

Mediated Collusion*

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Abstract

Cartels and bidding rings are often facilitated by intermediaries, who recommend prices/bids to firms, and can impose penalties (such as reverting to competitive behavior in future interactions) if these recommendations are not followed. Motivated by such cases, we study correlated equilibria in first-price auctions with complete information, where bidders who disobey their recommendations are penalized. Cartel-optimal profit is greater when more information about submitted bids is disclosed and when the maximum penalty is larger. When only the winner's identity is disclosed (or the winner's identity and bid), cartels do not benefit from mediation. Our main result characterizes the cartel-optimal equilibrium with two symmetric bidders when both bids are disclosed. When the maximum penalty is not too small, the optimal bid distribution is atomless on a connected subset of \mathbb{R}^2 , and is characterized by a double-continuum of binding downward incentive constraints. The optimal equilibrium involves high prices with positive probability, even when the maximum penalty is very small.

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1 Introduction

Cartels and bidding rings are often facilitated by intermediaries, which exert control over the information available to cartel participants. For example, Harrington (2006) and Marshall and Marx (2012) document that numerous industrial cartels uncovered by the European Commission were supported by industry groups, consultancies, or accounting firms that intentionally limited participants’ information concerning each other’s operations in various ways.¹ In an auction context, Asker (2010) studies a bidding ring of stamp dealers, who relied on an intermediary (a New York taxi driver) to privately collate bids in an internal, “knockout” auction, before bidding on behalf of the winning bidder in a target auction. A particularly striking case comes from Kawai, Nakabayashi, and Ortner (2022), who study a long-running bidding ring among construction firms in the town of Kumatori, Japan. Collusion in this ring was facilitated by a trade association—the Kumatori Contractors Cooperative—which privately recommended bids in procurement auctions to the various ring members. The ring members then submitted their own bids, which were later publicly disclosed following the auction.² In this setting, while the trade association could not directly control the firms’ bids, the auction environment provided some scope for the association to punish firms who did not bid according to their recommendations, in addition to limiting firms’ information about each other’s recommended bids. For instance, if a bidder placed a bid below the one the association recommended to her, the association could announce that a deviation occurred, and could recommend that the bidders revert to competitive bidding in future auctions.

In mediated auctions of this form, the intermediary (or *mediator*) is more powerful than a mere correlating device (Aumann, 1974; also called a *bid coordination mechanism* by Marshall and Marx, 2007), because she has some scope to punish bidders who disobey their recommendations. However, because these punishments are bounded—e.g., by the difference in a bidder’s continuation value under collusive and competitive bidding—the mediator

¹One well-known intermediary in this context was the Swiss consulting firm AC Treuhand (see, e.g., Marshall and Marx, 2012, pp. 138–140).

²First-price sealed-bid auctions where all bids are disclosed following the auction are common in procurement (e.g., Marshall and Marx, 2012, pp. 200–202). A classic empirical study of bid-rigging in such auctions is Porter and Zona (1993).

cannot fully control the bidders' bids: she is weaker than a *bid submission mechanism* that can directly submit ring members' bids or prevent some members from bidding (Marshall and Marx, 2007). The object of this paper is to analyze optimal collusion with such a mediator.³

We study correlated equilibria in a symmetric first-price procurement auction with complete information, where a mediator privately recommends a bid to each bidder, and can penalize bidders who deviate from their recommendations. (The complete information assumption is for tractability; as we will see, optimal mediated collusion can be quite complicated even with complete information.) A leading interpretation is that the bidders are playing a stationary collusive equilibrium in a repeated auction environment, where, following a deviation, the mediator can direct the bidders to revert to competitive bidding in future auctions. With this interpretation, this size of the penalty equals the difference between a bidder's continuation payoff under collusive and competitive bidding. Other interpretations are also possible: for example, Marshall and Marx (2012, p. 138) document several industrial cartels that required firms to post bonds to a common fund, where the bonds were forfeited if the firms deviated from the collusive agreement, while Clark and Houde (2013, p. 118) study a large gasoline cartel whose leaders relied on harassment, threats, and intimidation (in addition to price wars) to enforce compliance.

In this setting, we characterize the cartel-optimal equilibrium as a function of the size of the penalty and the type of bid information that is disclosed following the auction. We consider three different specifications of the bid disclosure policy: *winner's identity observed*, where only the identity of the winning bidder is disclosed; *winner's bid observed*, where the identity and the bid of the winning bidder are disclosed; and *all bids observed*, where the full vector of bids is disclosed.⁴ In the first two cases, the cartel-optimal equilibrium is relatively straightforward to characterize: the optimal equilibrium involves a deterministic winning bid, and in fact can be implemented by the cartel even without the assistance of the

³In practice, the line between bid coordination and submission can be blurry. For example, while the bidding ring studied by Asker (2010) superficially resembles a bid submission mechanism, the intermediary had no means of preventing dealers from directly bidding in the target auction as well as the knockout, so economically this setting is more like bid coordination.

⁴While disclosing all bids is probably the most common practice in procurement auctions, auctions where only the winner's identity and/or bid are disclosed are also well-studied (e.g., Skrzypacz and Hopenhayn, 2004), and are natural benchmarks for comparing the benefits and drawbacks of disclosing additional bid information.

mediator. In the third case—which forms the core of our analysis—the optimal equilibrium is more complex. Here, firms mix over a continuum of bids in a correlated manner, so the winning bid is stochastic, and mediation is essential for obtaining cartel-optimal profits. Each firm sometimes places a bid that it is certain will win the auction, but—in contrast to a *bid rotation scheme*, where the cartel agrees on a winner in advance (McAfee and McMillan, 1992; Kawai et al., 2022)—a firm never places a bid that it is certain will lose. The cartel obtains strictly higher profits when firms privately communicate with a mediator, as compared to the case where firms can only communicate in public, or cannot communicate at all. Our model is thus one where explicit (mediated) collusion is more profitable than tacit collusion. We also note that, *without* a mediator, cartel-optimal profit is the same in the winner’s bid observed and all bids observed cases. Thus, the classic intuition of Stigler (1964) that disclosing all bids facilitates collusion turns out to hold *only* for mediated cartels.

Our analysis of the observed-bids case (with mediation) proceeds as follows. We first derive simple upper and lower bounds on optimal cartel profit, as a function of the penalty size and the number of bidders. In the limit where the number of bidders is large, these bounds are tight, and optimal cartel profit coincides with that under bid rotation. Thus, while bid rotation is suboptimal for any number of bidders, it is approximately optimal with many bidders. Intuitively, the downside of bid rotation is that losing bidders know their bids are losing, and are thus tempted to deviate. However, with many bidders, almost all bidders expect to lose with high probability in any equilibrium, so the disadvantage of bid rotation is small.

Our main result (Proposition 6) fully characterizes the cartel-optimal equilibrium with two bidders, in the case where the constraint that a bidder is not tempted to deviate from her recommended bid to a *higher* bid is slack. Such *upward incentive constraints* are always slack when the penalty is not too small (in particular, when it is at least $1/3$ the size of the available surplus in a single auction), as well as when the mediator has the ability to deter upward deviations by placing a shill bid just above the recommended winning bid.⁵ When upward constraints are slack, we show that the optimal equilibrium is characterized

⁵Similar upward deviation-detering bids also arise in Marshall and Marx (2007) and Bernheim and Madsen (2017).

by a double-continuum of binding downward incentive constraints. Denoting the penalty size by x , the available surplus (i.e., the reserve price or consumer valuation) by 1, and the optimal cartel profit level by π^* , we show that the optimal bid distribution is atomless and is supported on a connected subset of the set $[2x, 1]^2$, so that $\pi^* \in (2x, 1)$. Even though the bid distribution is atomless, the bidders place the same bid with positive probability. When a bidder is recommended a bid p above π^* , she expects to win the auction with positive probability, and she is indifferent between following the recommendation and placing any bid in an interval $[\chi(p), p)$ (and facing the punishment, x), where $\chi(p) \in [2x, \pi^*]$ is the lowest bid that the other bidder is ever recommended when one bidder is recommended p . When a bidder is recommended a bid below π^* , she expects to win the auction for sure, and she strictly prefers to follow the recommendation. Methodologically, we cast the problem of finding the cartel-optimal equilibrium as an infinite-dimensional linear program, which we are able to solve analytically by solving a pair of ordinary differential equations that characterize the boundary of the support of the optimal bid distribution as well as the optimal multipliers on the downward incentive constraints.

When $x < 1/3$ and the mediator cannot place shill bids, upward incentive constraints also bind, and the optimal equilibrium is more complicated. In this case, we mostly rely on numerical solutions.⁶ However, a notable analytic result is that, no matter how small x is, the support of the optimal bid distribution contains the point $(1, 1)$: thus, the optimal equilibrium involves high prices with positive probability, even when the maximum penalty is very small. To see why this is a striking result, recall that the Nash equilibrium bid vector $(0, 0)$ is the unique correlated equilibrium in the first-price auction with complete information (Jann and Schottmuller 2015; Feldman, Lucier, and Nisan, 2016), and by upper hemi-continuity all equilibrium bid distributions in our model converge to this degenerate distribution as $x \rightarrow 0$. Thus, while the optimal bid distribution converges to a point mass on $(0, 0)$ as $x \rightarrow 0$, for any $x > 0$ there is a positive probability that the winning bid is arbitrarily close to 1. This result has the empirical implication that observing high prices in a single auction does not allow an observer to conclude that a cartel has substantial enforcement

⁶When upward incentive constraints bind, the optimal equilibrium can be characterized by a system of delay differential equations. However, the resulting equations are quite complicated, so we omit them.

power, or that prices will remain high in future auctions.

1.1 Related Literature

We contribute to the literature on mediation and correlated equilibrium in moral hazard problems and repeated games, as well as to the literature on correlated and communication equilibrium in auctions. In the former literature, Rahman and Obara (2010) and Rahman (2012) derive general results on incentive compatibility in mediated partnerships. Several papers study how a cartel can benefit from creating uncertainty about its members' current-period prices. These include Sugaya and Wolitzky (2017, 2018a), who show how cartel members can benefit from less precise monitoring of their competitors' past prices, since past prices can be informative of current prices⁷; Bernheim and Madsen (2017), who show how mixed equilibria outperform pure ones in repeated auctions with asymmetric costs; and Kawai, Nakabayashi, and Ortner (2022), who characterize optimal correlated equilibria within the class of *bid rotation equilibria*, where in each period the identity of the designated winning bidder is publicly announced before the auction.⁸ However, none of these papers attempts to characterize unrestricted optimal correlated equilibria. We also mention Awaya and Krishna (2016), who exhibit a class of repeated Bertrand games where explicit communication increases cartel-optimal profits. In their model, this occurs because communication facilitates improved monitoring; instead, in our model, (mediated) communication increases profits by creating uncertainty about competitors' current-period prices, which reduces a firm's gain from deviating.

A few papers study communication equilibria in one-shot first-price auctions (among other auction formats), although many results in this literature are negative ones. With complete information, Jann and Schottmuller (2015) and Feldman, Lucier, and Nisan (2016) show that the Nash equilibrium is the only correlated equilibrium in the first-price auction. With independent private values, Marshall and Marx (2007) show that a cartel cannot obtain the first-best surplus in any communication equilibrium; moreover, Lopomo, Marx, and

⁷Sugaya and Wolitzky (2018b) derive conditions under which optimal correlated equilibria in repeated games coincide with optimal Nash equilibria (see also Neyman, 1997; Ui, 2008). These conditions are not satisfied by the first-price auction game; otherwise, the cartel could not benefit from mediation in our model.

⁸The authors document that the Kumatori Contractors Cooperative relied on bid rotation equilibria.

Sun (2011) show that with two symmetric bidders, two types, and discrete bids, no communication equilibrium outperforms Nash. With general information structures, Bergemann, Brooks, and Morris (2017) characterize the best equilibrium for the bidders that can arise over all information structures; Bergemann, Brooks, and Morris (2021) analyze a related model where firms are assumed to know their own values. The classic papers of Graham and Marshall (1987), Mailath and Zemsky (1991), and McAfee and McMillan (1992) are less related, as they assume that the ring controls its members’ bids (i.e., they consider bid submission mechanisms). Such enforcement power arises endogenously in repeated auctions with patient players (e.g., Athey and Bagwell, 2001; Athey, Bagwell, and Sanchirico, 2004; Aoyagi 2003, 2007; Skrzypacz and Hopenhayn, 2004; Blume and Heidhues, 2006; Hörner and Jamison, 2007; Harrington and Skrzypacz, 2011). In contrast, we focus on the case where players are impatient, so the maximum penalty size imposes a binding constraint.

More speculatively, mediated collusion can be related to recent interest in algorithmic collusion (e.g., Harrington, 2018; Schwalbe, 2018; Calvano et al., 2020; Klein, 2021). Even if firms do not explicitly communicate through a mediator, in principle the same outcome could be obtained if firms use a common pricing algorithm or online price recommendation tool that mimics the mediator’s stochastic recommendations. Such an algorithm could support firm profits that are even higher than those attainable under tacit collusion, or under explicit collusion with only public communication. While we suspect that current pricing algorithms do not randomize in this way, the relationship between algorithms and mediation seems like an interesting issue for future study.⁹

2 Preliminaries

2.1 Model

Consider $n \geq 2$ firms (bidders) competing in a first-price procurement auction, with a reserve price normalized to 1. We assume that the firms have the same, commonly known production

⁹For an example of a price-recommendation website that is suspected of facilitating collusion, see “Company That Makes Rent-Setting Software for Apartments Accused of Collusion, Lawsuit Says” (ProPublica, 2022).

cost, which we normalize to 0. Firms simultaneously place bids $\mathbf{p} = (p_i)_{i=1}^n$, where without loss $p_i \in [0, 1]$ for each firm i . If a unique firm i places the lowest bid p_i , this firm wins the auction and receives a payment of p_i , while the other firms receive payment 0. In case of a tie, the winner is chosen uniformly at random.¹⁰

The firms are assisted by a mediator, who has two roles. Before the auction, the mediator privately recommends a bid to each firm. After the auction, the mediator may impose a *penalty* of size $x \in (0, 1)$ on each firm.¹¹ A firm's payoff is its payment in the auction, less the penalty if it is applied. In sum, the mediator cannot directly control the firms' bids, but it can impose a (limited) penalty on firms who deviate from their recommended bids, for example by triggering reversion to competitive bidding in future auctions.

