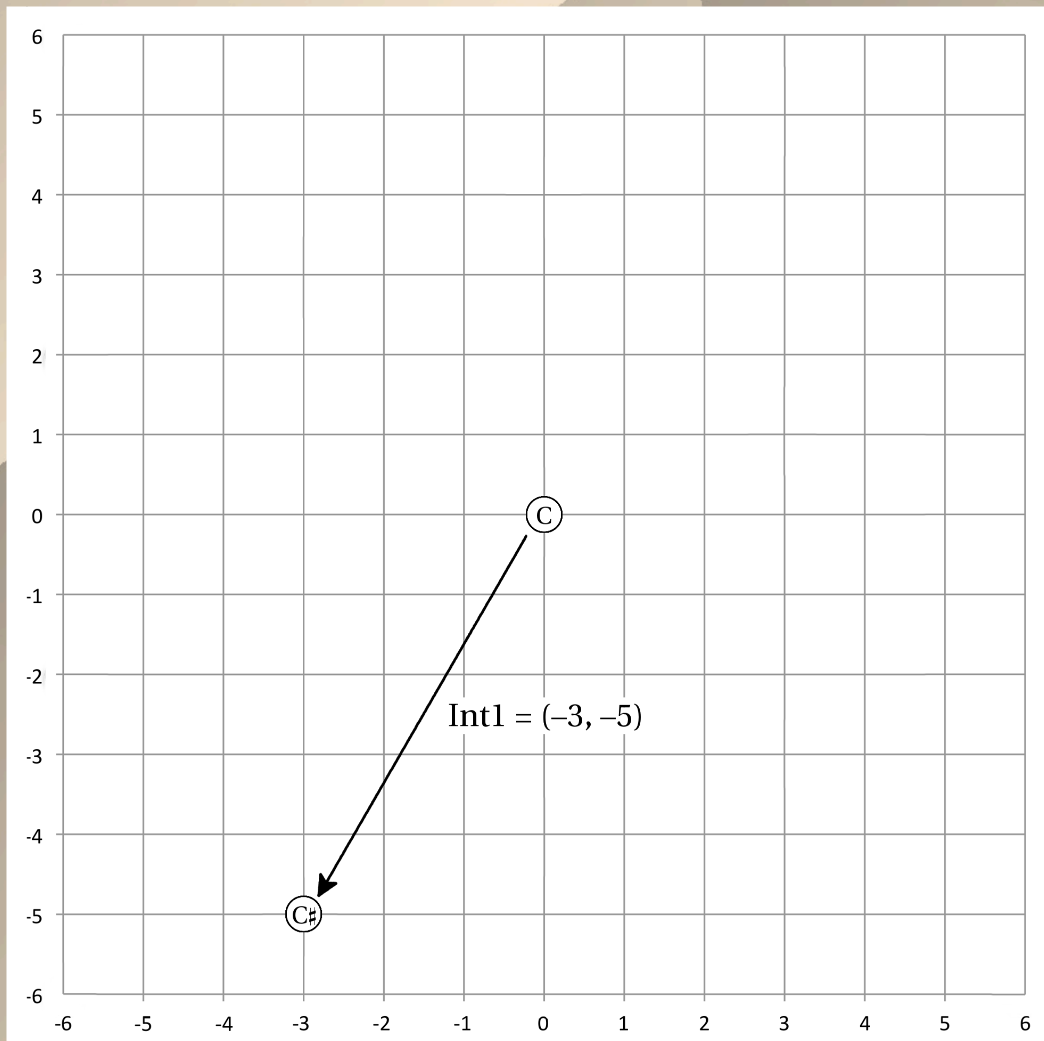
- For Tymoczko, the embedding three-dimensional space has an important role in the voice-leading intent of the figure, but to the transposability of vertices and the lattice it is superfluous.

General 2-dimensional *Tonnetz*

Construction of a two-dimensional *Tonnetz:*

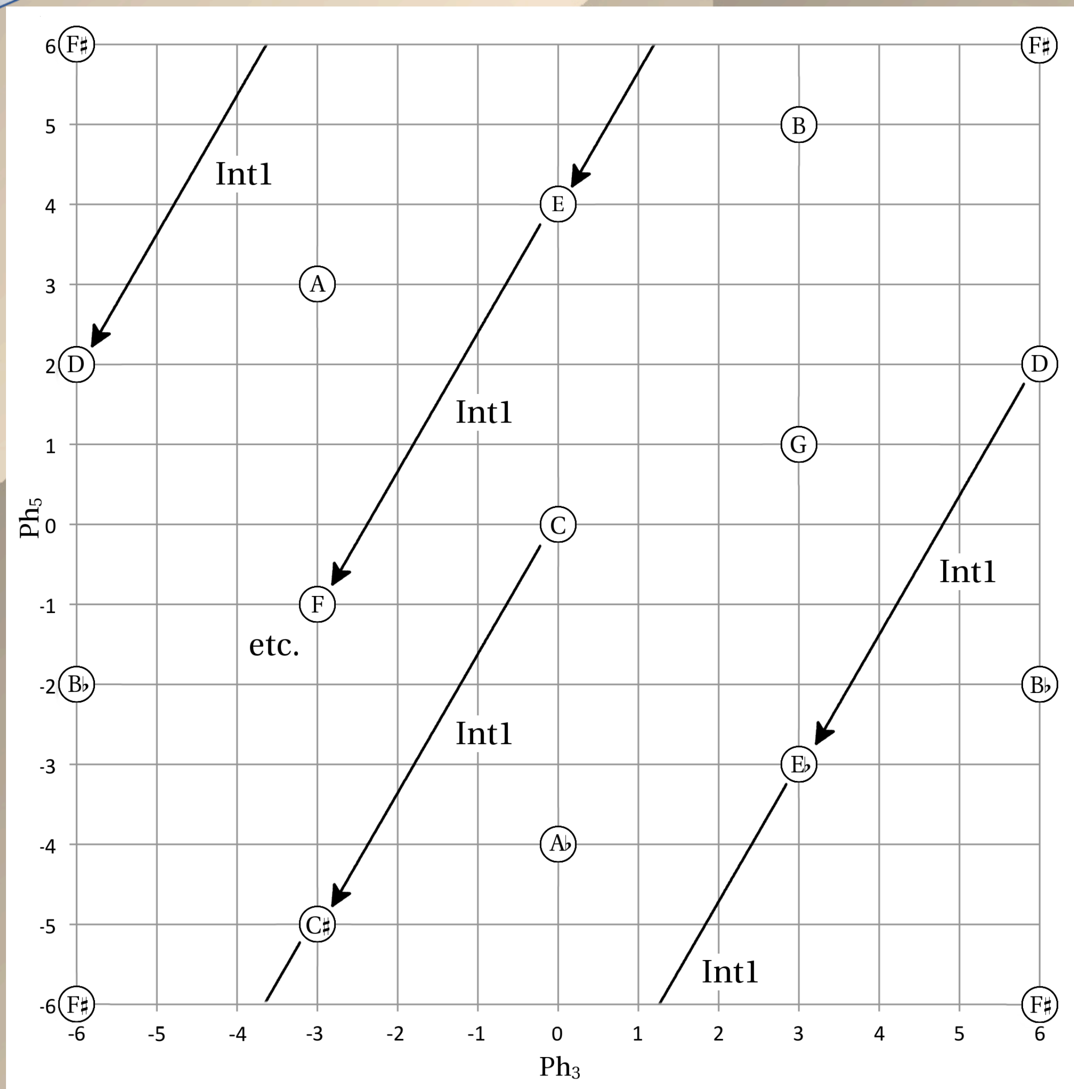
- Space is T_2 with dimensions scaled to $[0, u)$
- Choose (x, y) to represent interval 1 such that x, y, and u are mutually coprime integers.
- Choose two non-parallel lines through pc 0, add lines parallel to these through all pcs.
- Add parallel lines that bisect the resulting quadrilaterals.

Example



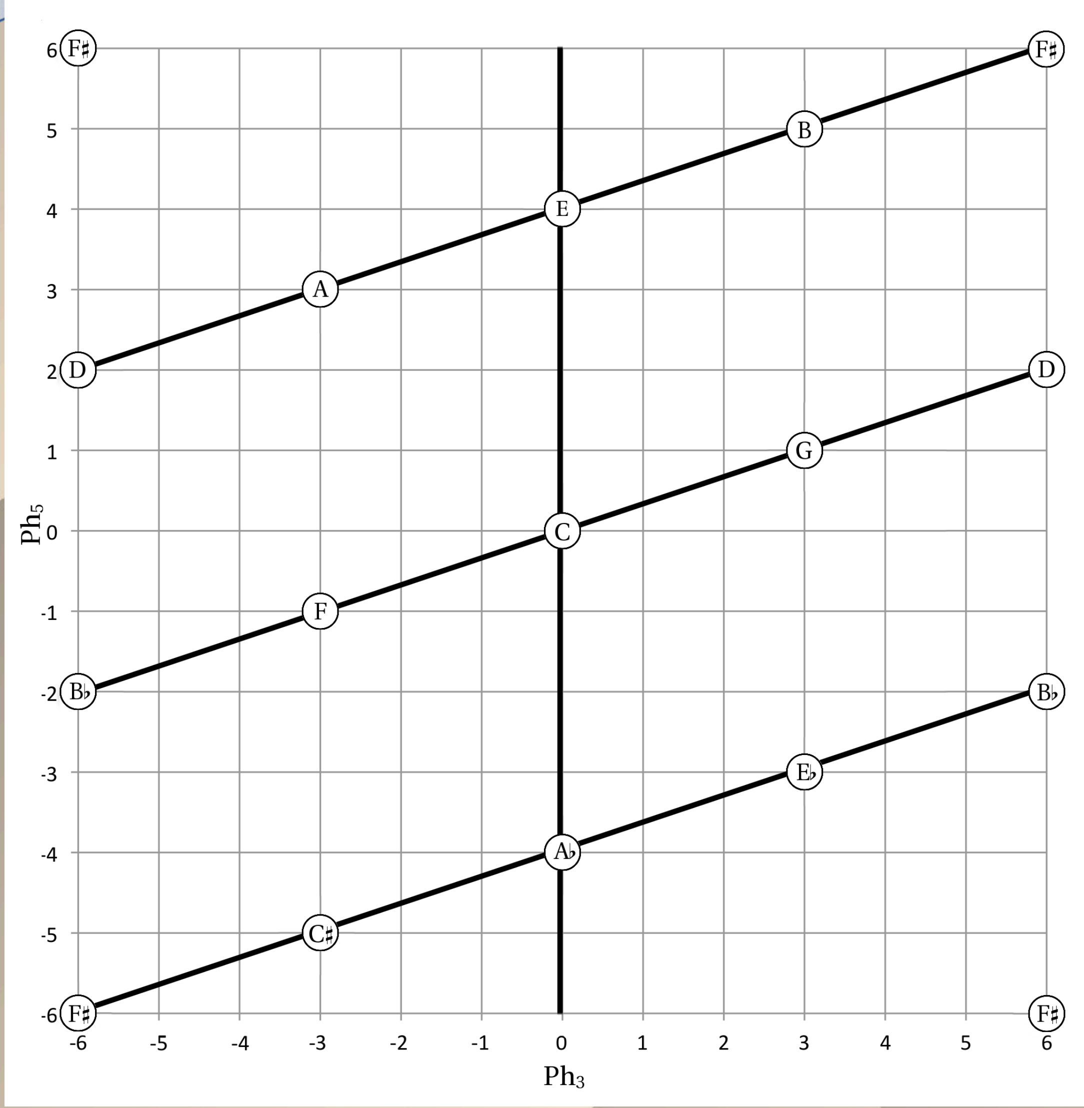
Define the semitone as the vector (–3, –5) in universe $u = 12$

Example



Place all pcs by reiterating this interval.

Example

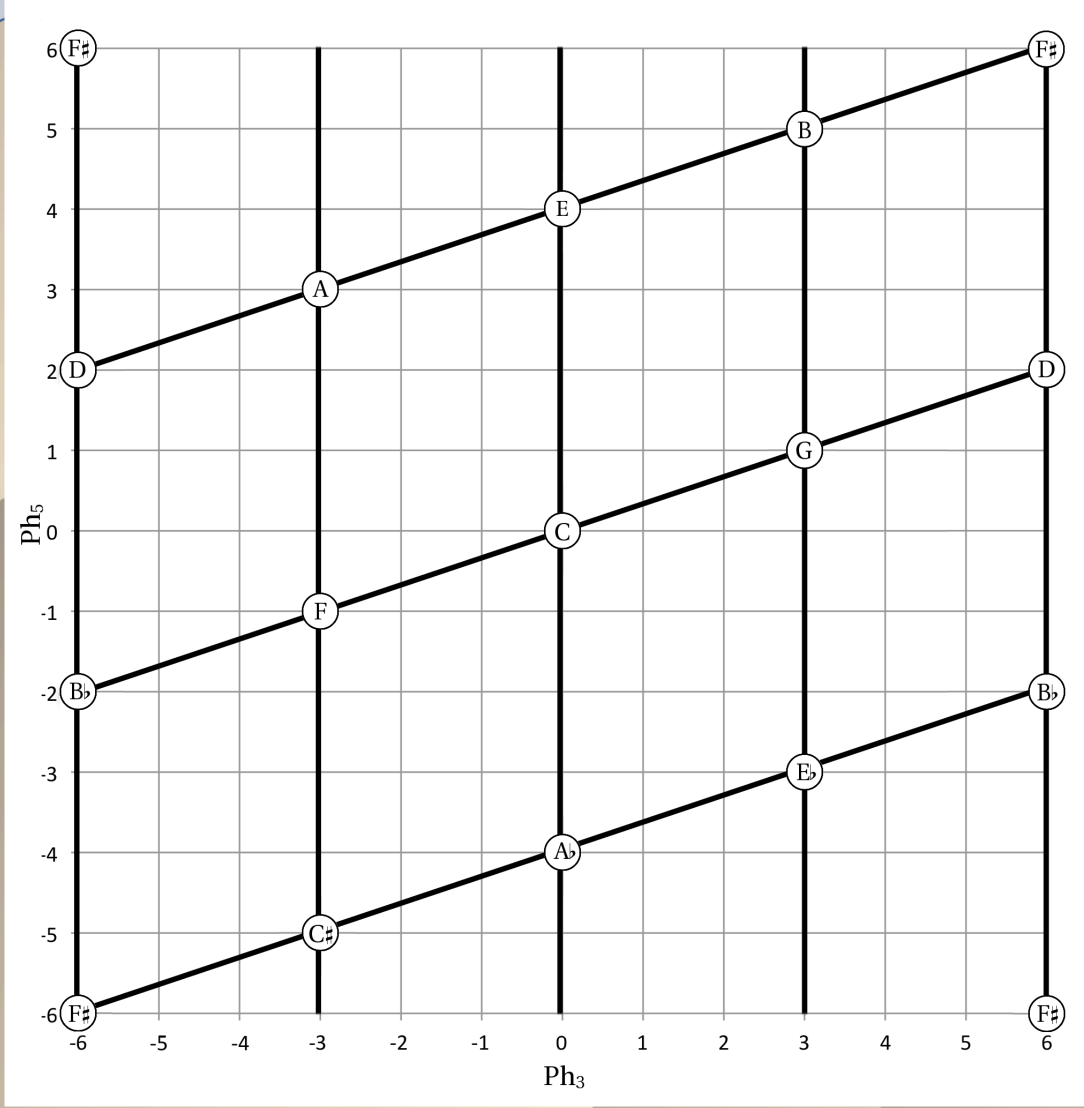


Choose two axes through C.

One corresponds to interval 5 or 7 and cycles through all pcs.

The other corresponds to interval 4 or 8 and goes through three pcs

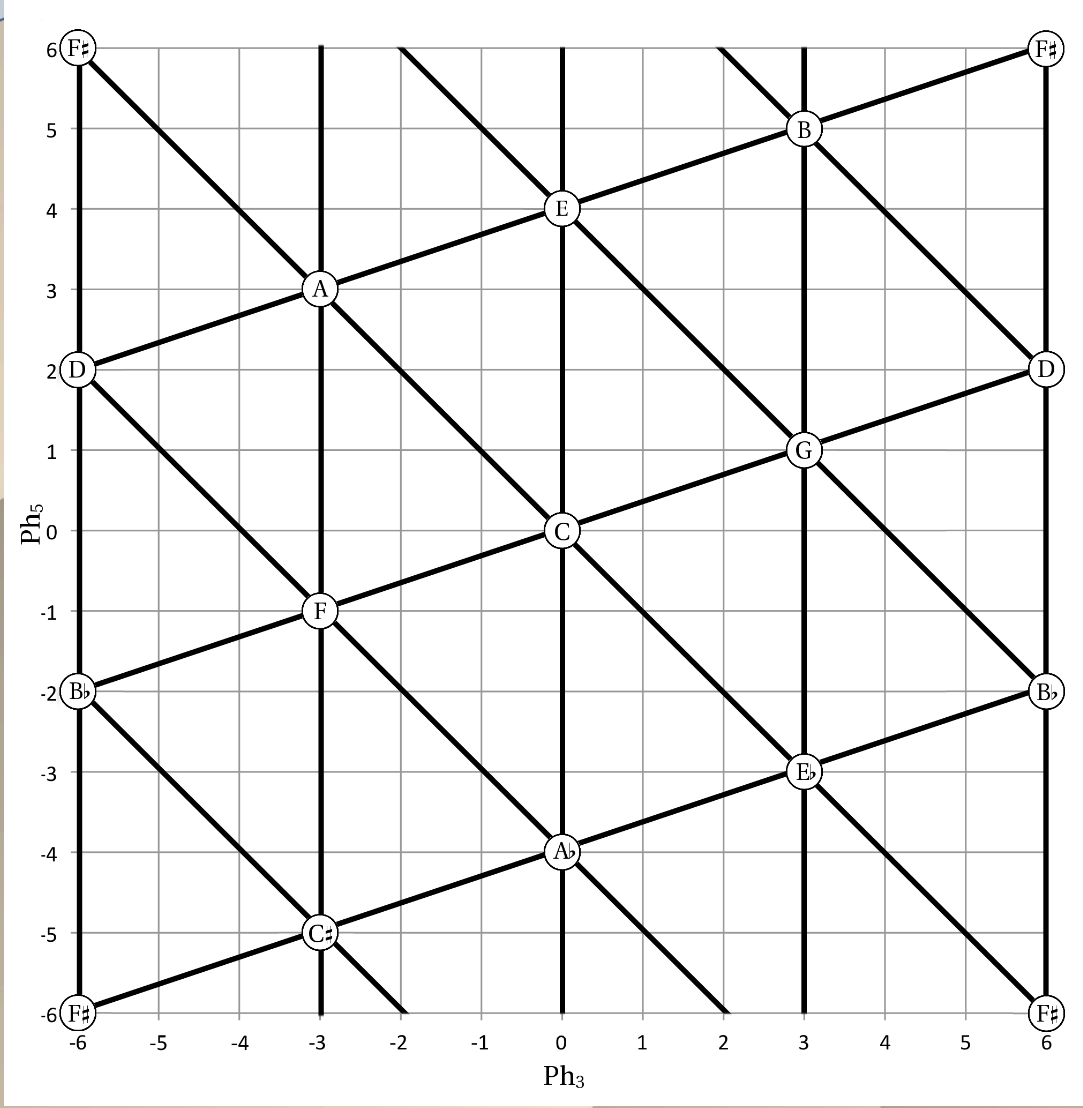
Example



Add lines parallel to the major-third axis to cover all pcs.

