

# Restoring the Structural Status of Keys through DFT Phase Space

Jason Yust

**Abstract** One of the reasons for the widely felt influence of Schenker’s theory is his idea of long-range voice-leading structure. However, an implicit premise, that voice leading is necessarily a relationship between chords, leads Schenker to a reductive method that undermines the structural status of keys. This leads to analytical mistakes as demonstrated by Schenker’s analysis of Brahms’s Second Cello Sonata. Using a spatial concept of harmony based on DFT phase space, this paper shows that Schenker’s implicit premise is in fact incorrect: it is possible to model long-range voice-leading relationships between objects other than chords. The concept of voice leading derived from DFT phases is explained by means of *triadic orbits*. Triadic orbits are then applied in an analysis of Beethoven’s *Heiliger Dankgesang*, giving a way to understand the ostensibly “Lydian” tonality and the tonal relationship between the chorale sections and “Neue Kraft” sections.

## 1 Long-Range Voice-Leading Structure without Reduction

### 1.1 Schenker’s Implicit Premise

As a voice-leading based approach that can address large-scale tonal structure, Schenkerian theory is widely regarded to be amongst the most sophisticated extant theories of tonality. However, when Schenker claimed that his theory of levels would supplant traditional notions of form and key, he overplayed his hand, creating conceptual tensions that persist in Schenkerian theory today. Schachter’s [8] insightful deconstruction of the Schenkerian perspective on keys stops short of denying their reality even as he claims that Schenkerian structures override them.

The conflict of Schenker’s theory with the traditional idea of keys as the objects of long-range structure comes from an implicit premise: that voice leading can only be a relationship between individual harmonies. This implies that a theory of long-range tonal structure based on voice leadings must posit that certain structural chords can be isolated from their contexts to relate directly at a deeper level. Specific tonic chords then must stand in for keys, undermining the important distinction between chord and key. While such an analytical approach often seems serviceable

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Jason Yust  
School of Music, Boston University, 855 Commonwealth Ave., Boston, MA, 02215, USA e-mail:  
jason.yust@gmail.com

despite the underlying conceptual flaws, in certain circumstances it prevents an accurate analysis.

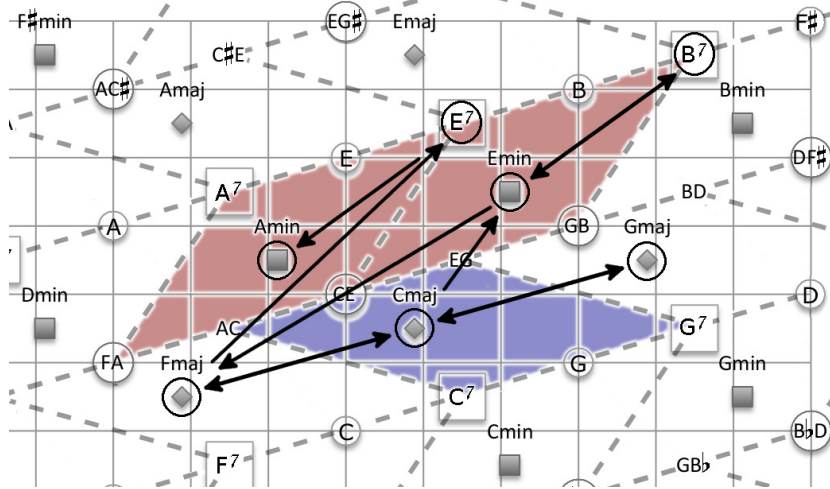
One such instance is the exposition of Brahms's F Major Cello Sonata, Op. 99, which Schenker analyzes in *Der Freie Satz* [9, Fig. 110d2]. The reduction in Figure 1 illustrates the problem posed by the piece. The subordinate theme begins in the standard subordinate key of C major. After a momentary deflection to E minor (confirmed by a weak cadence) the music enters a cadential phase where it toys with the possibility of ending in A minor or C major, tipping just at the last minute into A minor. Only an analysis that can show how C major is in play up to the last few measures can accurately reflect Brahms's tonal rhetoric here. A reductive analytical method, however, must proceed by first selecting out the most structural harmonies of the passage. As shown on the lower staff, the presence of cadences in E minor and A minor all but demand that these chords be selected as the most structural ones, which is exactly how Schenker analyzes the passage.