The usefulness of penalties for supporting collusion depends on the information that the mediator receives about the bids placed by the firms. We will consider three specifications of the mediator's information: *winner's identity observed*, where the mediator observes only the identity of the winning bidder; *winner's bid observed*, where the mediator observes the identity and the bid of the winning bidder; and *all bids observed*, where the mediator observes the full vector of bids. In each case, a *strategy profile* consists of a joint distribution of recommended bids (a probability distribution on $[0, 1]^n$), a bidding strategy for each firm i (a mapping from recommended bids $p_i \in [0, 1]$ to probability distributions over actual bids $p'_i \in [0, 1]$), and a punishment strategy for the mediator (a mapping from vectors of recommended bids and the mediator's subsequent observation to a probability of penalizing one or more bidders). The mediator is assumed to be able to commit to its strategy. Thus, a strategy profile is a (Bayes Nash) *equilibrium* if each firm's strategy maximizes its expected payoff, given the strategies of the other firms and the mediator.¹² We are interested in the *optimal equilibrium* from the firms' perspective: that is, the equilibrium that maximizes

¹⁰With symmetric firms, uniform tie-breaking is a harmless simplifying assumption.

¹¹Since the penalty will never be imposed on-path in an optimal equilibrium, our analysis is the same if the mediator has the option of penalizing only a subset of firms—e.g., by expelling them from the cartel—or is restricted to penalizing all firms simultaneously—e.g., by dissolving the cartel.

¹²With *all bids observed*, this solution concept is the same as (interim) ε -*correlated equilibrium* of the game without the mediator, with $\varepsilon = x$. Thus, our main technical contribution can be described as characterizing the optimal ε -correlated equilibrium of the symmetric first-price auction with complete information (or, equivalently, of symmetric Bertrand competition). However, as discussed below, our analytic results in this case mostly assume that $n = 2$ and only downward incentive constraints bind.

cartel profit, which is defined as the sum of the firms' expected payoffs, or equivalently the expected winning bid (taking for granted that the penalty is not imposed on path). Formally, the *optimal cartel profit* π^* is defined as the supremum of cartel profits over all equilibria, and an equilibrium is *optimal* if it attains the supremum.¹³ As we will see, optimal cartel profit is increasing and concave in the penalty size x .

When the mediator does not observe losing bids (i.e., with winner's identity observed or winner's bid observed), the optimal equilibrium turns out to be easy to characterize. In these cases, a bidder's deviation is detected only if she wins the auction, so a bidder's incentive constraint reduces to a simple comparison of the size of the penalty, the number of bidders, and the equilibrium winning bid. In particular, the optimal winning bid is deterministic, and the optimal equilibrium is implementable without the mediator's assistance. We analyze these simpler cases in Section 3. In contrast, with all bids observed, deviations are always detected, the optimal winning bid is typically stochastic, and mediation is essential. This leads to a much richer analysis, which we present in Section 4.

We briefly comment on the assumption that the mediator has commitment power. As we will see, the bid distribution that maximizes expected cartel profit can involve a stochastic winning bid. An intermediary who receives a share of cartel profit is thus tempted to recommend higher winning bids, upsetting the optimal equilibrium. We instead interpret the mediator as an outsider that is paid a flat fee, independent of the recommended bids or any deviations by the firms.¹⁴ Alternatively, the mediator can represent an automated system, such as a website or software package that recommends prices.

2.2 Canonical Equilibria

We start with two simple observations. First, for any equilibrium, there exists an equilibrium with the same joint distribution of bids where, on path, the bidders always follow their recommendations, and the penalty is never imposed. This follows by a revelation principle-

¹³Since the game has a continuum of strategies and discontinuous payoffs, we are not aware of a general result guaranteeing the existence of an optimal equilibrium. However, for most versions of the model we consider, we are able to explicitly construct an optimal equilibrium. Alternatively, existence could be guaranteed by letting the mediator select the auction's tie-breaking rule.

¹⁴For example, the intermediary in the bidding ring studied by Asker (2010) was paid \$30 per hour, plus an additional \$50 per auction from each participant.

like argument. Second, any level of cartel profit that is attainable in any equilibrium can be attained in a symmetric equilibrium, where the cdf of recommended bids $F(\mathbf{p})$ satisfies $F(p_1, \dots, p_n) = F(p_{\phi(1)}, \dots, p_{\phi(n)})$ for every permutation ϕ on $\{1, \dots, n\}$. This follows because, given any asymmetric equilibrium where bidders follow their recommendations, the strategy profile that results from randomly permuting the bidders' recommendations (and similarly permuting the bidders' identities in the mediator's punishment strategy) is symmetric (by construction), is an equilibrium (as each bidder's incentive constraint is an average of those in the original equilibrium), and has the same expected winning bid as in the original equilibrium. Given these observations, we henceforth restrict attention to symmetric equilibria where, on path, bidders follow their recommendations and the penalty is never imposed. We call such an equilibrium *canonical*.

2.3 Repeated Game Interpretation

A leading interpretation of the model is that the penalty x represents a firm's lost continuation payoff from switching from collusive to competitive play in a symmetric repeated-game equilibrium. To spell this out, consider any canonical equilibrium of our one-shot game with a penalty size of x . Let $\pi_i(x)$ denote a firm's expected payoff. (Recall that this is the same for each firm i .) If the firms repeatedly participate in identical auctions with a common discount factor δ , it is a (symmetric, stationary) equilibrium of the repeated game for the firms to play the symmetric equilibrium of the one-shot game in every period, where the penalty size x satisfies the fixed-point equation

$$x = \frac{\delta}{1 - \delta} \pi_i(x), \quad (1)$$

and the prescribed equilibrium play is enforced by the threat of reversion to the static Nash equilibrium $\mathbf{p} = \mathbf{0}$ following any deviation. Conversely, in any symmetric, stationary repeated-game equilibrium, play in every period corresponds to a canonical equilibrium of our one shot game, with a value of x that satisfies equation (1).

In some versions of our model, it will turn out that optimal cartel profits are linear in the penalty size x , and hence $\pi_i(x)$ is linear in x . In this case, the highest value of $\pi_i(x)$

that satisfies equation (1) is given by a corner solution, $\pi_i(x) \in \{0, 1/n\}$ (depending on the values of δ and the coefficient on x in the formula for $\pi_i(x)$), so the model becomes trivial. However, enriching the interpretation of the penalty recovers an interior solution to (1). For example, if the penalty consists of some exogenous component y in addition to the lost continuation payoff (e.g., bonds posted by the firms, the threat of harassment or intimidation), then equation (1) becomes $x = y + (\delta/(1-\delta))\pi_i(x)$, which has an interior solution whenever δ is sufficiently small. Another possibility is that the reserve price or production cost may be stochastic. In this case, the variable $\pi_i(x)$ in equation (1) should be interpreted as the expected profit prior to the realization of the stochastic variable. With this interpretation, $\pi_i(x)$ can be concave in x even if optimal cartel profit is linear in x in our baseline model. This again yields an interior solution for (1).¹⁵

3 Unobserved Losing Bids

This section characterizes the optimal equilibrium when the mediator does not observe losing bids (i.e., with winner's identity observed or winner's bid observed). We will see that in such cases the cartel does not benefit from mediation: the firms could obtain the same payoffs by observing a public randomizing device prior to the auction, and agreeing on a public rule for imposing penalties (e.g., reverting to competitive play) following an observed deviation. To establish this result, it turns out to be sufficient to consider a limited class of deviations for the bidders: *uniform downward deviations*, where, for some cutoff bid $p^* \in [0, 1]$, a bidder follows her recommendation p_i whenever $p_i \leq p^*$, and she deviates by bidding p^* whenever $p_i > p^*$.¹⁶ However, neither the ineffectiveness of mediation nor the sufficiency of uniform downward deviations will carry over when the mediator observes losing bids.

3.1 Winner's Identity Observed

When only the winner's identity is observed, we show that it is impossible to support bids greater than the penalty, x . Intuitively, since a deviation is detected only if it succeeds in

¹⁵Kawai, Nakabayashi, and Ortner (2022) work out a model along these lines.

¹⁶Uniform deviations also suffice to characterize optimal equilibria in some other auction games, such as the one considered by Bergemann, Brooks, and Morris (2017).

switching the firm from losing the auction to winning it, the maximum winning bid (which equals the gain from a successful deviation) cannot exceed the available punishment.¹⁷

Proposition 1 *With winner's identity observed, optimal cartel profit equals x . Moreover, the winning bid cannot exceed x with positive probability in any equilibrium.*

Proof. Here is a (canonical) equilibrium yielding cartel profit x : (1) The mediator randomly selects a designated winning bidder i and recommends bids $p_i = x$ and $p_j = x + \varepsilon$ for all $j \neq i$, where ε is a uniform $[0, \bar{\varepsilon}]$ random variable, and $\bar{\varepsilon}$ is any constant satisfying $\bar{\varepsilon} \in (0, \min\{x, 1 - x\})$. (2) Bidders follow their recommendations. (3) The mediator punishes the bidders iff the designated winner loses. To see that this is an equilibrium, note that a bidder who is recommended $p_i = x$ and bids $p'_i \in [x, x + \bar{\varepsilon}]$ wins with probability $\frac{x + \bar{\varepsilon} - p'_i}{\bar{\varepsilon}}$. Since $x = \operatorname{argmax}_{p'_i \in [x, x + \bar{\varepsilon}]} \left(\frac{x + \bar{\varepsilon} - p'_i}{\bar{\varepsilon}} \right) p'_i$, and bidding $p'_i \neq x$ only increases the probability of punishment, such a bidder follows her recommendation. Meanwhile, a bidder who is recommended $p_i > x$ gets an equilibrium payoff of 0, against a payoff of at most $x - x = 0$ from deviating to a winning bid and getting punished.

Conversely, fix any canonical equilibrium with bid profile cdf $F(\mathbf{p})$, and let $W(p)$ denote the corresponding cdf of the winning bid, $p = \min\{p_1, \dots, p_n\}$. Since the equilibrium is symmetric, each bidder's equilibrium expected payoff equals

$$\int_0^1 \frac{p}{n} dW(p).$$

In contrast, for any $p^* \in [0, 1]$ such that $W(p)$ is continuous at p^* , a uniform downward deviation with cutoff $p^* \in [0, 1]$ gives an expected payoff of

$$\int_0^{p^*} \frac{p}{n} dW(p) + (1 - W(p^*)) \left(p^* - \frac{n-1}{n} x \right). \quad (2)$$

Indeed, if the lowest recommended bid is $p < p^*$, with probability $\frac{1}{n}$ the firm wins with bid p , and with probability $\frac{n-1}{n}$ the firm loses but is not punished (as even if it deviated, the deviation is not detected). If instead all recommended bids are strictly above p^* , then the

¹⁷Jann and Schottmuller (2015) and Feldman, Lucier, and Nisan (2016) show that the Nash equilibrium of the symmetric, complete-information, first-price auction is also the unique correlated equilibrium. This result is similar to the $x = 0$ case of Proposition 1, and likewise is proved by considering uniform deviations.

firm wins with bid p^* , and the firm is punished with probability $\frac{n-1}{n}$ (since with probability $\frac{1}{n}$ the firm would have won even absent its deviation, so its deviation is not detected).¹⁸ Since this deviation must be unprofitable in equilibrium, we have

$$\begin{aligned} \int_0^{p^*} \frac{p}{n} dW(p) + (1 - W(p^*)) \left(p^* - \frac{n-1}{n}x \right) &\leq \int_0^1 \frac{p}{n} dW(p) &\iff \\ \int_{p^*}^1 \left(p^* - \frac{n-1}{n}x - \frac{p}{n} \right) dW(p) &\leq 0. \end{aligned}$$

Now, let $\bar{p} = \max \text{supp}(W)$ denote the highest winning bid. Since the above inequality must hold for $p^* = \bar{p} - \varepsilon$ for any $\varepsilon > 0$ such that $W(p)$ is continuous at p^* , taking a sequence $\varepsilon \downarrow 0$ we have

$$\bar{p} - \frac{n-1}{n}x - \frac{\bar{p}}{n} \leq 0 \quad \iff \quad \bar{p} \leq x,$$

as desired. ■

With winner's identity observed, mediation is not necessary to attain optimal cartel profit: it suffices that the firms observe a public randomizing device that selects the designated winner, and that all firms (or at least the winner) are punished if a different firm wins. This scheme is implementable without mediation: for example, as a symmetric repeated-game equilibrium where the players observe a public randomizing device in every period.

3.2 Winner's Bid Observed

Observing the winner's bid lets the cartel support somewhat higher bids, because now a firm that deviates to a winning bid is punished even if it would also have won by following its recommendation.

Proposition 2 *With winner's bid observed, optimal cartel profit equals $\min \left\{ \frac{n}{n-1}x, 1 \right\}$. Moreover, the winning bid cannot exceed $\min \left\{ \frac{n}{n-1}x, 1 \right\}$ with positive probability in any equilibrium.*

Proof. Here is a (canonical) equilibrium yielding cartel profit $\frac{n}{n-1}x$: (1) The mediator recommends bid $\frac{n}{n-1}x$ to all firms. (2) Bidders follow the recommendation. (3) The mediator

¹⁸The event that the lowest recommended price equals p^* occurs with probability 0 because $W(p)$ is continuous at p^* , and thus does not affect (2).

punishes the bidders iff the winning bid differs from $\frac{n}{n-1}x$. This is an equilibrium because a bidder's equilibrium expected payoff is $\frac{1}{n-1}x$ (by uniform tie-breaking), against a payoff of at most $\frac{n}{n-1}x - x = \frac{1}{n-1}x$ from deviating to a different bid and getting punished.

The proof of the converse follows the same line as the winner's identity observed case. The difference is that a uniform downward deviation with cutoff $p^* \in [0, 1]$ (where $W(p)$ is continuous at p^*) now gives an expected payoff of only

$$\int_0^{p^*} \frac{p}{n} dW(p) + (1 - W(p^*)) (p^* - x), \quad (3)$$

where the difference from (2) is that, now, if all recommended bids are strictly above p^* , the deviator is punished with probability 1 (rather than $\frac{n-1}{n}$ as in the winner's identity observed case). We thus have

$$\int_0^{p^*} \frac{p}{n} dW(p) + (1 - W(p^*)) (p^* - x) \leq \int_0^1 \frac{p}{n} dW(p) \quad \forall p^* \text{ s.t. } W(p) \text{ is continuous at } p^*.$$

Taking a sequence $p^* \uparrow \bar{p}$ gives

$$\bar{p} - x - \frac{\bar{p}}{n} \leq 0 \quad \Longleftrightarrow \quad \bar{p} \leq \frac{n}{n-1}x.$$

■

With winner's bid observed, mediation is again unnecessary: the optimal equilibrium is now attained when all firms place the same bid, and face punishment if anyone deviates.

