The Equilibrium Design of a Third-Party's Pricing Algorithm*

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Abstract

In markets such as apartment rentals, gasoline, and hotels, a data analytics company is supplying a pricing algorithm to competing suppliers. While there may be efficiencies from using the services of a third party, anticompetitive concerns emerge when a common agent recommends prices to competitors. To explore the properties of a third party's pricing algorithm, we consider a setting in which the third party is better able to condition price on a changing demand state - which is the efficiency it delivers - and designs the pricing algorithm to maximizes its profit from licensing it. Firms decide whether to pay the fee to gain access to the pricing algorithm and, upon receiving recommended prices, decide whether to implement them. Contrary to concerns expressed by some competition agencies, the third party does not build in a supracompetitive markup. However, price is more sensitive to the demand state compared to the competitive benchmark. This greater sensitivity is increasing in the adoption rate and the frequency with which demand changes.

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1 Introduction

There is a growing market for pricing algorithms supplied by data analytics companies such as RealPage, Yardi, A2i Systems, Kalibrate, Rainmaker, and others. A data analytics company is likely to offer a better pricing algorithm than would be created internally by a firm because it has access to more data, more expertise and experience, and stronger incentives to invest in their development (as the pricing algorithm can be licensed to many firms). Viewing a pricing algorithm as just another input, firms could well realize efficiencies from outsourcing this input. At the same time, concerns have rightly been expressed by the UK's Competition & Markets Authority,[1] the German Monopolies Commission,[2], the OECD,[3] and other institutions that there could be anticompetitive effect when competitors adopt a pricing algorithm from the same third party. According to plaintiffs in multiple class action suits, such concerns have been realized in the markets for apartment rentals, hotel rooms, and equipment rentals in the United States.[4] To add fuel to the fire, a recent study found evidence of both anticompetitive and procompetitive effects in the market for apartment rentals (Calder-Wang and Kim, 2024), and there is also evidence of anticompetitive effect in the supply of third-party pricing algorithms in the German retail gasoline market (Assad, Clark, Ershov, and Xu, 2024). In response to these concerns, the U.S. Senate has proposed legislation to constrain data analytics companies in how they use data[5] and similar restrictions have been proposed at the state and local level and implemented for housing markets in several U.S. cities.[6]

Clearly, if the third party and subscribing firms have a collusive agreement then one would indeed expect supracompetitive prices. But suppose that is not the case. Suppose the third party is designing the pricing algorithm to maximize its profits from licensing it and firms are independently deciding whether to contract with the third party and, should they do so, whether to implement the prices generated by the pricing algorithm. Will a third party design the pricing algorithm to recommend supracompetitive prices? One reason to think it might is that it will take into account that its pricing algorithm "competes against

[1] "If a sufficiently large proportion of an industry uses a single algorithm to set prices, this could result in a ... structure that may have the ability and incentive to increase prices." United Kingdom Competition & Markets Authority (2018) ¶ 5.21.

[2] A third party "knows or accepts [it] could contribute to a collusive market outcome [and] it is even conceivable that [they] see such a contribution as an advantage, as it makes the algorithm more attractive for users interested in profit maximization." German Monopolies Commission (2018) ¶ 263.

[3] "[C]oncerns of coordination would arise if firms outsourced the creation of algorithms to the same IT companies and programmers [and] [t]his might create a sort of 'hub and spoke' scenario where co-ordination is, willingly or not, caused by competitors using the same 'hub' for developing their pricing algorithms and end up relying on the same algorithms." OECD (2017) ¶ 68.

[4] See, for example, *Bason, et al. v. RealPage, Inc., et al.*, No. 3:22-cv-01611-WQH-MDD, U.S. District Court, Southern District of California, October 18, 2022; *Duffy, et al v. Yardi System, Inc., et al,* No. 23-cv-01391, U.S. District Court, Western District of Washington at Seattle, September 8, 2023; *Altman, et al. v. Caesars Entertainment, Inc., et al.*, No. 2 :23-cv-02536, U.S. District Court, District of New Jersey, May 9, 2023; and *Hanson Dai, et al v. SAS Institute, Inc., et al*, No. 3:24-cv-02537, U.S. District Court, Northern District of California, April 26, 2024; *Mack's Junk Removal, et al v Rouse Services, et al*, No. 2:25-cv-03565, U.S. District Court, Central District Court of California, April 22, 2025.

[5] "Preventing Algorithmic Collusion Act" https://www.congress.gov/bill/118th-congress/senate-bill/3686/text

[6] For a discussion of some of the competition law issues, see Harrington (2025a).

itself" when it licenses it to competitors. Thus, it may want to build in a supracompetitive markup to raise the profits for adopting firms if that would allow it to charge a higher fee.

To explore this and other issues, we develop and solve a model of the equilibrium design of a third party's pricing algorithm. The setting is one where there is demand variation across market segments and a third party's comparative advantage is that its pricing algorithm can tailor price to those market segments. The third party charges a fee for a firm to access the pricing algorithm's recommended prices and designs the pricing algorithm to maximize its profits from licensing it.

In assessing whether the third party's pricing algorithm is supracompetitive, the appropriate competitive benchmark is the pricing algorithm that firms would have independently designed if they had the ability to condition price on a market segment's demand state. Thus, our analysis will isolate the effect of the pricing algorithm's design being coordinated by the third party. Within our model, we show it is possible for the third party to program in a supracompetitive markup and induce firms to implement the recommended supracompetitive prices. However, the third party finds it optimal not to do so. The third party is able to charge a higher fee and earn higher profits by instead having the average price of its pricing algorithm equal the average price of the pricing algorithm under the competitive benchmark. Where the third party's pricing algorithm differs from what firms would have independently designed is that it has price being more sensitive to the demand state. Compared to the competitive benchmark, prices are higher in market segments with strong demand but are lower in market segments with weak demand. However, this property creates an incentive for a firm to price below the recommended price when demand is strong and above the recommended price when demand is weak. This problem is solved by the third party threatening to deny future service if a firm does not comply. Given this mechanism, the higher is a firm's discount factor, the more inclined they are to comply with the recommended prices and this allows the third party to make price more sensitive to the demand state. In sum, we find that the supply of a pricing algorithm by a third party to competitors does not raise the average price but does increase the variability of price as it is more responsive to local market conditions.

Section 2 introduces a standard static model of imperfect competition and characterizes equilibrium prices without a third party. Section 3 introduces a third party into that model. The third party designs a pricing algorithm to maximize its profit from licensing it and firms decide whether to pay the fee and adopt it where it is assumed an adopting firm implements the algorithm's recommended prices. This solution will provide a useful benchmark for the later analysis. In Section 4, the model of Section 3 is modified in two ways. First, an adopting firm now also decides whether to implement the algorithm's generated prices. Second, the static model becomes the stage game in an infinitely repeated game. Repetition will provide a mechanism through which the third party can incentivize adopting firms to implement the recommended prices. The equilibrium pricing algorithm is solved for in Section 5 where its properties are explored, and Section 6 investigates an adopting firm's surplus.

1.1 Literature Review

There is a growing theoretical literature exploring the implications of firms adopting and committing to pricing algorithms. One line of work has firms designing their pricing algorithms rather than having them supplied by a third party. Research includes Arunachaleswaran et al (2024), Brown and MacKay (2023, 2025), Johnson, Rhodes, and Wildenbeest (2023), Lamba and Zhuk (2024), Leisten (2024), Musolff (2024), and Salcedo (2015).[7] In the case of Musolff (2024), a third party is present in the form of repricers who supply pricing algorithm templates to firms. However, as it is the decision of a firm whether to make its pricing algorithm, say, aggressive or accommodating, the repricers' offerings do not determine the degree of competition as with the settings motivating the analysis of this paper.

Turning to the theoretical literature on the third-party supply of pricing algorithms, research has largely investigated how the pricing algorithm conditions on demand variation.[8] The third party's objective may be to maximize its profit from licensing the pricing algorithm (Harrington 2022, 2024, 2025b; Ichihashi, 2025) or from charging a commission which is proportional to firms' revenue (Hickok, 2024), or to maximize industry profit with the third party acting as an agent for adopting firms (Harrington, 2025b, 2025c; Sugaya and Wolitzky, 2025). When adopting firms are colluding with the assistance of the third party's pricing algorithm, Harrington (2025c) and Sugaya and Wolitzky (2025) show the resulting supracompetitive markup is increasing in the variability of demand. Information disclosure is more the focus in Hickok (2024) and Sugaya and Wolitzky (2025) where, under different assumptions, Hickok (2024) finds full information disclosure and Sugaya and Wolitzky (2025) finds less than full disclosure as the third party recommends the same price for a collection of strong demand states.

The papers most closely related to the current paper are Harrington (2025b) and Sugaya and Wolitzky (2025). Assuming the same market setting, the current paper and Harrington (2025b) differ with regards to firms' decisions. In both papers, firms decide whether to adopt the pricing algorithm but an adopting firm is given discretion regarding whether to implement the algorithm's recommended prices in the current paper, while they are assumed to implement those prices in Harrington (2025b).[9] As a result, the third party designs the pricing algorithm to maximizes its profit from licensing the algorithm subject to two incentive compatibility constraints: 1) firms are incentivized to adopt the pricing algorithm; and 2) firms are incentivized to implement the recommended prices. Only the first IC constraint is modelled in Harrington (2025b). When both IC constraints are allowed, we will see that it is the second IC constraint that binds and thus is determinative of the pricing algorithm's design. Sugaya and Wolitzky (2025) also allows an adopting firm to decide whether to implement the recommended prices. Its model differs from this paper's model in two ways. First, Sugaya and Wolitzky (2025) assume the third party's objective is to maximize adopting

[7] There is also an exploding literature exploring supracompetitive pricing when firms use learning algorithms such as Q-learning. See the seminal contributions of Waltman and Kaymak (2008) and Calvano, Calzolari, Denicolò, and Pastorello (2020) and subsequent papers citing them.

[8] A departure is Ichihashi (2025) where the model encompasses strategic uncertainty rather than demand uncertainty.

[9] Giving firms discretion over whether to comply with a third party's recommended prices is especially relevant in markets where prices tend to change infrequently so there is adequate time for a firm to evaluate an algorithm's recommendations, decide whether to implement them, and, if not, determine what prices to set in their place. For example, compliance may be more precarious in markets for apartment rentals than in gasoline markets. Apartment rents are probably not changed daily or even weekly which leaves time for a firm to intelligently override them. In contrast, gasoline prices often change over the course of a day and thus the adoption of a pricing algorithm's output may need to be automated.

firms' profits rather than its own profit from licensing the pricing algorithm. The setting is then one in which there is a collusive agreement between the third party and firms. Our interest is in when the third party and firms are acting independently. Second, the demand variation upon which the third party conditions price is intertemporal in Sugaya and Wolitzky (2025). In the current paper, it is cross sectional so the interpretation is that a third party can condition price on market segments. After we present our results, the role of the structure of demand variation is discussed.

2 Static Equilibrium without a Third-Party Developer

2.1 Model

Firms have symmetrically differentiated products and a common and constant marginal cost c. So as to allow for multiple firms in a tractable manner, there is a continuum of firms.[10] The market is partitioned into a continuum of submarkets with submarket $h \in [0,1]$. Firm demand in submarket h is assumed to be linear, $a^h - bp_i + dP_{-i}$, where p_i is a firm's own price in submarket h, P_{-i} is the average price of rival firms in submarket h, and $b > d > 0$.[11] a^h is a real random variable with finite support $\mathcal{A} \equiv \{a_1, ..., a_n\}$ where $0 < a_1 < ... < a_n$.[12] $g(a)$ is the probability $a^h = a$ and a^h has mean μ and variance σ^2.[13] Note that submarkets are ex ante identical as g does not depend on h. While there is demand uncertainty at the level of the individual submarket, there is certainty at the aggregate level represented by the same g. That is, in every period, the fraction of submarkets with $a^h = a$ is $g(a)$ for all $a \in \mathcal{A}$.

In each period, a^h is stochastically realized according to g. Firms are assumed not to observe a submarket's demand state. Given all submarkets have the same g then, in equilibrium, a firm will choose a common (or uniform) price across all submarkets. In contrast, the endowed advantage of the third party (and its raison d'être) is that it can track a submarket's demand state and thus condition price on it. One can imagine that, in each period, a function $\lambda : [0,1] \to \mathcal{A}$ is randomly selected which maps from submarkets to demand states. The third party observes that mapping λ. By our assumption of aggregate certainty, the probability function over demand states induced by any feasible λ is g.[14]

[10] This demand specification is the extension of a common specification - see Vives (1999), p. 146 - to when there is a continuum of firms. Having a firm be negligible enhances tractability because an individual firm's adoption decision will not affect the third party's design of the pricing algorithm. That property is a good approximation for when there are many firms in the market.

[11] To save on notation and without risk of causing confusion, prices will not be superscripted with h.

[12] The support is assumed to be sufficiently tight so that equilibrium prices always result in positive demand.

[13] It is only to ease some technical arguments that $\mathcal{A}$ is assumed to be finite. The paper's results depend only on μ and σ^2 and extend to when $\mathcal{A}$ is an interval $[\underline{a}, \overline{a}]$ and the cumulative distribution function on a^h is continuously differentiable.