A spatial concept of tonality can serve us better in this situation, as shown in Figure 2. The space used here is the *DFT phase space* described in [1] and [13]. The vertical axis of the space is the phase of the fifth Fourier coefficient of a pitch-class set, and the horizontal axis is the phase of the third coefficient. Dashed lines show the tonal regions derived in Section 8 of [13]. Any pitch-class set or multiset has a position in the space, including chords, scales, and single pitch classes. As Figure 2 shows, a trajectory may be drawn by plotting chords used in the passage, which are somewhat spread out in the space. However, averaging over multiple chords restricts the range of activity, and each pair of chords averages to a location within the appropriate tonal region. The summary of the progression—i.e., a long-range picture of tonal motion through the passage—shows the key a C major acting as an intermediary between excursions to the bordering regions of E minor and A minor, as Figure 3 shows. C major is central to the harmonic content of the entire passage, making this the principal key of the theme.

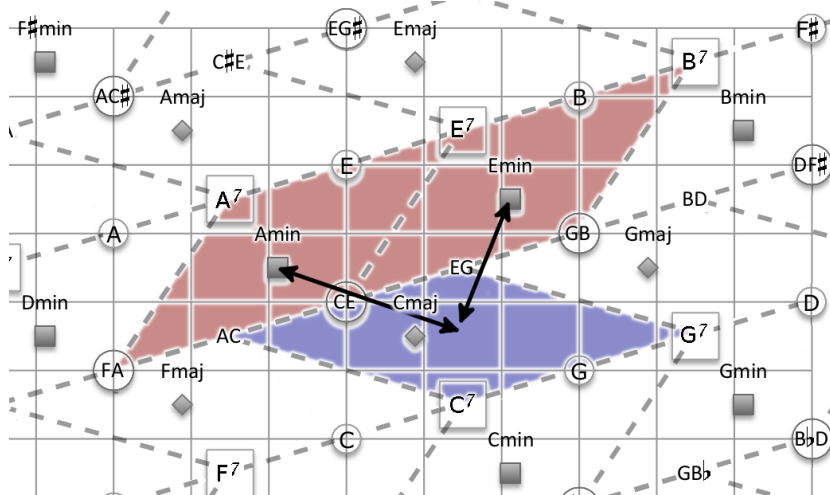
34 40 46 58

C maj.: I V (HC) I E min.: V i (weak PAC) C maj.: IV A min.: V i

**Fig. 1** A reduction of the subordinate theme in Brahms's Op. 99 Cello Sonata



**Fig. 2** Significant chords in the subordinate theme of Brahms's Op. 99 Cello Sonata



**Fig. 3** Averaging over a larger number of elements shows a large-scale progression between regions

### 1.2 Triadic Orbits

One of the compelling features of Schenkerian theory is its grounding of tonal structure in voice leading. Positions in Fourier phase space can also be understood to reflect voice-leading relationships. Since one can analyze harmony at different levels using the space by summarizing the harmonic content of larger or smaller amounts of music, as illustrated above, this means that Schenker's implicit premise is wrong.

It is in fact possible to conceive of voice-leading relationships between objects other than chords, including scales and keys. Therefore, we can theorize large-scale voice-leading processes without making reductive assumptions—that is, without asserting direct relationships between remote harmonic objects.