There are two ways to bisect the resulting parallelograms

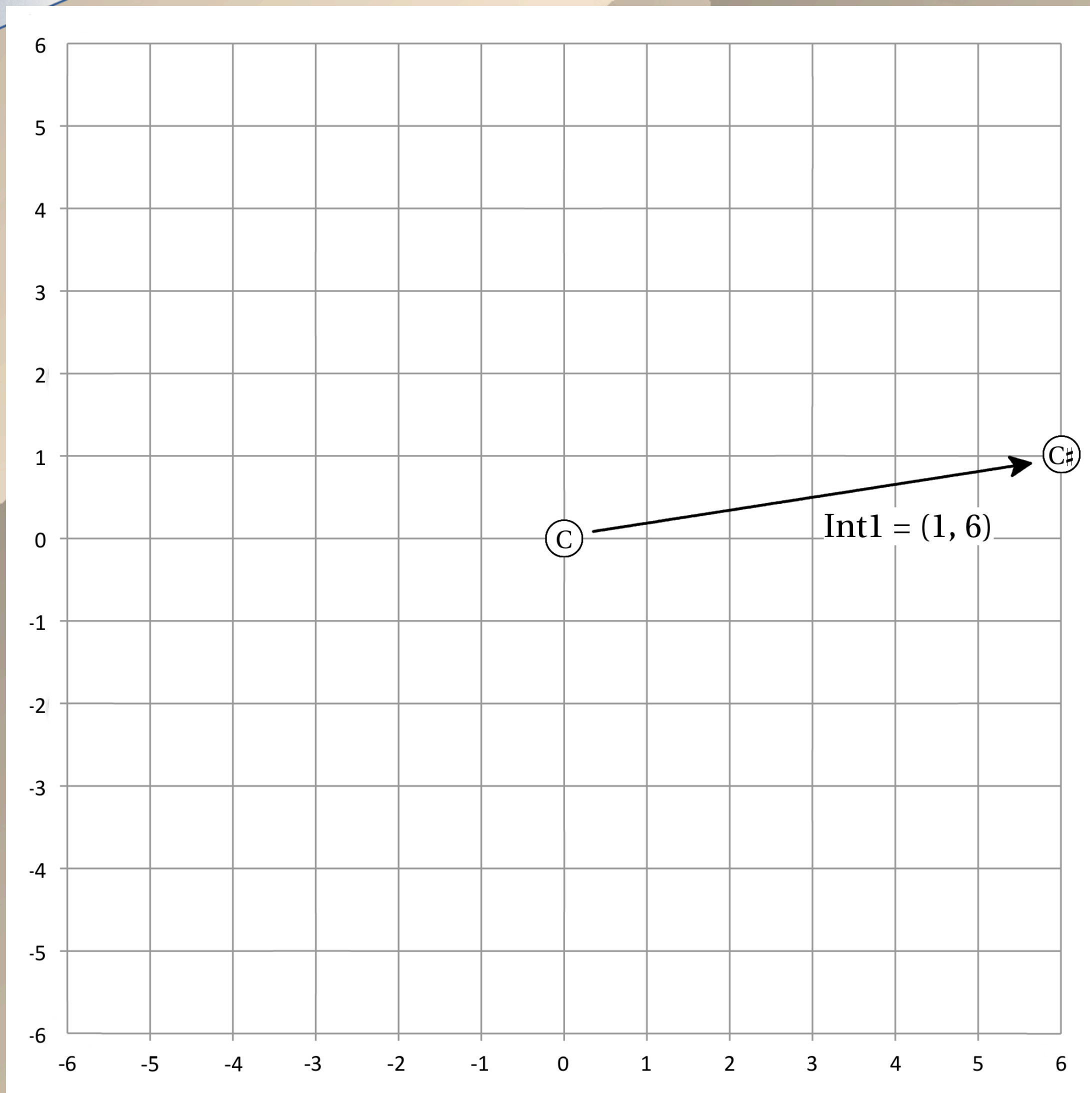
Example



Add a third set of parallel lines bisecting the parallelograms.

The choice of a minor-third axis results in a compact version of the standard triadic *Tonnetz*

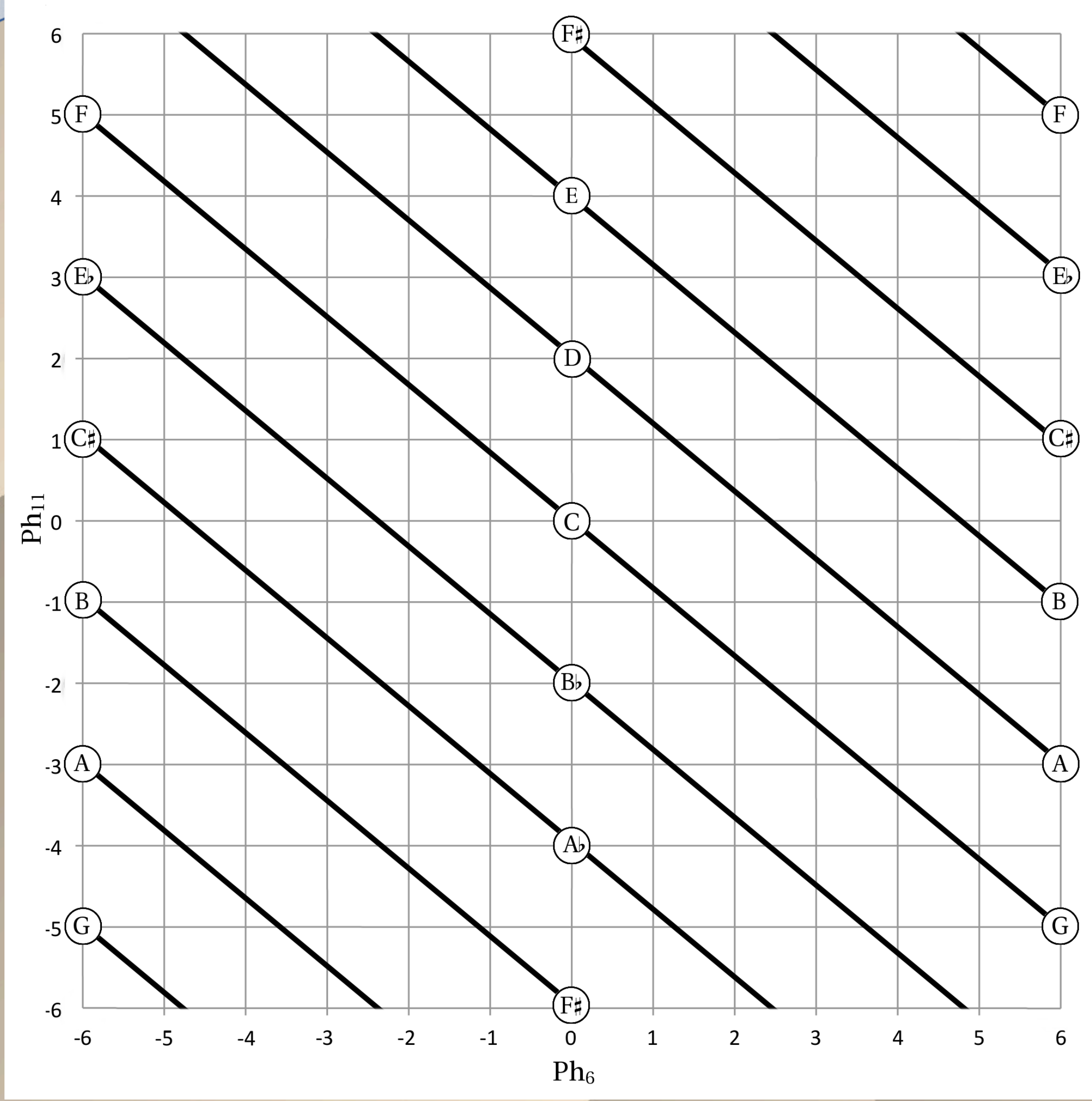
Example



What happens when a different choice is made for the representa-tive of interval 1?

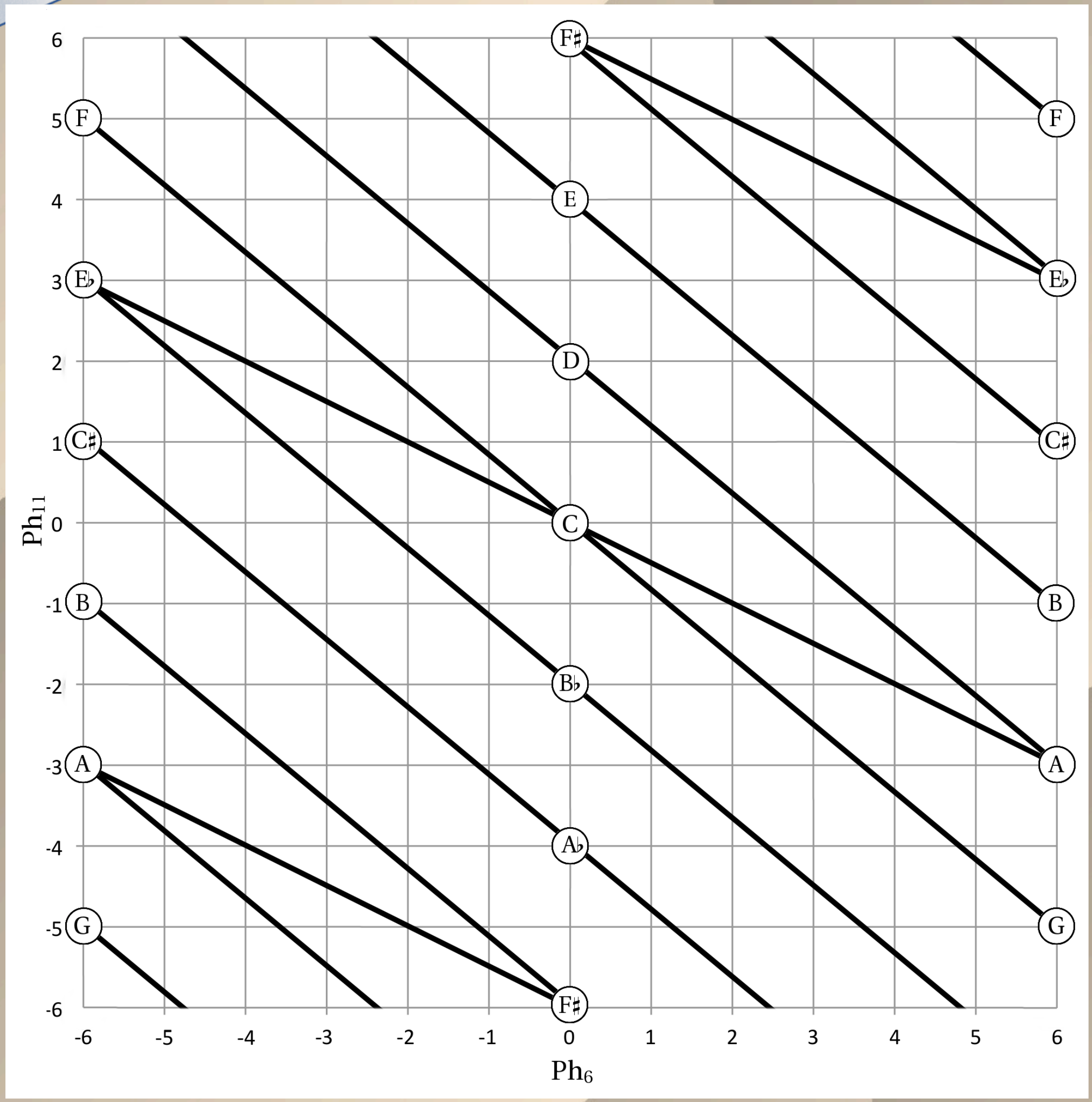
For example,
Let int1 = (1, 6).

Example



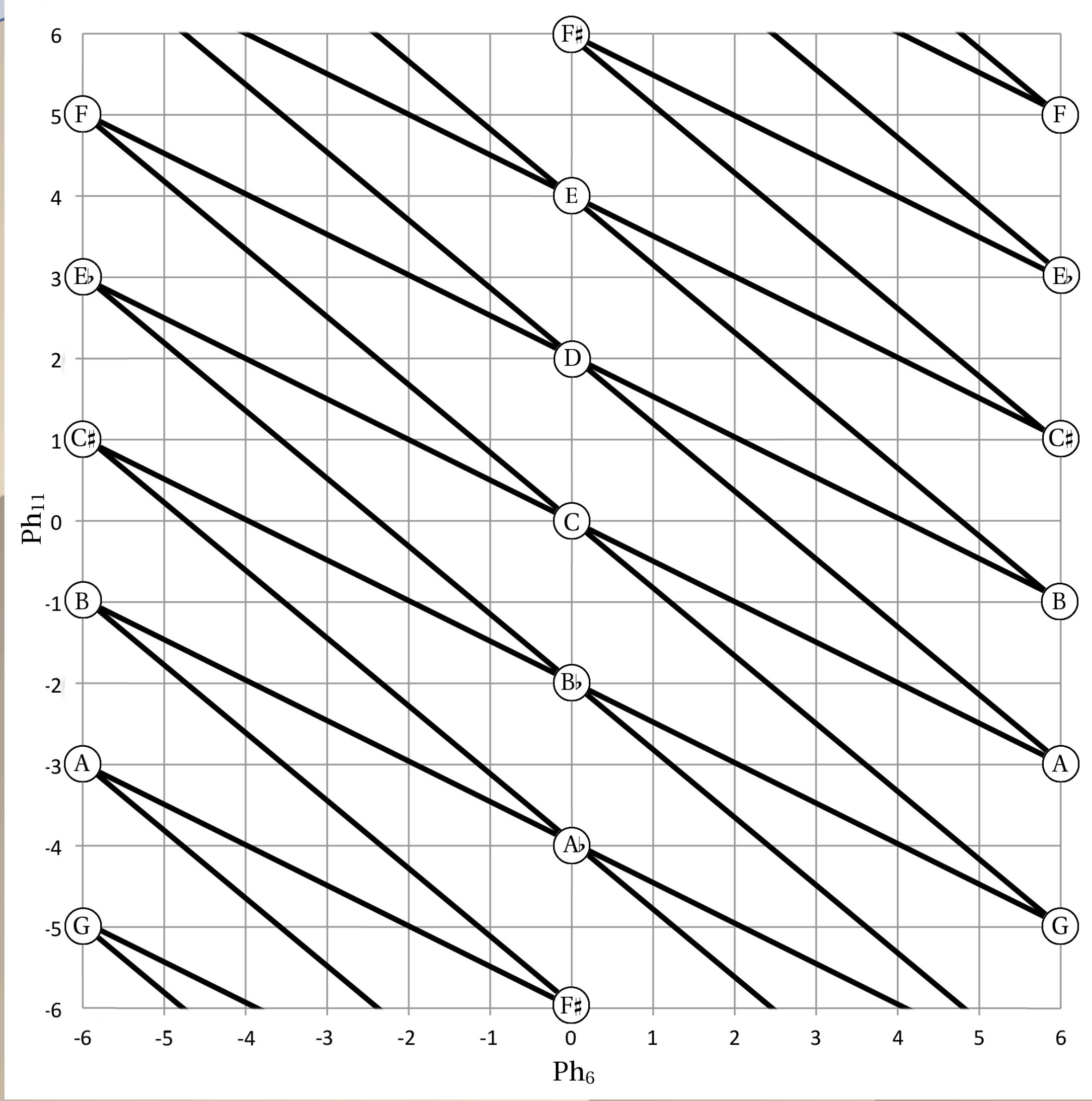
The fifths axis cycles through all pcs, but is also much longer here than in the other space.