4 All Bids Observed

Our main analysis concerns the case where all bids are observed. We proceed in four steps. First, we show that observing all bids does not allow the cartel to attain a profit above $\min\{\frac{n}{n-1}x, 1\}$ without mediation.¹⁹ Second, we derive simple upper and lower bounds on optimal cartel profit, where the upper bound comes from considering uniform downward deviations as in the previous section, and the lower bound comes from considering equilibria

¹⁹If $n > 2$, this result requires a restriction to equilibria where all firms make positive profits.

where at most one firm at a time bids below all the others. Unlike in the previous section, the profit bound implied by uniform downward deviations is not tight; however, these bounds are tight in the many-firm limit ($n \rightarrow \infty$), where they imply that bid rotation is asymptotically optimal. Third, restricting attention to the case of two firms for tractability, we fully characterize the optimal equilibrium when upward incentive constraints are slack: this is our main result. As we show, upward incentive constraints are slack if $x \geq 1/3$. Alternatively, if the mediator herself (or a proxy) can enter the auction and place a shill bid just above the lowest recommended bid, then upward deviations are always unprofitable. (Such shill bids always lose in equilibrium, so entering such a bid may be costless for the mediator in many auction settings.) Finally, we consider the case where both downward and upward constraints bind (again with $n = 2$): i.e., the case where $x < 1/3$ and the mediator cannot place shill bids. Here we prove that, no matter how small x is, the support of the distribution of winning bids contains the reserve price of 1. We also further characterize the optimal equilibrium numerically, showing that the support of the bid distribution takes a relatively simple form. We end with a figure (Figure 6) comparing cartel profit across the various settings and equilibria considered in this section, for two firms and any value for x .

4.1 No Mediation

We first consider a benchmark case where the firms can agree in advance on a punishment scheme as a function of the realized bid profile, but do not have access to a mediator who makes private bid recommendations. (Equivalently, a mediator is present but is restricted to public recommendations.) Here, we show that maximum cartel profit is again $\min\{\frac{n}{n-1}x, 1\}$, if $n = 2$ or if $n > 2$ and all firms make positive profits.²⁰ Together with the results in the previous section, this result shows that cartels benefit from mediation only if the mediator observes losing bids.

Proposition 3 *With all bids observed but no mediation, optimal cartel profit is at least $\min\{\frac{n}{n-1}x, 1\}$ and at most $\min\{2x, 1\}$. The lower bound of $\min\{\frac{n}{n-1}x, 1\}$ is exact if $n = 2$*

²⁰See Appendix B for an example where $n = 3$ but only two firms make serious bids, and cartel profit is above $\frac{3}{2}x$. In general, asymmetric equilibria can potentially outperform symmetric ones without mediation, but not with mediation (cf. Section 2.2).

or if attention is restricted to equilibria where all firms make positive expected profits.

Proof. The lower bound follows as it remains an equilibrium for all firms to bid $\min \left\{ \frac{n}{n-1}x, 1 \right\}$, while facing punishment if anyone deviates.

For the upper bound, fix an equilibrium bid profile cdf F , and let $S \subseteq \{1, \dots, n\}$ denote the set of firms that make positive expected profits. We consider separately the cases where $|S| = 1$ and where $|S| \geq 2$, and show that in each case cartel profit is at most $\min \{2x, 1\}$.

If $|S| = 1$, note that the winning firm $i \in S$ must bid at or below x with positive probability, as otherwise another firm could obtain a positive profit by bidding just above x and facing punishment. Since without mediation the winning firm must be indifferent among all bids in the support of its equilibrium strategy, its profit—and hence cartel profit—is at most x , which is less than $\min \{2x, 1\}$.

Now suppose that $|S| \geq 2$. For each firm $i \in S$, let $\pi_i > 0$ be the firm's expected payoff, and let $\bar{p}_i = \max \text{supp}(F_i)$ be the firm's highest equilibrium bid. (Here F_i denotes the marginal of F on p_i .) Since $\pi_i > 0$ for each $i \in S$, there exists \bar{p} such that $\bar{p}_i = \bar{p}$ for all $i \in S$, as otherwise there exists a firm $i \in S$ for which bidding $p_i = \bar{p}_i$ is optimal and yet this bid never wins. Let $\alpha_i = \Pr(p_j = \bar{p} \forall j \in S : j \neq i)$. We have $\frac{\alpha_i}{|S|}\bar{p} \geq \alpha_i\bar{p} - x$ (as firm i weakly prefers bidding \bar{p} to bidding just below \bar{p}) and $\frac{\alpha_i}{|S|}\bar{p} \geq \pi_i$ (as bidding \bar{p} gives firm i its equilibrium payoff of π_i , possibly following some punishment). By the first inequality, $\alpha_i\bar{p} \leq \frac{|S|}{|S|-1}x$. Hence, by the second inequality, $\pi_i \leq \frac{1}{|S|-1}x$. Therefore, cartel profit π satisfies $\pi = \sum_{i \in S} \pi_i \leq \min \left\{ \frac{|S|}{|S|-1}x, 1 \right\}$. Finally, since $\min \left\{ \frac{|S|}{|S|-1}x, 1 \right\}$ is decreasing in $|S|$, and $|S| \geq 2$ by hypothesis, we have $\pi \leq \min \{2x, 1\}$.

If $n = 2$, then the lower and upper bounds coincide. Moreover, if all firms make positive profits, then $|S| = n$, and hence $\pi \leq \min \left\{ \frac{n}{n-1}x, 1 \right\}$, so again the lower and upper bounds coincide. ■

It may be surprising that, in the absence of mediation, the cartel does not strictly benefit from the disclosure of the losing bids. Indeed, this finding contrasts somewhat with the classic intuition of Stigler, who argued that, “The system of sealed bids, publicly opened with full identification of each bidder's price and specifications, is the ideal instrument for the detection of price-cutting,” (Stigler, 1964, p. 48). Propositions 2 and 3 qualify this intuition by noting that, without mediation, disclosing all bids is no more favorable to

collusion than is disclosing only the winner's bid and identity.²¹ However, we will see that mediated cartels can obtain strictly higher profits when all bids are disclosed, so Stigler's intuition is vindicated for mediated cartels.

4.2 Profit Bounds and Comparison to Bid Rotation

The upper bound on cartel profit that results from considering uniform downward deviations is as follows.

Proposition 4 *With all bids observed, cartel profit cannot exceed*

$$\frac{n}{n-1} \left(x + ((n-1)x)^{\frac{n-1}{n}} (n-1-x)^{\frac{1}{n}} \right) - nx.$$

For example, when $n = 2$, cartel profit cannot exceed $2\sqrt{x - x^2}$.

The proof shows that in the optimal bid distribution that deters all uniform downward deviations, with probability 1 at least $n - 1$ firms bid 1: that is, at most one firm at a time bids below 1. The intuition is that increasing all losing bids to 1 increases the marginal distribution of each bidder's bid—and hence increases the probability that she is punished following a uniform downward deviation—without affecting the distribution of the winning bid (and hence without affecting a bidder's equilibrium payoff or her probability of winning the auction following a uniform downward deviation). However, this bid distribution does not deter non-uniform downward deviations, where a bidder deviates to a bid $p < 1$ when she is recommended a bid of 1, but does not deviate when she is recommended a bid in between p and 1. For this reason (and in contrast to the unobserved losing bids case), the upper bound on cartel profit that results from considering only uniform downward deviations is not tight. The proof (along with all other omitted proofs) is deferred to Appendix A.

We obtain a lower bound on optimal cartel profit by considering equilibria of a similar form, where there exists a bid $\bar{p} \in [0, 1]$ such that, with probability 1, at least $n - 1$ firms bid \bar{p} . We say that such an equilibrium has *almost-uniform bids*.

²¹This result relies on the assumption of complete information about costs and values, so that optimal collusion without mediation implies a fixed winning bid. However, we conjecture that the same result would also hold with incomplete information if the bidders held a knockout auction before the target auction, as in this case the prescribed winning bid in the target auction would again be commonly known by the bidders.

Proposition 5 *With all bids observed, the optimal equilibrium among those with almost-uniform bids gives cartel profit equal to*

$$\min \left\{ \frac{(n-1)x}{n-1-x} \left(1 - \frac{n}{n-1} \log \left(\frac{nx}{n-1} \right) \right), \frac{(2n-1)x}{2n-2} \left(1 - \frac{n}{n-1} \log \left(\frac{n}{2n-1} \right) \right) \right\}.$$

Alternatively, if the mediator can deter upward deviations by placing a shill bid, the optimal equilibrium with almost-uniform bids gives cartel profit equal to $\frac{(n-1)x}{n-1-x} \left(1 - \frac{n}{n-1} \log \left(\frac{nx}{n-1} \right) \right)$.

Proposition 5 comes from piecing together two cases. We show that, if one imposes only downward incentive constraints, the best bid distribution with almost-uniform bids has a maximum bid of 1 (regardless of x) and gives cartel profit equal to $\frac{(n-1)x}{n-1-x} \left(1 - \frac{n}{n-1} \log \left(\frac{nx}{n-1} \right) \right)$.²² This bid distribution also satisfies all upward incentive constraints iff $x \geq \frac{n-1}{2n-1}$, or alternatively if the mediator can deter upward deviations by placing a shill bid. If instead $x < \frac{n-1}{2n-1}$ (and shill bids are infeasible), then the best equilibrium with almost-uniform bids has a maximum bid of $\frac{2n-1}{n-1}x$ and gives cartel profit equal to $\frac{(2n-1)x}{2n-2} \left(1 - \frac{n}{n-1} \log \left(\frac{n}{2n-1} \right) \right)$. Note that $\frac{2n-1}{2n-2} \left(1 - \frac{n}{n-1} \log \left(\frac{n}{2n-1} \right) \right) > \frac{n}{n-1}$, so profits in the latter case are linear in x , as in the equilibrium where both firms always bid $\frac{n}{n-1}x$ (which, recall, is the optimal equilibrium without mediation), but with a strictly higher slope.²³

Intuitively, the best equilibrium with almost-uniform bids is characterized by the maximum bid \bar{p} , together with the condition that a firm that is recommended to bid \bar{p} is indifferent between bidding \bar{p} and any bid p in between \bar{p} and $\frac{n}{n-1}x$, which is the lowest bid in the support of the equilibrium bid distribution. In turn, if $x \geq \frac{n-1}{2n-1}$, then the maximum bid \bar{p} is optimally set to equal 1. If instead $x < \frac{n-1}{2n-1}$, then the maximum bid cannot equal 1, because a bidder who is recommended the minimum bid of $\frac{n}{n-1}x$ would prefer to deviate upward to a bid just below 1. In this case, the maximum bid is set to satisfy this upward incentive constraint, so that $\frac{n}{n-1}x = \bar{p} - x$, or $\bar{p} = \frac{2n-1}{n-1}x$.

²²Note that this is lower than the bound from Proposition 4, because here the corresponding bid distribution deters *all* downward deviations, not just uniform ones.

²³With $n = 2$, Kawai, Nakabayashi, and Ortner (2022) construct an equilibrium where firms mix independently between prices $2x$ and 1. Such equilibria yield strictly lower cartel profit than that given in Proposition 5. Kawai, Nakabayashi, and Ortner's main focus is on *bid rotation equilibria*, where the identity of the winning bidder is fixed in advance. The best bid rotation equilibrium has a winning bid distribution of $W(p) = 1 - x/p$ for $p \in [x, 2x)$ and $W(2x) = 1$, which gives expected cartel profit $(1 + \log 2)x$. Thus, when $n = 2$ the best bid rotation equilibrium (with mediation) is even less profitable than the best equilibrium without mediation. (See Figure 6.)

Combining Propositions 4 and 5 lets us characterize optimal cartel profit with a large number of firms. This exercise requires taking a stance on how x varies with n . We assume that x is non-increasing in n , so that either x approaches a positive lower bound $\underline{x} > 0$ as $n \rightarrow \infty$ (e.g., the penalty is harassment by other cartel members, which is similarly unpleasant in small and large cartels), or $x \rightarrow 0$ as $n \rightarrow \infty$ (e.g., the penalty is reversion to competitive play, whence each firm loses their $1/n$ share of cartel profits).

Corollary 1 *Suppose that the penalty size with n firms is given by a non-increasing function $x(n)$ satisfying $\lim_{n \rightarrow \infty} x(n) = \underline{x} \geq 0$. Let $\pi^*(n)$ denote optimal cartel profit with n firms and penalty size $x(n)$. Then $\lim_{n \rightarrow \infty} \pi^*(n) = \underline{x}(1 - \log \underline{x})$, with convention $0 \log 0 = 0$.*

Proof. Note that $\lim_{n \rightarrow \infty} \frac{n}{n-1} \left(\underline{x} + ((n-1)\underline{x})^{\frac{n-1}{n}} (n-1-\underline{x})^{\frac{1}{n}} \right) - n\underline{x} = \underline{x}(1 - \log \underline{x})$. Hence, by Proposition 4, $\limsup_{n \rightarrow \infty} \pi^*(n) \leq \underline{x}(1 - \log \underline{x})$. Next, note that $\lim_{n \rightarrow \infty} \frac{(n-1)\underline{x}}{n-1-\underline{x}} \left(1 - \frac{n}{n-1} \log \left(\frac{n\underline{x}}{n-1} \right) \right) = \underline{x}(1 - \log \underline{x})$. Hence, by Proposition 5, if $\underline{x} \geq 1/2$ then $\liminf_{n \rightarrow \infty} \pi^*(n) \geq \underline{x}(1 - \log \underline{x})$. In Appendix A, we show that if $\underline{x} < 1/2$ then again $\liminf_{n \rightarrow \infty} \pi^*(n) \geq \underline{x}(1 - \log \underline{x})$. Thus, the \liminf and \limsup must both equal $\underline{x}(1 - \log \underline{x})$. ■

There is a simple intuition for the limit profit of $\underline{x}(1 - \log \underline{x})$. This is the profit that results from a bid-rotation scheme where one firm (the designated winner) bids 1 with probability \underline{x} , and otherwise bids between \underline{x} and 1 according to the unit-elastic cdf $F(p) = 1 - \underline{x}/p$, while all other firms place losing bids. Under this scheme, losing bidders cannot profitably deviate, because bidding just below 1 wins with probability \underline{x} but results in punishment, and (by construction) bidding anywhere between \underline{x} and 1 gives the same expected payoff. Hence, this scheme is an equilibrium whenever upward incentive constraints hold. Moreover, the corresponding winning bid distribution is the limit as $n \rightarrow \infty$ of both the distribution in the proof of Proposition 4 and that in the proof of Proposition 5. For fixed n , both of these distributions put more weight on higher bids, because a bidder that is recommended to bid 1 wins with positive probability, and so is less tempted to deviate as compared to a designated loser under bid rotation. However, as $n \rightarrow \infty$, a bidder that is recommended to bid 1 wins with probability approaching 0, so both distributions converge to the one just

described.²⁴ In sum, Corollary 1 shows that bid rotation is asymptotically optimal.