[14] If $\{a^h\}_{h\in[0,1]}$ are *iid* with each having probability function g then, by the exact law of large numbers (i.e., the exact form of the Glivenko-Cantelli Theorem), the realized probability function of demand states over submarkets is also g. Here, we appeal to Theorem 2.8 in Sun (2006). In applying this theorem, the index set I is the set of markets $[0,1]$, $(\Omega, \mathcal{F}, P)$ is some probability space, and X is the set of demand states $\mathcal{A}$. f is a function from $I \times \Omega$ to X so that, for any $\omega \in \Omega$, $f(\cdot, \omega)$ maps submarkets to demand states.

2.2 Equilibrium

In the absence of a third party, the only information that firms have when they set their prices is that each submarket's demand state is distributed according to g. Consequently, equilibrium has a firm setting the same price for all submarkets and, furthermore, this price is the same across firms. A symmetric Nash equilibrium price p^N is defined by:

$$p^N \equiv \arg\max_{p_i \in \Re_+} \sum_{a \in \mathcal{A}} (p_i - c)\left(a - bp_i + dp^N\right) g(a) \Leftrightarrow p^N = \frac{\mu + bc}{2b - d}, \tag{1}$$

with expected profit

$$\pi^N \equiv \sum_{a \in \mathcal{A}} \left(\frac{\mu + bc}{2b - d} - c\right)\left(a - (b - d)\left(\frac{\mu + bc}{2b - d}\right)\right) g(a) = \frac{b\left(\mu - (b - d)c\right)^2}{(2b - d)^2}. \tag{2}$$

A useful benchmark for subsequent analysis is when a fraction θ of firms observe the demand state for each submarket and condition price on it, while the remaining firms cannot and thus set a uniform price. Rather than describing an informed firm's pricing rule as assigning a price to a submarket, we will equivalently describe it as assigning a price to a demand state a. When $a^h = a$, it sets the price for submarket h associated with demand state a. It is straightforward to show that the pricing rule for the firms who can condition price on the demand state is[15]

$$\phi^B(a) \equiv \frac{d(1 - \theta)\mu + (2b - d\theta)bc}{(2b - d\theta)(2b - d)} + \left(\frac{1}{2b - d\theta}\right) a, \tag{3}$$

and the fraction $1 - \theta$ of firms who set a uniform price choose $\frac{\mu + bc}{2b - d}$. Note that

$$\mathbb{E}\left[\phi^B\right] = \frac{\mu + bc}{2b - d} \tag{4}$$

so average price is the same for all firms. It also equals the uniform price in (1) when firms cannot condition on the demand state.

3 Static Equilibrium with a Third-Party Developer

3.1 Model

Let us now introduce a single third-party developer whose business model is to design and license pricing algorithms. The efficiency delivered by the third-party developer is that its pricing algorithm can track a submarket's demand state and condition price on it. While the

Because ω is randomly drawn, $f(\cdot, \omega)$ is the function λ assigning demand states to submarkets. If one does not want to impose independence of demand states aross submarkets, Proposition 2 in Feldman and Gilles (1985) shows individual uncertainty (at the submarket level) and aggregate certainty (across a continuum of submarkets) are consistent in that one can find a probability space and random variables delivering those properties.

[15] The derivation can be found in the appendix of Harrington (2025b). B refers to benchmark.

third party is assigning a price to each submarket, it will be more straightforward to represent the problem as assigning a price to a demand state and the price applies to all submarkets with that demand state. A generic pricing algorithm is denoted $\phi \in \Phi$ where Φ is the space of functions mapping $\mathcal{A}$ to $\mathbb{R}_+$. The third party charges a fixed fee f when it licenses a firm to use its pricing algorithm. Given (ϕ, f) , the third party's profit (or, equivalently, revenue, as there is assumed to be zero marginal cost to selling a pricing algorithm) is simply the number of firms that buy ϕ multiplied by f.[16]

This section considers a static setting and characterizes the equilibrium pricing algorithm when firms, upon adopting the pricing algorithm, are committed to implementing the prices generated by it. For ease of analysis, we will initially restrict attention to increasing affine pricing algorithms so $\phi(a) = \alpha + \gamma a$. Thus, a third party chooses $(\alpha, \gamma, f) \in \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}_+$. This pricing algorithm is a useful benchmark as it is what the third party would choose if it was not constrained to incentivizing the firms to implement the recommended prices. Come Section 4, we will turn to a dynamic model and give adopting firms discretion as to whether to implement the prices recommended by the algorithm, which is the setting of interest. Furthermore, Section 4 will not restrict pricing algorithms to being affine in the demand state but it will be shown that the equilibrium pricing algorithm is, in fact, affine.[17]

It is assumed that only some firms have the option of paying the fee and adopting the third party's pricing algorithm. We will refer to them as "eligible" firms and they will make up a fraction $\theta \in (0, 1]$ of all firms. If an eligible firm adopts the pricing algorithm then it implements the algorithm-generated prices. If an eligible firm does not adopt the pricing algorithm then it chooses a uniform price (i.e., one that does not condition on the demand state). The remaining $1 - \theta$ fraction of firms are referred to as "ineligible" and they decide on a uniform price to charge. In motivating the existence of ineligible firms, one might imagine that some managers are simply not open to the idea of delegating pricing to a third party or they face a high cost in terms of integrating the pricing algorithm into their pricing systems.[18] Our main results hold for all $\theta \in (0, 1]$ and having θ as a parameter allows us to perform comparative statics with respect to the adoption rate.[19]

With the third party and a continuum of firms as players, the extensive form game is as follows.

Stage 1: Third party chooses a pricing algorithm and fee, (α, γ, f) .

Stage 2: Firms observe (α, γ, f) . Eligible firms make adoption decisions where adoption means paying the fee f and committing to the pricing algorithm $\alpha + \gamma a$.

[16] The cost of designing the pricing algorithm is assumed to be independent of the pricing algorithm's properties and is sufficiently low so it is always profitable for a third party to participate in the market.

[17] The restriction to affine pricing algorithms is without loss of generality. Allowing the third party to choose any mapping from $\mathcal{A}$ to $\mathbb{R}_+$, it can be shown the equilibrium pricing algorithm is affine for the static model. We do not provide that proof here because it is contained in the proof that the equilibrium pricing algorithm for the dynamic model is affine, so its addition would be repetitive.

[18] When there is a firm-specific adoption cost which varies across firms, it is shown in Harrington (2024) that the third party may set the fee so that some eligible firms do not adopt in equilibrium.

[19] An alternative interpretation is that θ firms have a zero adoption cost (excluding the fee) and $1-\theta$ firms have a prohibitive adoption cost (so adoption is not optimal even with a zero fee). Thus, changing θ literally means changing the fraction of firms with zero adoption cost and is equivalent to changing the adoption rate.

Stage 3: The adoption rate $\theta^o \in [0, \theta]$ is observed by all firms (and the third party, though that will not be of consequence) and the assignment of demand states to submarkets is realized and observed only by the third party. Firms make simultaneous price decisions where adopting firms are constrained to price according to the pricing algorithm and non-adopting firms choose a uniform price.

A stylized feature of this model is that firms observe the third party's pricing algorithm when, in practice, a third party keeps its pricing algorithm confidential. As firms are faced with deciding whether to pay the fee, they will have to determine their willingness-to-pay (WTP) and it eases tractability to assume they know the pricing algorithm. One can imagine this observational assumption as representing firms learning about the pricing algorithm over time.

This model is the same as in Harrington (2022, 2025b) with some minor differences. As opposed to a continuum of firms, there are two firms in the model in Harrington (2022). The model in Harrington (2025b) has a continuum of firms but a different extensive form. As described above, the third party's pricing algorithm is observed by the firms prior to them making their adoption decisions. That sequential-move structure is not in Harrington (2025b). In all three models, the equilibrium pricing algorithm is the same.[20]

3.2 Equilibrium

We will derive subgame perfect equilibria through backward induction and perform a selection along the way. Come stage 3, the third party will have chosen a pricing $\phi(a) = \alpha + \gamma a$ in stage 1 and θ^o firms will have adopted it in stage 2. By assumption, those θ^o firms will price according to $\alpha + \gamma a$. The only decisions to be made in stage 3 are with regards to the uniform prices for the $1 - \theta^o$ non-adopting firms. Their symmetric equilibrium price, $\psi(\phi, \theta^o)$, is defined by:

$$\begin{aligned}
\psi(\phi, \theta^o) &= \arg\max_p \sum_{a \in \mathcal{A}} \pi\left(p, \theta^o \phi(a) + (1-\theta^o)\psi(\phi, \theta^o)\right) g(a) \\
&= \arg\max_p (p - c)\left(\mu - bp + d\left(\theta^o \mathbb{E}[\phi] + (1-\theta^o)\psi(\phi, \theta^o)\right)\right) \\
&= \frac{\mu + bc + d\theta^o \mathbb{E}[\phi]}{2b - d(1-\theta^o)} = \frac{\mu + bc + d\theta^o(\alpha + \gamma\mu)}{2b - d(1-\theta^o)}.
\end{aligned}$$

Turning to stage 2, the θ eligible firms decide whether to pay f and adopt the pricing algorithm $\alpha + \gamma a$ or instead plan to price at $\psi(\phi, \theta^o)$ in stage 3 (which will depend on the adoption rate θ^o coming out of stage 2). In making an adoption decision when it expects an adoption rate of θ^o, a firm will compare the profit from adopting (less the fee) with the profit from not adopting. For convenience, it is assumed a firm adopts when it is indifferent. Defining

$$\pi(p_i, P_{-i}, a) \equiv (p_i - c)(a - bp_i + dP_{-i})$$

[20] With regards to Harrington (2022), the equilibrium pricing algorithm is the same as long as the variance of the demand state σ^2 is above a threshold value. For a description of equilibrium when σ^2 is below that threshold value, the reader is referred to Harrington (2022).

as a firm's profit where p_i is a its own price, P_{-i} is the average price of other firms, and a is the demand state, adoption is optimal if and only if (iff)

$$\begin{aligned} & \sum_{a \in \mathcal{A}} \pi\left(\phi\left(a\right), \theta^{o}\phi\left(a\right) + (1-\theta^{o})\psi\left(\phi, \theta^{o}\right), a\right) g(a) - f \\ \geq \; & \sum_{a \in \mathcal{A}} \pi\left(\psi\left(\phi, \theta^{o}\right), \theta^{o}\phi\left(a\right) + (1-\theta^{o})\psi\left(\phi, \theta^{o}\right), a\right) g(a) \end{aligned} \tag{5}$$

where the LHS is the expected profit from adopting less the fee and the RHS is the expected profit from not adopting. Equivalent to (5) is that the fee does not exceed the WTP:

$$\begin{aligned} f \;\; \leq \;\; & WTP\left(\phi, \theta^{o}\right) \equiv \sum_{a \in \mathcal{A}} \pi\left(\phi\left(a\right), \theta^{o}\phi\left(a\right) + (1-\theta^{o})\psi\left(\phi, \theta^{o}\right), a\right) g(a) \\ & - \sum_{a \in \mathcal{A}} \pi\left(\psi\left(\phi, \theta^{o}\right), \theta^{o}\phi\left(a\right) + (1-\theta^{o})\psi\left(\phi, \theta^{o}\right), a\right) g(a). \end{aligned}$$

Assuming $\phi\left(a\right) = \alpha + \gamma a$, it is shown in the Appendix that

$$WTP\left(\phi, \theta^{o}\right) = \gamma\left(1 - (b - d\theta^{o})\gamma\right)\sigma^{2} - \frac{b\left(\mu + bc - (2b-d)(\alpha + \gamma\mu)\right)^{2}}{\left(2b - d(1-\theta^{o})\right)^{2}}. \tag{6}$$

Note that the WTP is increasing in the adoption rate,

$$\frac{\partial WTP\left(\phi, \theta^{o}\right)}{\partial \theta^{o}} = d\sigma^{2}\gamma^{2} + \frac{2bd\left(\mu + bc - (2b-d)\left(\alpha + \gamma\mu\right)\right)^{2}}{\left(2b - d(1-\theta^{o})\right)^{3}} > 0,$$

so adoption decisions are strategic complements. As a result, equilibrium either has all or no (eligible) firms adopting. More specifically, a stage 2 equilibrium adoption rate is: i) $\theta^{o} = 0$ when $f > WTP(\phi, 0)$; and ii) $\theta^{o} = \theta$ when $f \leq WTP(\phi, \theta)$. Given $WTP(\phi, 0) < WTP(\phi, \theta)$, if $f \leq WTP(\phi, 0)$ then the unique equilibrium adoption rate is $\theta^{o} = \theta$; if $WTP(\phi, 0) < f \leq WTP(\phi, \theta)$ then there are two equilibrium adoption rates, $\theta^{o} \in \{0, \theta\}$; and if $f > WTP(\phi, \theta)$ then the unique equilibrium adoption rate is $\theta^{o} = 0$. As it maximizes the third party's payoff, we will make a selection of the equilibrium with maximal adoption.[21]

Finally, we come to stage 1 and the third party's choice of a pricing algorithm and fee. Clearly, the third party will prefer to set the fee so all eligible firms adopt in the stage 2 equilibrium, as opposed to the alternative of no firms adopting. The third party's problem is to design the pricing algorithm and set the fee to maximize its revenue subject to inducing the θ eligible firms to adopt:

$$\max_{\alpha, \gamma, f} \theta \times f \;\; \text{s.t.} \;\; f \leq \gamma\left(1 - (b - d\theta)\gamma\right)\sigma^{2} - \frac{b\left(\mu + bc - (2b-d)(\alpha + \gamma\mu)\right)^{2}}{\left(2b - d + d\theta\right)^{2}}.$$

[21] One could imagine the third party engaging in some marketing activity to create the expectation among firms that other firms will adopt. Furthermore, upon solving for the stage 1 equilibrium, the equilibrium with complete adoption in stage 2 weakly Pareto dominates the equilibrium with no adoption in stage 2. The third party has a higher payoff from selling the pricing algorithm to all firms compared to not selling it, while firms receive the same payoff because the third party extracts the full surplus from adoption through its fee.