Example



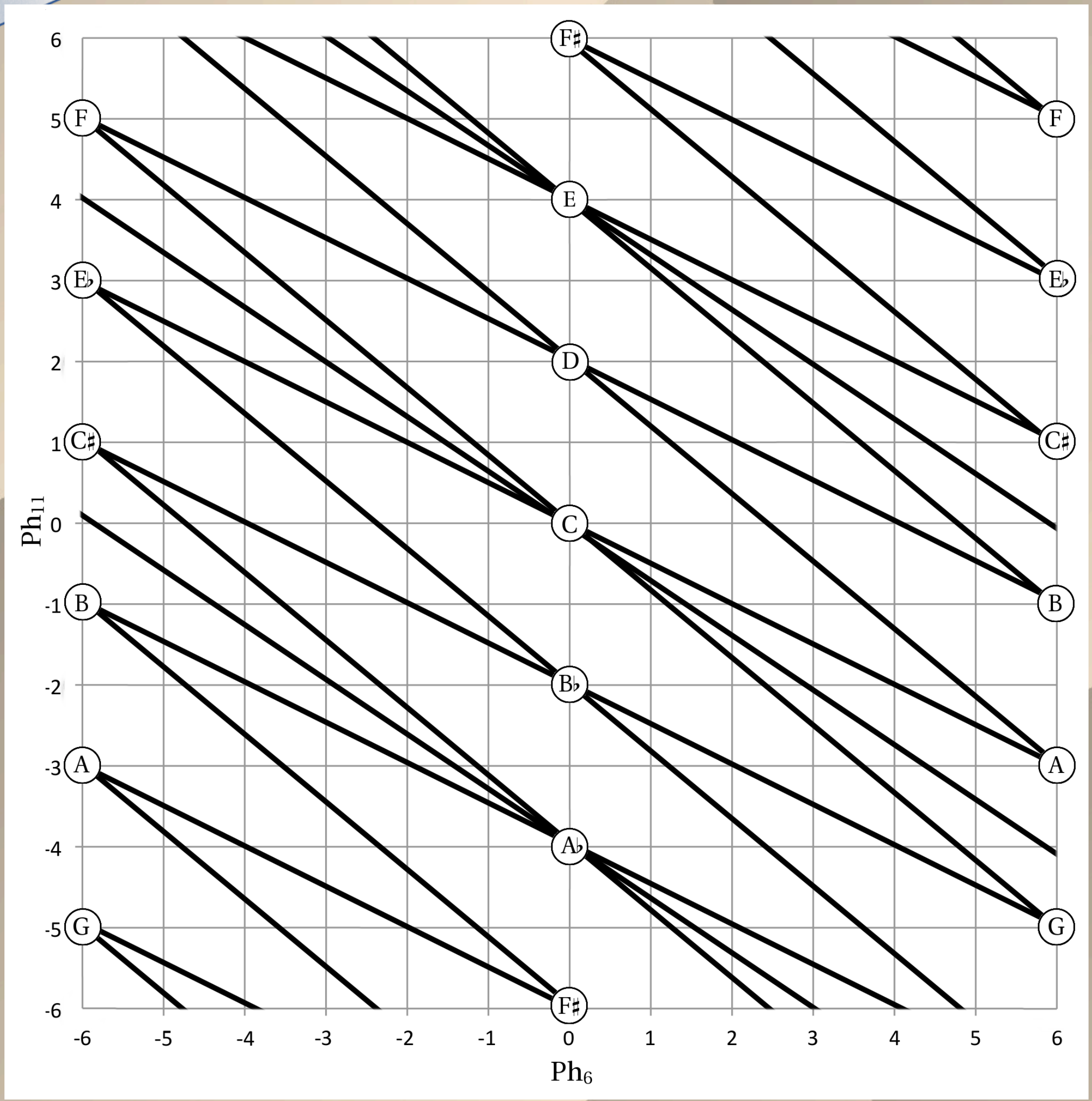
Add a minor-third axis.

Example



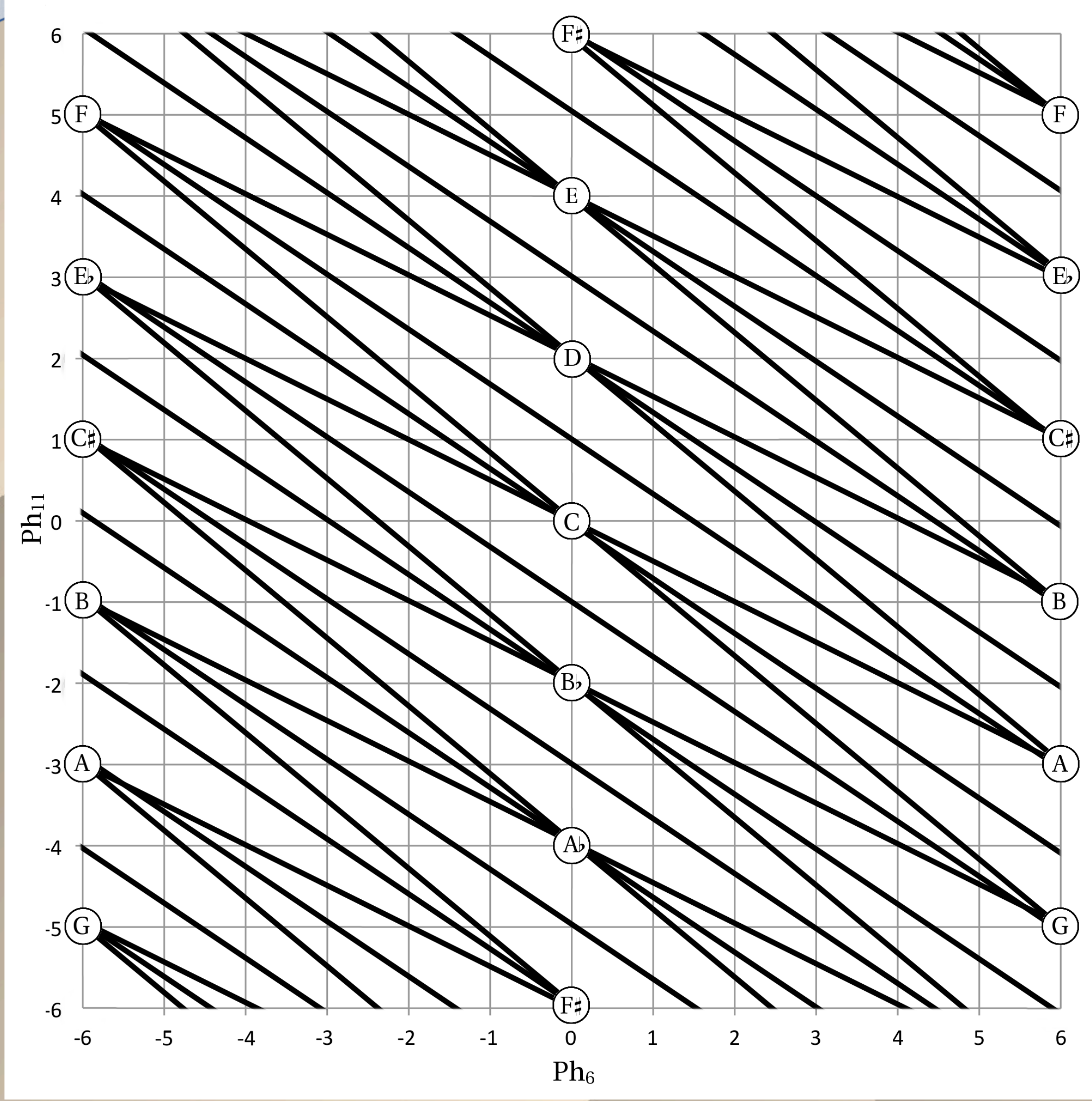
Add the other parallel minor-third axes.

Example



The resulting parallelograms can be bisected by a major-third axis.

Example



Add the other major-third axes.

This *Tonnetz* is the same as the previous one skewed. The fit of space to *Tonnetz* is poor, resulting in non-compact regions.

General 2-dimensional *Tonnetz*

Resulting space:

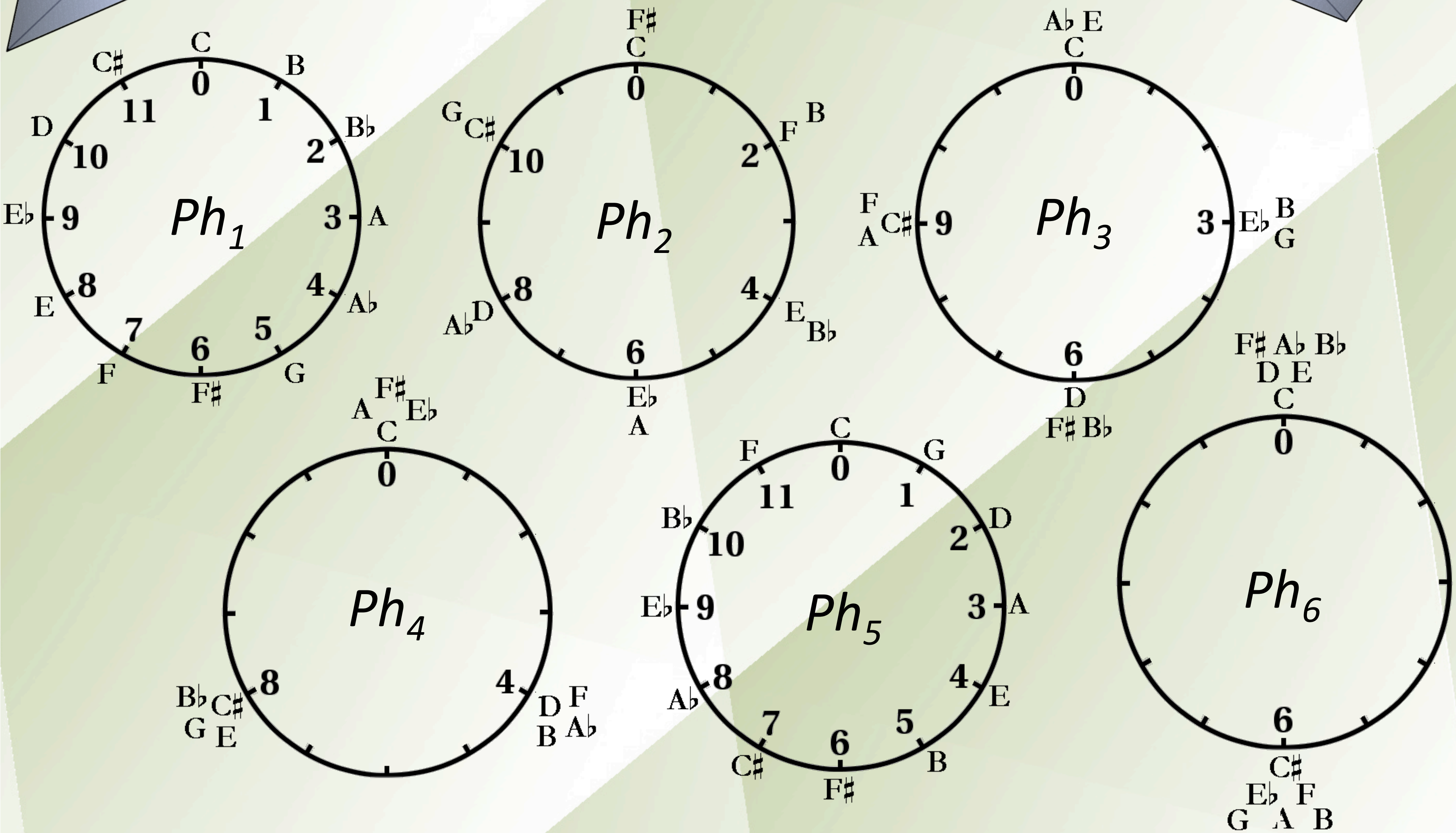
Equivalent to some *Fourier phase space* (Amiot *MCM* 2013, 2016; Yust *JMT* 2015, *JMT* 2016)
I.e., each dimension represents one of the interval cycles of universe *u*.

Triangulation into 2*u* regions, where triangles represent all instances of a single set class.

Any set class is possible (as long as *u* is its minimal embedding universe).

(2) Optimizing Spaces and *Tonnetze* with Intervallic Duplications

One-Dimensional Phase Spaces



Optimizing Spaces

Available unique two-dimensional spaces (u = 12):

Ph_1-Ph_1	Ph_1-Ph_2	Ph_1-Ph_3	Ph_1-Ph_4	Ph_1-Ph_5	Ph_1-Ph_6
~~Ph_2-Ph_2~~	Ph_2-Ph_3	~~Ph_2-Ph_4~~	Ph_2-Ph_5	~~Ph_2-Ph_6~~	
~~Ph_3-Ph_3~~	Ph_3-Ph_4	Ph_3-Ph_5	~~Ph_3-Ph_6~~		
~~Ph_4-Ph_4~~	Ph_4-Ph_5	~~Ph_4-Ph_6~~			
Ph_5-Ph_5	Ph_5-Ph_6				
~~Ph_6-Ph_6~~					

Optimizing Spaces

Possible optimizing criteria for a given trichord type:

- Minimize total length of intervallic axes
- Maximize the Fourier coefficients of trichord

These generally agree, with the second method being more sensitive

Note: Trichords with duplicated intervals are special cases!

Optimizing Spaces

Trichord	Best space(s)	Other
(012)**	Ph_1-Ph_6	
(013)	Ph_1-Ph_4	Ph_1-Ph_5
(014)	Ph_1-Ph_3	Ph_3-Ph_4
(015)	Ph_2-Ph_3	
(016)	Ph_1-Ph_2, Ph_2-Ph_3, Ph_2-Ph_5	
(025)	Ph_4-Ph_5	Ph_1-Ph_5
(027)**	Ph_5-Ph_6	
(037)	Ph_3-Ph_5	Ph_3-Ph_4

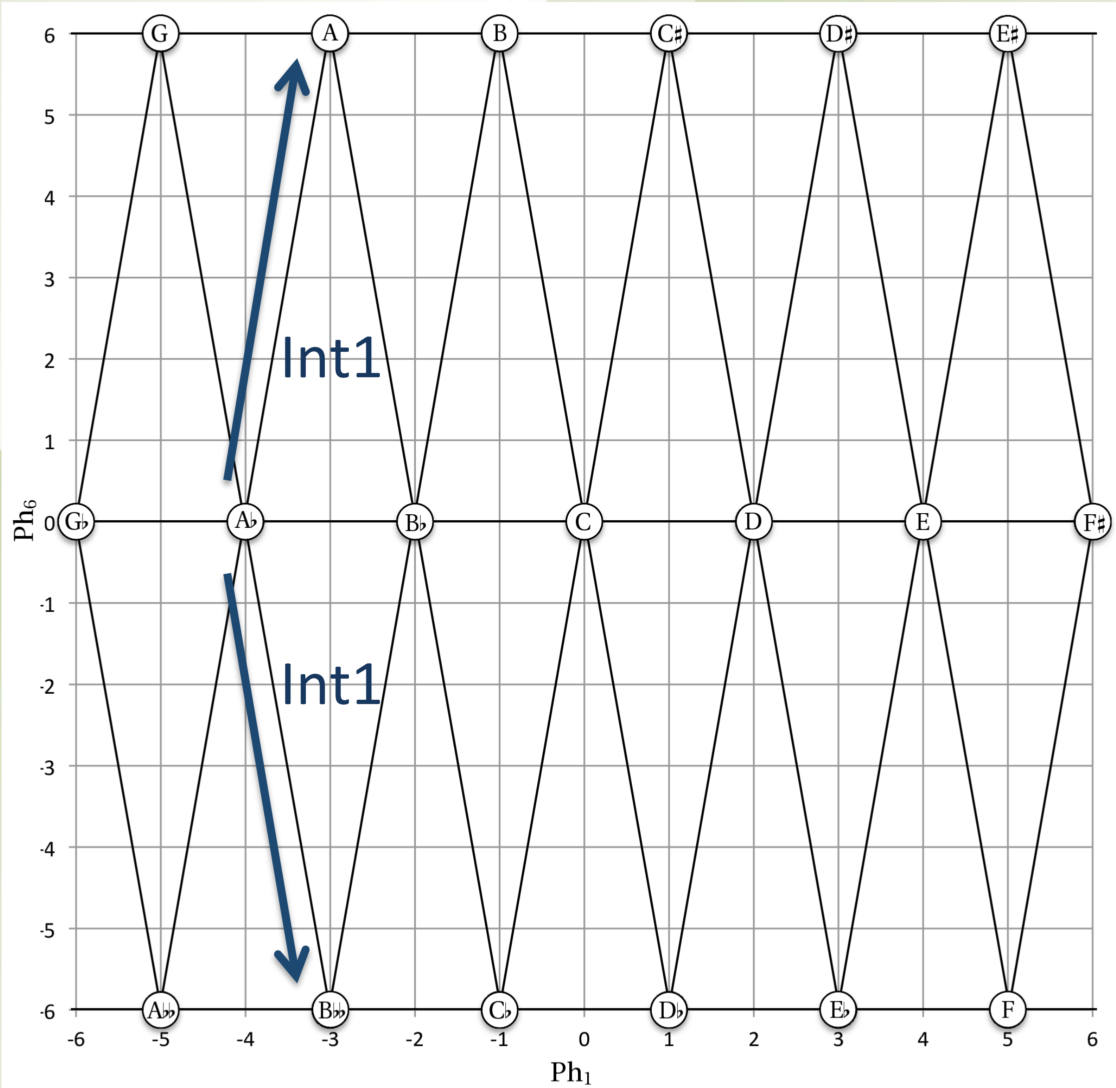
Duplicated Intervals

Sets (012) and (027) have duplicated intervals (int1 and int5)

Nonetheless, toroidal (012) and (027) *Tonnetze* are possible because we can draw multiple distinct axes for the same interval.

The optimization strategy is then to minimize *two* axes for the same interval, meaning one dimension should *minimize* the Fourier coefficient for this interval (to spread it out).