An interesting implication of Corollary 1 is that if the reserve price r is taken as a free parameter (rather than being normalized to 1), cartel profit in the $n \rightarrow \infty$ limit equals $x(1 - \log(x/r))$, which diverges as $r \rightarrow \infty$. The intuition is that when the winning bid is randomized over a wide range, a small penalty is enough to deter deviations by losing bidders.

While the bounds in Propositions 4 and 5 coincide only in the limit, they are already quite close together when $n = 2$ and $x \geq 1/3$ (so upward incentive constraints are slack). In this case, the upper bound exceeds the lower bound by less than 5%.

4.3 Optimal Equilibrium with Downward Incentive Constraints

For the rest of the paper, we assume that $n = 2$. We now characterize the optimal equilibrium, when only downward incentive constraints are considered. We will see that the resulting strategy profile is a genuine equilibrium—and hence the optimal one—if $x \geq 1/3$, or if shill bids are feasible.

To understand the structure of the optimal equilibrium, first recall the optimal equilibrium with almost-uniform bids, ignoring upward incentive constraints. In this equilibrium, the higher of the two bids is always equal to 1, and the lower bid is distributed on the interval $[2x, 1]$, so that a firm that is recommended a bid of 1 is indifferent among all bids in this interval. Observe that one way to improve cartel profit relative to this equilibrium is to recommend the bid profile $p_1 = p_2 = p$ with small probability, for any price $p < 1$ that is greater than the cartel profit in the original equilibrium. This follows because the resulting bid distribution yields higher expected profit by construction, and it remains an equilibrium because a firm that is recommended a bid of p expects to win the auction with high probability. The same logic implies that to obtain the optimal cartel profit π^* , the bid profile $p_1 = p_2 = p$ should be recommended with positive weight for every price p in the interval $[\pi^*, 1]$. Moreover, to increase the weight that can be placed on such profiles, for each bid

²⁴The distribution in the proof of Proposition 4 is not an equilibrium for any finite n , because a bidder that is recommended to bid 1 has a profitable non-uniform downward deviation. However, as the probability that each bidder is recommended to bid 1 converges to 1, the expected payoff difference between uniform and non-uniform deviations vanishes, so this distribution is an ε -equilibrium, for ε converging to 0 as $n \rightarrow \infty$.

$p \geq \pi^*$ there should exist a bid $\chi(p) \in [2x, \pi^*]$ such that a firm that is recommended a bid of p is indifferent among all bids in the interval $[\chi(p), p]$. To support this indifference condition, the bid $\chi(p)$ is also the lowest bid in the support of the conditional distribution $F_j(p_j|p_i = p)$; that is, for $p \geq \pi^*$, the function $\chi(p)$ describes the lower boundary of the support of the optimal joint bid distribution. For $p \geq \pi^*$, the conditional distribution $F_j(p_j|p_i = p)$ is thus supported on the interval $[\chi(p), 1]$; it turns out to have a positive density for all $p_j \neq p_i$, with an atom at $p_j = p_i$. Finally, given the function $\chi(p)$, the double-continuum of binding downward incentive constraints (from each $p \geq \pi^*$ to each $p' \in [\chi(p), p)$), together with symmetry, determines the optimal distribution with this support.²⁵

Specifically, let $\hat{\pi}$ denote the unique solution for π in the interval $[2x, 1]$ to the equation

$$2x \log 2x + (\pi - x) \log \frac{\pi(\pi - x)}{x} = 0. \quad (4)$$

We show that $\hat{\pi}$ is the optimal cartel profit when only downward incentive constraints are considered, and that the corresponding bid distribution is determined as the solution to certain relatively simple equations involving the function $\chi(p)$ (equations (8) and (9) below). Formally, we establish the following result.

Proposition 6 *With all bids observed and $n = 2$, the optimal equilibrium when only downward incentive constraints are considered gives cartel profit $\hat{\pi}$. The optimal joint distribution of bids $F(p_1, p_2)$ is determined by the marginal distribution $F_i(p_i)$ defined in equation (8) and the family of conditional distribution $F_j(p_j|p_i)$ defined in equation (9). The distribution $F(p_1, p_2)$ is atomless on a connected subset of $[2x, 1]^2$, and has the following properties:*

1. *The marginal $F_i(p_i)$ is supported on the interval $[2x, 1]$. It is atomless with a positive density $f_i(p_i)$ on $[2x, 1]$, which satisfies $\lim_{p_i \rightarrow 1} f_i(p_i) = \infty$.*
2. *A bidder that is recommended $p_i \geq \hat{\pi}$ wins with positive probability. In particular, for each $p_i \geq \hat{\pi}$, the conditional $F_j(p_j|p_i)$ is supported on the interval $[\chi(p_i), 1]$, where $\chi(p)$*

²⁵As this discussion indicates, while the optimal distribution maximizes the expected winning bid (by definition), it does not maximize the winning bid distribution in terms of first-order stochastic dominance. For example, the distribution given by $\Pr(\max\{p_1, p_2\} = 1) = 1$ and the same conditional distribution $F_j(p_j|p_i = 1)$ as in the optimal distribution (which is also the same conditional distribution as in the proof of Proposition 5) gives a higher probability that $\min\{p_1, p_2\} = 1$.

is a decreasing function satisfying $\chi(\hat{\pi}) = \hat{\pi}$ and $\chi(1) = 2x$, defined in equation (6). The conditional $F_j(p_j|p_i)$ has a positive density $f_j(p_j|p_i)$ at all $p_j \neq p_i$ in $[\chi(p_i), 1)$, which satisfies $\lim_{p_j \rightarrow 1} f_j(p_j|p_i) = \infty$, and $F_j(p_j|p_i)$ has an atom at $p_j = p_i$.

3. A bidder that is recommended $p_i < \hat{\pi}$ wins with probability 1. In particular, for each $p_i < \hat{\pi}$, the conditional $F_j(p_j|p_i)$ is supported on the interval $[\chi^{-1}(p_i), 1]$. It is atomless with a positive density $f_j(p_j|p_i)$ on $[\chi^{-1}(p_i), 1)$, which satisfies $\lim_{p_j \rightarrow 1} f_j(p_j|p_i) = \infty$.

Moreover, $F(p_1, p_2)$ also satisfies all upward incentive constraints—and hence is the optimal equilibrium—if either $x \geq 1/3$ or the mediator can place a shill bid.

Figure 1 is a heat map for the optimal joint bid distribution when $x = .35$ and recommended bids are restricted to multiples of .01.²⁶ The qualitative features described in Proposition 6 are readily apparent. The marginal $F_i(p_i)$ is supported on the interval $[2x, 1]$; the conditional $F_j(p_j|p_i)$ is supported on an interval that includes 1, and that is wider when p_i is higher; and more probability mass is assigned to bid pairs (p_1, p_2) where either $p_1 = p_2$ or $\max\{p_1, p_2\} = 1$ than to other pairs. (When recommended bids are continuous, this corresponds to the conditional $F_j(p_j|p_i)$ having an atom at $p_j = p_i$, and to the marginal density $f_i(p_i)$ diverging to infinity as $p_i \rightarrow 1$.) In addition, the optimal cartel profit $\hat{\pi}$ is equal to the smallest price p such that $(p, p) \in \text{supp } F(p_1, p_2)$, which is approximately .93. (This can be seen by counting down seven grid points along the diagonal, starting from $(1, 1)$.)

We mention a few more properties of the optimal equilibrium. First, a bidder who is recommended a bid $p < \hat{\pi}$ expects to win the auction for sure, and hence has an expected payoff (conditional on her recommendation) of p ; while a bidder who is recommended a bid $p \geq \hat{\pi}$ expects to win with positive probability, and has an expected payoff of $\chi(p) - x$. Since $\chi(p)$ is a decreasing function satisfying $\chi(\hat{\pi}) = \hat{\pi}$ and $\chi(1) = 2x$, we see that a bidder's expected payoff conditional on her recommendation p is inverse-U shaped in p , with a supremum of $\hat{\pi}$ (attained at $p \uparrow \hat{\pi}$), a discontinuity at $p = \hat{\pi}$, and a minimum of x (attained at $p = 1$). In particular, unlike a bid rotation scheme, a bidder is never recommended a bid that she knows is losing. Second, whenever the winning bid is below $\hat{\pi}$, this bid is “isolated”:

²⁶The distribution is a true equilibrium: i.e., deviations to all bids in $[0, 1]$ are unprofitable. For heat maps for some other values for x , see Appendix C.



Figure 1: The optimal bid distribution when $x = .35$. The figure is generated by restricting recommended bids to multiples of .01. The color scheme is as follows: gray cells have 0 mass; green cells have small positive mass; yellow cells have mass .01; red cells have mass at least .1; and colors in between green and yellow, (resp., yellow and red), interpolate between between mass 0 and .01 (resp., .01 and .1). The only red cell in the figure is at $p_1 = p_2 = 1$; this single bid pair is recommended with probability approximately .42. (The second-heaviest cell is $p_1 = p_2 = .99$, which is recommended with probability approximately .03.)

i.e., there is a bounded gap between the winning and losing bid. The optimal equilibrium thus displays the “missing bids” pattern documented in Chassang et al. (2022). In contrast, when the winning bid is above π^* , the bids are tied with with positive probability. Third, the support of the optimal bid distribution expands when the penalty x shrinks. Loosely speaking, “less enforcement power” for the cartel is associated with “less predictable” bids.

To prove Proposition 6, we set up the problem of finding the optimal equilibrium with only downward incentive constraints as an infinite-dimensional linear program, and solve it using duality. We first construct the function $\chi(p)$ as the solution to an ordinary differential equation (equation (7)). Next, we define the marginal distribution of a firm’s bid, $F_i(p_i)$, as the solution to an integral equation involving the function $\chi(p)$ (equation (8)). The characterization of the optimal joint distribution $F(p_1, p_2)$ is then completed by specifying the conditional distributions $F_j(p_j|p_i = p)$ so that downward incentive constraint from any $p \geq \hat{\pi}$ to any $p' \in [\chi(p), p)$ binds (equation (9)), and using symmetry. Finally, we prove that this joint distribution is indeed optimal by constructing multipliers $\lambda(p'|p)$ on the downward incentive constraints, which are feasible for the dual linear program and yield the same value in the dual program as $F(p_1, p_2)$ does in the primal program. (Here, $\lambda(p'|p)$ denotes the multiplier on the constraint that it is unprofitable for a firm that is recommended a bid of p to deviate to a bid of p' .) By weak duality, the existence of such multipliers implies that $F(p_1, p_2)$ is optimal in the primal program, which completes the proof.

In sum, the proof proceeds by first guessing equations that the value $\hat{\pi}$, the function $\chi(p)$, and the joint distribution $F(p_1, p_2)$ should satisfy; then constructing $\hat{\pi}$, $\chi(p)$, and $F(p_1, p_2)$ to satisfy these equations; and then constructing multipliers $\lambda(p'|p)$ that certify the optimality of $F(p_1, p_2)$. This proof approach works, but it does not explain where the equations that determine $\hat{\pi}$, $\chi(p)$, $F(p_1, p_2)$, and $\lambda(p'|p)$ come from. These equations can be derived as follows: Start with the properties that $\chi(\hat{\pi}) = \hat{\pi}$, $\chi(1) = 2x$, downward incentives constraints from any $p \geq \hat{\pi}$ to any $p' \in [\chi(p), p)$ bind, and the multipliers $\lambda(p'|p)$ take the form

$$\lambda(p'|p) = \begin{cases} \kappa(p') & \text{if } p \geq \hat{\pi} \text{ and } \chi(p) \leq p' < p, \\ 0 & \text{otherwise,} \end{cases}$$

for some function $\kappa(p')$. (In particular, $\lambda(p'|p) = \lambda(p'|\hat{p})$ for all $p, \hat{p} \geq \hat{\pi}$ such that $p' \in$

$[\chi(p), p] \cap [\chi(\hat{p}), \hat{p}]$. This property must hold in order for (p, p') , $(p, p' + \varepsilon)$, (\hat{p}, p') , and $(\hat{p}, p' + \varepsilon)$ to all satisfy the dual constraint with equality, which in turn is necessary for these points to all lie in the support of the optimal bid distribution.) For any candidate optimal profit level π , the equation $\chi(\pi) = \pi$, together with the above properties for $\chi(p)$ and $\lambda(p|\hat{p})$, “inductively” determines differential equations in p for $\chi(p)$ and $\lambda(p|\hat{p})$ (where $\hat{p} > p$ is an arbitrary price satisfying $p \in [\chi(\hat{p}), \hat{p}]$), starting from $p = \pi$ and ending at $p = 1$. The optimal profit level $\hat{\pi}$ is then determined as the value for π such that initializing these equations at $\chi(\pi) = \pi$ yields the required terminal condition $\chi(1) = 2x$, and the optimal functions $\chi(p)$ and $\lambda(p|\hat{p})$ are given by the corresponding solutions. Next, given the optimal function $\chi(p)$, the joint distribution $F(p_1, p_2)$ is obtained in three steps. First, binding downward IC from $p \geq \hat{\pi}$ to $p' \in [\chi(p), p]$ implies that, for all $p \geq \hat{\pi}$, $F_j(p'|p) = 1 - \frac{\chi(p)}{p'}$ for all $p' \in [\chi(p), p]$, and $F_j(p|p) = 1 - \frac{\chi(p) - 2x}{p}$. This pins down the conditional distributions $F_j(p'|p)$ for $p \geq \hat{\pi}$ and $p' \leq p$. Second, by symmetry, we have $f_j(p'|p)f_i(p) = f_j(p|p')f_i(p')$ for all p and $p' \leq p$, where $f_i(p)$ is the marginal density and $f_j(p'|p)$ is the conditional density. This symmetry condition pins down $F_j(p'|p)$ for $p' > p$. Third, the marginal distribution $F_i(p)$ is determined by the equation

$$(1 - F_j(p|p))f_i(p) = \int_p^1 f_j(p|p')f_i(p')dp' \quad \forall p \in \text{supp } F_i,$$

which again holds by symmetry. Together, the marginal distribution $F_i(p_i)$ and the conditional distributions $F_j(p_j|p_i)$ determine the joint distribution $F(p_1, p_2)$.

Extending Proposition 6 to $n \geq 3$ firms seems challenging. For any n , downward incentive constraints depend only on the joint distribution of a bidder’s own recommended bid and the minimum (winning) recommended bid. When $n = 2$, the optimal (*own bid, winning bid*) distribution that satisfies downward incentive constraints is always implementable by a symmetric bid distribution. In contrast, when $n \geq 3$, we have verified numerically that the optimal such (*own bid, winning bid*) distribution is not always implementable by a symmetric distribution. Thus, when $n \geq 3$ global constraints on the set of implementable (*own bid, winning bid*) distributions bind. These constraints seem difficult to handle analytically.