The DFT reparameterizes a pcset by modeling it with sinusoidal pitch-class distributions that divide the octave evenly into 1–11 parts. The phase of these components indicates which perfectly even distribution the pcset best approximates. The third component approximates the pcset with a distribution of three evenly spaced peaks, while the fifth component approximates it with a distribution of five or seven evenly spaced peaks. Motion between relatively even three-note chords (triads) in the horizontal dimension of the phase space in Figures 2 and 3 reflects the direction of the most efficient voice leading. Motion in the vertical dimension on relatively even seven-note collections (scales) reflects the direction of scalar voice leading, or change of key signature, in the sense of Hook [4] or Tymoczko [11]. However, since pcsets of any cardinality appear in the same space, near to harmonically related pcsets of other cardinalities, we can also speak of scalar voice leadings between chords, or triadic voice leadings between scales. Roughly speaking, a scalar voice leading between chords is the average change between scales that contain each chord. Similarly a triadic voice leading between scales is the average voice leading between the possible tonics of that scale.

It is useful to interpret phases of the third component as *triadic orbits*. Figure 4 shows a sinusoid for the third component of a C diatonic scale. The peaks of the sinusoid are aligned as closely as possible to notes of the scale while the troughs avoid them. The troughs of the sinusoid divide the pitch class circle up into three triadic orbits, with the peaks at the center of each orbit. We can interpret notes that fall in the center of the orbits as triadically stable, and notes towards the periphery as unstable, drawn to the center of their respective orbits by a force of triadic resolution. A voice leading within orbits shifts them in the direction of the voice leading, but a voice leading that crosses orbit boundaries shifts them in the opposite direction.

In analytical application of Fourier phase space we may relate pcsets in two ways: a path from *A* to *B* may indicate “*A* in the context of *B*” or a motion from *A* to *B*. Mathematically these are equivalent: if a motion from *A* to *B* has a descending voice leading, then *A* has an upper-neighbor quality in the context of *B* (its notes tend to be high in the triadic orbits of *B*). Conceptually, however, these two kinds of relation

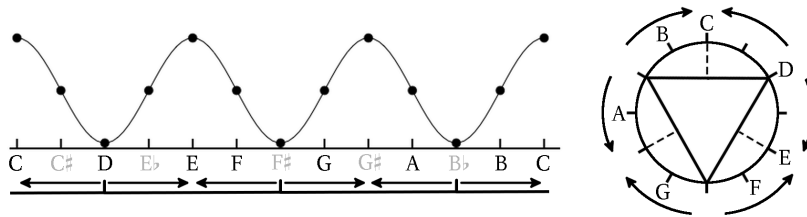


Fig. 4 The third Fourier component for a C diatonic scale, and its triadic orbits

are quite different and tend to apply to different kinds of objects. For example, if  $A$  is a single pitch class and  $B$  is a scale, we are more likely to talk about  $A$  in the context of  $B$ . Nonetheless, it is theoretically possible to speak of a triadic voice leading from a single pitch class to a scale (e.g., to explain a common-tone modulation).

The idea of orbits can also be applied to the fifth Fourier coefficient (the vertical dimension of the phase spaces in Figs. 2 and 3). The tendency tones of a key (such as leading tones) and chromatic notes would be unstable, with their orbits indicated by the letter name of their spelling.

## 2 Beethoven’s *Heiliger Dankgesang*

### 2.1 Tonal Contexts and Triadic Orbits

Beethoven’s *Heiliger Dankgesang*, the third movement of his late A minor String Quartet, op. 132, remains inspiring yet enigmatic nearly two centuries after its composition. Its biographic resonances, play of musical topics, and misprision of antiquated contrapuntal styles have all been well explored (e.g., [3], [10]). But current theories of harmony are not well tooled to address one of its most puzzling features, the status of tonality in this nominally “Lydian mode” work. The piece begins *in* F major, but ends only *on* F major, because its tonal context has shifted to C major. The meaning and purpose of this unusual tonal design is inaccessible to a theory that reduces the tonal contexts out of the middleground representation. We can overcome the problem without throwing the proverbial baby—the idea of deep structural voice leading—out with the bathwater with the spatial concept of tonality and harmony outlined above and the use of triadic orbits to construe this space in voice-leading terms.