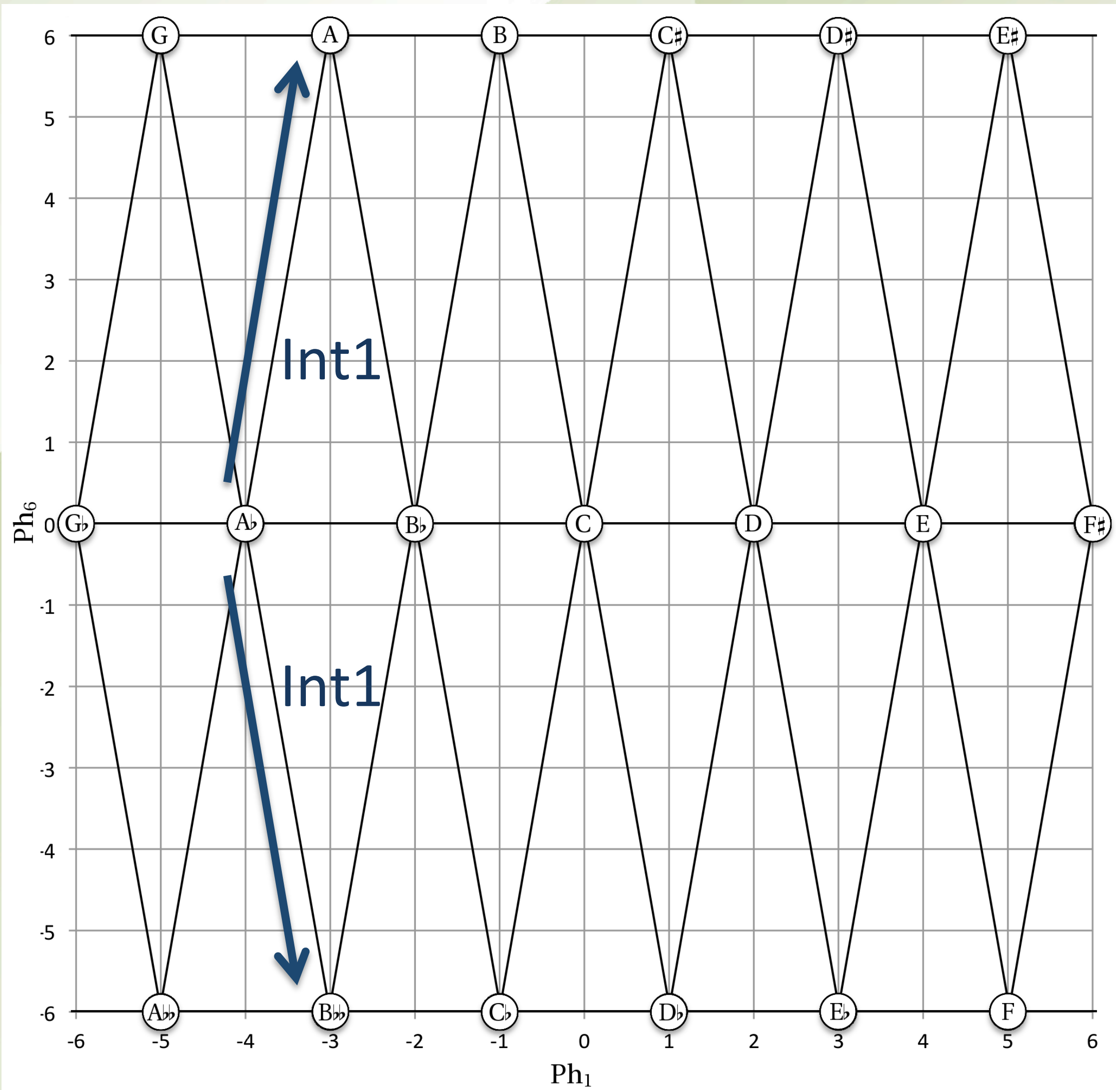
(012) *Tonnetz*



Ph_1 maximizes Fourier coefficients, but *Ph_6* minimizes it for int1.

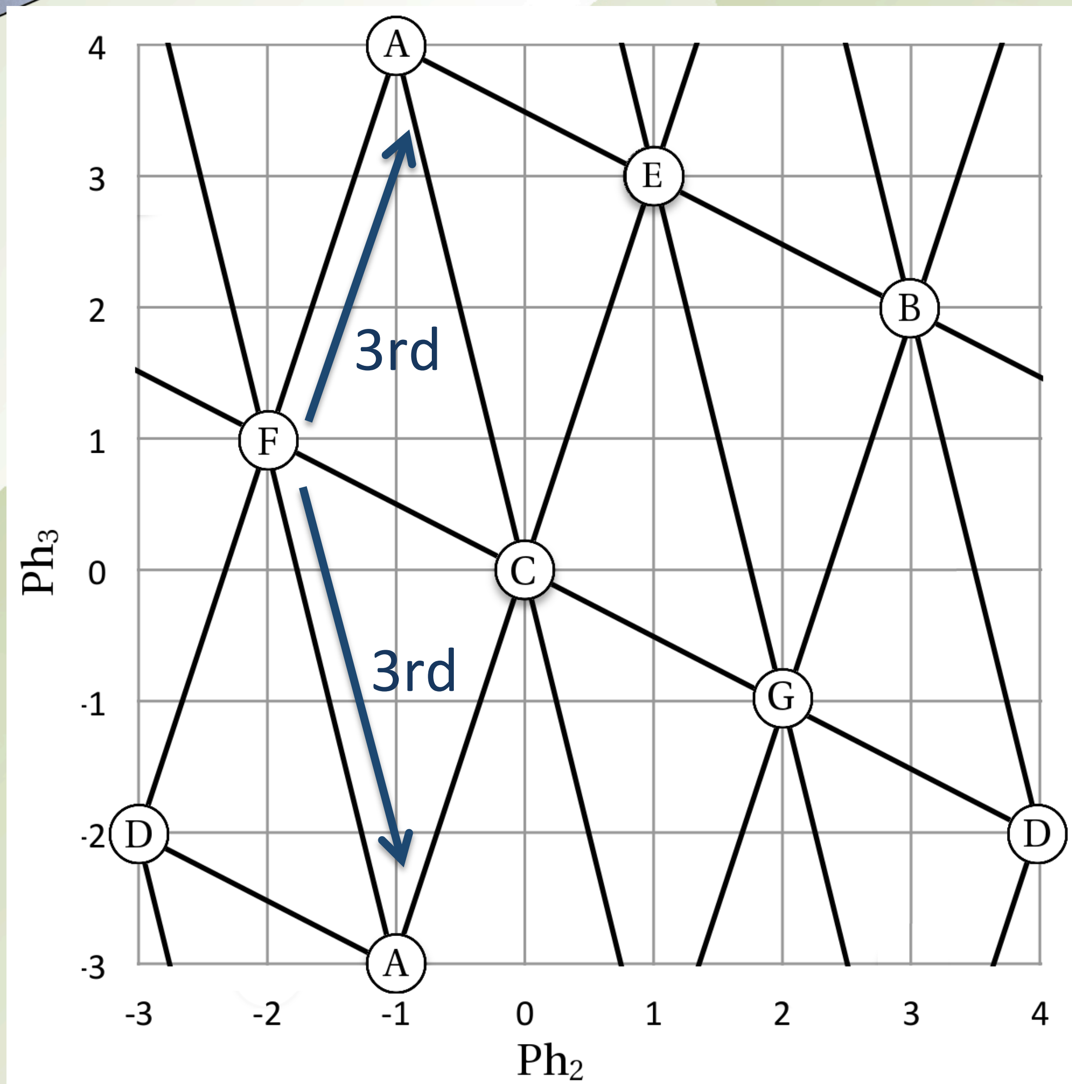
The spread of int1 in the *Ph_6* dimension allows for two equally good int1-axes (up 6 or down 6).

(012) *Tonnetz*



The two distinct axes may be used to represent distinct forms of the given interval. For instance, here the ic1s are spelled as chromatic or diatonic semitones (enharmonic distinctions).

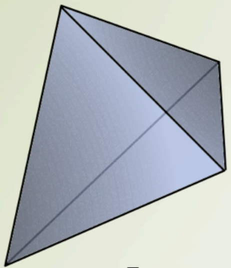
7-equal triad *Tonnetz*

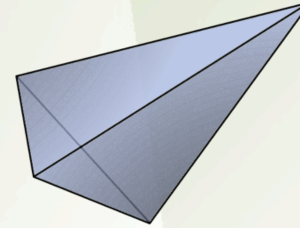


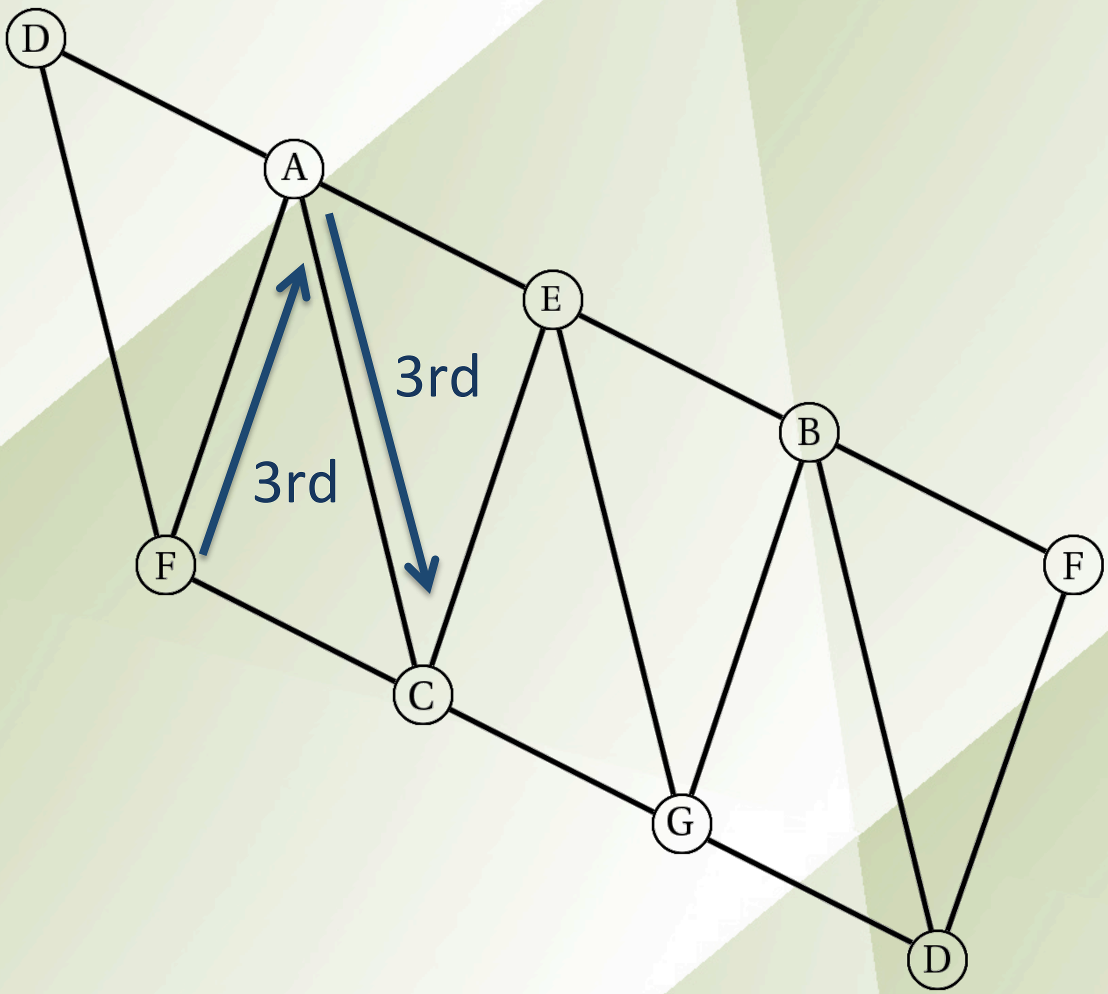
In a scale of 7 equally spaced notes, triads have a duplicated interval (the 3rd).

The *Tonnetz* therefore has two distinct intervallic axes for 3rds.

7-equal triad *Tonnetz*

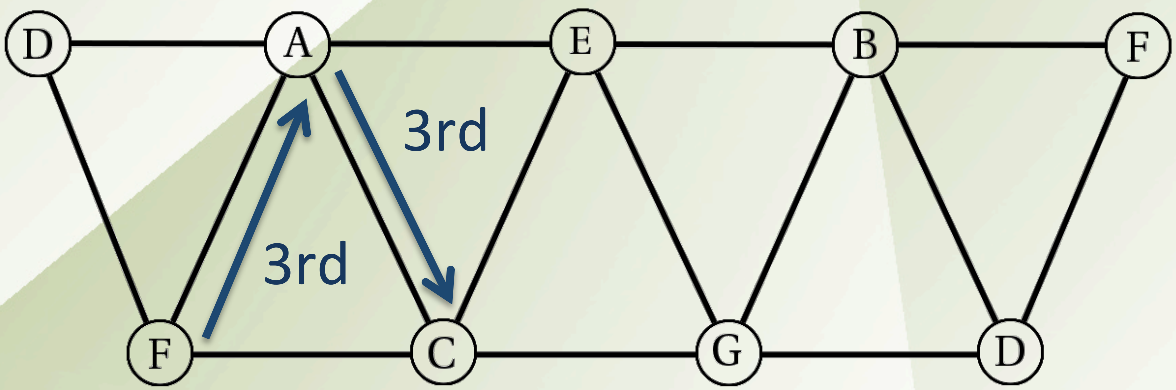






The space can be sheared and folded to equate the two kinds of thirds.

7-equal triad *Tonnetz*



The space can be sheared and folded to equate the two kinds of thirds.

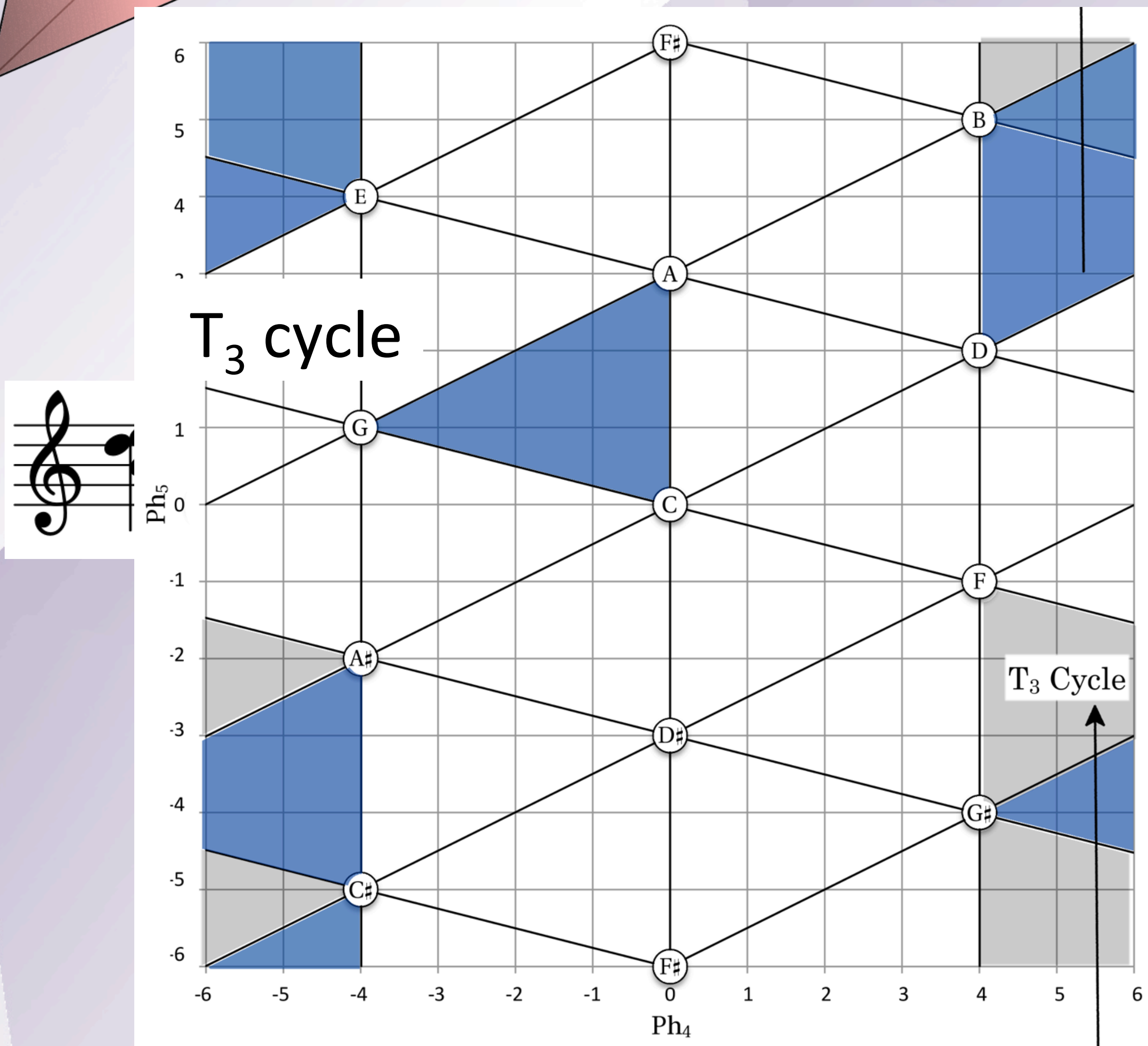
The result is Muzzulini (1995) and Mazzola's (2002) Möbius strip *Tonnetz*.