4.3.1 Proof of Proposition 6

Primal Linear Program Let \mathcal{F} denote the set of all symmetric cdfs on $[0, 1]^2$. The primal program \mathbf{P} that characterizes the optimal equilibrium (considering only downward incentive constraints) is

$$\begin{aligned} & \sup_{F \in \mathcal{F}} \Pi(F) \\ \text{s.t.} \quad & IC(p, p'; F) \geq 0 \quad \forall p \in \text{supp}(F_i), p' < p, \end{aligned} \tag{5}$$

where $\Pi(F) := \int_{p, q \leq p} (2q - p \mathbf{1}_{p=q}) dF(p, q)$ is cartel profit when firms follow their recommendations, F_i is the marginal of F over p_i and, for all $p > p'$ and $F \in \mathcal{F}$,

$$IC(p, p'; F) := p \left(1 - \frac{1}{2} (F_j^-(p|p_i = p) + F_j(p|p_i = p)) \right) - p' (1 - F_j(p'|p_i = p)) + x,$$

where $F_j(p|p_i)$ denotes conditional probability under F , and $F_j^-(p|p_i) := \lim_{p_j \uparrow p} F_j(p_j|p_i)$.²⁷ To understand the constraint, note that a firm that follows a recommendation to bid p wins the auction with probability $1 - \frac{1}{2} (F_j^-(p|p_i = p) + F_j(p|p_i = p))$ (by uniform tie-breaking), while if the firm bids $p' < p$ it wins with probability $1 - \frac{1}{2} (F_j^-(p'|p_i = p) + F_j(p'|p_i = p))$ and is penalized.²⁸ However, since it is better to deviate to a bid just below any mass point of the conditional distribution $F_j(p_j|p_i)$, it is equivalent to impose the apparently tighter constraint where the latter probability is replaced by $1 - F_j(p'|p_i = p)$.

The program characterizing the optimal equilibrium with both downward and upward constraints differs from \mathbf{P} only in that $IC(p, p'; F) \geq 0$ is also imposed for all $p' > p$. We will see that when $x \geq 1/3$ the solution to \mathbf{P} also satisfies these extra constraints.

Optimal Bid Distribution We define a joint distribution of bids $F(p_1, p_2)$, which we will later verify is optimal. Recall that $\hat{\pi}$ is defined as the unique solution to equation (4) in the interval $[2x, 1]$. (This is well-defined because the LHS of (4) is strictly increasing in π over

²⁷Formally, we require that there is a version of the conditional probability $F_j(p'|p)$ that satisfies the constraint for all p , or equivalently that the constraint holds for almost all p for every version of the conditional probability.

²⁸Penalizing any firm that deviates from its recommendation is without loss, as this only relaxes incentive constraints.

this range, and takes a negative value at $\pi = 2x$ and a positive value at $\pi = 1$.)

Given the definition of $\hat{\pi}$, the function $\chi(p)$ is defined in the following lemma.²⁹

Lemma 1 *There exists a unique function $\chi : [\hat{\pi}, 1] \rightarrow [0, 1]$ such that, for each $p \in [\hat{\pi}, 1]$,*

$$\frac{2x \log(\chi(p)) + (\hat{\pi} - x) \log(x + \hat{\pi} - \chi(p))}{x + \hat{\pi}} = \log p + \frac{2x \log(\hat{\pi}) + (\hat{\pi} - x) \log(x)}{x + \hat{\pi}} - \log(\hat{\pi}). \quad (6)$$

The function $\chi(p)$ is strictly decreasing and satisfies $\chi(\hat{\pi}) = \hat{\pi}$ and $\chi(1) = 2x$. Moreover, at each $p \in [\hat{\pi}, 1)$, $\chi(p)$ is differentiable, and its derivative $\chi'(p)$ satisfies

$$\frac{p\chi'(p)}{\chi(p)} = \frac{\chi(p) - (x + \hat{\pi})}{\chi(p) - 2x}. \quad (7)$$

Given the definitions of $\hat{\pi}$ and $\chi(p)$, the marginal distribution $F_i(p)$ is defined in the following lemma.

Lemma 2 *There exists a unique function $F_i : [0, 1] \rightarrow [0, 1]$ such that $F_i(2x) = 0$, $F_i(1) = 1$, and $F_i(p)$ is differentiable at each $p \in [0, 1)$, with derivative $f_i(p)$ satisfying*

$$f_i(p) = \begin{cases} 0 & \text{if } p \in [0, 2x), \\ \frac{1}{p^2} \int_{\chi^{-1}(p)}^1 \chi(p') f_i(p') dp' & \text{if } p \in [2x, \hat{\pi}), \\ \frac{1}{(\chi(p) - 2x)p} \int_{p'}^1 \chi(p') f_i(p') dp' & \text{if } p \in [\hat{\pi}, 1], \end{cases} \quad \forall p \in [0, 1). \quad (8)$$

Given the definitions of $\hat{\pi}$, $\chi(p)$, and $F_i(p)$, the conditional distribution $F_j(p'|p)$ is defined as follows: for all $p, p' \in [2x, 1]$,

$$F_j(p'|p) = \begin{cases} 0 & \text{if } p \geq \hat{\pi}, p' < \chi(p) \text{ or } p < \hat{\pi}, p' < \chi^{-1}(p), \\ 1 - \frac{\chi(p)}{p'} & \text{if } p \geq \hat{\pi}, p' \in [\chi(p), p), \\ 1 - \frac{\chi(p) - 2x}{p} & \text{if } p \geq \hat{\pi}, p' = p, \\ 1 - \frac{\chi(p) - 2x}{p} + \frac{1}{f_i(p)} \int_p^{p'} \frac{\chi(\hat{p})}{p^2} f_i(\hat{p}) d\hat{p} & \text{if } p \geq \hat{\pi}, p' > p, \\ \frac{1}{f_i(p)} \int_{\chi^{-1}(p)}^{p'} \frac{\chi(\hat{p})}{p^2} f_i(\hat{p}) d\hat{p} & \text{if } p < \hat{\pi}, p' > \chi^{-1}(p). \end{cases} \quad (9)$$

²⁹Proofs of all lemmas are deferred to Appendix A.

Note that, for all $p \in [2x, 1]$, $F_j(p'|p)$ is indeed a cdf. To see this, consider first $p \in [\hat{\pi}, 1]$. By (8) and (9), $F_j(p'|p)$ is increasing and right-continuous in p' for $p' \in [\chi(p), 1]$, with $F_j(\chi(p)|p) = 0$ and $F_j(1|p) = 1$. Similarly, for $p \in [2x, \hat{\pi})$, by (8) and (9) $F_j(p'|p)$ is increasing and continuous in p' for $p' \in [\chi^{-1}(p), 1]$, with $F_j(\chi^{-1}(p)|p) = 0$ and $F_j(1|p) = 1$.

Finally, let $F(p_1, p_2) \in \mathcal{F}$ be the (unique) joint cdf with marginal distribution $F_1(p_1)$ given by (8), and conditional distributions $F_2(p_2|p_1)$ given by (9). Note that

$$\text{supp}(F) = \{(p_i, p_j) \in [0, 1]^2 : p_i \in [2x, \hat{\pi}), p_j \in [\chi^{-1}(p), 1] \text{ or } p_i \in [\hat{\pi}, 1], p_j \in [\chi(p), 1]\}.$$

We verify that F is feasible for \mathbf{P} . We first show that F is symmetric.

Lemma 3 $F(p, p') = F(p', p)$ for all $p, p' \in [0, 1]$.

We now verify that $IC(p, p'; F) \geq 0$ for all $p \geq 2x, p' < p$. This shows that F is feasible for \mathbf{P} : i.e., F satisfies all downward incentive constraints. Moreover, if $x \geq 1/3$ then in addition $IC(p, p'; F) \geq 0$ for all $p \geq 2x, p' > p$, so F is an equilibrium bid distribution.

Lemma 4 $IC(p, p'; F) \geq 0$ for all $p \geq 2x, p' < p$, with equality for all $p \geq \hat{\pi}, p' \in [\chi(p), p)$. Moreover, if $x \geq 1/3$ then in addition $IC(p, p'; F) \geq 0$ for all $p \geq 2x, p' > p$.

Dual Linear Program Let $\mathcal{B}[0, 1]$ denote the set of Borel subsets of $[0, 1]$, and let \mathcal{M} denote the set of all bounded, measurable functions $\Lambda : [0, 1] \times \mathcal{B}[0, 1] \rightarrow \mathbb{R}_+$ such that, for every $p \in [0, 1]$, the induced mapping $P \mapsto \Lambda(p, P)$ (henceforth written as $\Lambda(P|p)$) is a (finite) measure on $\mathcal{B}[0, 1]$. Consider the dual linear program \mathbf{D} , given by

$$\begin{aligned} & \inf_{\Lambda \in \mathcal{M}, \mu \in \mathbb{R}_+} \mu \\ \text{s.t.} \quad & G(p, q; \Lambda, \mu) \geq 0 \quad \forall p, q \leq p, \end{aligned} \tag{10}$$

where, for all $p > q$, Λ , and $\mu \geq 0$,

$$\begin{aligned} G(p, p; \Lambda, \mu) &:= \int_{p' \leq p} \left(-\frac{p}{2} - x + p' \right) d\Lambda(p'|p) + \mu - p \quad \text{and} \\ G(p, q; \Lambda, \mu) &:= \int_{p' \leq p} (-x + p' \mathbf{1}_{p' \leq q}) d\Lambda(p'|p) + \int_{p' \leq q} (-q - x + p') d\Lambda(p'|q) + 2(\mu - q). \end{aligned}$$

The interpretation of the dual program is that μ is cartel profit (which also equals the shadow value of probability mass on an optimal bid pair), $d\Lambda(p'|p)$ is the value of relaxing the incentive constraint that it is unprofitable for a bidder who is recommended a bid of p to deviate to a bid of p' , and $G(p, q; \Lambda, \mu)$ is the cost of increasing the probability of recommending the bid pair (p, q) , with $q < p$, where this cost is given by the effect of such an increase on the downward incentive constraints for each $p' < p$ (the first integral in the formula for $G(p, q; \Lambda, \mu)$), plus the effect on the downward incentive constraints for each $p' < q$ (the second integral in the formula for $G(p, q; \Lambda, \mu)$), plus twice the difference between equilibrium cartel profit (equal to μ) and profit at the bid pair (p, q) (equal to q).

Lemma 5 *Let F be feasible for \mathbf{P} , and let (Λ, μ) be feasible for \mathbf{D} . We have:*

(i) **Weak duality:** $\Pi(F) \leq \mu$.

(ii) **Complementary slackness:** *If F and Λ satisfy (1) $IC(p, p'; F) = 0$ for all $p, p' < p$ with $p \in \text{supp}(F_i)$ and $p' \in \text{supp}(\Lambda(\cdot|p))$, and (2) $G(p, q; \Lambda, \mu) = 0$ for all $p, q \in \text{supp}(F)$ with $q \leq p$, then $\Pi(F) = \mu$, F is optimal for \mathbf{P} , and (Λ, μ) is optimal for \mathbf{D} .*

Optimal Dual Variables Given Lemma 5, we prove Proposition 6 by finding a function Λ such that $(\Lambda, \hat{\pi})$ is feasible for \mathbf{D} , and F and $(\Lambda, \hat{\pi})$ satisfy the complementary slackness conditions in Lemma 5(ii).

For each $p \in [0, 1]$, let $\Lambda(\cdot|p)$ be the unique Borel measure on $\mathcal{B}([0, 1])$ such that

$$\Lambda([p', p'']|p) = \int_{p'}^{p''} \lambda(q|p) dq \quad \forall p', p'' \in [0, 1], p' < p'',$$

where for each $p, p' \in [0, 1]$,

$$\lambda(p'|p) = \begin{cases} \frac{2(\chi(p') - x)(-\chi'(p'))}{x\chi(p')} & \text{if } p \geq \hat{\pi}, p' \in [\hat{\pi}, p), \\ \frac{2}{p'} & \text{if } p \geq \hat{\pi}, p' \in [\chi(p), \hat{\pi}), \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

We verify that $(\Lambda, \hat{\pi})$ is feasible for \mathbf{D} .

Lemma 6 $\Lambda \in \mathcal{M}$, and $G(p, q; \Lambda, \hat{\pi}) \geq 0$ for all $p \geq q$, with equality if $p \geq \hat{\pi}$ and $q \in [\chi(p), p]$.

Complementary Slackness Finally, we verify that F and $(\Lambda, \hat{\pi})$ satisfy the conditions in Lemma 5(ii). For each $p \in \text{supp}(F_i) = [2x, 1]$, we have $\text{supp}(\Lambda(\cdot|p)) = \emptyset$ if $p < \hat{\pi}$, and $\text{supp}(\Lambda(\cdot|p)) = [\chi(p), p]$ if $p \geq \hat{\pi}$. By Lemma 4, we have $IC(p, p'; F) = 0$ for all $p \in \text{supp}(F_i)$ and $p' \in \text{supp}(\Lambda(\cdot|p))$. By Lemma 6, $G(p, q; \Lambda, \hat{\pi}) = 0$ for all $(p, q) \in \text{supp}(F)$ with $p \geq q$. Hence, by Lemma 5(ii), $\Pi(F) = \mu$, F is optimal for \mathbf{P} , and (Λ, μ) is optimal for \mathbf{D} .

4.4 Upward Incentive Constraints

We finally consider optimal equilibria when both upward and downward incentive constraints bind. As we have seen, this case arises when $x < 1/3$ and shill bids are infeasible. (For example, this case may apply to auctions with a fixed set of officially registered bidders.) We first show that for any $x > 0$, the support of the distribution of winning bids includes 1, the highest possible winning bid. Thus, an arbitrarily small amount of cartel “enforcement power” implies a positive probability that the winning bid is as high as the reserve price. We then illustrate numerically how the winning bid distribution depends on x . We find that if x is just below $1/3$, then the support of the bid distribution takes a similar form as in the $x \geq 1/3$ case, and in particular consists of a single connected subset of the set $[2x, 1]^2$. If instead x is farther below $1/3$, then the support of the bid distribution consists of the union of such a connected set and an interval where both firms place the same low bid. Intuitively, introducing a small probability that the firms place the same low bid deters a firm with a low recommendation from deviating upward. Finally, we sketch how the optimal distribution with $x < 1/3$ can be characterized as the solution to a system of differential equations. Since the characterization in this case is quite complicated, we omit the details.

We first show that, without loss of optimality, the distribution of winning bids includes the bid $p = 1$: that is, the support of the bid profile distribution includes the profile $p_1 = p_2 = 1$.³⁰

³⁰We conjecture the stronger result that there is a unique equilibrium that yields profit exactly π^* , and that in this distribution the support of the winning bid distribution includes $p = 1$. This result holds when $x \geq 1/3$ as a consequence of Proposition 6. But proving it when $x < 1/3$ seems complicated, so we content ourselves with the stated version of Proposition 7.