As optimality requires the constraint to bind, the third party will design the pricing algorithm to maximize the WTP and set the fee equal to the WTP. The third party's design problem is then

$$\max_{\alpha,\gamma} \gamma\left(1-(b-d\theta)\gamma\right)\sigma^2 - \frac{b\left(\mu+bc-(2b-d)(\alpha+\gamma\mu)\right)^2}{\left(2b-d+d\theta\right)^2}.$$

In examining the objective, the first term is the additional profit from being able to condition price on the demand state; it is maximized at $\gamma = \frac{1}{2(b-d\theta)}$. The second term is the net benefit that a non-adopter obtains from setting its prices optimally, given the algorithm's average price $\alpha+\gamma\mu$. It is non-positive and is minimized by having $\alpha+\gamma\mu = \frac{\mu+bc}{2b-d}$. Substituting for the equilibrium value of γ, the equilibrium value of α solves:

$$\begin{aligned}
\alpha+\gamma\mu &= \frac{\mu+bc}{2b-d} \Leftrightarrow \alpha = \frac{\mu+bc}{2b-d} - \gamma\mu = \frac{\mu+bc}{2b-d} - \left(\frac{1}{2(b-d\theta)}\right)\mu \\
\alpha &= \frac{2bc(b-d\theta)+d\mu(1-2\theta)}{2\left(b-d\theta\right)\left(2b-d\right)}.
\end{aligned}$$

Summing up, the equilibrium pricing algorithm is

$$\phi^S(a) \equiv \frac{2bc(b-d\theta)+d\mu(1-2\theta)}{2\left(b-d\theta\right)\left(2b-d\right)} + \left(\frac{1}{2(b-d\theta)}\right)a, \tag{7}$$

where S refers to the "static" solution. Non-adopting firms set a uniform price of

$$\psi\left(\phi^S,\theta\right) = \frac{\mu+bc+d\theta E[\phi^S]}{2b-d(1-\theta)} = \frac{\mu+bc}{2b-d}. \tag{8}$$

Substituting (7) into (6) and simplifying, the equilibrium WTP is:

$$WTP\left(\phi^S,\theta\right) = \frac{\sigma^2}{4(b-d\theta)}. \tag{9}$$

Based on the preceding analysis, the subgame perfect equilibrium strategy profile is described below.

- Third party's strategy:

$$\phi(a) = \frac{2bc(b-d\theta)+d\mu(1-2\theta)}{2\left(b-d\theta\right)\left(2b-d\right)} + \left(\frac{1}{2(b-d\theta)}\right)a,\ \ f = \frac{\sigma^2}{4(b-d\theta)}.$$

- Eligible firm's strategy
  - Given fee f and pricing algorithm $\alpha+\gamma a$, it adopts iff

$$f \le \gamma\left(1-(b-d\theta)\gamma\right)\sigma^2 - \frac{b\left(\mu+bc-(2b-d)(\alpha+\gamma\mu)\right)^2}{\left(2b-d+d\theta\right)^2}.$$

  - If it adopted then its price in submarket h is $\alpha+\gamma a$ when $a^h = a$ (by assumption).

– If it did not adopt and θ^o is the fraction of firms that adopted then its uniform price is

$$\frac{\mu + bc + d\theta^o(\alpha + \gamma\mu)}{2b - d(1 - \theta^o)}.$$

- Non-eligible firm's strategy: If θ^o is the fraction of firms that adopted then its uniform price is [22]

$$\frac{\mu + bc + d\theta^o(\alpha + \gamma\mu)}{2b - d(1 - \theta^o)}.$$

4 Dynamic Model

Thus far, the analysis has assumed an adopting firm is committed to setting the prices generated by the third party's pricing algorithm. It is as if the firm delegates pricing to the third party. We will now interpret the pricing algorithm's output as a *recommendation* and leave it to a firm to decide whether to implement the recommended prices or instead override them and charge something else. Having firms retain pricing authority raises a number of questions. How can firms be incentivized to implement recommended prices? How does the need to induce implementation of the price recommendations affect the design of the pricing algorithm? And, most crucially, does the third party build a supracompetitive markup into its pricing algorithm? These questions will be addressed in the context of an infinite (or indefinite) horizon where firms and the third party interact over time.

In each period, the third party designs a pricing algorithm and, for a fee, offers to license it to firms for the current period. Eligible firms decide whether or not to pay the fee in order to have access to the pricing algorithm's recommendations. The extensive form of the stage game is as in Section 3.1 except that we will give the third party the right to deny its services to any firm (which occurs between stages 2 and 3).[23] In any period, the third party will know all its past pricing algorithms, fees, firms' adoption decisions, and adopting firms' prices. A firm will know all past pricing algorithms, fees, adoption rates, its own adoption decisions, and recommended prices in periods it adopted. A firm's payoff is the sum of discounted expected profits where $\delta \in [0, 1)$ is a firm's discount factor. Given the stationarity of the problem, the equilibrium that is characterized will not require specifying the third party's time preferences.

In our setting, it is assumed firms are competing and thus do not have an agreement among themselves to comply with the third party's recommended prices.[24] Consequently, if a firm is to be induced to implement the recommended prices, it will be due to the third party's conduct. We will focus on the following device: If an adopting firm ever fails to implement recommended prices then the third party prohibits it from using the third party's services thereafter. Given firms are negligible, the threat is credible as executing

[22] Results do not change if all past prices are publicly observed.

[23] If this is not clear, the extensive form is provided at the start of Section 8.2.

[24] An analysis of collusion between the third party and adopting firms is explored in Harrington (2025b, 2025c), when firms decide on adoption and are committed to charging the prices generated by the pricing algorithm, and Sugaya and Wolitsky (2025), when firms are assumed to adopt and decide whether to charge the prices generated by the pricing algorithm.

it has no effect on the third party's profit. This punishment is the most severe from the class of punishments that only affect the noncompliant firm. To make that point, think about the third party having two levers by which to punish a firm: the pricing algorithm it supplies to the firm and the fee it charges the firm. A punishment of denying services can be interpreted as charging a sufficiently high fee to the noncompliant firm so it chooses to no longer subscribe. Alternatively, it can change the design of the pricing algorithm supplied to the noncompliant firm (while setting the fee so that it would pay it). However, as the noncompliant firm can always ignore the price recommendations of the algorithm and post the same price across all submarkets, their payoff must be at least as high as that for the punishment denying the third party's services. Thus, if there is a more severe punishment for a noncompliant firm, it must entail changing the design of the pricing algorithm for compliant firms. The third party could revert to the stage game equilibrium which has it supply ϕ^B (which is the equilibrium pricing algorithm when $\delta = 0$) and charge a fee equal to the WTP. However, the punishment payoff is the same as barring the noncompliant firm from accessing the third party's pricing algorithm and continuing to supply ϕ^D to compliant firms.[25] Thus, a lower punishment payoff can be delivered only by changing the pricing algorithm supplied to compliant firms so prices are lower than ϕ^B. Furthermore, the third party could induce adoption of such a pricing algorithm by appropriately lowering the fee. But such a punishment is harmful to the third party and it seems unrealistic for a single firm's noncompliance to affect the pricing algorithm for all firms. Balancing severity and realism, we find a punishment of barring a noncompliant firm from the third party's services to be the most compelling.

Towards describing the third party's optimization problem, consider an adopting firm who considers not implementing the algorithm's generated prices. Having been informed of what price to set for each submarket and given knowledge of the pricing algorithm, a firm updates its beliefs on a submarket's demand state. More specifically, given a recommended price $p' \in \{\phi(a) : a \in \mathcal{A}\}$ for submarket h and knowing the third party is using ϕ, a firm infers demand state $a^h \in \{a \in \mathcal{A} : \phi(a) = p'\}$.[26] Updating its beliefs on the demand state, the optimal deviation price for that submarket is

$$\xi(p') \equiv \arg\max_{p} \sum_{a' \in \{a \in \mathcal{A}:\phi(a)=p'\}} \pi(p, \theta p' + (1-\theta)\psi(\phi), a') \left(\frac{g(a')}{\sum_{a \in \{a \in \mathcal{A}:\phi(a)=p'\}} g(a)} \right). \tag{10}$$

Given the punishment is the same regardless of how many recommended prices are overridden, the best deviation will entail setting the profit-maximizing price in all submarkets given the firm's updated beliefs on a submarket's demand state. Thus, total deviation profit is

$$\sum_{a \in \mathcal{A}} \pi(\xi(\phi(a)), \theta\phi(a) + (1-\theta)\psi(\phi), a)\, g(a).$$

[25] The "denying service" punishment results in the noncompliant firm having profit of π^N as it sets a uniform price of $\frac{\mu+bc}{2b-d}$ and the other firms' average price is $\frac{\mu+bc}{2b-d}$. While firms earn higher profit when they all use ϕ^B - as their prices adjust to the demand state and average price is still $\frac{\mu+bc}{2b-d}$ - the third party extracts that additional profit with its fee so net profit is again π^N.

[26] While it is stylized to suppose firms are able to make such inferences on the demand state from the recommended price, it is one way to capture the reasonable property that a firm will infer something about the demand state from the recommended price when it goes about overriding the recommended price.

The third party's problem is to design the pricing algorithm and set the fee to maximize its revenue subject to incentivizing eligible firms to adopt the pricing algorithm and then implement the recommended prices.

$$\sup_{\phi \in \Phi, f \in \mathbb{R}_+} \quad \theta \times f \text{ s.t.} \tag{11}$$

$$\begin{aligned} & \sum_{a \in \mathcal{A}} \pi\left(\phi\left(a\right), \theta\phi\left(a\right) + (1-\theta)\psi\left(\phi\right), a\right) g(a) - f \\ \geq & \sum_{a \in \mathcal{A}} \pi\left(\psi\left(\phi\right), \theta\phi\left(a\right) + (1-\theta)\psi\left(\phi\right), a\right) g(a) \end{aligned} \tag{12}$$

$$\begin{aligned} & \sum_{a \in \mathcal{A}} \pi\left(\phi\left(a\right), \theta\phi\left(a\right) + (1-\theta)\psi\left(\phi\right), a\right) g(a) \\ & + \left(\frac{\delta}{1-\delta}\right)\left(\sum_{a \in \mathcal{A}} \pi\left(\phi\left(a\right), \theta\phi\left(a\right) + (1-\theta)\psi\left(\phi\right), a\right) g(a) - f\right) \\ \geq & \sum_{a \in \mathcal{A}} \pi\left(\xi\left(\phi\left(a\right)\right), \theta\phi\left(a\right) + (1-\theta)\,\psi\left(\phi\right), a\right) g(a) \\ & + \left(\frac{\delta}{1-\delta}\right) \sum_{a \in \mathcal{A}} \pi\left(\psi\left(\phi\right), \theta\phi\left(a\right) + (1-\theta)\,\psi\left(\phi\right), a\right) g(a) \end{aligned} \tag{13}$$

and $\psi\left(\phi\right)$ is the equilibrium uniform price for non-adopting firms,

$$\psi\left(\phi\right) = \arg\max_{p} \sum_{a \in \mathcal{A}} \pi\left(p, \theta\phi\left(a\right) + (1-\theta)\psi\left(\phi\right), a\right) g(a) \Rightarrow \psi\left(\phi\right) = \frac{\mu + bc + d\theta\mathbb{E}[\phi]}{2b - d(1-\theta)}.$$

(12) is the incentive compatibility constraint (ICC) ensuring an eligible firm will want to pay f for access to the third party's pricing algorithm. The LHS of the inequality is the profit to a firm adopting (and complying with) the pricing algorithm minus the fee and the RHS is the profit to a firm not adopting the pricing algorithm. Having adopted the pricing algorithm, (13) is the ICC ensuring a firm will implement the recommended prices. The LHS of the inequality is the current period's profit from implementing recommended prices (and note the fee is absent as it is a sunk cost) plus the present value of the future profit stream from adopting and implementing the pricing algorithm. On the RHS of the inequality is the maximal deviation profit plus the present value of the future profit stream from not having access to the pricing algorithm.