(3) Examples of Music Analysis with Non-Triadic *Tonnetze*

- (025) *Tonnetz,* Stravinsky “Owl and the Pussycat”
  - (013) *Tonnetz*, Shostakovich String Quartet 12
- (014) *Tonnetz,* Beethoven String Quartet Op. 132

Stravinsky: “Owl and the Pussycat”

Stravinsky: "Owl and the Pussycat"

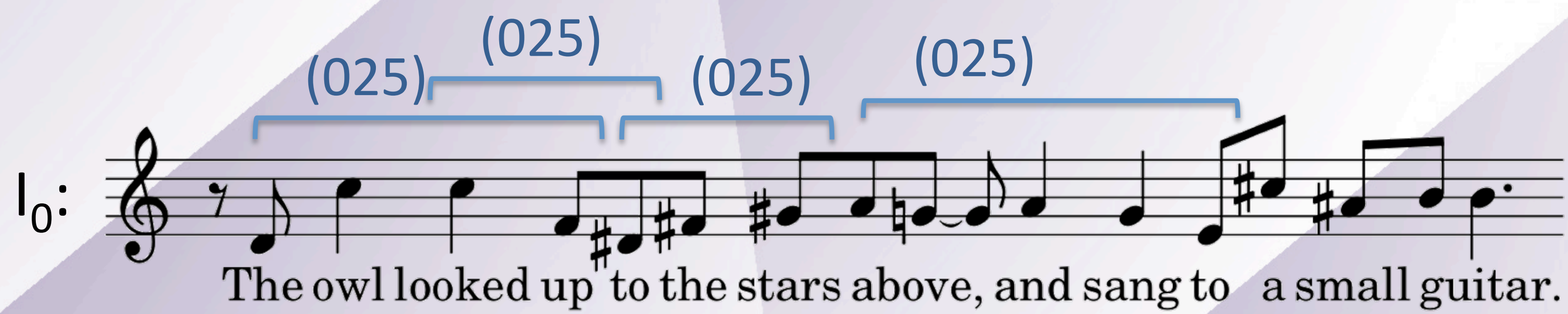


(025)s in P_0 in blue

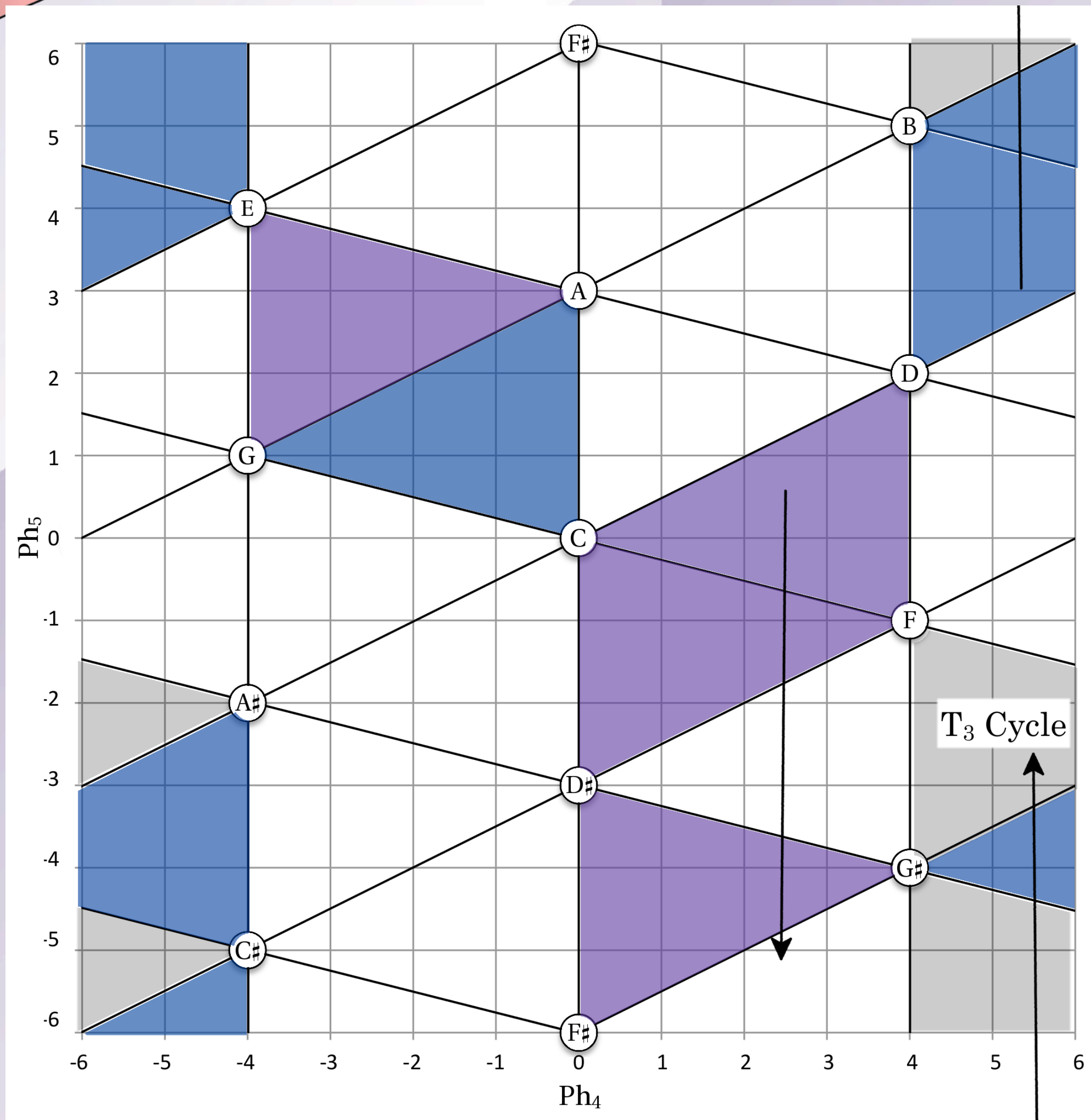
Vertical (025)s in gray

T_3 cycle (Octatonic)

Stravinsky: “Owl and the Pussycat”



Stravinsky: “Owl and the Pussycat”



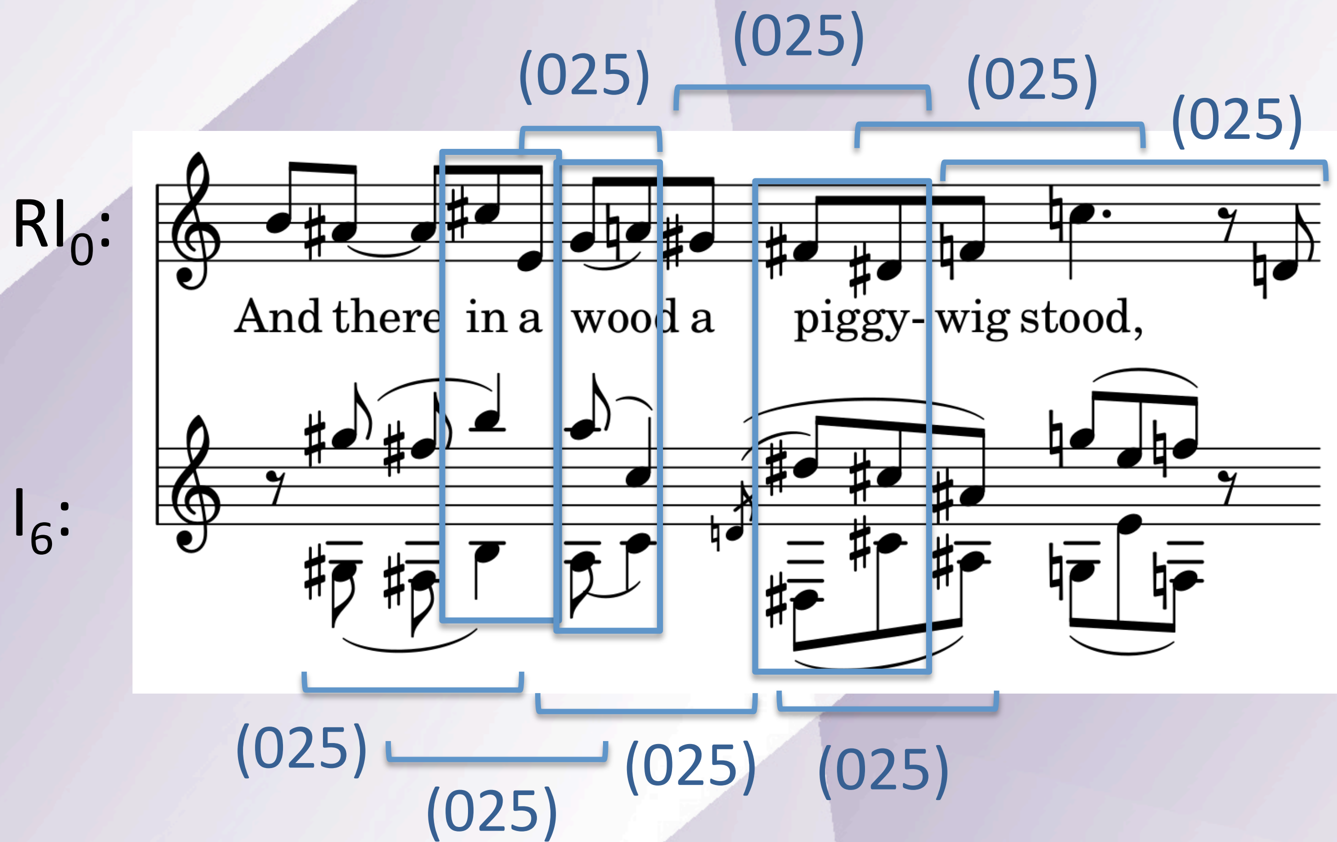
(025)s in P_0 in blue

Vertical (025)s in gray

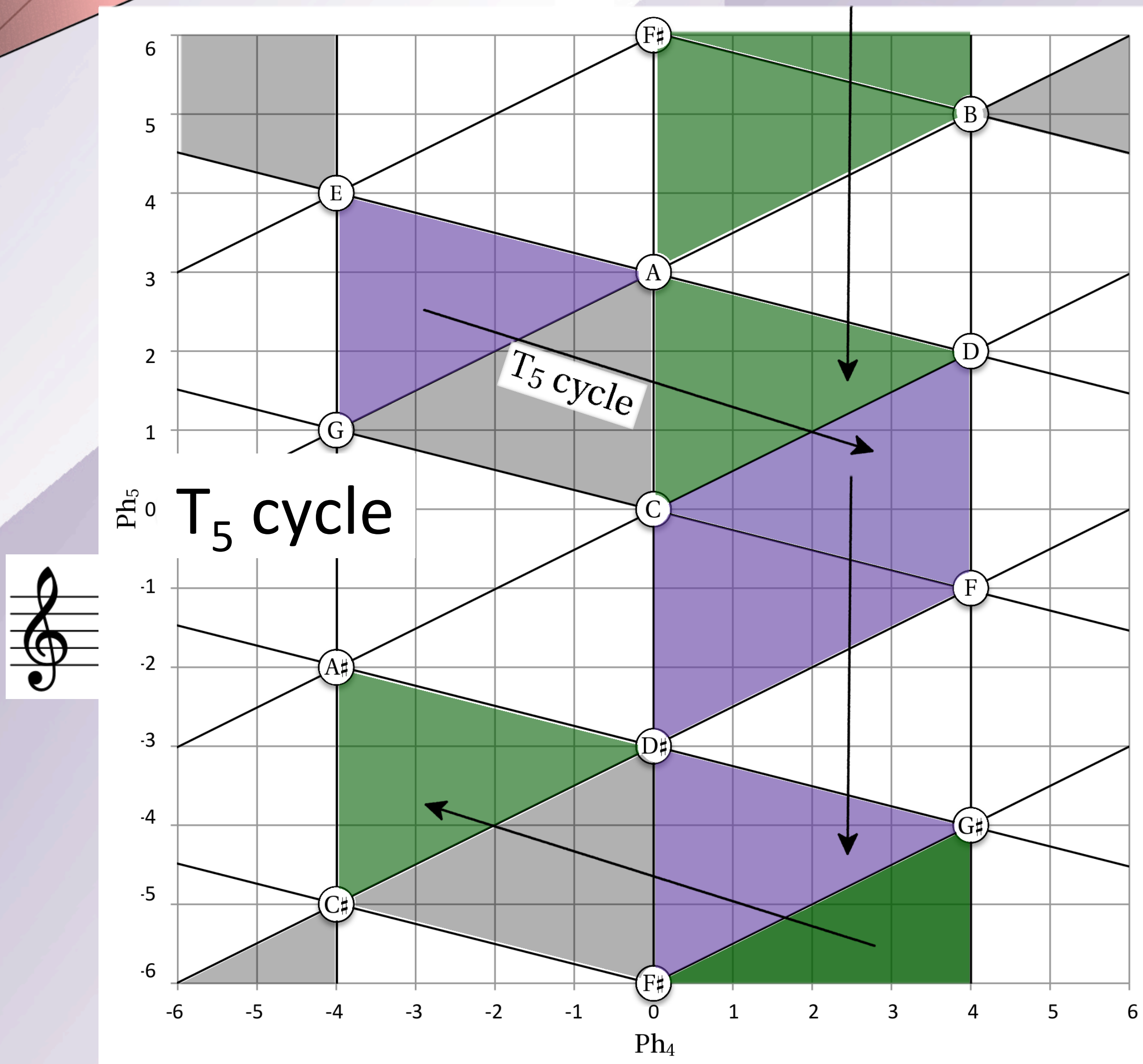
(025)s in I_0 in purple

Stravinsky: "Owl and the Pussycat"

An isolated use of I_6 occurs at the pivot in the narrative



Stravinsky: "Owl and the Pussycat"



(025)s in I_0 in purple

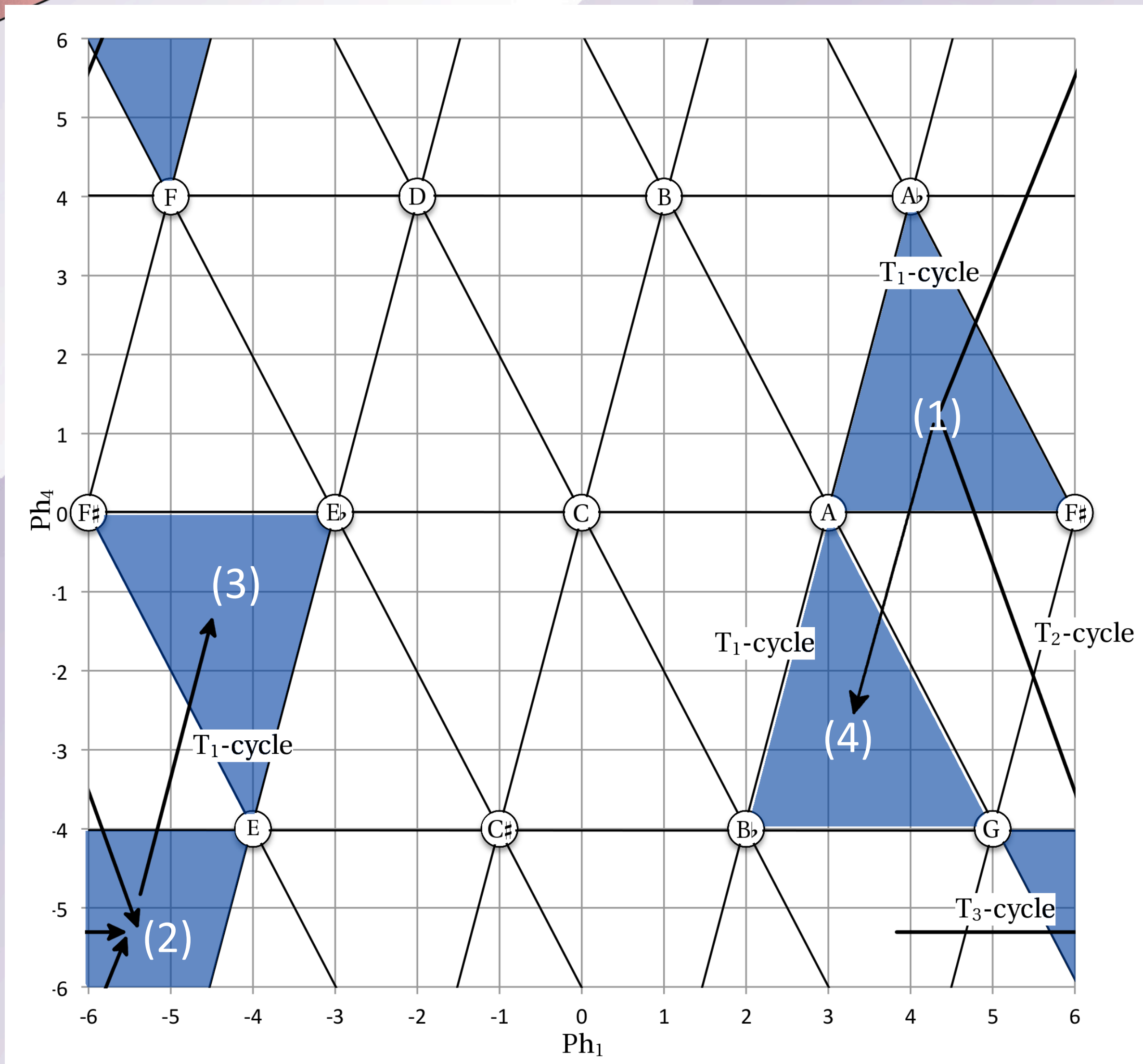
(025)s in I_6 in green

Vertical (025)s in gray

Shostakovich: String Quartet 12

Beginning of second movement, (013)s:

Shostakovich: String Quartet 12



The melody initially focuses on *disjunct* and T_1 relationships between (013)s

We can identify cycles (T_1 or T_2) for the disjunct relationship

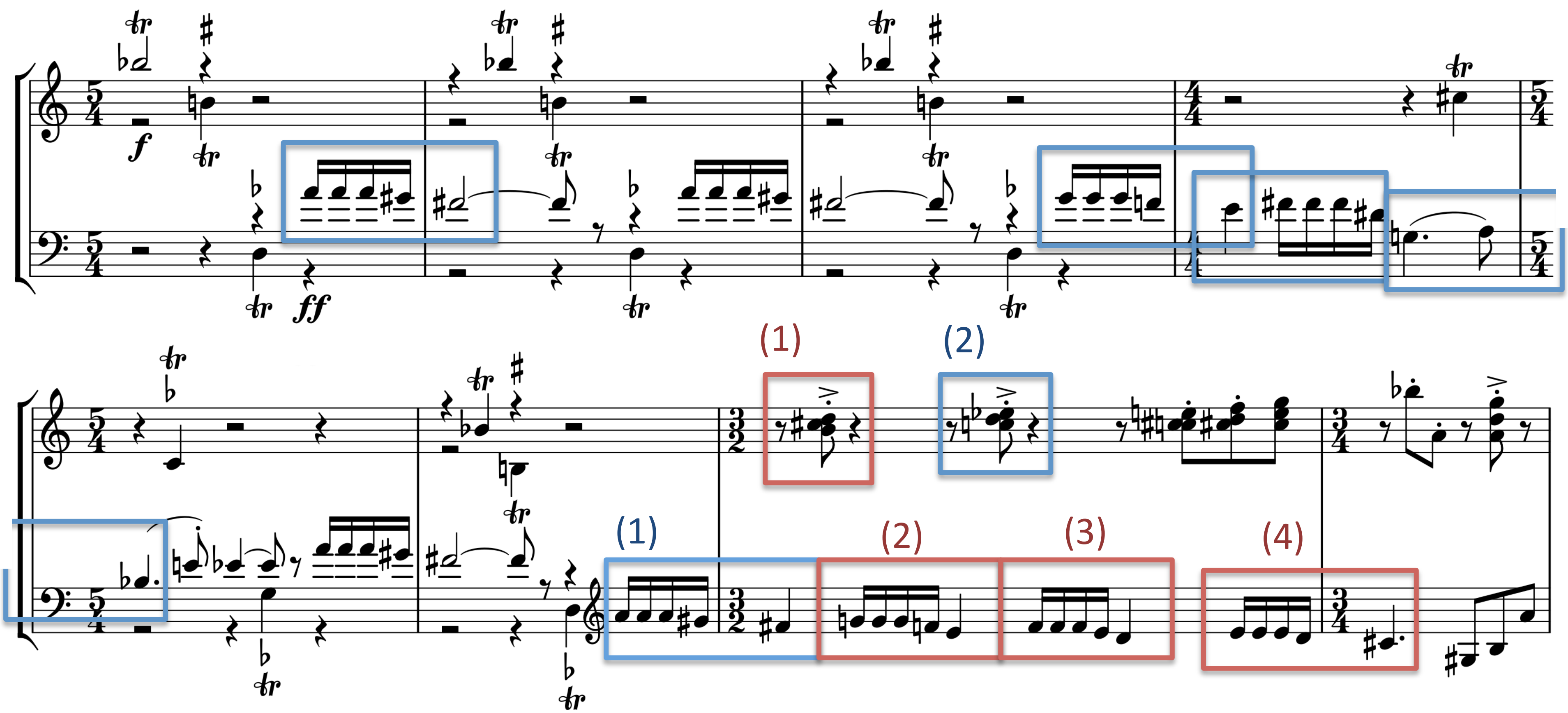
Shostakovich: String Quartet 12

T_2 cycle:

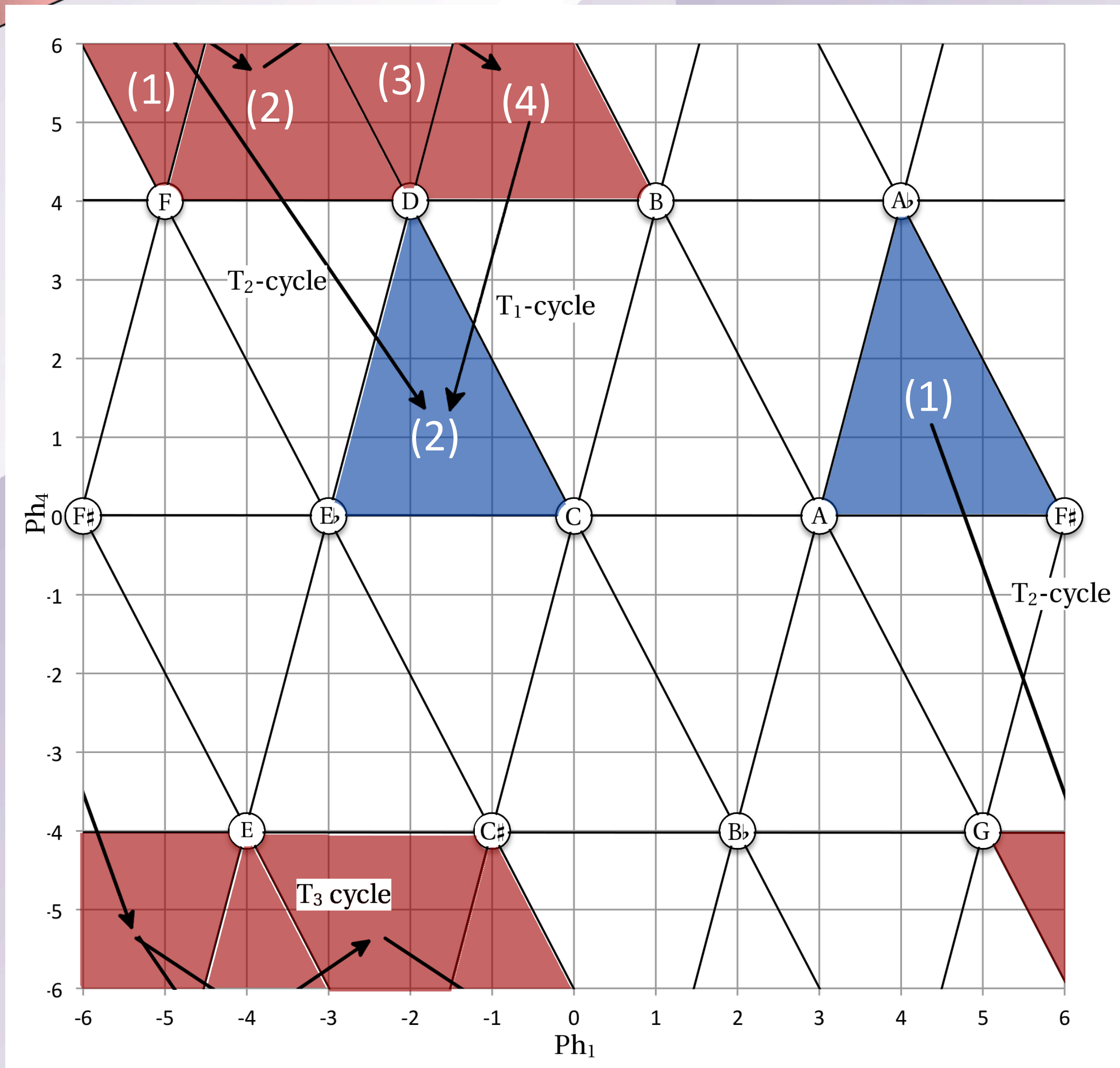
T_1 cycle:

Shostakovich: String Quartet 12

Beginning of second movement, (013)s:



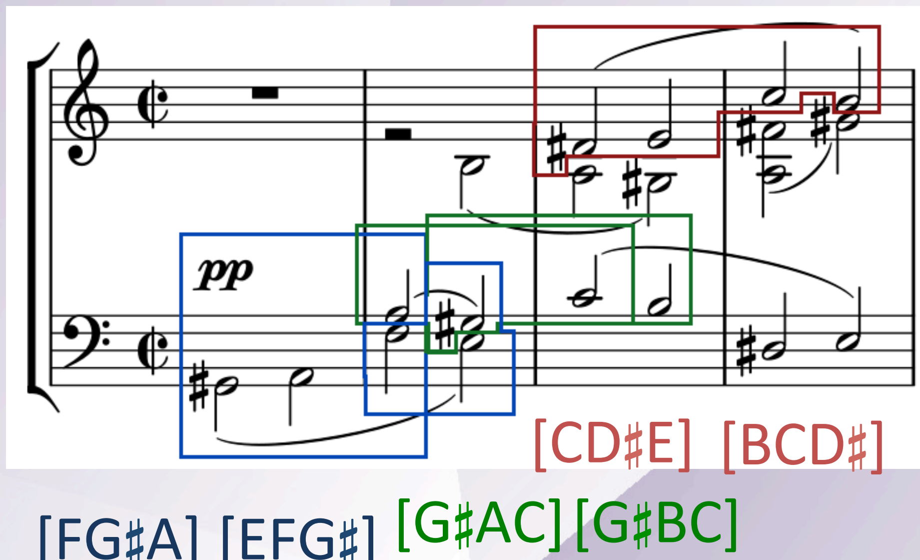
Shostakovich: String Quartet 12



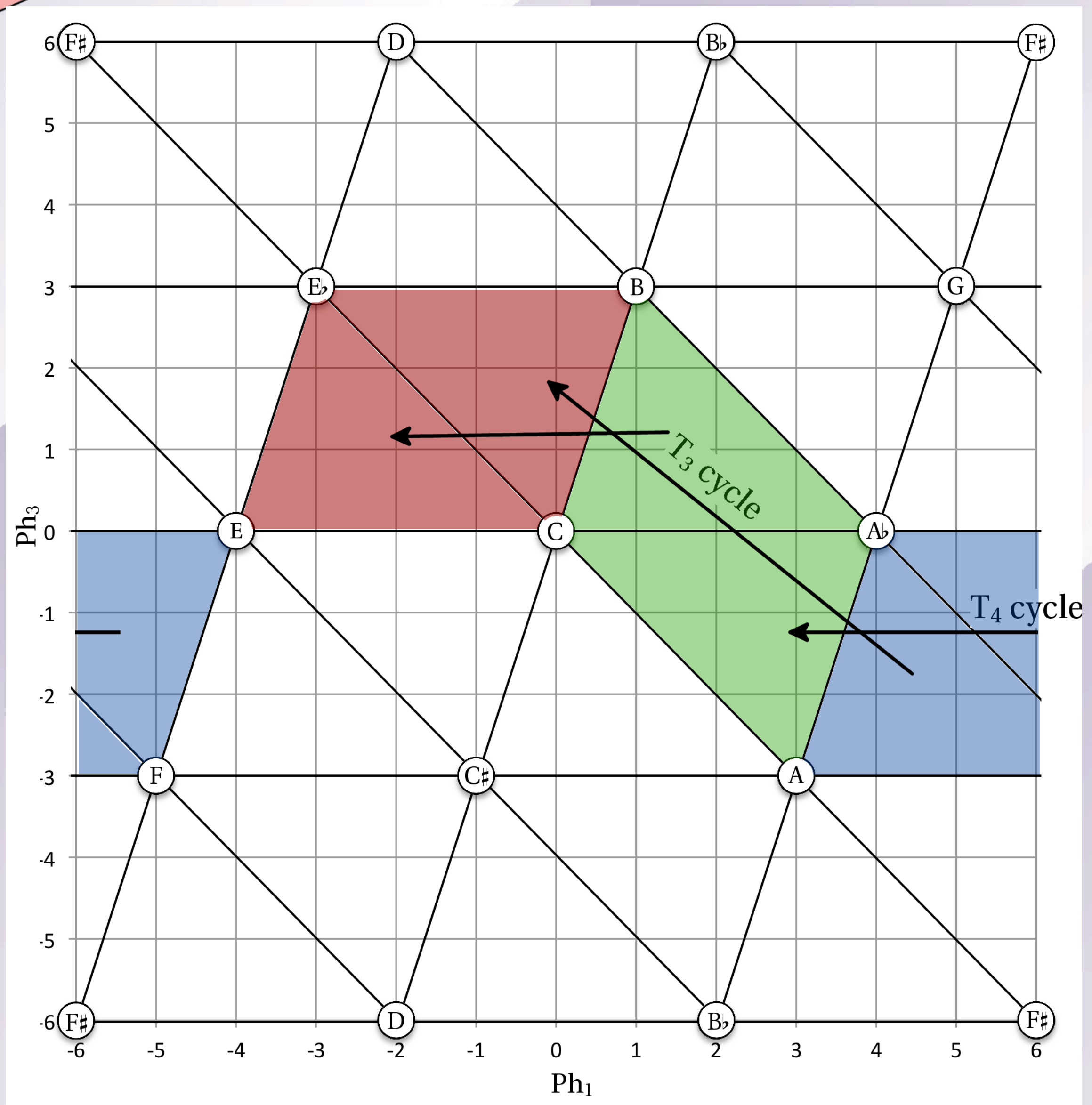
Meas. 7 extends the previous relationships and also expresses part of a conjunct T_3 (octatonic) cycle.

Beethoven, Op. 132 String Quartet

Network of (014)s in the opening of the first movement

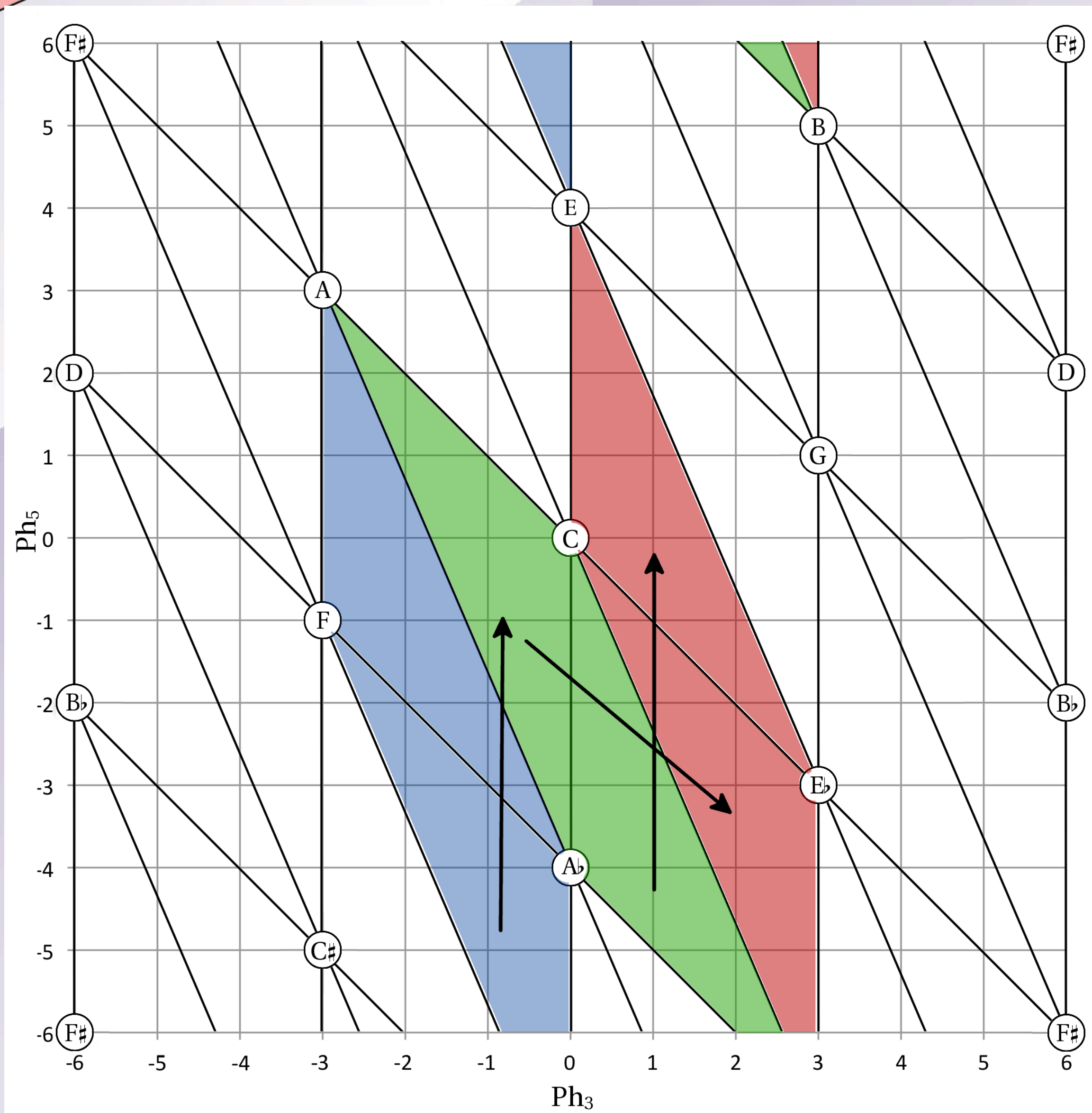


Beethoven, Op. 132 String Quartet



Chain of (014)s in optimal space, Ph_1-Ph_3

Beethoven, Op. 132 String Quartet



Chain of (014)s in non-optimal space, Ph_3-Ph_5

(4) Three-dimensional *Tonnetze*

Tetrachordal *Tonnetz*

Previous approaches (Gollin *JMT* 1998, Childs *JMT* 1998, Bigo *CMJ* 2015) take it as basic that a *Tonnetz* involves a single set class.

The result is networks of tetrahedra representing tetrachords, usually (0258)s, that intersect mostly in edges. Geometrically this means that there is empty space between the tetrahedra.

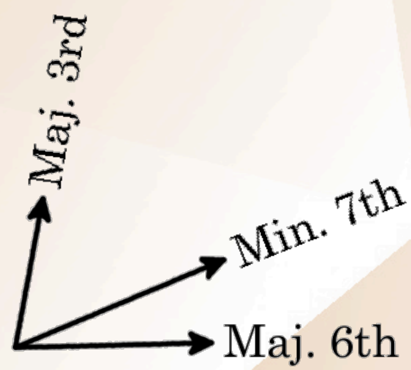
To make the *Tonnetz* a *simplicial decomposition* we instead completely fill the space with tetrahedra: this requires *three* set classes intersecting in shared trichords.