Proposition 7 *For every $\varepsilon > 0$, there exists an equilibrium that yields profit at least $\pi^* - \varepsilon$ in which the support of the distribution of winning bids includes the bid $p = 1$.*

The intuition for Proposition 7 is as follows. First, for every $\varepsilon > 0$, there exists an equilibrium that yields profit at least $\pi^* - \varepsilon$ where all recommended bids lie on a sufficiently fine grid. Second, among such equilibria, there exists an optimum where the firms are simultaneously recommended the maximum price in the support, \bar{p} , with positive probability. This follows because it is suboptimal to recommend a bid that is surely losing. Third, if \bar{p} is not the highest grid element below 1, then the bid distribution can be modified so that the probability mass on the bid recommendation pair (\bar{p}, \bar{p}) is split among the pairs (\bar{p}, \bar{p}) , $(\bar{p}, \bar{p} + \eta)$, $(\bar{p} + \eta, \bar{p})$, and $(\bar{p} + \eta, \bar{p} + \eta)$, where η is the grid size, in a way that increases profit and preserves incentive compatibility.

Next, we illustrate numerically how the optimal bid distribution depends on x . Figures 2–5 displays heat maps for the optimal bid distribution for four values of x : .1, .2, .3, and .32.³¹ When $x = .32$, the shape of the optimal distribution is similar to that when $x > 1/3$ (cf. Figure 1), except that the support of the conditional $F_j(p_j|p_i)$ is now larger for the smallest values of p_i in the support of the marginal $F_i(p_i)$ than it is for slightly larger values of p_i . Intuitively, upward incentive constraints now bind for the smallest recommended bids, and recommending a wider range of opposing bids relaxes these constraints.

When $x \in \{.1, .2, .3\}$, the optimal bid distribution again has a similar shape, except that now $(p, p) \in \text{supp } F(p_1, p_2)$ for an interval of bids p including $2x$. Intuitively, occasionally recommending the pair (p, p) is another way to deter upward deviations from p , and when x is small this can be the most efficient way to deter such deviations. As the figures show, if x is above a threshold (approximately equal to .31), $F(p_1, p_2)$ is supported on a single connected subset of \mathbb{R}^2 , which includes the point $(1, 1)$; while if x is below this threshold, $F(p_1, p_2)$ is supported on the union of two connected subsets of \mathbb{R}^2 , of which one includes $(1, 1)$ and the other lies on the “diagonal” $\{(p_1, p_2) : p_1 = p_2\}$. Note also that as x shrinks, the distribution $F(p_1, p_2)$ converges (in distribution) toward a point mass on $p_1 = p_2 = 0$, but the set $\text{supp } F(p_1, p_2) \cup \{(p_1, p_2) : p_1 = p_2\}$ only expands, consistent with Proposition 7.

³¹These figures are all given for the case where skill bids are infeasible, so upward incentive constraints bind. Analogous figures with only downward incentive constraints—which illustrate the bid distribution derived in Proposition 6—are given in Appendix C.



Figure 2: The optimal bid distribution when $x = .1$. The color scheme in Figures 2–5 is the same as in Figure 1. When $x = .1$, the bid pair $p_1 = p_2 = 1$ is recommended only with probability approximately .02.



Figure 3: The optimal bid distribution when $x = .2$. Bid pair $p_1 = p_2 = 1$ is recommended with probability .11.



Figure 4: The optimal bid distribution when $x = .3$. Bid pair $p_1 = p_2 = 1$ is recommended with probability .29.

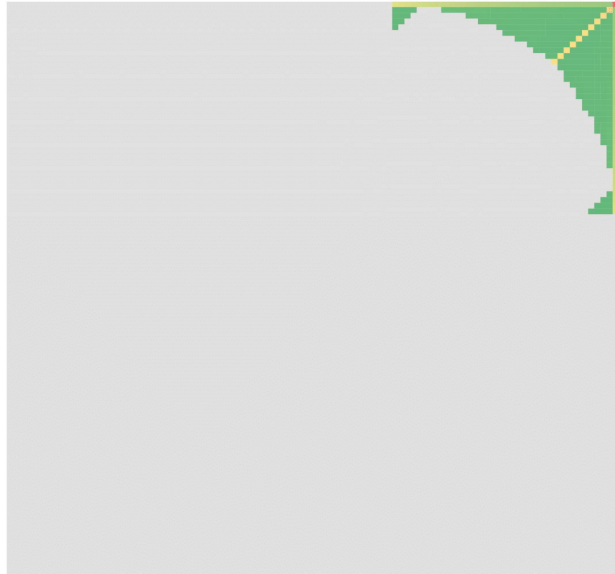


Figure 5: The optimal bid distribution when $x = .32$. Bid pair $p_1 = p_2 = 1$ is recommended with probability .34. Note that the support of the bid distribution is connected.

We now sketch the analytic characterization of the optimal bid distribution when $x < 1/3$ (and shill bids are infeasible). In this case, the distribution derived in Proposition 6 violates upward incentive constraints for recommended bids sufficiently close to $2x$ (i.e., the lowest bids in the support of the optimal bid distribution). To characterize the optimal distribution, we first conjecture that the multipliers $\lambda(p'|p)$ take the form

$$\lambda(p'|p) = \begin{cases} \kappa(p') & \text{if } p \geq \pi^* \text{ and } \chi(p) \leq p' < p, \\ \psi(p') & \text{if } 2x \leq p \leq \bar{p} \text{ and } \varphi(p) \leq p' \leq \chi^{-1}(p), \\ 0 & \text{otherwise,} \end{cases} \quad (12)$$

where \bar{p} satisfies $\bar{p} + x = \chi^{-1}(\bar{p})$, and $\varphi(p)$ is the smallest bid above p to which a bidder who is recommended p is tempted to deviate. Intuitively, as in the $x \geq 1/3$ case, the multiplier on the constraint that a bidder does not gain by deviating from p to $p' < p$ is the same for all p such that $p \geq \pi^*$ and $\chi(p) \leq p'$; moreover, the multiplier on the constraint that a bidder does not gain by deviating from p to $p' > p$ is the same for all p such that $2x \leq p \leq \bar{p}$ and $\varphi(p) \leq p' \leq \chi^{-1}(p)$. (Note that if $x \geq 1/3$ then $p + x \geq 1 \geq \chi^{-1}(p)$ for all $p \geq 2x$, so $\lambda(p'|p) = 0$ for all $p' \geq p$.) The conjecture that the dual variables $\lambda(p'|p)$ take this form, together with strong duality, implies a system of (delay) differential equations for π^* , $\chi(p)$, $\varphi(p)$, and $\lambda(p'|p)$. Unlike in the $x \geq 1/3$ case, these equations cannot be solved analytically; however, they can be shown to admit a solution, which characterizes optimal cartel profit π^* . The solution also reveals that the support of the optimal bid distribution contains the pair (p, p) for all $p \leq p_0$, where p_0 solves

$$\pi^* - p_0 = \left(\frac{p_0}{2} + x\right) \int_{p=\varphi(\chi^{-1}(p_0))}^{\chi^{-1}(p_0)} \kappa(p) dp.$$

Intuitively, this equation corresponds to the equation $G(p_0, p_0; \Lambda, \pi^*) = 0$ in the $x \geq 1/3$ case. We then have $G(p, p; \Lambda, \pi^*) = 0$ for all $p \leq p_0$, so the support of the optimal bid distribution contains the pair (p, p) , by complementary slackness. The resulting shape of the support of the optimal bid distribution matches that found numerically above.

Figure 6 compares cartel profit across the different settings and equilibria considered in this section, for two firms and any value for x . The top (orange) curve is the upper bound

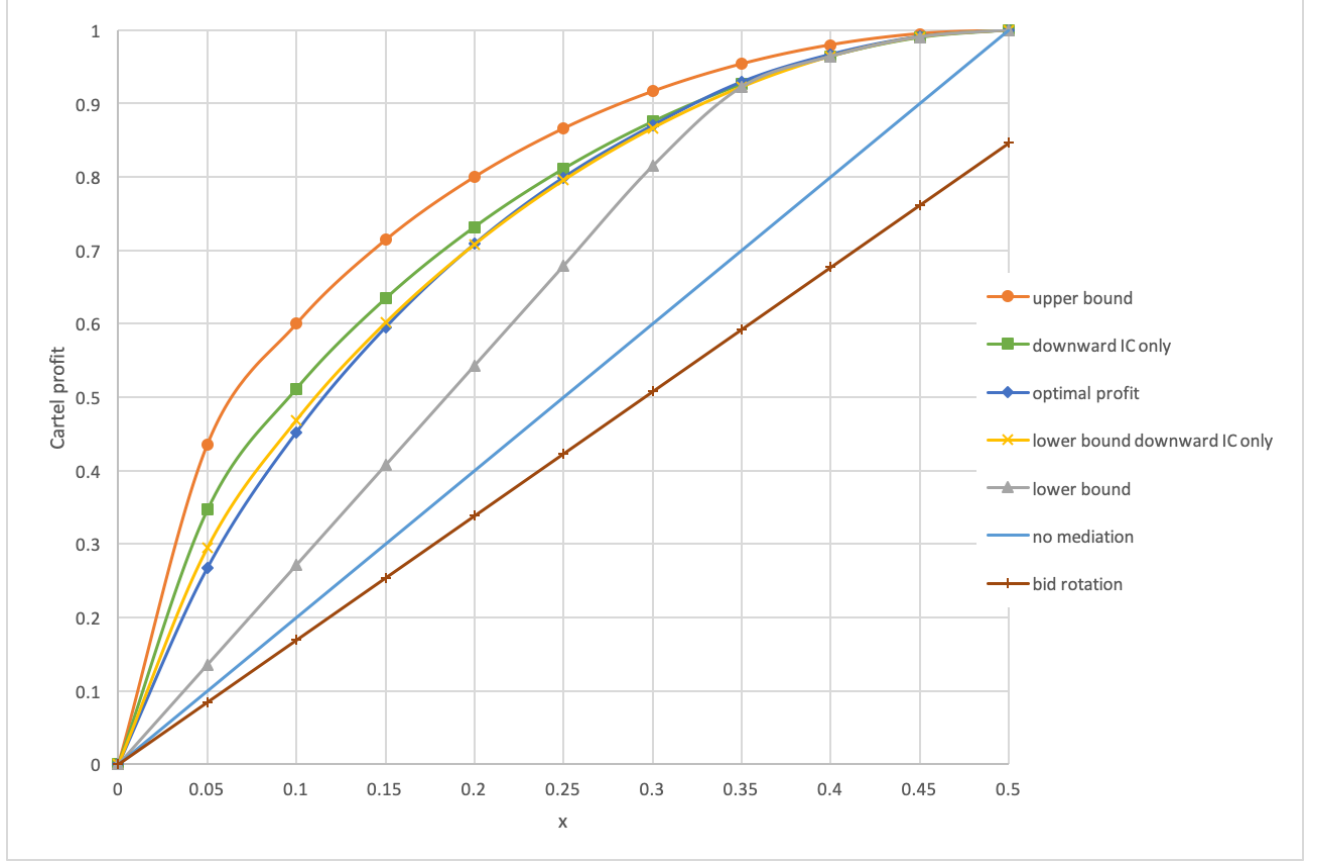


Figure 6: Cartel profit with two bidders.

$2\sqrt{x - x^2}$ derived in Proposition 4. The next (green) curve is the solution to equation (4), which by Proposition 6 is optimal cartel profit with only downward incentive constraints. The next two curves are optimal profit with all incentives constraints (the dark blue curve, which coincides with the green curve for $x \geq 1/3$) and the optimal equilibrium with almost-uniform bids with only downward incentive constraints (the yellow curve, which is below the dark blue curve for x above approximately .2, and is always below the green curve). The next (grey) curve is the optimal equilibrium with almost-uniform bids with all incentive constraints, derived in Proposition 5 (which coincides with the yellow curve for $x \geq 1/3$). The next (light blue) curve is the line $2x$, which equals optimal cartel profit without mediation by Proposition 3. The value of mediation for the cartel is thus the gap between the light blue curve and the dark blue curve (if shill bids are infeasible) or the green curve (if shill bids are feasible). The last (red) curve is the line $(1 + \log 2)x$, which equals optimal cartel profit under bid rotation, as observed in footnote 23.

5 Conclusion

This paper has introduced and analyzed the problem of how colluding firms maximize profit when they are assisted by an intermediary that can privately recommend prices/bids, and can punish firms that disobey their recommendations. The cartel strictly benefits from the intermediary's assistance when all bids are disclosed at auction; if instead only the winner's identity and/or bid is disclosed, the cartel can do just as well without the intermediary. Among other results, we are able to fully characterize the cartel-optimal bid distribution with two symmetric firms, when bids are disclosed and upward incentive constraints are slack. When upward incentive constraints bind—which occurs when the maximum penalty is small and the auction environment precludes shill bidding—the optimal bid distribution can be found numerically. Interestingly, no matter how small the maximum penalty, the winning bid is close to the reserve price with positive probability.

Our characterization of the cartel-optimal equilibrium with disclosed bids relies on the strong assumption that there are two firms with identical and commonly-known production costs. It may be possible to relax these assumptions in future work, although the analytic characterization of the optimal equilibrium is likely quite complicated. Another direction for future work is introducing additional frictions in the auction environment, which could rationalize why governments often choose auction formats that disclose so much bid information, despite the risk of facilitating collusion. For example, if the government is concerned about corruption on the part of the auctioneer (e.g., Compte, Lambert-Mogiliansky, and Verdier, 2005), disclosing bids may be optimal, both for the usual reason that this can allow the government to monitor the auctioneer for corruption, and also because it allows the cartel mediator to monitor cartel members for collusion with the auctioneer. While the latter effect can increase cartel profit by facilitating collusion among bidders, this could still be cheaper for society than allowing collusion between bidders and a corrupt auctioneer, which cedes rents to the auctioneer in addition to the bidders.