These two ICCs can be rearranged to:

$$\begin{aligned} f \quad \leq & \sum_{a \in \mathcal{A}} \pi\left(\phi\left(a\right), \theta\phi\left(a\right) + (1-\theta)\psi\left(\phi\right), a\right) g(a) \\ & - \sum_{a \in \mathcal{A}} \pi\left(\psi\left(\phi\right), \theta\phi\left(a\right) + (1-\theta)\psi\left(\phi\right), a\right) g(a) \end{aligned} \tag{14}$$

$$f \leq \begin{pmatrix} \sum_{a\in\mathcal{A}} \pi\left(\phi\left(a\right), \theta\phi\left(a\right) + (1-\theta)\psi\left(\phi\right), a\right) g(a) \\ -\sum_{a\in\mathcal{A}} \pi\left(\psi\left(\phi\right), \theta\phi\left(a\right) + (1-\theta)\psi\left(\phi\right), a\right) g(a) \end{pmatrix} \tag{15}$$
$$-\left(\frac{1-\delta}{\delta}\right)\begin{pmatrix} \sum_{a\in\mathcal{A}} \pi\left(\xi\left(\phi\left(a\right)\right), \theta\phi\left(a\right) + (1-\theta)\,\psi\left(\phi\right), a\right) g(a) \\ -\sum_{a\in\mathcal{A}} \pi\left(\phi\left(a\right), \theta\phi\left(a\right) + (1-\theta)\psi\left(\phi\right), a\right) g(a) \end{pmatrix}.$$

Note that the RHS of (14) is the same as the first of the two expressions on the RHS of (15). Given that the second expression on the RHS of (15) is negative then: If (15) holds then (14) holds. That is, if an adopting firm finds it optimal to comply with the pricing algorithm then an eligible firm will find it optimal to adopt the pricing algorithm. Hence, the third party's problem is to choose (ϕ, f) to maximize $\theta \times f$ subject to (15).

As the constraint (15) will be binding at the optimal solution - as the third party will want to set the fee as high as possible controlling for demand for the pricing algorithm - the problem can be recast as designing the pricing algorithm to maximize the maximum fee satisfying it:

$$\max_{\phi\in\Phi} \begin{pmatrix} \sum_{a\in\mathcal{A}} \pi\left(\phi\left(a\right), \theta\phi\left(a\right) + (1-\theta)\psi\left(\phi\right), a\right) g(a) \\ -\sum_{a\in\mathcal{A}} \pi\left(\psi\left(\phi\right), \theta\phi\left(a\right) + (1-\theta)\psi\left(\phi\right), a\right) g(a) \end{pmatrix} \tag{16}$$
$$-\left(\frac{1-\delta}{\delta}\right)\begin{pmatrix} \sum_{a\in\mathcal{A}} \pi\left(\xi\left(\phi\left(a\right)\right), \theta\phi\left(a\right) + (1-\theta)\,\psi\left(\phi\right), a\right) g(a) \\ -\sum_{a\in\mathcal{A}} \pi\left(\phi\left(a\right), \theta\phi\left(a\right) + (1-\theta)\psi\left(\phi\right), a\right) g(a) \end{pmatrix}.$$

5 Dynamic Equilibrium

The equilibrium pricing algorithm is described in Theorem 1 with the proof in the appendix (Section 8.1). A subgame perfect equilibrium strategy profile supporting the outcome characterized in Theorem 1 is described in the appendix (Section 8.2).

Theorem 1 *The solution to (16) is $\phi^D(a) \equiv \alpha^D + \gamma^D a$ where*

$$\alpha^D = \frac{2b(2bc(b-d\theta) + d\mu(1-2\theta)) + (1-\delta)d\theta\left(\mu(2b-d(1-\theta)) + bd\theta c\right)}{(2b-d)\left(4b(b-d\theta) + d^2\theta^2(1-\delta)\right)}$$
$$\gamma^D = \frac{2b-d\theta(1-\delta)}{4b(b-d\theta)+d^2\theta^2(1-\delta)},$$

and the fee is

$$f^D = \frac{b\sigma^2}{4b(b-d\theta)+d^2\theta^2(1-\delta)}.$$

In equilibrium, all eligible firms adopt so the adoption rate is θ.

There are two key steps to showing the equilibrium pricing algorithm is an increasing affine function of the demand state. The first step shows that the optimal pricing algorithm is linear in the conditional mean of the demand state: the price $\phi\left(a_k\right)$ recommended at demand state a_k is a linear function of the conditional mean $\mathbb{E}[a|\phi(a) = \phi\left(a_k\right)]$. An implication is that the third party's expected profits depend on the pricing algorithm ϕ only through the distribution of conditional means induced by ϕ. The second step shows that the third party's profits are convex in the conditional mean of the demand state. As a result, pricing algorithms that generate a more dispersed distribution of conditional means yield higher profits. In particular, the profit-maximizing pricing algorithm does not pool states and thus is strictly increasing in the demand the state.

The first point to make is that the third party's pricing algorithm is robust to giving firms the option not to implement the recommended prices. As $\delta \to 1$, ϕ^D converges to the pricing algorithm when firms fully delegate to the third party; that is, $\lim_{\delta\to 1}\phi^D = \phi^S$, where ϕ^S is defined in (7). Thus, if firms sufficiently value future profits then the third party chooses a pricing algorithm that is approximately the same as when the third party has control over firms' prices. The reason is that the threat of being excluded from accessing the pricing algorithm makes it easy to incentivize firms to comply with the recommended prices. Next note that if firms are myopic then the only pricing algorithm for which firms will implement the recommended prices is the one that they would have chosen if they could design it; that is, if $\delta = 0$ then $\phi^D = \phi^B$ where ϕ^B is defined in (3).

To consider whether the third party's pricing algorithm has a supracompetitive markup, recall that the competitive benchmark is when firms independently design their pricing algorithms to condition price on the demand state. This is the pricing algorithm ϕ^B so the average price of the competitive equilibrium pricing algorithm is $\mathbb{E}[\phi^B] = \frac{\mu+bc}{2b-d}$. Taking the expectation of the pricing algorithm in Theorem 1, we find that the third party does not build in a supracompetitive markup:

$$\begin{aligned}\mathbb{E}[\phi^D] \quad &= \quad \frac{2b(2bc(b-d\theta)+d\mu(1-2\theta))+(1-\delta)d\theta\left(\mu(2b-d(1-\theta))+bd\theta c\right)}{(2b-d)\left(4b(b-d\theta)+d^2\theta^2(1-\delta)\right)} \\ &\quad + \left(\frac{2b-d\theta(1-\delta)}{4b(b-d\theta)+d^2\theta^2(1-\delta)}\right)\mu = \frac{\mu+bc}{2b-d}.\end{aligned}$$

It is true that a higher average price for the pricing algorithm would raise the profits that a third party delivers for an adopting firm. That follows from all θ adopting firms pricing closer to the joint profit-maximizing price. However, it would also raise the profits to a non-adopting firm because θ rival firms would be pricing higher. Given a non-adopting firm optimizes its uniform price against the average price of adopting firms while an adopting firm does not (as it is just sets the prices generated by the pricing algorithm), a non-adopting firm's profit rises *more* than an adopting firm's profit in response to an increase in the average price of the pricing algorithm. That lowers a firm's WTP which would cause the third party to reduce its fee and thereby earn lower revenue. While a third party could design the pricing algorithm to have a supracompetitive markup, it is optimal for it not to do so. This result is consistent with the finding in Harrington (2022) when an adopting firm is forced to implement the recommended prices.

Compared to when firms independently design their pricing algorithms, the third party's pricing algorithm is more sensitive to the demand state:

$$\frac{\partial\phi^D}{\partial a} = \frac{2b-d\theta(1-\delta)}{4b(b-d\theta)+d^2\theta^2(1-\delta)} > \frac{1}{2b-d\theta} = \frac{\partial\phi^B}{\partial a}.$$

The reason is that the third party is internalizing the effect of all adopting firms' prices when it designs the pricing algorithm. A stronger demand state implies the optimal price is higher. Given prices are strategic complements, the best price for an adopting firm is then higher when other adopting firms raise their prices. The third party internalizes that effect so its pricing algorithm has price respond more to a stronger demand state compared to when each firm independently designs its pricing algorithm. Of course, by the same logic, one might have thought that the third party would internalize the effect of an adopting firm's price on all adopting firms' profits and thus build in a supracompetitive markup. We did not find that to be the case because non-adopting firms would exploit it by undercutting the higher average price of adopting firms. In contrast, adopting firms cannot exploit the third party making price more sensitive to the demand state because non-adopting firms cannot condition price on the demand state; that only comes from using the third party's pricing algorithm. The third party can make price more sensitive to the demand state relative to ϕ^B and, as long as it keeps average price fixed (i.e., raising γ and adjusting α so average price is unchanged), it will raise an adopting firm's profit while leaving a non-adopting firm's profit unchanged; hence, the WTP rises and with it the third party's fee.

This greater sensitivity of price to the demand state from the third party's pricing algorithm is accentuated when the adoption rate is higher as then there are more firms for whom the third party internalizes the effect of their prices. In response to a higher adoption rate, the third party's pricing algorithm is more sensitive to the demand state in absolute terms,

$$\frac{\left(\partial\phi^D/\partial a\right)}{\partial\theta} = \frac{d\left(4b(b(1+\delta)-d\theta(1-\delta))+d^2\theta^2(1-\delta)^2\right)}{\left(4b(b-d\theta)+d^2\theta^2(1-\delta)\right)^2} > 0,$$

and relative to the competitive pricing algorithm (as long as $\delta > 0$),

$$\frac{\partial\left(\frac{\partial\phi^D}{\partial a}-\frac{\partial\phi^B}{\partial a}\right)}{\partial\theta} = \frac{4bd\delta\left(4b^3-3bd^2\theta^2+d^3\theta^3(1-\delta)+bd^2\theta^2\delta\right)}{(2b-d\theta)^2\left(4b(b-d\theta)+d^2\theta^2(1-\delta)\right)^2} > 0$$

Both $\frac{\partial\phi^D}{\partial a}$ and $\frac{\partial\phi^B}{\partial a}$ are increasing in the adoption rate because prices are strategic complements. A firm prices high (low) when demand is strong (weak) and if more firms are pricing in such a manner then prices will be even higher (lower) when demand is strong (weak). The impact of the adoption rate on the responsiveness of the pricing algorithm to the demand state is stronger with ϕ^D because the third party is internalizing this effect across all adopting firms.

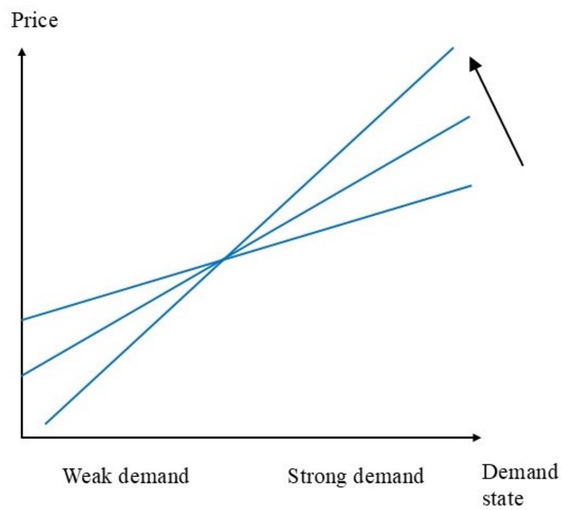
In designing its pricing algorithm, recall that the third party is constrained by the need to incentivize adopting firms to implement the prices generated by the algorithm. When demand is strong, the recommended price is above the competitive pricing algorithm - $\phi^D(a) > \phi^B(a)$ - and this means it is above a firm's best response so an adopting firm has a short-run incentive to price lower; and, when demand is weak, the recommended price is below the competitive pricing algorithm - $\phi^D(a) < \phi^B(a)$ - which means it is below a firm's best response so an adopting firm has a short-run incentive to price higher. The third party induces compliance through the threat of denying a firm access to its pricing algorithm in the future should it not implement the algorithm's generated prices. As a firm's discount factor is higher, it is then easier to incentivize compliance with the recommended prices, and that allows the third party to increase the sensitivity of price to the demand state. Thus, as the discount factor is increased, the third party's pricing algorithm is more sensitive to the demand state in absolute terms,

$$\frac{\left(\partial\phi^D/\partial a\right)}{\partial\delta} = \frac{2bd\theta(2b-d\theta)}{\left(4b(b-d\theta)+d^2\theta^2(1-\delta)\right)^2} > 0,$$

and relative to the competitive pricing algorithm since ϕ^B does not depend on δ.

Figure 1 depicts the effect on the third party's pricing algorithm as the adoption rate or the discount factor increase. The pricing algorithm becomes more sensitive to the demand state as θ or δ rises - as reflected in a steeper slope - and it rotates around the mean demand state μ. Thus, prices are higher when demand is above the mean and lower when demand is below the mean with average price remaining fixed at the competitive level.

Figure 1: Pricing Algorithm's Sensitivity to the Demand State



As θ or δ increase, slope increases.