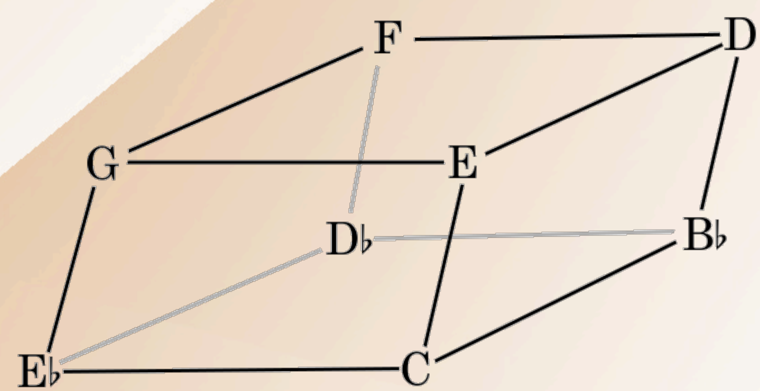
Tetrachordal *Tonnetz:* Construction

(1) Choose a three-dimensional phase space and any three non-parallel sets of intervallic axes

This partitions the space into parallelepipeds

Here are axes for intervals 2/10, 3/9, and 4/8 (in $\mathbf{Z}_{12}$)



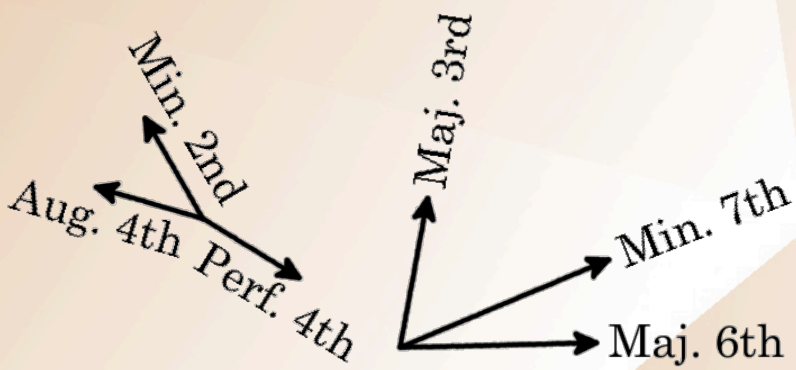


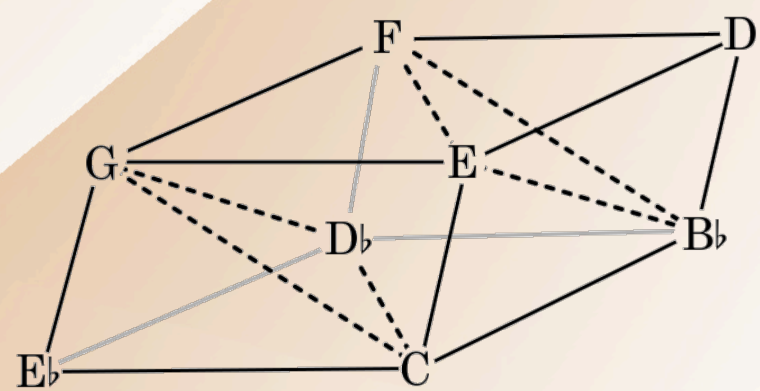
Tetrachordal *Tonnetz:* Construction

(2) Add three intervals to bisect the faces

The plane defined by these intervals creates tetrahedra for a single set class separated by octahedral regions. (Like Gollin's tetrachordal *Tonnetz* but with no shared faces!)

The new intervals in this example are 1/11, 6, and 5/7. The set class is (0137) (an all-interval tetrachord).

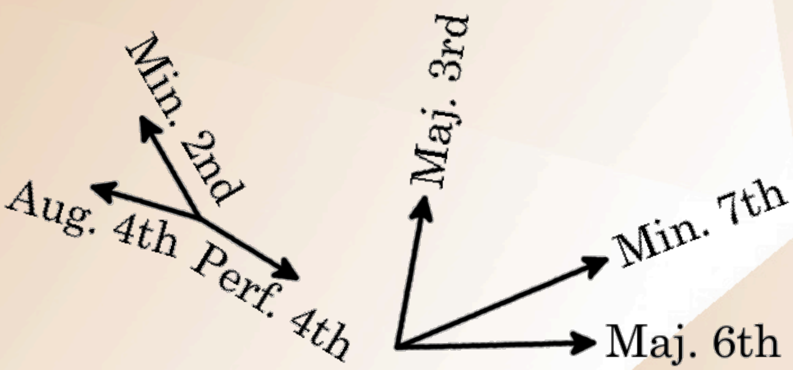


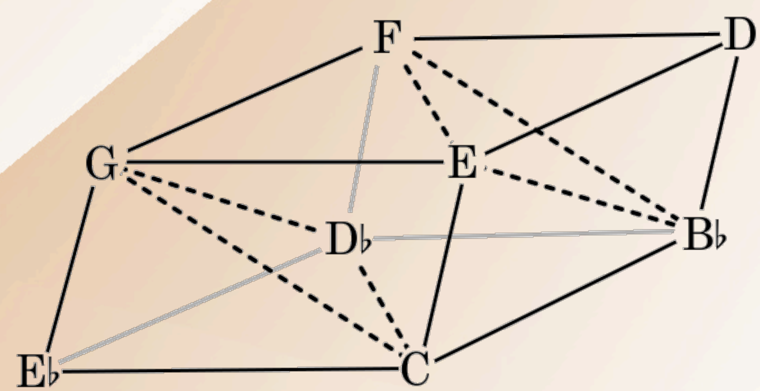


Tetrachordal *Tonnetz:* Construction

The space is now partitioned by four (sets of) planes, each of which is cut by the others into a two-dimensional *Tonnetz* based on one of the trichordal subsets of the given tetrachord.

In this example, there are planes with (037), (013), (026) and (016) *Tonnetze*.

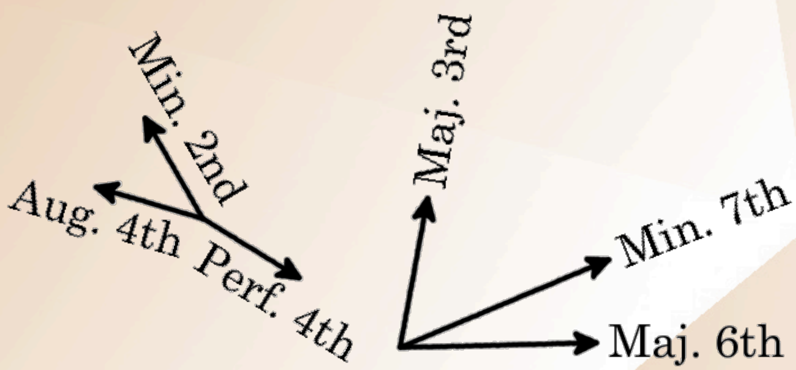


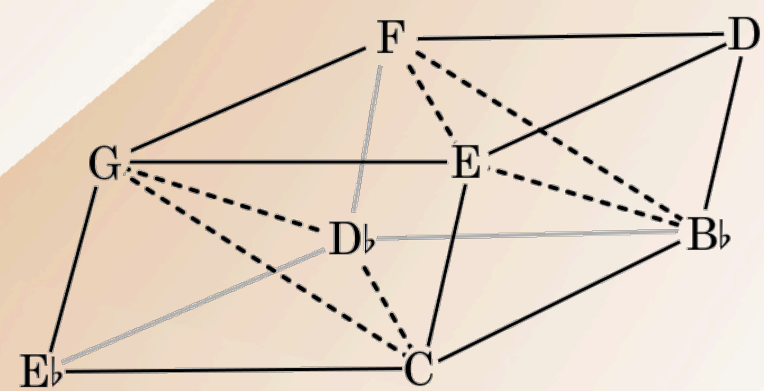


Tetrachordal *Tonnetz:* Construction

(3) The octahedral regions have three internal quadrilaterals. Bisecting *any two* of these with a single interval completes the partition into tetrachords. There are three possibilities.

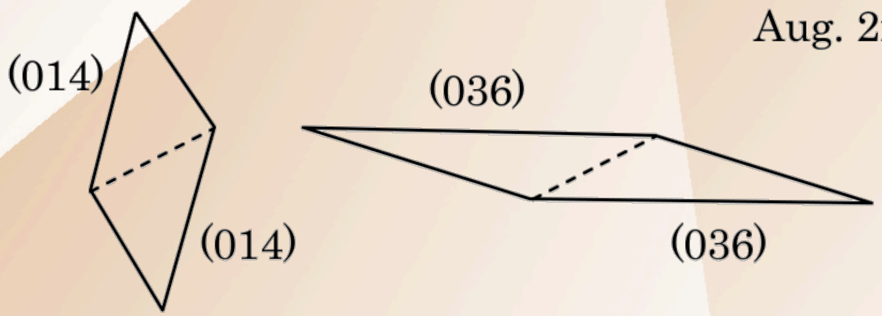
The internal quadrilaterals in this example are (FECD♭), (EB♭D♭G), and (FB♭CG).

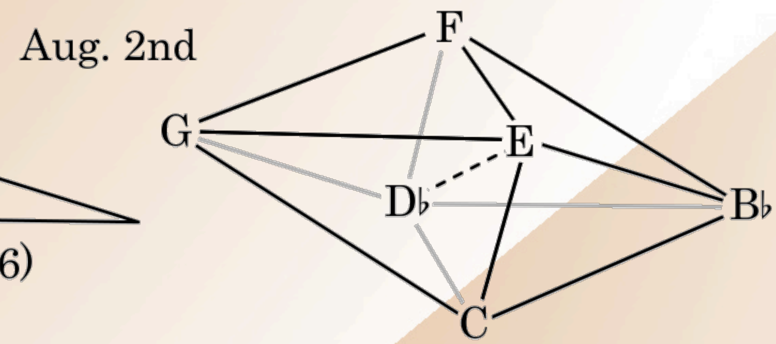


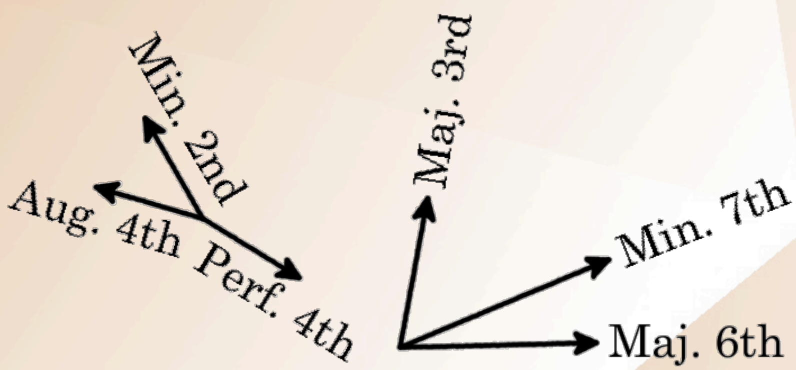


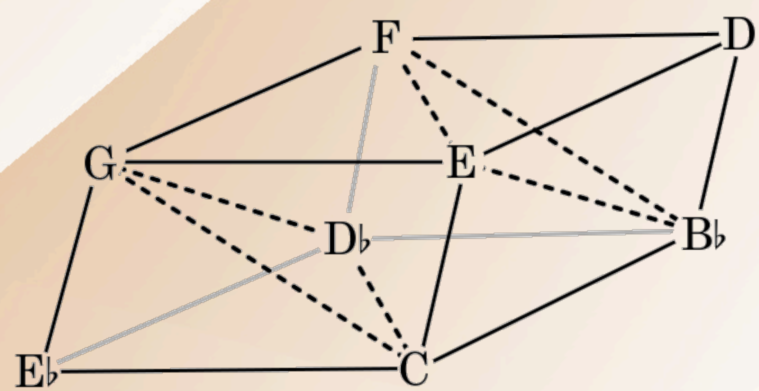
Tetrachordal *Tonnetz:* Construction

The first possibility adds an aug-2nd axis (distinct from the min-3rd axis!) creating (014) and (036) planes.



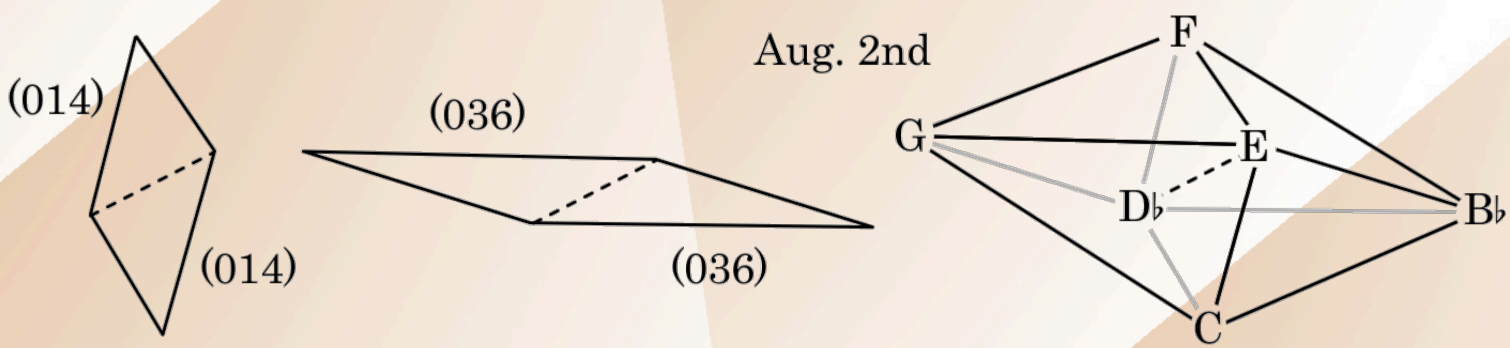


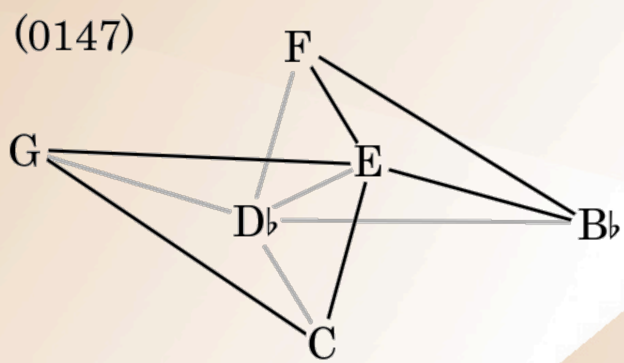


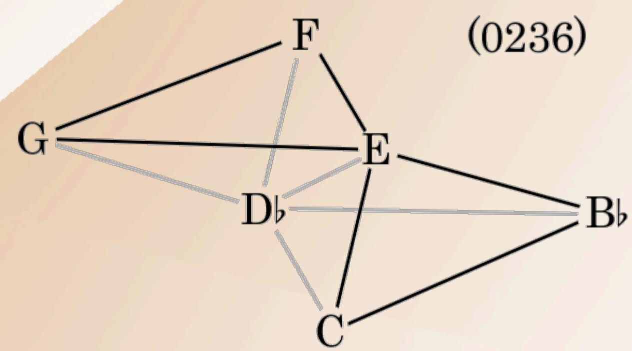


Tetrachordal *Tonnetz:* Construction

These add new tetrahedral (0147) and (0236) regions

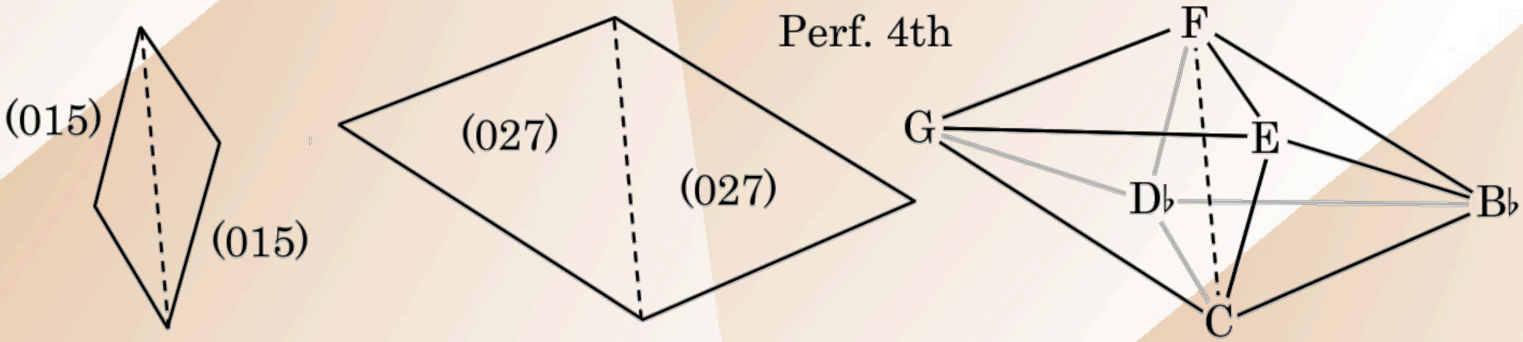






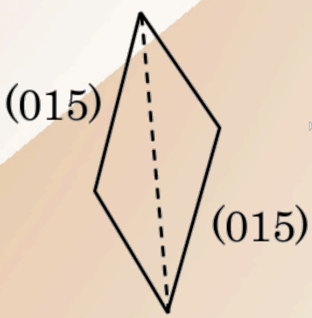
Tetrachordal *Tonnetz:* Construction

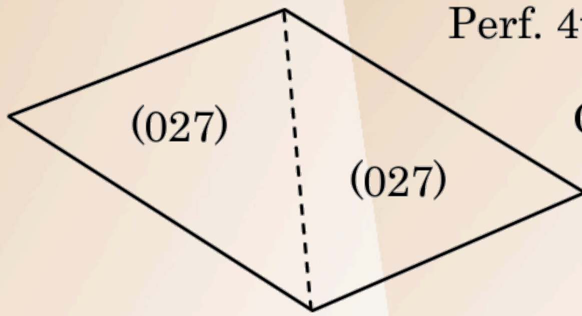
The second possibility adds a distinct perfect-fourth axis creating (027) and (015) planes.