A Omitted Proofs

A.1 Proof of Proposition 4

Fix a canonical equilibrium with bid profile cdf $F(\mathbf{p})$ and winning bid cdf $W(p)$. A uniform downward deviation with cutoff $p^* \in [0, 1]$ (where $W(p)$ is continuous at p^*) now gives an expected payoff of

$$\int_0^{p^*} \frac{p}{n} dW(p) + (1 - W(p^*))p^* - (1 - F_i(p^*))x, \quad (13)$$

where the difference from (2) and (3) is that now the firm is punished with probability $1 - F_i(p^*)$. Since this deviation must be unprofitable in equilibrium, we have

$$\begin{aligned} \int_0^{p^*} \frac{p}{n} dW(p) + (1 - W(p^*))p^* - (1 - F_i(p^*))x &\leq \int_0^1 \frac{p}{n} dW(p) &\iff \\ 1 - \frac{1}{p^*} \left(\int_{p^*}^1 \frac{p}{n} dW(p) + (1 - F_i(p^*))x \right) &\leq W(p^*). \end{aligned} \quad (14)$$

Note next that for all p , $W(p) = \Pr(\exists i : p_i \leq p) \leq nF_i(p)$, by union bound and symmetry, so $F_i(p) \geq \frac{1}{n}W(p)$. Using this in (14), we get that for all p^* where $W(\cdot)$ is continuous,

$$\begin{aligned} 1 - \frac{1}{p^*} \left(\int_{p^*}^1 \frac{p}{n} dW(p) + (1 - \frac{1}{n}W(p^*))x \right) &\leq W(p^*) &\iff \\ np^* - 1 - nx + \int_{p^*}^1 W(p)dp &\leq W(p^*)(np^* - p^* - x), \end{aligned} \quad (15)$$

where the second line follows since, by integration by parts, $\int_{p^*}^1 p dW(p) = 1 - W(p^*)p^* - \int_{p^*}^1 W(p)dp$. Moreover, note that since (15) holds for all p^* where W is continuous, it must also hold for all $p^* \in (x/(n-1), 1]$ where W is discontinuous.³²

³²Suppose W is discontinuous at $\hat{p} \in (x/(n-1), 1]$, so that $W(\hat{p}^-) < W(\hat{p})$. Since (15) holds for all continuity points $p^* < \hat{p}$, taking the limit $p^* \uparrow \hat{p}$ on both sides of (15), we get $n\hat{p} - 1 - nx + \int_{\hat{p}}^1 W(p)dp \leq W(\hat{p}^-)(\hat{p}(n-1) - x) < W(\hat{p})(\hat{p}(n-1) - x)$, where the last inequality uses $\hat{p} > x/(n-1)$.

Now define an operator Φ , mapping cdf's on $[0, 1]$ to cdf's on $[0, 1]$, as

$$\Phi(W)(p) = \begin{cases} 0 & \text{if } p \leq \frac{x}{n-1} \\ \max \left\{ 0, \min \left\{ \frac{1}{np-p-x} \left(np - 1 - nx + \int_p^1 W(\tilde{p}) d\tilde{p} \right), 1 \right\} \right\} & \text{if } p \in \left(\frac{x}{n-1}, 1 \right) \\ 1 & \text{if } p = 1 \end{cases}$$

By (15), for any winning bid distribution W ,

$$W(p) \geq \Phi(W)(p) \quad \forall p \in [0, 1]. \quad (16)$$

In particular, profits under $\Phi(W)$ are weakly larger than under W . Note also that the operator Φ is monotone: i.e., $W \geq \hat{W} \implies \Phi(W) \geq \Phi(\hat{W})$.

Consider the problem of finding the winning bid distribution W that maximizes cartel profit, subject to (16). Since any equilibrium winning bid distribution satisfies (16), the solution to this problem gives an upper bound for π^* . We now show that the solution to this problem is a cdf W^* that satisfies $W^* = \Phi(W^*)$, and that cartel profit under W^* equals $\frac{n}{n-1} \left(x + ((n-1)x)^{\frac{n-1}{n}} (n-1-x)^{\frac{1}{n}} \right) - nx$.

Fix any winning bid distribution W satisfying $W \geq \Phi(W)$, and consider the sequence of cdfs (W^k) with $W^0 = W$ and $W^{k+1} = \Phi(W^k)$ for all $k \geq 0$. Note that, since Φ is monotone, $W^{k+1} \geq W^k$ for all k . We now show that sequence (W^k) converges in distribution to a cdf W^* , independent of the initial W . Hence, W^* solves our relaxed problem.

Since (W^k) is a decreasing sequence, we have that for all $p \in (x/(n-1), 1)$ and all $k \geq 1$,

$$\begin{aligned} W^{k+2}(p) &= \max \left\{ 0, \min \left\{ \frac{1}{np-p-x} \left(np - 1 - nx + A^{k+1}(p) \right), 1 \right\} \right\} \\ &\geq W^{k+1}(p) = \max \left\{ 0, \min \left\{ \frac{1}{np-p-x} \left(np - 1 - nx + A^k(p) \right), 1 \right\} \right\} \end{aligned}$$

where, for each k , $A^k(p) = \int_p^1 W^k(\tilde{p}) d\tilde{p}$. Since $A^k(p)$ is decreasing in k and bounded (because (W^k) is decreasing), it converges to some A^* . Hence, for all $p \in (x/(n-1), 1)$,

$$\lim_{k \rightarrow \infty} W^k(p) = W^*(p) = \max \left\{ 0, \min \left\{ \frac{1}{np-p-x} \left(np - 1 - nx + A^*(p) \right), 1 \right\} \right\}.$$

Moreover, it is clear that $\lim_{k \rightarrow \infty} W^k(p) = W^*(p) = 0$ for $p \leq x/(n-1)$, and that $\lim_{k \rightarrow \infty} W^k(1) = W^*(1) = 1$. Note next that, by Dominated Convergence, and since (W^k) converges pointwise to W^* ,

$$A^* = \lim_{k \rightarrow \infty} A^k(p) = \lim_{k \rightarrow \infty} \int_p^1 W^k(\tilde{p}) d\tilde{p} = \int_p^1 W^*(\tilde{p}) d\tilde{p}.$$

Hence, $W^* = \Phi(W^*)$.

The unique solution to the equation $W^* = \Phi(W^*)$ is

$$W^*(p) = \begin{cases} 0 & \text{if } p < \underline{p} \\ 1 - \frac{(n-1)x(n-1-x)^{\frac{1}{n-1}}}{(np-p-x)^{\frac{n}{n-1}}} & \text{if } p \in [\underline{p}, 1) \\ 1 & \text{if } p = 1, \end{cases}$$

where $\underline{p} = \frac{1}{n-1} \left(x + ((n-1)x)^{\frac{n-1}{n}} (n-1-x)^{\frac{1}{n}} \right)$ is the lowest point in the support of W^* .

This follows because, for any p such that $W(p) \in (0, 1)$, W is differentiable with derivative satisfying $W'(p) = n(1 - W(p)) / (np - p - x)$, and $\lim_{p \rightarrow 1} W(p) = (n-1 - nx) / (n-1-x)$, and solving this differential equation yields the desired equation. We thus have

$$\begin{aligned} W^*(\underline{p}) = 0 &= \Phi(W^*)(\underline{p}) = \frac{1}{\underline{p}(n-1) - x} \left(n\underline{p} - 1 - nx + \int_{\underline{p}}^1 W^*(p) dp \right) \iff \\ \int_{\underline{p}}^1 p dW^*(p) &= n(\underline{p} - x) = \frac{n}{n-1} \left(x + ((n-1)x)^{\frac{n-1}{n}} (n-1-x)^{\frac{1}{n}} \right) - nx, \end{aligned}$$

where the second line uses $\int_{\underline{p}}^1 W^*(p) dp = 1 - \int_{\underline{p}}^1 p dW^*(p)$. Thus, $\frac{n}{n-1} \left(x + ((n-1)x)^{\frac{n-1}{n}} (n-1-x)^{\frac{1}{n}} \right) - nx$ is the solution to the relaxed problem, and hence is an upper bound for π^* .

A.2 Proof of Proposition 5

Fix an equilibrium with almost-uniform bids. Let $\alpha = \Pr(p_j = \bar{p} \mid \forall j \neq i, p_i = \bar{p})$, the probability that a bidder who is recommended price \bar{p} has the lowest price, and let $\beta = \Pr(p_i = \bar{p} \mid \forall i)$.

Note that Bayes' rule gives $\beta = \frac{(n-1)\alpha}{n-\alpha}$, because

$$\alpha = \Pr(p_j = \bar{p} \forall j \neq i | p_i = \bar{p}) = \frac{\Pr(p_j = \bar{p} \forall j)}{\Pr(p_i = \bar{p})} = \frac{\beta}{\beta + \frac{n-1}{n}(1-\beta)},$$

where $\Pr(p_i = \bar{p}) = \beta + \frac{n-1}{n}(1-\beta)$ by symmetry and the assumption that at most one bidder bids below \bar{p} .

Let \bar{F} denote the cdf of the random variable $\min_{j \neq i} p_j$ conditional on the event $p_i = \bar{p}$. Since a firm that is recommended bid \bar{p} wins the auction with equilibrium probability α/n , the incentive constraint for this firm is

$$\frac{\alpha \bar{p}}{n} \geq (1 - \bar{F}(p)) p - x \quad \text{for all } p < \bar{p}. \quad (17)$$

For fixed values of x and \bar{p} , the greatest distribution \bar{F} (in the FOSD sense) that satisfies this constraint is given by $\alpha = \frac{n}{n-1} \frac{x}{\bar{p}}$ and

$$\bar{F}(p) = \begin{cases} 0 & \text{if } p < \frac{n}{n-1}x, \\ 1 - \frac{n}{n-1} \frac{x}{p} & \text{if } p \in [\frac{n}{n-1}x, \bar{p}), \\ 1 & \text{if } p \geq \bar{p}. \end{cases}$$

Therefore, in an optimal equilibrium with an almost-uniform bid of \bar{p} , the conditional distribution of $\min_{j \neq i} p_j$ is given by this distribution \bar{F} .

Now note that

$$\mathbb{E} \left[\min_{j \neq i} p_j | \min_{j \neq i} p_j < \bar{p} \right] = \frac{1}{1-\alpha} \int_0^{\bar{p}} p d\bar{F}(p) = \frac{1}{1-\alpha} \int_{\frac{n}{n-1}x}^{\bar{p}} \frac{n}{n-1} \frac{x}{p} dp = \frac{n}{n-1} \frac{x}{1-\alpha} \left(-\log \left(\frac{n}{n-1} \frac{x}{\bar{p}} \right) \right).$$

Therefore, cartel profit in an optimal equilibrium with an almost-uniform bid of \bar{p} are given by

$$\pi = \beta \bar{p} + (1-\beta) \frac{n}{n-1} \frac{x}{1-\alpha} \left(-\log \left(\frac{n}{n-1} \frac{x}{\bar{p}} \right) \right) = \frac{(n-1)x\bar{p}}{(n-1)\bar{p}-x} \left(1 - \frac{n}{n-1} \log \left(\frac{n}{n-1} \frac{x}{\bar{p}} \right) \right). \quad (18)$$

Note that (18) is increasing in \bar{p} whenever $\bar{p} \geq \frac{n}{n-1}x$. Therefore, the optimal equilibrium

with almost-uniform bids is given by maximizing \bar{p} , subject to the remaining incentive constraints. The remaining constraints are those for a firm that is recommended $p < \bar{p}$. In an equilibrium with almost-uniform bids, such a firm knows that its opponent bids \bar{p} , so the most tempting deviation is to bid just below \bar{p} . In turn, this deviation is most tempting for a firm with the lowest possible recommendation, $\frac{n}{n-1}x$. Hence, the remaining binding incentive constraint is $\frac{n}{n-1}x \geq \bar{p} - x$, or equivalently $\bar{p} \leq \frac{2n-1}{n-1}x$. Therefore, if $x \geq \frac{n-1}{2n-1}$, the optimal equilibrium is given by $\bar{p} = 1$, and cartel profit is given by $\frac{(n-1)x}{n-1-x} \left(1 - \frac{n}{n-1} \log \left(\frac{nx}{n-1}\right)\right)$. If instead $x < \frac{n-1}{2n-1}$, then the optimal equilibrium is given by $\bar{p} = \frac{2n-1}{n-1}x$, and cartel profit is given by $\frac{2n-1}{2n-2}x \left(1 - \frac{n}{n-1} \log \frac{n}{2n-1}\right)$.

A.3 Proof of Corollary 1

It remains to show that $\liminf_{n \rightarrow \infty} \pi^*(n) \geq \underline{x}(1 - \log \underline{x})$, with convention $0 \log 0 = 0$. This is obvious if $\underline{x} = 0$, so suppose that $\underline{x} > 0$. We show that for any $\varepsilon > 0$, if n is sufficiently large then optimal cartel profit with n firms and penalty size \underline{x} is at least $(\underline{x} - \varepsilon)(1 - \log(\underline{x} - \varepsilon))$. Since optimal cartel profit is non-decreasing in x and $x(n)$ is non-increasing in n , this completes the proof.

Fix $\varepsilon > 0$, and take any n such that $(1 - \varepsilon + \varepsilon \underline{x})^{n-1} - \underline{x} < 0$. We construct a bid rotation equilibrium, where bidder 1 always wins, as follows:

Bidder 1's recommendation p_1 is drawn with cdf $F(p_1) = 1 - \frac{\underline{x} - \varepsilon}{p_1}$.

For each bidder $i \neq 1$, bidder i 's recommendation is drawn as follows, independently across bidders $i \neq 1$: With probability $1 - \varepsilon$, p_i is drawn uniformly from $[1, 2]$. With probability ε , a uniform $[0, 1]$ random variable q_i is drawn, and then p_i is determined as follows: If $q_i > p_1$, then $p_i = q_i$; otherwise, p_i is drawn uniformly from $[1, 2]$.

This distribution gives cartel profit $(\underline{x} - \varepsilon)(1 - \log(\underline{x} - \varepsilon))$. It thus remains to check that it is an equilibrium.

Player 1 does not have incentive to deviate downward, since she always wins. To see that she also does not have incentive to deviate upward, note that if she deviates to $q > p_1$, her net payoff gain is $\Pr(\min_{i \neq 1} p_i \geq q)q - \underline{x} - p_1$. This gain is non-positive if $q < p_1 + \underline{x}$. If

$q \geq p_1 + \underline{x}$, it is no more than

$$\Pr \left(\min_{i \neq 1} p_i \geq p_1 + \underline{x} \right) - \underline{x} - p_1 \leq (1 - \varepsilon + \varepsilon \underline{x})^{n-1} - \underline{x} - p_1 \leq (1 - \varepsilon + \varepsilon \underline{x})^{n-1} - \underline{x} < 0.$$

(The first inequality holds since, for each $i \neq 1$, $p_i \geq p_1 + \underline{x}$ only if (i) with probability $1 - \varepsilon$, p_i is drawn uniformly from $[1, 2]$, or (ii) with probability ε , q_i is drawn uniformly from $[0, 1]$ and q_i is not included in $[p_i, p_i + \underline{x}]$.)