Finally, there is a complementarity between the adoption rate and the discount factor, as we will show

$$\frac{\partial^2(\partial\phi^D/\partial a)}{\partial\theta\partial\delta} > 0.$$

Since

$$\frac{\partial^2(\partial\phi^D/\partial a)}{\partial\theta\partial\delta} = \frac{\partial^2\gamma^D}{\partial\delta\partial\theta} = \frac{4bd\left(4b^3-d^2\theta^2(1-\delta)(3b-d\theta)\right)}{\left(4b(b-d\theta)+d^2\theta^2(1-\delta)\right)^3},$$

it follows that

$$sign\left\{\frac{\partial^2\gamma^D}{\partial\delta\partial\theta}\right\} = sign\left\{4b^3-d^2\theta^2(1-\delta)(3b-d\theta)\right\}.$$

The expression in brackets is increasing in δ and decreasing in θ, so it is minimized at $\delta=0$ and $\theta=1$. Thus, a sufficient condition for it to be positive is

$$4b^3-d^2(3b-d)>0 \Leftrightarrow b(4b^2-3d^2)+d^3>0,$$

which holds. As the adoption rate rises, the third party wants to make price more responsive to the demand state because the value from internalizing firms' prices being strategic complements is larger when there are more firms subscribing to the pricing algorithm. However, the extent to which the third party can do so depends on its ability to induce adopting firms to implement recommended prices, and that becomes easier as the discount factor increases. Thus a higher adoption rate and a higher discount factor complement each other to make the pricing algorithm more responsive to the demand state.

In closing this section, let us compare the equilibrium pricing algorithm with that derived in Sugaya and Wolitzky (2025). Theorem 1 shows that the third party's pricing algorithm fully discloses its demand information as the pricing algorithm is monotonically increasing in the demand state. In contrast, there is less than full information disclosure in Sugaya and Wolitzky (2025). Sugaya and Wolitzky (2025) embed a third party into a repeated oligopoly game with stochastic demand along the lines of Rotemberg and Saloner (1986). The third party privately observes the demand state each period, and chooses what public information to disclose. They show that the optimal information disclosure policy pools high demand states, so as to reduce firms' incentives to deviate from the recommended price. The critical difference with our paper is in the incentive compatibility constraints faced by the third party.[27] In Sugaya and Wolitzky (2025), the third party must satisfy a continuum of constraints, one for each public signal it sends. These constraints bind only when the demand state is above a threshold, which makes profits concave around the threshold and renders pooling optimal. By contrast, in our setting with a continuum of submarkets and no aggregate uncertainty, there is a single incentive compatibility constraint ensuring that firms do not gain from deviating across all submarkets. Because third-party profits are convex in the conditional mean of the demand state, the optimal algorithm is fully revealing.

6 Adopting Firm's Surplus

As shown in Section 3, when firms are committed to implementing a pricing algorithm's prices, the third party charges a fee equal to the firms' WTP and extracts the entire surplus. That is not the case when the third party has to incentivize adopting firms to implement the recommended prices. To see this property, let us first derive the WTP evaluated at the equilibrium pricing algorithm. Inserting $\left(\alpha^D, \gamma^D\right)$ into (6) and simplifying, we find:[28]

$$WTP(\phi^D) = \frac{b\sigma^2\left(4b(b-d\theta)+d^2\theta^2(1-\delta^2)\right)}{\left(4b(b-d\theta)+d^2\theta^2(1-\delta)\right)^2}.$$

[27] Though not relevant to the point being made here, another distinction is the third party's objective. The third party is assumed to maximize firms' profits in Sugaya and Wolitzky (2025), while we assume it maximizes its profit from licensing the pricing algorithm.

[28] In equilibrium, $WTP(\phi^D)$ is an adopting firm's profit minus a non-adopting firm's profit. Note that a non-adopting firm's profit depends only on the mean of a rival firm's price. Given the mean price is the same for the equilibrium with a third party and the equilibrium without a third party then the equilibrium non-adopting firm's profit is equal to the static Nash equilibrium profit π^N, see (2). Hence, $WTP(\phi^D)$ also measures an eligible firm's gain from the presence of a third party in the market.

Netting out the equilibrium fee, the surplus to an adopting firm is

$$\begin{aligned} S(\delta) &\equiv WTP(\phi^D) - f^D = \frac{b\sigma^2\left(4b(b-d\theta)+d^2\theta^2(1-\delta^2)\right)}{\left(4b(b-d\theta)+d^2\theta^2(1-\delta)\right)^2} - \frac{b\sigma^2}{4b(b-d\theta)+d^2\theta^2(1-\delta)} \\ &= \frac{bd^2\theta^2(1-\delta)\delta\sigma^2}{\left(4b(b-d\theta)+d^2\theta^2(1-\delta)\right)^2}. \end{aligned}$$

While $S(0)=0=S(1)$, $S(\delta)>0$ for all $\delta\in(0,1)$ so surplus is left to firms.

Towards understanding the determination of the surplus, let us first show $S(\delta)$ is quasi-concave in δ. Given

$$S'(\delta) = \frac{bd^2\theta^2\sigma^2}{\left(4b(b-d\theta)+d^2\theta^2(1-\delta)\right)^3}\left(d^2\theta^2 - 8b^2\delta + 4b^2 - 4bd\theta - d^2\theta^2\delta + 8bd\theta\delta\right),$$

then

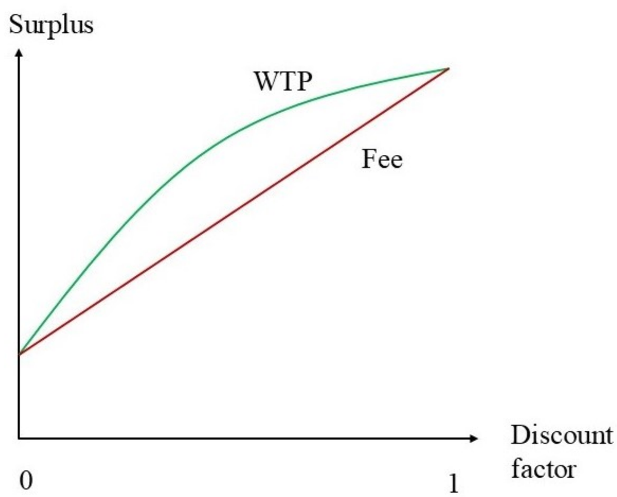
$$S'(\delta) \gtreqless 0 \text{ as } d^2\theta^2 - 8b^2\delta + 4b^2 - 4bd\theta - d^2\theta^2\delta + 8bd\theta\delta \gtreqless 0$$

or

$$S'(\delta) \gtreqless 0 \text{ as } \delta \lesseqgtr \widehat{\delta} \equiv \frac{d^2\theta^2 + 4b(b-d\theta)}{d^2\theta^2 + 8b(b-d\theta)} \in (0,1).$$

Thus, an adopting firm's surplus is maximized at some intermediate value of δ, as shown in Figure 2.

Figure 2: Firm's Surplus



When $\delta=0$, firms care only about current profit and thus cannot be incentivized to depart from what which maximizes it. The third party then chooses pricing algorithm ϕ^B and extracts the entire surplus with its fee. As $\delta\to 1$, firms care only about future profits so incentivizing them is easy, in which case the third party chooses the first-best solution ϕ^S and extracts the entire surplus with its fee. When $\delta\in(0,1)$, the third party needs to leave firms with some surplus in order to induce them to comply with the pricing algorithm. The tension at work is the following. Collectively, firms are better off with price being highly sensitive to the demand state but individually want to make price less sensitive to demand by pricing below the recommended price when demand is strong and above the recommended price when demand is weak. While the third party could choose ϕ^B and set the fee equal to the WTP, it does better by making the pricing algorithm more responsive to the demand state. That will raise the WTP to adopt and, even after leaving enough surplus to induce adopting firms io implement the recommended prices, its fee is higher. This property is further reflected in the equilibrium fee being increasing in firms' discount factor,

$$\frac{\partial f^D}{\partial \delta} = \frac{bd^2\theta^2\sigma^2}{\left(4b(b-d\theta)+d^2\theta^2(1-\delta)\right)^2} > 0.$$

Given it is easier to induce compliance when firms attach more value to future profits, the third party can increase joint adopters' profits - by making price more sensitive to the demand state - and charge a higher fee. Note that a higher value of δ captures a shorter period and thus a situation in which demand (and, along with it, price) changes more frequently.

As the adoption rate rises, the collective benefit from making price more responsive to the demand state is greater which means the third party wants the pricing algorithm to depart more from what is individually optimal. This requires giving firms more of the surplus in order to incentivize them to comply with the pricing algorithm,

$$\frac{\partial S}{\partial \theta} = \frac{2bd^2\theta\sigma^2\delta\,(1-\delta)\left(4b^2-d^2\theta^2(1-\delta)\right)}{\left(4b(b-d\theta)+d^2\theta^2(1-\delta)\right)^3} > 0.$$

Even after delivering more surplus to subscribing firms, the third party is able to charge a higher fee when a higher fraction of firms adopt its pricing algorithm,

$$\frac{\partial f^D}{\partial \theta} = \frac{\left(2b-d\theta(1-\delta)\right)bd\sigma^2}{\left(4b(b-d\theta)+d^2\theta^2(1-\delta)\right)^2} > 0.$$

In conclusion, we find that both the third party's fee and a subscribing firm's surplus are increasing in the adoption rate. Thus, they have a common incentive to see that the adoption rate is higher. However, their preferences depart with regards to the frequency with which demand changes, as captured by δ. While the third party's payoff is increasing in firms' discount factor, a firm's payoff is non-monotonic in its discount factor.

7 Concluding Remarks

A fundamental assumption in modelling a firm's prices is that the firm chooses its pricing rule to maximize its profits and, in the absence of a collusive agreement, competitors choose their pricing rules independently. That assumption is now being upended with the emergence of data analytics companies who will design a pricing algorithm for a firm. This has two key implications. First, the pricing algorithm is being designed to maximize the profits to a third party from licensing it. While a firm would only pay a fee to the third party if the use of its pricing algorithm resulted in higher profits, that does not necessarily imply the pricing algorithm is designed to maximize the profits of the firm that is using it. Second, competitors' pricing algorithms are not being independently designed if they are using the same third party. The third party coordinates the design decisions and, in particular, takes into account that the pricing algorithm will be competing against itself.

This paper explores this setting to understand the incentives of a third party and its implications for properties of its pricing algorithm. The market environment is one in which there is demand variation and the efficiency of a third party is that it can observe that variation and tailor price to market segments. Given a third party would like to design its pricing algorithm to result in higher profits for those firms who adopt it - as that would allow it to charge a higher fee - it would seem to suggest that it would build in a supracompetitive markup. However, a pricing algorithm that generates high prices is ripe for exploitation by non-adopting firms. While a non-adopting firm loses out by not being able to condition price on the demand state - and instead must set a uniform price across market segments - it benefits from being able to undercut the high average price of rival firms who are using the third party's pricing algorithm. We showed that this prospect of exploitation of a supracompetitive pricing algorithm induces the third party to design the pricing algorithm so that its average price is at the competitive level, so there is no supracompetitive markup. Where the third party's pricing algorithm departs from the competitive pricing algorithm is that it makes price more sensitive to the demand state. As a result, firms price higher when demand is strong and lower when demand is weak compared to when the pricing algorithm is independently designed by firms. By internalizing the strategic complementarity of firms' prices, the third party makes the pricing algorithm more attractive because it raises adopting firms' profits. As non-adopting firms cannot condition price on the demand state, the heightened sensitivity of price to the demand state is not exploitable. However, making price very sensitive to the demand state creates an incentive for an adopting firm to override the recommended price as it earns higher profit by pricing below the recommended price when demand is strong and above the recommended price when demand is weak. We show how the third party can induce compliance though in doing so it is forced to limit the pricing algorithm's sensitivity to the demand state.

There is much more work to be done to understand the market for pricing algorithms and we'll close with two research directions. A feature of our theory, as well as Sugaya and Wolitzky (2025), is that the equilibrium pricing algorithm results in all recommended prices being implemented. In practice, some firms have been found to regularly override the recommended prices.[29] There are at least two reasons why we need a theory which results in some but not all price recommendations being implemented. First, a theory should fit the facts and this is a feature of the data. Second, a firm's price may be influenced by the recommended price even when it does not adopt the latter. A theory is needed to tell us how recommended prices would influence actual prices.

A second important research direction is to allow for competing third parties. Though the data analytics companies supplying a pricing algorithm to a particular market are typically

[29] "RealPage defines an acceptance as where the final floor plan price is within 1% of the recommended floor plan price. According to that definition, the average acceptance rate across all landlords nationally for new leases between January 2017 and June 2023 is between 40-50%. ... Widening the definition of acceptance even slightly to account for partial acceptances illustrates the influence of recommendations: nearly 60% of final floor plan prices are within 2.5% of RealPage's recommendation, and more than 85% are within 5% of RealPage's recommendation." *United States et al v. RealPage Inc. et al*, U.S. District Court for the Middle District of North Carolina, amended complaint, Case No. 1:24-cv-00710-LCB-JLW, January 7, 2025, ¶ 73-74

few in number, it is still often more than one. For example, in the German retail gasoline market, both A2i Systems and Kalibrate offer pricing algorithms. In the U.S. apartment rentals markets, there is RealPage and Yardi, though RealPage dominates with a market share exceeding 80%.[30] What is the effect of competition among suppliers of pricing algorithms on competition among firms using those pricing algorithms? To explore that question, we may want to move past the simple model of this paper - where a third party is assumed to observe the demand state - and explicitly model information and data sharing. One could assume each firm receives a demand signal which it shares with the third party that it has contracted. The third party's information about the demand state is based on the signals of its subscribing firms. Consequently, a third party can have a competitive advantage from having more subscribing firms as it will be better informed about the demand state. Such a structure could create network effects and result in a third party dominating, as we are witnessing in the apartment rental market. To what extent competition among third parties will result in outcomes that significantly depart from the case of a single third party supplier is then an open question.