The C-D interval is a prominent motivic element of the movement, manifest at many levels, and brought the forefront especially in the final chorale section. The status of this interval constitutes one of the most significant differences between the triadic orbits of the F major tonality suggested by the initial intonation of the chorale sections, and the C major tonality established in the later phrases of the chorale. This shift is already indicated by the first chorale phrase, as shown in Figure 5. In the initial contexts of {FGAC} in the intonation and {CDEFGA} in the first part of the chorale, D is an upper neighbor in the triadic orbit of C. The first cadence introduces the full C diatonic context, in which D is ambiguously positioned between two orbits. In later chorale phrases, the greater centrality of C major and its dominant shift the triadic orbits further in this direction so that D crosses over into an orbit where it strives upward, away from C. This is also reflected in the melodic shape of the first two chorale phrases, where D resists the descent to C that would complete the F–C tetrachordal space.<sup>1</sup>

<sup>1</sup> Korsyn’s [5] motivic analysis highlights motives of the chorale tune involving D, including C–D and D–E–F, the latter representative of D’s resistance to downward resolution.



The figure shows a musical score for piano and cello/bass, marked 'Molto adagio'. The score consists of two staves. Below the score, two circular diagrams illustrate triadic orbits. The left diagram, labeled 'Central', shows a circle with notes A, B, C#, D, E, F#, G. The right diagram, labeled 'Upper neighbor', shows a circle with notes A, C, D, E, F, G. A dashed line connects the D in the 'Central' diagram to the D in the 'Upper neighbor' diagram. The musical score has a dynamic marking 'p' at the beginning and another 'p' further along, with arrows indicating the scope of these dynamics.

Fig. 7 The transition from the *Neue Kraft* interlude back to the chorale

## 2.2 Strength and Weakness

A central metaphor to the *Heiliger Dankgesang* movement is the concept of strength versus weakness. Beethoven finds a musical analogue to this dichotomy ready at hand: strong and weak harmonic progressions, particularly cadential progressions. The feeling of strength or lack thereof in a tonal harmonic progression can be understood through triadic orbits: if the voice leading of the progression crosses triadic orbits, it is strong. The model of harmonic strength, the PAC, does just this: the melodic descent from  $\hat{2}$  crosses a triadic orbit. If voices instead remain within their orbits, acquiescing to their gravitational forces, the progression will tend to feel weak, as in the typical neighboring I–V–I or I–IV–I progressions so often used to tread water at the beginning of a Classical theme.

The third and last chorale section of the piece, an extended contrapuntal interweaving of the first intonation and first chorale phrase, ends with a C major cadence that is set up by the most intense dynamic and registral crescendo that the ensemble can muster. This willful cadence in C major, despite the clear formal requirements of a beginning in F, are an unprecedented tonal representation of human agency. Korsyn [5] notes how this section synthesizes registral, dynamic, and rhythmic traits of the *Neue Kraft* music with the material of the chorale sections. The D–C motion that has operated motivically at numerous levels in the piece is set in relief at the cadence by its register, the dramatic triple suspension, and the use of double stops in all instruments other than the first violin (Fig. 8). The chorale tune returns in the cello after this cadence, and here, for the first time, the melody completes the F–C tetrachord with a D–C descent. The context, however, has changed: following the cadence, the note B is completely absent for the remaining ten measures of the movement. The D–C motion in the cello is weak in this tonal context, the same one shown for the beginning of the movement in Figure 5. This juxtaposition of two D–C motions, one strong, the other weak, exhibits the distance that has been traveled by the tonal process of the chorale, and summarizes the main idea of the movement,

D - C (Strong)

PAC: C major    C-A hexachord    D - C (Weak)

**Fig. 8** Strong and weak C-D motions at the end of the third chorale section

gratitude for the strength granted in life, expressed at its fullest in the final moments before the surrender to death.

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