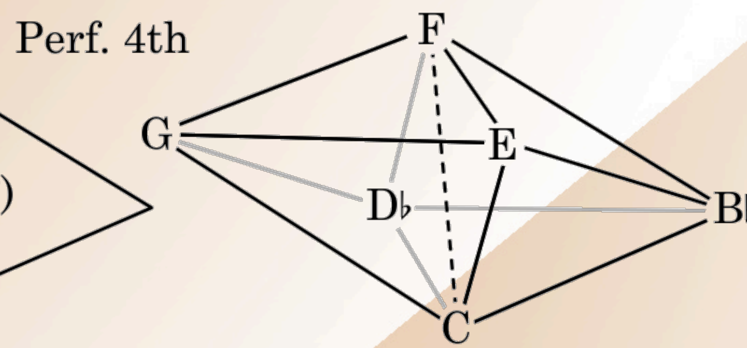


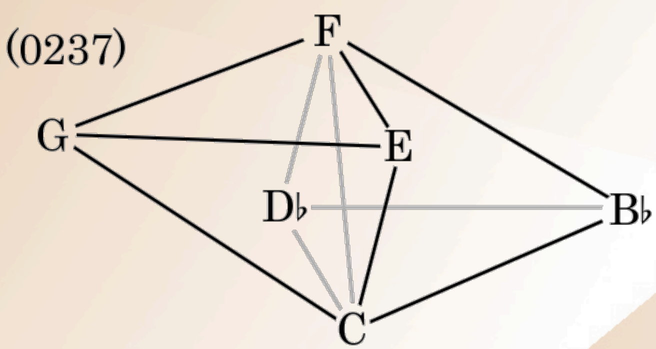
Tetrachordal *Tonnetz:* Construction

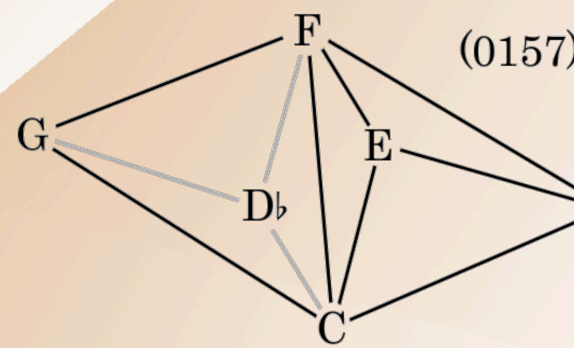
The other tetrahedral regions for this *Tonnetz* are (0237) and (0157)





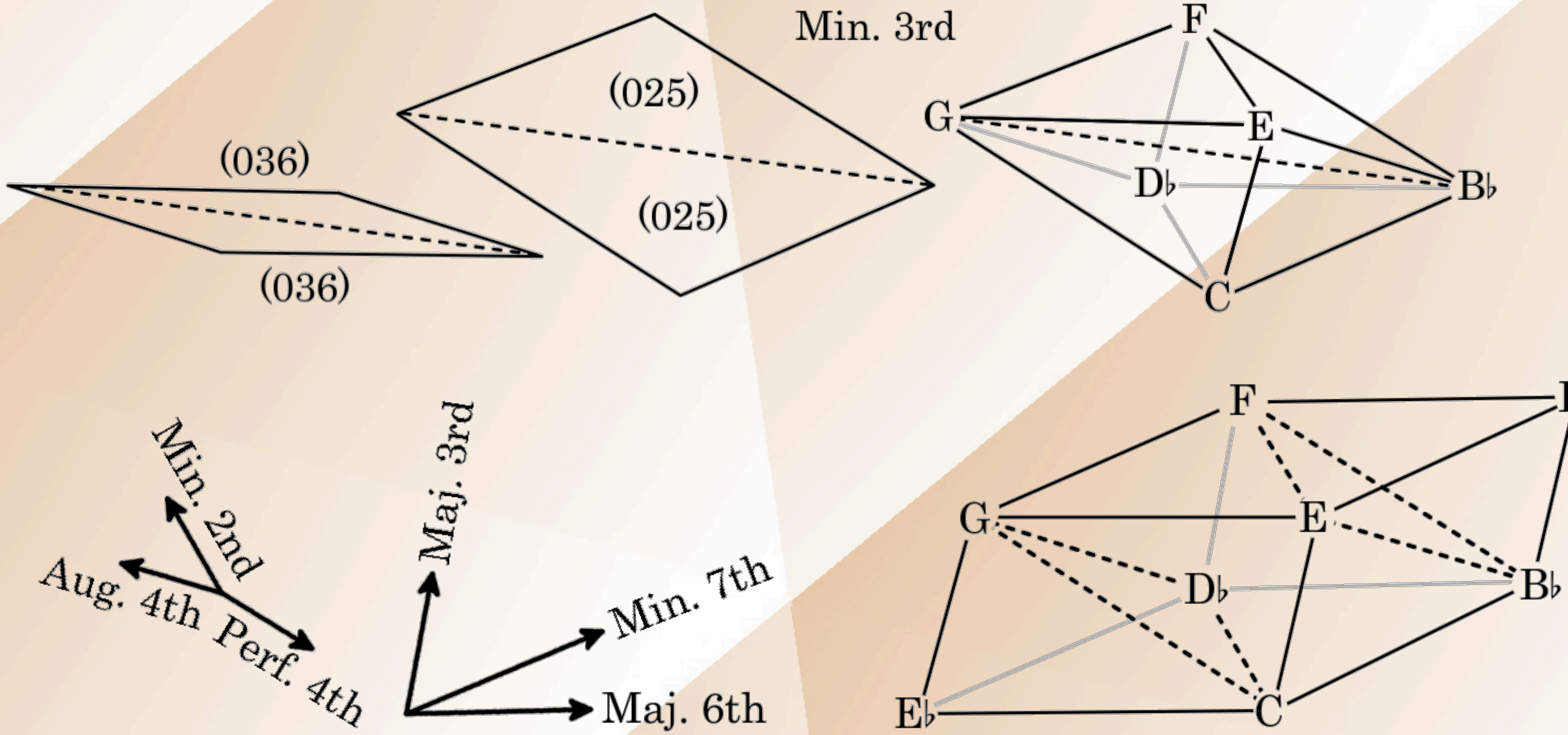






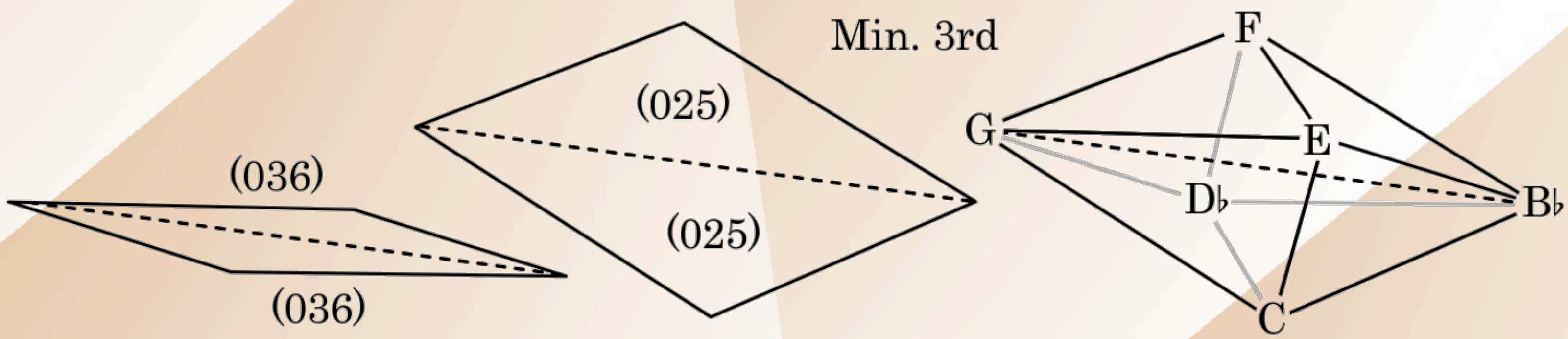
Tetrachordal *Tonnetz:* Construction

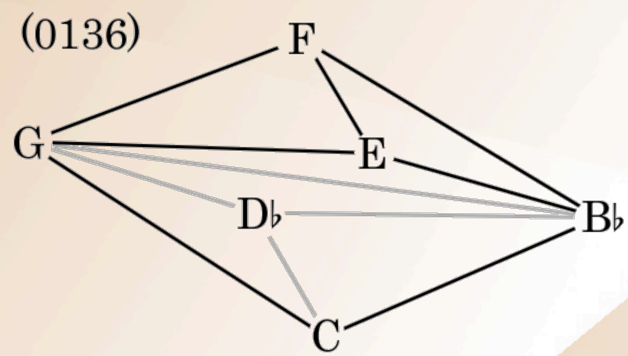
The third possibility also adds a distinct minor-thirds axis creating (036) and (025) planes.

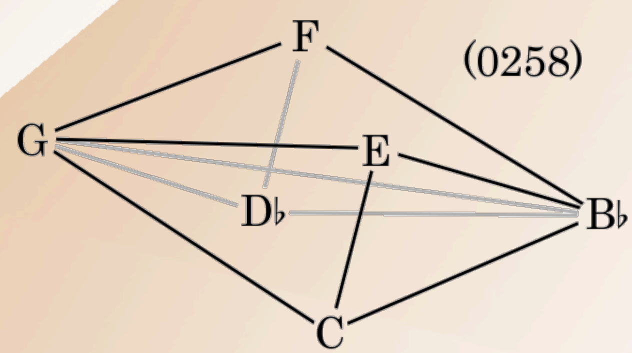


Tetrachordal *Tonnetz:* Construction

The resulting *Tonnetz* has (0136) and (0258) regions.





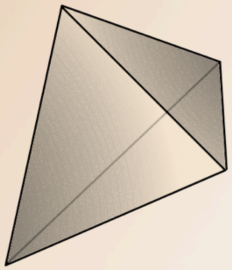


Properties and Classes of Three-Dimensional *Tonnetze*

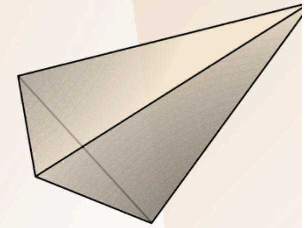
- Three tetrachord types
- Six planes with trichordal *Tonnetze:*
  Each trichord is shared by two tetrachords
- Seven intervallic axes
  - –Four are shared by all tetrachords and three trichordal planes
  - –Three are shared by two tetrachords and two trichordal planes

In $\mathbf{Z}_{12}$ there are only six interval classes, so at least one must be duplicated.

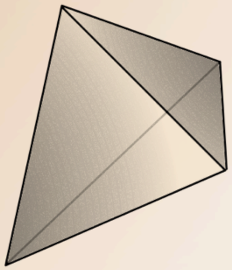
Tonnetze can be classified according to where (in which group of intervallic axes) duplications occur.



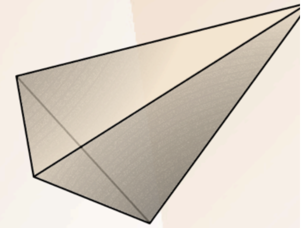
Classes of 3-D *Tonnetz* in $\mathbf{Z}_{12}$



Class	Properties	Tetrachords	Duplications	Omits	Space
A	One sym. trichord	(0125) (0126) (0146)	ic1		$Ph_{1,2,6}$
		(0237) (0157) (0137)	ic5		$Ph_{2,5,6}$
		(0136) (0236) (0146)	ic3		$Ph_{1,2,4}$
		(0147) (0236) (0137)	ic3		$Ph_{2,3,4}/Ph_{3,4,6}$
		(0147) (0258) (0146)	ic3		$Ph_{2,3,4}/Ph_{3,4,6}$
		(0136) (0258) (0137)	ic3		$Ph_{2,4,5}$
B	Two sym. trichords	(0124) (0125) (0135)	ic1, ic2	ic6	$Ph_{1,3,6}$
		(0135) (0237) (0247)	ic5, ic2	ic6	$Ph_{3,5,6}$
C	Two sym. and one dup. trichord	(0126) (0127) (0157)	ic1, ic5, (016)	ic3	$Ph_{1,2,6}$ or $Ph_{2,5,6}$



Classes of 3-D *Tonnetz* in $\mathbf{Z}_{12}$



Class	Properties	Tetrachords	Duplications	Omits	Space
D	Dup. tetrachord	(0134), (0236) x 2	ic1, ic3, (013), (014)	ic5	$Ph_{1,4,6}$
	+ one sym. trichord	(0145), (0125) x 2	ic1, ic4, (014), (015)	ic6	$Ph_{1,3,6}$
		(0156), (0126) x 2	ic1, ic5, (015), (016)	ic3	$Ph_{1,2,6}$
		(0156), (0157) x 2	ic1, ic5, (015), (016)	ic3	$Ph_{2,5,6}$
		(0235), (0135) x 2	ic2, ic3, (013), (025)	ic6	$Ph_{1,3,6}$ or $Ph_{3,5,6}$
		(0235), (0136) x 2	ic2, ic3, (013), (025)	ic4	$Ph_{2,3,4}$
		(0347), (0147) x 2	ic3, ic4, (014), (037)	ic2	$Ph_{2,3,4}$
		(0158), (0237) x 2	ic4, ic5, (015), (037)	ic6	$Ph_{3,5,6}$
		(0358), (0258) x 2	ic3, ic5, (025), (037)	ic1	$Ph_{4,5,6}$

Classes of 3-D *Tonnetz* in $\mathbf{Z}_{12}$

Class	Properties	Tetrachords	Duplications	Omits	Space
E	Dup. tetrachord	(0145), (0148) x 2	ic4 (x 3), ic1, (014), (015)	ic2, ic6	$Ph_{1,2,6}$
	& Aug. triad	(0347), (0148) x 2	ic4 (x 3), ic3, (014), (037)	ic2, ic6	$Ph_{1,4,6}$ / $Ph_{4,5,6}$
		(0158), (0148) x 2	ic4 (x 3), ic5, (015), (037)	ic2, ic6	$Ph_{2,5,6}$
F	Dup. tetrachord &	(0134), (0124) x 2	ic1, ic3, ic2, (013), (014)	ic5, ic6	$Ph_{1,3,6}$
	two sym. trichords	(0358), (0247) x 2	ic5, ic3, ic2, (025), (037)	ic1, ic6	$Ph_{3,5,6}$
G	Dup. tetrachord &	(0123), (0124) x 2	ic1 (x 3), ic2, (012), (013)	ic5, ic6	$Ph_{1,5,6}$
	sym. tetrachord	(0257), (0247) x 2	ic5 (x 3), ic2, (025), (027)	ic1, ic6	$Ph_{1,5,6}$

Example: Aug-2nd *Tonnetz*

Class A: (0137), (0147), (0236)

Optimal space: Ph_2-Ph_3-Ph_4 or Ph_3-Ph_4-Ph_6

Duplicated ic3s may be interpreted as min 3rd/aug 2nd

(0137): Diatonic subset, maj. triad + $\sharp$4th (IV + $\hat{7}$) or
min. triad + $\flat$2nd ($V^{7(\text{no5th})}$ + $\hat{3}$)

(0147): Non-diatonic, maj. triad + $\flat$2nd (V + $\flat\hat{6}$) or
min. triad + $\sharp$4th (iv + $\sharp\hat{7}$)

(0236): Non-diatonic, harmonic minor scale segment:
$\flat\hat{6}$-$\hat{7}$-$\hat{1}$-$\hat{2}$ or $\hat{4}$-$\hat{5}$-$\flat\hat{6}$-$\hat{7}$

Example: JI scalar *Tonnetz*

Class D: (0235), (0135) x 2

Optimal space: Ph_2-Ph_3-Ph_6

Duplicated ic2s and ic3s may be interpreted as just vs. Pythagorean intervals (10/9 vs. 9/8 and 6/5 vs. 32/27)

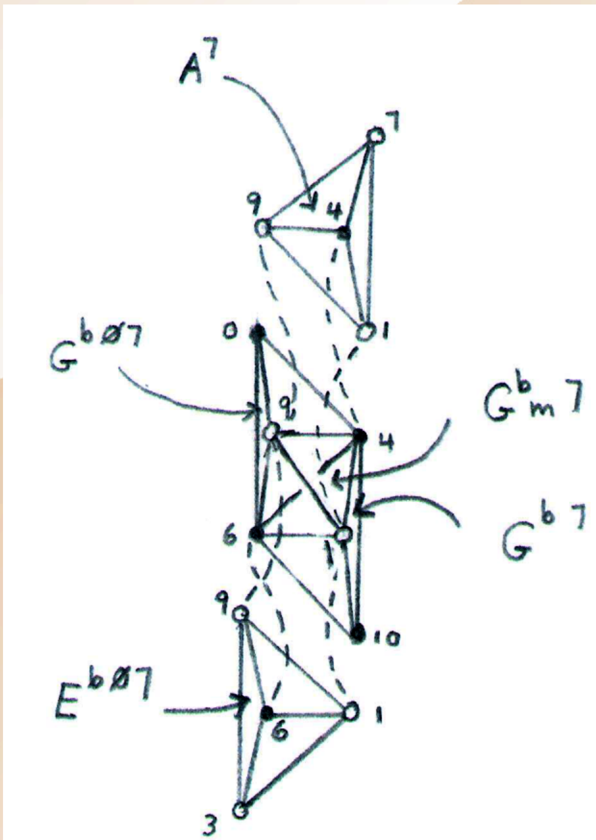
j(0135): $\underbrace{s + p2}_{j3} + j2$ or $j2 + \underbrace{p2 + s}_{j3}$

p(0135): $\underbrace{s + j2}_{p3} + p2$ or $p2 + \underbrace{j2 + s}_{p3}$

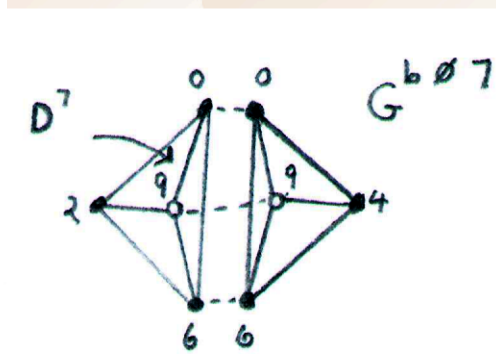
(0235): $j2 + s + p2$ or $p2 + s + j2$

p3 j3 j3 p3

Example: Douthett’s Tetrahedral *Tonnetz*



This can be derived from class D *Tonnetz* on (0358), (0258) x 2 by equating duplicated ic3 and ic5 axes. The result is cyclic in two dimensions with the (026)-*Tonnetz* planes forming a boundary in the third dimension.



From unpublished ms., 1997

Summary

- Toroidal topology is central to the *Tonnetz* idea generally, although it can be modified to reflect other kinds of topology.
- The possible toroidal geometries of any *Tonnetz* are equivalent to Fourier phase spaces.
- Toroidal spaces can duplicate intervals—i.e., multiple axes representing the same interval.
- *Tonnetze* may be understood as simplicial decompositions of toroidal spaces. Three-dimensional *Tonnetze* are then based on *three* tetrachord types intersecting in shared faces.
- Because three-dimensional *Tonnetz* have seven intervallic axes, interval duplications are unavoidable in $\mathbf{Z}_{12}$.

Geometrical Realizations of Two- and Three-Dimensional Generalized *Tonnetze*

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