[30] "As early as 2019, a RealPage presentation for clients stated that RealPage 'has around 80% of the Revenue Management market share.' ... In 2023, during a sales pitch to a property owner, a RealPage representative noted that '[RealPage] has 80% to 85% of the market share with the closest competitor around 12%.'" *United States et al v. RealPage Inc. et al*, U.S. District Court for the Middle District of North Carolina, amended complaint, Case No. 1:24-cv-00710-LCB-JLW, January 7, 2025, ¶ 224

8 Appendix

8.1 Expression for WTP in Equation (6)

$$\begin{aligned} WTP\left(\phi,\theta^{o}\right) \equiv & \sum_{a\in\mathcal{A}}\pi\left(\phi\left(a\right),\theta^{o}\phi\left(a\right)+(1-\theta^{o})\psi\left(\phi,\theta^{o}\right),a\right)g(a) \\ & -\sum_{a\in\mathcal{A}}\pi\left(\psi\left(\phi,\theta^{o}\right),\theta^{o}\phi\left(a\right)+(1-\theta^{o})\psi\left(\phi,\theta^{o}\right),a\right)g(a). \end{aligned}$$

On the RHS, the first term is

$$\begin{aligned} & \sum_{a\in\mathcal{A}}\pi\left(\phi\left(a\right),\theta^{o}\phi\left(a\right)+(1-\theta^{o})\psi\left(\phi,\theta^{o}\right),a\right)g(a) \qquad (17) \\ = & \sum_{a\in\mathcal{A}}(\alpha+\gamma a-c)\left(a-(b-d\theta^{o})(\alpha+\gamma a)+(1-\theta^{o})\psi\left(\phi,\theta^{o}\right)\right)g(a) \\ = & (\alpha+\gamma\mu-c)\left(\mu-(b-d\theta^{o})\left(\alpha+\gamma\mu\right)+d(1-\theta^{o})\psi\left(\phi,\theta^{o}\right)\right)+\gamma\left(1-(b-d\theta^{o})\gamma\right)\sigma^{2} \\ = & (\alpha+\gamma\mu-c)\left(\mu-(b-d\theta^{o})\left(\alpha+\gamma\mu\right)+d(1-\theta^{o})\left(\frac{\mu+bc+d\theta^{o}(\alpha+\gamma\mu)}{2b-d(1-\theta^{o})}\right)\right) \\ & +\gamma\left(1-(b-d\theta^{o})\gamma\right)\sigma^{2} \end{aligned}$$

and the second term is

$$\begin{aligned} & \sum_{a\in\mathcal{A}}\pi\left(\psi\left(\phi,\theta^{o}\right),\theta^{o}\phi\left(a\right)+(1-\theta^{o})\psi\left(\phi,\theta^{o}\right),a\right)g(a) \qquad (18) \\ = & \left(\psi\left(\phi,\theta^{o}\right)-c\right)\left(\mu-\left(b-d\left(1-\theta^{o}\right)\right)\psi\left(\phi,\theta^{o}\right)\right)+d\theta^{o}(\alpha+\gamma\mu)) \\ = & \left(\frac{\mu+bc+d\theta^{o}(\alpha+\gamma\mu)}{2b-d(1-\theta^{o})}-c\right)\left(\mu-\left(b-d\left(1-\theta^{o}\right)\right)\left(\frac{\mu+bc+d\theta^{o}(\alpha+\gamma\mu)}{2b-d(1-\theta^{o})}\right)+d\theta^{o}(\alpha+\gamma\mu)\right) \\ = & \left(\frac{1}{2b-d(1-\theta^{o})}\right)^{2}\left(\mu-(b-d(1-\theta^{o}))c+d\theta^{o}(\alpha+\gamma\mu)\right)\times \\ & \left(\left(\mu+d\theta^{o}(\alpha+\gamma\mu)\right)\left(2b-d(1-\theta^{o})\right)-\left(b-d\left(1-\theta^{o}\right)\right)\left(\mu+bc+d\theta^{o}(\alpha+\gamma\mu)\right)\right). \end{aligned}$$

Using (18)-(17),

$$\begin{aligned} & WTP\left(\phi,\theta^{o}\right) \\ = & (\alpha+\gamma\mu-c)\left(\mu-(b-d\theta^{o})\left(\alpha+\gamma\mu\right)+d(1-\theta^{o})\left(\frac{\mu+bc+d\theta^{o}(\alpha+\gamma\mu)}{2b-d(1-\theta^{o})}\right)\right) \\ & +\gamma\left(1-(b-d\theta^{o})\gamma\right)\sigma^{2}-\left(\frac{1}{2b-d(1-\theta^{o})}\right)^{2}\left(\mu-(b-d(1-\theta^{o}))c+d\theta^{o}(\alpha+\gamma\mu)\right)\times \\ & \left(\left(\mu+d\theta^{o}(\alpha+\gamma\mu)\right)\left(2b-d(1-\theta^{o})\right)-\left(b-d\left(1-\theta^{o}\right)\right)\left(\mu+bc+d\theta^{o}(\alpha+\gamma\mu)\right)\right). \end{aligned}$$

Simplifying the first and third terms,

$$\begin{aligned}
&(\alpha+\gamma\mu-c)\left(\mu-(b-d\theta^o)(\alpha+\gamma\mu)+d(1-\theta^o)\left(\frac{\mu+bc+d\theta^o(\alpha+\gamma\mu)}{2b-d(1-\theta^o)}\right)\right)\\
&-\left(\frac{1}{2b-d(1-\theta^o)}\right)^2(\mu-(b-d(1-\theta^o))c+d\theta^o(\alpha+\gamma\mu))\times\\
&((\mu+d\theta^o(\alpha+\gamma\mu))(2b-d(1-\theta^o))-(b-d(1-\theta^o))(\mu+bc+d\theta^o(\alpha+\gamma\mu)))\\
=\;&-\frac{b}{(2b-d+d\theta^o)^2}(\mu-2b\alpha+d\alpha+bc-2b\gamma\mu+d\gamma\mu)^2,
\end{aligned}$$

we then have

$$WTP(\phi,\theta^o)=\gamma(1-(b-d\theta^o)\gamma)\sigma^2-\frac{b(\mu+bc-(2b-d)(\alpha+\gamma\mu))^2}{(2b-d(1-\theta^o))^2}.$$

8.2 Proof of Theorem 1

Recall that a has support $\mathcal{A}\equiv\{a_1,...,a_n\}$ and $\mathbb{E}[a]=\mu$. To save on notation, let $g_k\equiv g(a_k)$ denote the probability $a=a_k$. $\phi:A\to\mathbb{R}_+$ denotes a pricing algorithm and, again to save on notation, let $\phi_k\equiv\phi(a_k)$. Conditional on θ firms using ϕ, the equilibrium price for non-adopting firms is

$$\psi(\phi)\equiv\frac{\mu+bc+d\theta\mathbb{E}[\phi]}{2b-d(1-\theta)}. \tag{19}$$

By its definition in (10), the optimal deviation price in a submarket for which the recommended price is ϕ_k can be shown to take the form

$$\xi(\phi_k)=\frac{1}{2b}(\mathbb{E}[a|\phi(a)=\phi_k]+bc+d(\theta\phi_k+(1-\theta)\psi(\phi))). \tag{20}$$

$\mathbb{E}[a|\phi(a)=\phi_k]$ captures a firm's inference about the demand state after receiving recommendation ϕ_k. Substituting (19) and (20) into (16), the third party's objective function is

$$\begin{aligned}
W(\phi)=&\frac{1}{\delta}\sum_k g_k(\phi_k-c)\big(a_k-(b-d\theta)\phi_k+d(1-\theta)\psi(\phi)\big) \qquad (21)\\
&-\frac{1-\delta}{\delta 4b}\sum_k g_k\Big(\mathbb{E}[a\mid\phi(a)=\phi_k]+d(\theta\phi_k+(1-\theta)\psi(\phi))-bc\Big)^2\\
&-(\psi(\phi)-c)\big(\mu-b\psi(\phi)+d(\theta\mathbb{E}[\phi]+(1-\theta)\psi(\phi))\big).
\end{aligned}$$

Letting Φ denote the set of functions $\phi:\mathcal{A}\to\mathbb{R}_+$, our program is

$$\sup_{\phi\in\Phi}W(\phi). \tag{22}$$

Theorem 1 states that the solution to (22) takes the form $\phi_k=\alpha^d+\gamma^d a_k$. The proof of Theorem 1 is in two parts. The first part uses necessary conditions to show that if a solution exists then it must be $\alpha^d + \gamma^d a$. The second part shows the objective function $W(\phi)$ is concave in ϕ, and so the necessary conditions are also sufficient. Hence, the pricing algorithm $\alpha^d + \gamma^d a$ is the unique solution to (22).

For future reference, note that the following inequality holds for all $\theta, \delta \in (0,1)$ and all $d \in (0, b)$:

$$\gamma(1-\gamma(b-d\theta)) > \frac{1-\delta}{4b}(1+d\theta\gamma)^2. \tag{23}$$

In order to prove Theorem 1, we first need to establish two preliminary results.

Lemma 2 *If ϕ is a solution to (22) then $\mathbb{E}[\phi] = \frac{\mu+bc}{2b-d}$.*

Proof. Let ϕ be a solution to (22). Partition $\{1, ..., n\}$ into sets $K_1, K_2, ..., K_j$ such that, for each l, $\phi_k = \phi_l$ for all $k \in K_l$, and $\phi_k \neq \phi_{k'}$ for all $k \in K_l$, $k' \notin K_l$. For each l, let $\overline{a}_l \equiv \mathbb{E}[a_k | k \in K_l] = \mathbb{E}[a|\phi(a) = \phi_l]$, and let $G_l \equiv \sum_{k \in K_l} g_k$.

Note that, holding partition $K_1, K_2, ..., K_j$ fixed, function $W(\phi)$ is differentiable in ϕ_l. Thus, if ϕ is a solution to (22), the first-order condition $\frac{\partial W(\phi)}{\partial \phi_l} = 0$ must hold for $l = 1, ..., j$. Hence, for $l = 1, ..., j$,

$$\begin{aligned}
\frac{\partial W(\phi)}{\partial \phi_l} = 0 = G_l \frac{1}{\delta}\left(\overline{a}_l - 2(b-d\theta)\phi_l + d(1-\theta)\psi(\phi) + (b-d\theta)c\right) + \frac{1}{\delta}\left(\sum_{l'} G_{l'}(\phi_{l'} - c)\frac{d(1-\theta)d\theta G_l}{2b-(1-\theta)d}\right) \\
- \frac{1-\delta}{\delta 2b} G_l \left(\overline{a}_l + d(\theta\phi_l + (1-\theta)\psi(\phi)) - bc\right) d\theta \\
- \frac{1-\delta}{\delta 2b} \sum_{l'} G_{l'}\left(\overline{a}_{l'} + d(\theta\phi_{l'} + (1-\theta)\psi(\phi)) - bc\right)\frac{d(1-\theta)d\theta G_l}{2b-(1-\theta)d} \\
- \frac{d\theta}{2b-d(1-\theta)} G_l\left(\mu - \psi(\phi)(b-d(1-\theta)) + d\theta\mathbb{E}[\phi]\right) \\
+ (\psi(\phi)-c)\left(\frac{d\theta G_l}{2b-d(1-\theta)}(b-d(1-\theta))\right) - (\psi(\phi)-c)d\theta G_l
\end{aligned}$$

where we used the fact that $\frac{\partial\psi(\phi)}{\partial\mathbb{E}[\phi]}\frac{\partial\mathbb{E}[\phi]}{\partial\phi_l} = G_l\frac{d\theta}{2b-(1-\theta)d}$. Using $\sum_{l'} G_{l'} = 1$, $\sum_{l'} G_{l'}\phi_{l'} = \mathbb{E}[\phi]$ and $\sum_{l'} G_{l'}\overline{a}_{l'} = \mu$ yields

$$\begin{aligned}
\frac{\partial W(\phi)}{\partial \phi_l} = 0 = G_l \frac{1}{\delta}\left(\overline{a}_l - 2(b-d\theta)\phi_l + d(1-\theta)\psi(\phi) + (b-d\theta)c\right) + \frac{1}{\delta}\left((\mathbb{E}[\phi] - c)\frac{d(1-\theta)d\theta G_l}{2b-(1-\theta)d}\right) \\
- \frac{1-\delta}{\delta 2b} G_l \left(\overline{a}_l + d(\theta\phi_l + (1-\theta)\psi(\phi)) - bc\right) d\theta \\
- \frac{1-\delta}{\delta 2b}\left(\mu + d(\theta\mathbb{E}[\phi] + (1-\theta)\psi(\phi)) - bc\right)\frac{d(1-\theta)d\theta G_l}{2b-(1-\theta)d} \\
- \frac{d\theta}{2b-d(1-\theta)} G_l\left(\mu - \psi(\phi)(b-d(1-\theta)) + d\theta\mathbb{E}[\phi]\right) \\
+ (\psi(\phi)-c)\left(\frac{d\theta G_l}{2b-d(1-\theta)}(b-d(1-\theta))\right) - (\psi(\phi)-c)d\theta G_l.
\end{aligned}$$

Multiplying all terms by δ, summing over $l = 1, ..., j$, and using $\sum_l G_l = 1$, $\sum_l G_l \overline{a}_l = \mu$ and $\sum_l G_l \phi_l = \mathbb{E}[\phi]$, yields

$$\begin{aligned}
0 = &(\mu - 2(b - d\theta)\mathbb{E}[\phi] + d(1-\theta)\psi(\phi) + (b - d\theta)c) + \left((\mathbb{E}[\phi] - c)\frac{d(1-\theta)d\theta}{2b - (1-\theta)d}\right) \\
&- \frac{1-\delta}{2b}\left(\mu + d(\theta\mathbb{E}[\phi] + (1-\theta)\psi(\phi)) - bc\right)d\theta \\
&- \frac{1-\delta}{2b}\left(\mu + d(\theta\mathbb{E}[\phi] + (1-\theta)\psi(\phi)) - bc\right)\frac{d(1-\theta)d\theta}{2b - (1-\theta)d} \\
&- \delta\frac{d\theta}{2b - d(1-\theta)}\left(\mu - \psi(\phi)(b - d(1-\theta)) + d\theta\mathbb{E}[\phi]\right) \\
&+ \delta(\psi(\phi) - c)\left(\frac{d\theta}{2b - d(1-\theta)}(b - d(1-\theta))\right) - \delta(\psi(\phi) - c)d\theta.
\end{aligned}$$

Substituting $\psi(\phi) = \frac{\mu + bc + d\theta\mathbb{E}[\phi]}{2b - d(1-\theta)}$ and solving for $\mathbb{E}[\phi]$ given the desired result: $\mathbb{E}[\phi] = \frac{\mu + bc}{2b - d}$. ■

Lemma 3 *If ϕ is a solution to (22) then $\psi(\phi) = \frac{\mu + bc}{2b - d}$.*

Proof. From Lemma 2, $\mathbb{E}[\phi] = \frac{\mu + bc}{2b - d}$. Using this in $\psi(\phi) = \frac{\mu + bc + d\theta\mathbb{E}[\phi]}{2b - d(1-\theta)}$ delivers the result. ■

We now turn to proving that necessary conditions imply a solution to (22) is of the form in Theorem 1. We start by showing that any solution ϕ to (22) does not pool: $\phi_k \neq \phi_{k'}$ for $k \neq k'$. Towards proving a contradiction to that claim, suppose that ϕ solves (22), and that there exists $K_l \subset \{1, ..., n\}$ with $|K_l| \geq 2$ such that $\phi_k = \phi_l$ for all $k \in K_l$ and $\phi_{k'} \neq \phi_l$ for all $k' \notin K_l$. Let $\overline{a}_l \equiv \mathbb{E}[a|\phi(a) = \phi_l]$. We now show that, given this pooling structure, the optimal ϕ_l takes the form $\phi_l = \alpha + \gamma\overline{a}_l$.

As in the proof of Lemma 2, it must be that

$$\begin{aligned}
\frac{\partial W(\phi)}{\partial \phi_l} = 0 = &G_l\frac{1}{\delta}\left(\overline{a}_l - 2(b - d\theta)\phi_l + d(1-\theta)\psi(\phi) + (b - d\theta)c\right) + \frac{1}{\delta}\left((\mathbb{E}[\phi] - c)\frac{d(1-\theta)d\theta G_l}{2b - (1-\theta)d}\right) \\
&- \frac{1-\delta}{\delta 2b}G_l\left(\overline{a}_l + d(\theta\phi_l + (1-\theta)\psi(\phi)) - bc\right)d\theta \\
&- \frac{1-\delta}{\delta 2b}\left(\mu + d(\theta\mathbb{E}[\phi] + (1-\theta)\psi(\phi)) - bc\right)\frac{d(1-\theta)d\theta G_l}{2b - (1-\theta)d} \\
&- \frac{d\theta}{2b - d(1-\theta)}G_l\left(\mu - \psi(\phi)(b - d(1-\theta)) + d\theta\mathbb{E}[\phi]\right) \\
&+ (\psi(\phi) - c)\left(\frac{d\theta G_l}{2b - d(1-\theta)}(b - d(1-\theta))\right) - (\psi(\phi) - c)d\theta G_l.
\end{aligned}$$

By the same arguments as in Lemmas 2 and 3, summing the first-order conditions for each l gives $\mathbb{E}[\phi] = \psi(\phi) = \frac{\mu + bc}{2b - d}$. Substituting $\mathbb{E}[\phi] = \psi(\phi) = \frac{\mu + bc}{2b - d}$ in the equality above, dividing by G_l and solving for ϕ_l we get that $\phi_l = \alpha + \gamma\overline{a}_l$.

Let $\phi' : \mathcal{A} \to \mathbb{R}_+$ be such that $\phi'_k = \phi_k$ for all $k \notin K_l$, and such that $\phi'_k = \alpha + \gamma a_k$ for all $k \in K_l$. Note that ϕ' is such that $\mathbb{E}[\phi'_k | k \in K_l] = \alpha + \gamma\overline{a}_l = \phi_l = \mathbb{E}[\phi_k | k \in K_l]$. Hence

$\mathbb{E}[\phi'] = \mathbb{E}[\phi]$ and $\psi(\phi) = \psi(\phi') = \psi$. This implies that the last term in $W(\phi)$ is equal to the last term in $W(\phi')$, and so

$$\delta(W(\phi') - W(\phi)) = \sum_{k \in K_l} g_k \left((\phi'_k - c)\big(a_k - (b - d\theta)\phi'_k + d(1-\theta)\psi\big) - (\phi_l - c)\big(\overline{a}_l - (b - d\theta)\phi_l + d(1-\theta)\psi\big) \right)$$
$$- \frac{1-\delta}{4b} \sum_{k \in K_l} g_k \left(\Big(a_k + d(\theta\phi_k + (1-\theta)\psi) - bc\Big)^2 - \Big(\overline{a}_l + d(\theta\phi_l + (1-\theta)\psi) - bc\Big)^2 \right).$$

Consider the first term. Using $\phi'_k = \alpha + \gamma a_k$ and $\phi_l = \alpha + \gamma \overline{a}_l$, we get

$$\sum_{k \in K_l} g_k \left((\phi'_k - c)\big(a_k - (b - d\theta)\phi'_k + d(1-\theta)\psi\big) - (\phi_l - c)\big(\overline{a}_l - (b - d\theta)\phi_l + d(b-\theta)\psi\big) \right)$$
$$= \gamma(1 - \gamma(b - d\theta)) \sum_{k \in K_l} g_k (a_k^2 - \overline{a}_l^2) = \gamma(1 - \gamma(b - d\theta)) G_l \mathrm{Var}(a_k | a_k \in A_l),$$

where $A_l \equiv \{a_k : k \in K_l\}$ and $\mathrm{Var}(a | a \in A_l)$ is the conditional variance of a, which is strictly positive since A_l contains at least two elements.

Consider next the second term. Using $\phi'_k = \alpha + \gamma a_k$ and $\phi_l = \alpha + \gamma \overline{a}_l$, we get

$$\frac{1-\delta}{4b} \sum_{k \in K_l} g_k \left(\Big(a_k + d(\theta\phi_k + (1-\theta)\psi) - bc\Big)^2 - \Big(\overline{a}_l + d(\theta\phi_l + (1-\theta)\psi) - bc\Big)^2 \right)$$
$$= \frac{1-\delta}{4b} \sum_{k \in K_l} g_k \left(\Big(a_k + d(\theta(\alpha + \gamma a_k) + (1-\theta)\psi) - bc\Big)^2 - \Big(\overline{a}_l + d(\theta(\alpha + \gamma \overline{a}_l) + (1-\theta)\psi) - bc\Big)^2 \right)$$
$$= \frac{1-\delta}{4b} \mathrm{Var}(a_k + d(\theta(\alpha + \gamma a_k) + (1-\theta)\psi) - bc | a_k \in A_l)$$
$$= \frac{1-\delta}{4b} (1 + d\theta\gamma)^2 G_l \mathrm{Var}(a_k | a_k \in A_l).$$

By inequality (23), $\gamma(1 - \gamma(b - d\theta)) > \frac{1-\delta}{4b}(1 + d\theta\gamma)^2$. Since $\mathrm{Var}(a_k | a_k \in A_l) > 0$, we have $W(\phi') > W(\phi)$, a contradiction to ϕ being optimal. Hence, any solution to (22) has $\phi_k \neq \phi_{k'}$ for all $k \neq k'$.

Lastly, we show that $\phi_k = \alpha + \gamma a_k$ for all k. Since $\phi_k \neq \phi_{k'}$ for all $k \neq k'$, and using the same arguments as in the proof of Lemma 2, an optimal ϕ must satisfy the following first-order condition for all k:

$$\frac{\partial W(\phi)}{\partial \phi_k} = 0 = g_k \left(a_k - 2(b - d\theta)\phi_k + d(1-\theta)\psi(\phi) + (b - d\theta)c \right) + \left((\mathbb{E}[\phi] - c) \frac{d(1-\theta)d\theta g_k}{2b - (1-\theta)d} \right)$$
$$- \frac{1-\delta}{2b} g_k \left(a_k + d(\theta\phi_l + (1-\theta)\psi(\phi)) - bc \right) d\theta$$
$$- \frac{1-\delta}{2b} \left(\mu + d(\theta\mathbb{E}[\phi] + (1-\theta)\psi(\phi)) - bc \right) \frac{d(1-\theta)d\theta g_k}{2b - (1-\theta)d}$$
$$- \delta \frac{d\theta}{2b - d(1-\theta)} g_k \left(\mu - \psi(\phi)(b - d(1-\theta)) + d\theta\mathbb{E}[\phi] \right)$$
$$+ \delta(\psi(\phi) - c) \left(\frac{d\theta g_k}{2b - d(1-\theta)} (b - d(1-\theta)) \right) - \delta(\psi(\phi) - c) d\theta g_k$$

By the same arguments as in Lemmas 2 and 3, summing the first-order conditions for all k gives $\mathbb{E}[\phi] = \psi(\phi) = \frac{\mu+bc}{2b-d}$. Lastly, substituting $\mathbb{E}[\phi] = \psi(\phi) = \frac{\mu+bc}{2b-d}$ in the first-order condition for each k and solving for ϕ_k yields $\phi_k = \alpha^d + \gamma^d a_k$.

We now turn to the second part of the proof which is showing that the solution satisfies second-order conditions.

Lemma 4 *For any fixed pooling structure $K_1, ..., K_j$, $W(\phi)$ is concave in ϕ.*

Proof. Let $x = \mathbb{E}[\phi] = \sum_k g_k\phi_k$ and note that $\psi(\phi) = \kappa + \beta x$ with

$$\kappa = \frac{\mu + bc}{2b - d(1-\theta)} > 0, \qquad \beta = \frac{d\theta}{2b - d(1-\theta)} > 0.$$

To establish the Lemma, we show that the first term in $W(\phi)$ is concave and the last two terms are convex. Since the last two terms appear with a negative sign, this implies that W is the sum of concave functions, and is therefore concave.

Step 1: The first term is concave.

Let

$$T_1(\phi) = \frac{1}{\delta}\sum_k g_k(\phi_k - c)\big(a_k - (b - d\theta)\phi_k + d(1-\theta)\psi(\phi)\big).$$

Ignoring all the terms in $T_1(\phi)$ that are affine in ϕ, we get that the quadratic part of $T_1(\phi)$ is

$$Q_1(\phi) = \frac{1}{\delta}\left(-(b - d\theta)\sum_k g_k\phi_k^2 + d(1-\theta)\beta\, x^2\right),$$

where, recall, $x = \sum_k g_k\phi_k$. The Hessian of T_1 (and of Q_1) is

$$H_1 = \frac{2}{\delta}\Big(-(b - d\theta)\, G + d(1-\theta)\beta\, gg^\top\Big).$$

where $G = \mathrm{diag}(g_1, \dots, g_n)$, and $g = (g_1, \dots, g_n)^\top$. For any $z \in \mathbb{R}^n$,

$$z^\top H_1 z = \frac{2}{\delta}\left(-(b - d\theta)\sum_k g_k z_k^2 + d(1-\theta)\beta\Big(\sum_k g_k z_k\Big)^2\right).$$

By the Cauchy–Schwarz inequality, and using $\sum_k g_k = 1$, $(\sum_k g_k z_k)^2 \le \sum_k g_k z_k^2$. Hence,

$$z^\top H_1 z \le \frac{2}{\delta}\big[-(b - d\theta) + d(1-\theta)\beta\big]\sum_k g_k z_k^2.$$

Using $\beta = \frac{d\theta}{2b-d(1-\theta)}$, it follows that $d(1-\theta)\beta \le b - d\theta$, and so $z^\top H_1 z \le 0$ for all z. Thus H_1 is negative semidefinite and T_1 is concave in ϕ.

Step 2: The second term is convex.

Let

$$T_2(\phi) = \sum_k g_k\Big(\mathbb{E}[\phi] + d(\theta\phi_k + (1-\theta)\psi(\phi)) - bc\Big)^2.$$

For a fixed pooling structure (so $\mathbb{E}[a|\phi(a) = \phi_k]$ is constant), each inner term is an affine function of ϕ, because $\psi(\phi) = \kappa + \beta x$ and $x = \mathbb{E}[\phi]$ is linear in ϕ. Thus, T_2 is the sum of squares of affine functions, and hence convex. Since T_2 appears in W multiplied by $-(1-\delta)/(\delta 4b) < 0$, it enters W as a concave term.

Step 3: The third term is convex.

Let

$$T_3(\phi) = (\psi(\phi) - c)\big(\mu - b\psi(\phi) + d(\theta x + (1-\theta)\psi(\phi))\big),$$

where, recall, $x = \mathbb{E}[\phi]$. Since $\psi(\phi) = \kappa + \beta x$, the function $T_3(\phi)$ depends only on x. Thus $T_3(\phi) = G(x)$, with

$$G(x) = (A + \beta x)(B + Kx).$$

and

$$A = \kappa - c, \qquad B = \mu - b\kappa + d(1-\theta)\kappa, \qquad K = -b\beta + d\theta + d(1-\theta)\beta.$$

Using $\beta = \frac{d\theta}{2b - d(1-\theta)}$,

$$K = \frac{bd\theta}{2b - d(1-\theta)} > 0.$$

Hence

$$G''(x) = 2\beta K = \frac{2bd^2\theta^2}{(2b - d(1-\theta))^2} > 0,$$

so G is strictly convex in x. Because $x = \mathbb{E}[\phi]$ is affine in ϕ, the map $\phi \mapsto T_3(\phi)$ is convex. Since T_3 appears in W with a negative sign, it enters W as a concave term.

Combining Steps 1-3, W is the sum of concave functions and is therefore concave. ■

Finally, let us solve for the equilibrium fee. For any (α, γ), it can be shown[31]

$$\begin{aligned} W(\alpha,\gamma) &= \gamma\left(1 - (b - d\theta)\gamma\right)\sigma^2 - \frac{b\left(\mu + bc - (2b-d)(\alpha + \gamma\mu)\right)^2}{\left(2b - d(1-\theta)\right)^2} \\ &- \left(\frac{1-\delta}{\delta}\right)\left(\frac{b\left(\mu + bc - (2b-d)(\alpha + \gamma\mu)\right)^2}{\left(2b - d(1-\theta)\right)^2} + \left(\frac{\left(1 - (2b - d\theta)\gamma\right)^2}{4b}\right)\sigma^2\right). \end{aligned}$$

Given $f^d = W\left(\alpha^d, \gamma^d\right)$, inserting $\left(\alpha^d, \gamma^d\right)$ into the above expression and simplifying yields the equilibrium fee:

$$f^d = \frac{b\sigma^2}{4b(b - d\theta) + d^2\theta^2(1-\delta)}.$$

This completes the proof of Theorem 1.

8.3 Subgame Perfect Equilibrium Supporting the Outcome in Theorem 1

The extensive form game is:

Stage 1: Third party chooses a pricing algorithm and fee, (ϕ, f).

[31] The proof is available on request.

Stage 2: Firms observe (ϕ, f). Eligible firms make requests to adopt the pricing algorithm and pay the fee.

Stage 2.5: The third party observes all adoption requests and either accepts or declines each firm's adoption request.

Stage 3: The adoption rate is observed by all firms (and the third party) and the assignment of demand states to submarkets is realized and observed only by the third party. The third party recommends prices to adopting firms according to the pricing algorithm chosen in stage 1. Adopting firms learn recommended prices and then choose a price for each submarket. Non-adopting firms choose a uniform price.

When referring to a firm "adopting" the third party's strategy, it means it is requesting adoption and, if approved by the third party, pays the fee and has access to the pricing algorithm's recommendations. We will refer to a firm being "excluded" if the third party has decided to deny a firm's request to adopt the third party's pricing algorithm. In the ensuing description, the fraction of firms that have not been excluded is $\widetilde{\theta} \in [0, \theta]$ and the fraction of firms that adopted in the current period is $\theta^o \in \left[0, \widetilde{\theta}\right]$. For the strategy profile described below, the only element of the history that affects conduct is whether a firm is excluded.

- Third party's strategy
  - In stage 1, given the mass of eligible non-excluded firms is $\widetilde{\theta}$, it chooses

$$\begin{aligned} \phi(a) &= \frac{2b(2bc(b-d\widetilde{\theta})+d\mu(1-2\widetilde{\theta}))+(1-\delta)d\widetilde{\theta}\left(\mu(2b-d(1-\widetilde{\theta}))+bd\widetilde{\theta}c\right)}{(2b-d)\left(4b(b-d\widetilde{\theta})+d^2\widetilde{\theta}^2(1-\delta)\right)} \\ &\quad + \left(\frac{2b-d\widetilde{\theta}(1-\delta)}{4b(b-d\widetilde{\theta})+d^2\widetilde{\theta}^2(1-\delta)}\right)a \\ f &= \frac{b\sigma^2}{4b(b-d\widetilde{\theta})+d^2\widetilde{\theta}^2(1-\delta)}. \end{aligned} \tag{24}$$

    The third party chooses the pricing algorithm and fee that maximizes current revenue assuming all eligible non-excluded firms adopt and comply. Given its current period choice does not affect firms' future conduct (according to firms' strategies, as described below), this pricing algorithm and fee are optimal.
  - In stage 2.5, it accepts a request of a firm to adopt iff the firm is not excluded. As long as the mass of firms that are excluded is measure zero (which it will be in equilibrium), this choice is trivially optimal.
  - At the end of the period, it excludes a firm iff a firm adopted, the ICC for implementing recommended prices (hereafter, referred to as ICC-price) is satisfied,

which means

$$f \leq \begin{pmatrix} \sum_k \pi\left(\phi_k, \theta^o\phi_k + (1-\theta^o)\psi\left(\phi\right), a_k\right) g_k \\ -\sum_k \pi\left(\psi\left(\phi\right), \theta^o\phi_k + (1-\theta^o)\psi\left(\phi\right), a_k\right) g_k \end{pmatrix} - \left(\frac{1-\delta}{\delta}\right) \begin{pmatrix} \sum_k \pi\left(\xi\left(\phi_k\right), \theta^o\phi_k + (1-\theta^o)\psi\left(\phi\right), a_k\right) g_k \\ -\sum_k \pi\left(\phi_k, \theta^o\phi_k + (1-\theta^o)\psi\left(\phi\right), a_k\right) g_k \end{pmatrix}. \tag{25}$$

and the firm did not comply, which means there exists a such that a firm's price is not $\phi(a)$. Note that ICC-price is evaluated at the current period's adoption rate θ^o. As long as the mass of firms that are to be excluded is measure zero (which it will be in equilibrium), this choice is trivially optimal.

  - Note: If, in stage 2.5, the mass of firms excluded has positive measure then, in light of firms' strategies, it need not be optimal to deny requests. If, in stage 3, the mass of firms to be excluded has positive measure then it need not be optimal to exclude them. To handle those events, the third party's strategy can be modified to take all firms off the excluded list or not exclude them when the set of such firms has positive measure. That modification would not impact the other equilibrium conditions because it would leave unaffected the incentives of an individual firm regarding compliance because, in equilibrium, the set of excluded firms is measure zero.

- Provided below is a firm's strategy which is roughly summarized here. Suppose the firm is eligible and not excluded. If ICC-price is satisfied, it adopts and implements the recommended prices. If ICC-price is not satisfied, it adopts if it is more profitable to price according to ϕ^b than to set the optimal uniform price assuming all eligible non-excluded firms adopt and price according to ϕ^b; and, having adopted, it prices according to ϕ^b. For all other cases, it does not adopt. If a firm did not adopt (either because it is ineligible, excluded, or chose not to adopt) then it sets the optimal uniform price given the expected average price of rival firms. Depending on the history, the adoption rate used in defining ϕ^b will vary, which is made explicit below.

- Eligible firm's strategy

  - Suppose it is not excluded.

    * In stage 2, given pricing algorithm ϕ and fee f, it adopts iff

$$f \leq \begin{pmatrix} \sum_k \pi\left(\phi_k, \widetilde{\theta}\phi_k + (1-\widetilde{\theta})\psi\left(\phi\right), a_k\right) g_k \\ -\sum_k \pi\left(\psi\left(\phi\right), \widetilde{\theta}\phi_k + (1-\widetilde{\theta})\psi\left(\phi\right), a_k\right) g_k \end{pmatrix} - \left(\frac{1-\delta}{\delta}\right) \begin{pmatrix} \sum_k \pi\left(\xi\left(\phi_k\right), \widetilde{\theta}\phi_k + \left(1-\widetilde{\theta}\right)\psi\left(\phi\right), a_k\right) g_k \\ -\sum_k \pi\left(\phi_k, \widetilde{\theta}\phi_k + (1-\widetilde{\theta})\psi\left(\phi\right), a_k\right) g_k \end{pmatrix}. \tag{26}$$

or

$$\begin{pmatrix} \sum_k \pi\left(\phi_k, \widetilde{\theta}\phi_k + (1-\widetilde{\theta})\psi(\phi), a_k\right) g_k \\ -\sum_k \pi\left(\psi(\phi), \widetilde{\theta}\phi_k + (1-\widetilde{\theta})\psi(\phi), a_k\right) g_k \end{pmatrix} \tag{27}$$
$$-\left(\frac{1-\delta}{\delta}\right)\begin{pmatrix} \sum_k \pi\left(\xi(\phi_k), \widetilde{\theta}\phi_k + \left(1-\widetilde{\theta}\right)\psi(\phi), a_k\right) g_k \\ -\sum_k \pi\left(\phi_k, \widetilde{\theta}\phi_k + (1-\widetilde{\theta})\psi(\phi), a_k\right) g_k \end{pmatrix}$$
$$< \quad f \leq \sum_k \pi\left(\phi^b\left(a_k, \widetilde{\theta}\right), \widetilde{\theta}\phi^b\left(a_k, \widetilde{\theta}\right) + (1-\widetilde{\theta})\psi\left(\phi^b, \widetilde{\theta}\right), a_k\right) g_k - \pi^n.$$

(26) says ICC-price is satisfied in which case it expects all $\widetilde{\theta}$ rival firms to adopt and use ϕ. Thus, it is optimal to adopt. (27) says ICC-price is not satisfied (first inequality) and the ICC for pricing according to $\phi^b\left(\cdot, \widetilde{\theta}\right)$ (when $\widetilde{\theta}$ rival firms are expected to similarly do so) is satisfied (second inequality). Thus, it is optimal to adopt and then not comply (as firms will not be excluded) and price according to $\phi^b\left(\cdot, \widetilde{\theta}\right)$.

  * In stage 3, if a firm adopted and the adoption rate is θ^o then it prices according to ϕ (i.e, it complies) if

$$f \;\; \leq \;\; \begin{pmatrix} \sum_k \pi(\phi_k, \theta^o\phi_k + (1-\theta^o)\psi(\phi), a_k) g_k \\ -\sum_k \pi(\psi(\phi), \theta^o\phi_k + (1-\theta^o)\psi(\phi), a_k) g_k \end{pmatrix} \tag{28}$$
$$-\left(\frac{1-\delta}{\delta}\right)\begin{pmatrix} \sum_k \pi(\xi(\phi_k), \theta^o\phi_k + (1-\theta^o)\psi(\phi), a_k) g_k \\ -\sum_k \pi(\phi_k, \theta^o\phi_k + (1-\theta^o)\psi(\phi), a_k) g_k \end{pmatrix}.$$

  and prices according to $\phi^b(\cdot, \theta^o)$ otherwise. (28) makes compliance incentive compatible given θ^o adopting rival firms comply. If (28) does not hold then compliance is not incentive compatible and it is optimal to price according to $\phi^b(\cdot, \theta^o)$ given θ^o adopting rival firms use $\phi^b(\cdot, \theta^o)$.

- Suppose a firm is excluded or is not excluded and did not adopt. In stage 3, if the adoption rate is θ^o and
  * (28) is satisfied then it sets a price of

$$\frac{\mu + bc + d\theta^o\mathbb{E}[\phi]}{2b - d(1-\theta^o)}. \tag{29}$$

  This price is optimal given θ^o firms price according to ϕ and all other firms set a price of (29).

  * (28) is not satisfied then it sets a price of

$$\frac{\mu + bc}{2b - d}. \tag{30}$$

  This price is optimal given θ^o firms price according to ϕ^b and all other firms set a price of (30).

- Ineligible firm's strategy - In stage 3, if the adoption rate is θ^o and
  - (28) is satisfied then it sets a price of (29). This price is optimal given θ^o firms price according to ϕ and all other firms set a price of (29).
  - (28) is not satisfied then it sets a price of (30). This price is optimal given θ^o firms price according to ϕ^b and all other firms set a price of (30).

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