Player $i \neq 1$ does not have incentive to deviate upward since she always loses. To see that she also does not have incentive to deviate downward, suppose she is recommended $p_i \geq 1$. Then, for each $q \leq 1$,

$$\Pr(p_1 \geq q | p_i) = \frac{(1 - \varepsilon) \Pr(p_1 \geq q) + \varepsilon \Pr(p_1 \geq q) \Pr(q_i \leq p_1 | p_1 \geq q)}{1 - \varepsilon + \varepsilon \Pr(q_i \geq p_1)} \geq \frac{\Pr(p_1 \geq q)}{1 - \varepsilon},$$

and hence the net payoff gain from bidding q is no more than

$$\frac{\Pr(p_1 \geq q)}{1 - \varepsilon} q - \underline{x} = \frac{\underline{x} - \varepsilon}{(1 - \varepsilon) q} - \underline{x} = -\frac{\varepsilon(1 - \underline{x})}{1 - \varepsilon} < 0.$$

Next, suppose she is recommended $p_i < 1$. Then, since $q_i = p_i$ for sure in this case, for each $q \leq 1$, (i) if $q \leq p_i$ then $\Pr(p_1 \geq q | p_i) = 0$, and (ii) otherwise,

$$\Pr(p_1 \geq q | p_i) = \frac{\Pr(p_i \geq p_1 \geq q)}{\Pr(p_i \geq p_1)} = \frac{\frac{\underline{x} - \varepsilon}{q} - \frac{\underline{x} - \varepsilon}{p_i}}{1 - \frac{\underline{x} - \varepsilon}{p_i}}.$$

Hence, the net payoff gain from bidding q is no more than

$$\frac{\frac{\underline{x} - \varepsilon}{q} - \frac{\underline{x} - \varepsilon}{p_i}}{1 - \frac{\underline{x} - \varepsilon}{p_i}} q - \underline{x} = \frac{1 - \frac{q}{p_i}}{1 - \frac{\underline{x} - \varepsilon}{p_i}} (\underline{x} - \varepsilon) - \underline{x}.$$

Since $p_i \geq \underline{x} - \varepsilon$ (the lowest possible value of p_1), this in turn is no more than $\frac{1 - \frac{q}{p_i}}{1 - \frac{\underline{x} - \varepsilon}{p_i}} (\underline{x} - \varepsilon) - \underline{x} \leq -\varepsilon$, as desired.

A.4 Proof of Lemma 1

To see that $\chi(p)$ is well defined, note that the LHS of (6) is strictly concave in χ over the range $(0, \hat{\pi} + x)$, and attains its maximum at $\chi = 2x$. Hence, as long as

$$\frac{2x \log(2x) + (\hat{\pi} - x) \log(\hat{\pi} - x)}{x + \hat{\pi}} \geq \log p + \frac{2x \log(\hat{\pi}) + (\hat{\pi} - x) \log(x)}{x + \hat{\pi}} - \log(\hat{\pi}) \quad \forall p \in [\hat{\pi}, 1], \quad (19)$$

equation (6) admits at least one solution $\chi \in [2x, \hat{\pi} + x]$. Since the LHS of (19) is independent of p and the RHS is increasing in p , it suffices that (19) holds for $p = 1$. In turn, this holds by the definition of $\hat{\pi}$. Finally, that $\chi(p)$ is decreasing and differentiable, with derivative satisfying (7) on $[\hat{\pi}, 1)$, follows from the implicit function theorem.

A.5 Proof of Lemma 2

For all $p \in [\hat{\pi}, 1]$, define the functions

$$L(p) = \exp \left(\int_{\hat{\pi}}^p -\frac{\chi(p')}{(\chi(p') - 2x)p'} dp' \right),$$

$$M(p) = - \int_{\hat{\pi}}^p \frac{1}{L(p')} \frac{\chi(p')}{(\chi(p') - 2x)p'} dp'.$$

Note that $L(p)$ and $M(p)$ are bounded for all $p \in [\hat{\pi}, 1]$. Indeed, using (7) we have

$$L(p) = \exp \left(\int_{\hat{\pi}}^p \frac{\chi'(p')}{\hat{\pi} + x - \chi(p')} dp' \right),$$

$$M(p) = \int_{\hat{\pi}}^p \frac{1}{L(p')} \frac{\chi'(p')}{\hat{\pi} + x - \chi(p')} dp'.$$

Since $\hat{\pi} + x - \chi(p) \in [x, \hat{\pi} - x]$ (as $\chi(p) \in [2x, \hat{\pi}]$) and $\int_{\hat{\pi}}^p \chi'(p') dp' \in [2x - \hat{\pi}, 0]$ (as $\chi'(p') < 0$ and $\int_{\hat{\pi}}^1 \chi'(p') dp' = 2x - \hat{\pi}$), we have that $L(p)$ and $M(p)$ are bounded for all $p \in [\hat{\pi}, 1]$.

Next, for all $p \in [\hat{\pi}, 1]$, define the function

$$H(p) = L(p) \left(\frac{1 + L(1)(M(1) - M(p))}{1 + L(1)M(1)} \right).$$

Note that $H(p)$ is differentiable at each $p \in [\hat{\pi}, 1)$, with derivative given by

$$H'(p) = L'(p) \left(\frac{1 + L(1)(M(1) - M(p))}{1 + L(1)M(1)} \right) - \frac{L(p)L(1)M'(p)}{1 + L(1)M(1)} = \frac{\chi(p)}{(\chi(p) - 2x)p} (H(1) - H(p)),$$

where the last equality uses

$$L'(p) = -\frac{L(p)\chi(p)}{(\chi(p) - 2x)p}, \quad M'(p) = \frac{1}{L(p)} \frac{\chi(p)}{(\chi(p) - 2x)p}, \quad H(1) = \frac{L(1)}{1 + L(1)M(1)}.$$

We now show that $H(1) < H(\hat{\pi})$. This implies that H is decreasing over $[\hat{\pi}, 1)$.³³ Note that $L(\hat{\pi}) = 1$ and $M(\hat{\pi}) = 0$, so $H(\hat{\pi}) = 1$. We thus must show that $H(1) < 1$, or equivalently $L(1)(1 - M(1)) < 1$. Define $J(p) = L(p)(1 - M(p))$. Note that $J(\hat{\pi}) = 1$, and that, for all $p \in [\hat{\pi}, 1)$,

$$J'(p) = L'(p)(1 - M(p)) - L(p)M'(p) = -L(p) \frac{\chi(p)}{(\chi(p) - 2x)p} (1 - M(p)) - \frac{\chi(p)}{(\chi(p) - 2x)p} < 0,$$

where the strict inequality follows since $\chi(p) > 2x$, $M(p) \leq 0$, and $L(p) > 0$, for all $p \in [\hat{\pi}, 1)$. Hence, we have $J(1) = L(1)(1 - M(1)) < J(\hat{\pi}) = 1$, and so $H(1) < H(\hat{\pi})$.

Now define

$$f_i(p) = \begin{cases} 0 & \text{if } p < 2x, \\ \frac{K}{p^2} (H(\chi^{-1}(p)) - H(1)) & \text{if } p \in [2x, \hat{\pi}), \\ \frac{K}{\chi(p)} (-H'(p)) = \frac{K}{(\chi(p) - 2x)p} (H(p) - H(1)) & \text{if } p \in [\hat{\pi}, 1), \end{cases} \quad (20)$$

where $K > 0$ is a constant to be determined shortly. Using the fact that, for all $p \in [\hat{\pi}, 1)$, $K(H(p) - H(1)) = \int_p^1 -KH'(p')dp' = \int_p^1 \chi(p')f_i(p')dp'$, one can verify that f_i satisfies (8).

Finally, we determine the value of $K > 0$. Since F_i should satisfy $\int_{2x}^1 f_i(p)dp = F_i(1) - F_i(2x) = 1$, by (20) we have

$$1 = K \left[\int_{2x}^{\hat{\pi}} \frac{1}{p^2} (H(\chi^{-1}(p)) - H(1))dp + \int_{\hat{\pi}}^1 \frac{1}{(\chi(p) - 2x)p} (H(p) - H(1))dp \right]. \quad (21)$$

³³Indeed, $H(1) < H(\hat{\pi})$ implies $h(\hat{\pi}) < 0$ (since $\chi(p) > 2x$ for all $p \in [\hat{\pi}, 1)$). So, it must be that either $H'(p) < 0$ for all $p \in [\hat{\pi}, 1)$, or there exists some $\hat{p} \in (\hat{\pi}, 1)$ such that $H'(p) < 0$ for all $p \in [\hat{\pi}, \hat{p})$ and $H'(p) = 0$ for all $p \in [\hat{p}, 1)$.

Since both terms inside the square brackets are positive and bounded (as $H(p)$ is decreasing and bounded), there exists a unique $K > 0$ satisfying (21).

A.6 Proof of Lemma 3

Without loss, let $p < p'$. We consider two cases: $p \in [2x, \hat{\pi})$ and $p \in [\hat{\pi}, 1]$.

Suppose that $p \in [2x, \hat{\pi})$. We have

$$\begin{aligned} F(p, p') &= \int_{\hat{p} \leq p} F_j(p'|\hat{p}) f_i(\hat{p}) d\hat{p} = \int_{\chi(p')}^p \int_{\chi^{-1}(\hat{p})}^{p'} \frac{\chi(\tilde{p})}{\hat{p}^2} f_i(\tilde{p}) d\tilde{p} d\hat{p} \\ &= \int_{\chi^{-1}(p)}^{p'} \int_{\chi(\tilde{p})}^p \frac{\chi(\tilde{p})}{\hat{p}^2} d\hat{p} f_i(\tilde{p}) d\tilde{p} = \int_{\chi^{-1}(p)}^{p'} \left(1 - \frac{\chi(\tilde{p})}{p}\right) f_i(\tilde{p}) d\tilde{p} \\ &= \int_{\tilde{p} \leq p'} F_j(p|\tilde{p}) f_i(\tilde{p}) d\tilde{p} = F(p', p), \end{aligned}$$

where the second equality follows from (9) and the fact that $F(p'|\hat{p}) = 0$ for all $\hat{p} < \chi(p')$; the third equality follows from reversing the order of integration; and the fifth equality again follows from (9).

Now suppose that $p \in [\hat{\pi}, 1]$. We have

$$\begin{aligned} F(p, p') &= \int_{\hat{p} \leq p} F_j(p'|\hat{p}) f_i(\hat{p}) d\hat{p} \\ &= \int_{\hat{p} \leq p} F_j(p|\hat{p}) f_i(\hat{p}) d\hat{p} + \int_{\hat{p} \leq p} (F_j(p'|\hat{p}) - F_j(p|\hat{p})) f_i(\hat{p}) d\hat{p} \\ &= F(p, p) + \int_{\hat{p} \leq p} (F_j(p'|\hat{p}) - F_j(p|\hat{p})) f_i(\hat{p}) d\hat{p}. \end{aligned} \tag{22}$$

Similarly, we have

$$\begin{aligned} F(p', p) &= \int_{\hat{p} \leq p'} F_j(p|\hat{p}) f_i(\hat{p}) d\hat{p} \\ &= \int_{\hat{p} \leq p} F_j(p|\hat{p}) f_i(\hat{p}) d\hat{p} + \int_p^{p'} F_j(p|\hat{p}) f_i(\hat{p}) d\hat{p} \\ &= F(p, p) + \int_p^{p'} F_j(p|\hat{p}) f_i(\hat{p}) d\hat{p} \\ &= F(p, p) + \int_p^{p'} \left(1 - \frac{\chi(\hat{p})}{p}\right) f_i(\hat{p}) d\hat{p}, \end{aligned} \tag{23}$$

where the last equality uses (9). Since $F_j(p'|\hat{p}) = 0$ for all $\hat{p} < \chi(p')$ and $F_j(p|\hat{p}) = 0$ for all $\hat{p} < \chi(p)$, and again using (9), we have

$$\begin{aligned}
\int_{\hat{p} \leq p} (F_j(p'|\hat{p}) - F_j(p|\hat{p})) f_i(\hat{p}) d\hat{p} &= \int_{\chi(p')}^{\chi(p)} F_j(p'|\hat{p}) f_i(\hat{p}) d\hat{p} + \int_{\chi(p)}^p (F_j(p'|\hat{p}) - F_j(p|\hat{p})) f_i(\hat{p}) d\hat{p} \\
&= \int_{\chi(p')}^{\chi(p)} \int_{\chi^{-1}(\hat{p})}^{p'} \frac{\chi(\tilde{p})}{\hat{p}^2} f_i(\tilde{p}) d\tilde{p} d\hat{p} + \int_{\chi(p)}^p \int_p^{p'} \frac{\chi(\tilde{p})}{\hat{p}^2} f_i(\tilde{p}) d\tilde{p} d\hat{p} \\
&= \int_p^{p'} \int_{\chi(\tilde{p})}^{\chi(p)} \frac{\chi(\tilde{p})}{\hat{p}^2} d\hat{p} f_i(\tilde{p}) d\tilde{p} + \int_p^{p'} \int_{\chi(p)}^p \frac{\chi(\tilde{p})}{\hat{p}^2} d\hat{p} f_i(\tilde{p}) d\tilde{p} \\
&= \int_p^{p'} \left(1 - \frac{\chi(\tilde{p})}{p}\right) f_i(\tilde{p}) d\tilde{p}.
\end{aligned}$$

Together with (22) and (23), we have $F(p, p') = F(p', p)$.

A.7 Proof of Lemma 4

The lemma follows from checking the various cases in (9). In particular, if $p \geq \hat{\pi}$ and $p' \in [\chi(p), p)$, then $F_j(p|p) = 1 - \frac{\chi(p) - 2x}{p}$ and $F_j(p'|p) = 1 - \frac{\chi(p)}{p'}$, and hence

$$\begin{aligned}
IC(p, p'; F) &= p \left(1 - \frac{1}{2} (F_j^-(p|p_i = p) + F_j(p|p_i = p))\right) - p' (1 - F_j(p'|p_i = p)) + x \\
&= p \left(\frac{\chi(p) - x}{p}\right) - p' \left(\frac{\chi(p)}{p'}\right) + x = 0.
\end{aligned}$$

If $p \geq \hat{\pi}$ and $p' < \chi(p)$, then $F_j(p'|p) = 0$, and hence

$$IC(p, p'; F) = p \left(\frac{\chi(p) - x}{p}\right) - p' + x = \chi(p) - p' > 0.$$

Finally, if $p < \hat{\pi}$ and $p' < p$, then $F_j(p|p) = F_j(p'|p) = 0$, and hence $IC(p, p'; F) = x > 0$.

Now suppose that $x \geq 1/3$. If $p \in [2x, \hat{\pi})$ and $p' > p$, then

$$IC(p, p'; F) = p - p'(1 - F_j(p'|p)) + x \geq 3x - 1 \geq 0,$$

where the first inequality uses $p \geq 2x$ and $p' \leq 1$, and the second uses $x \geq 1/3$. If $p \geq \hat{\pi}$ and

$p' > p$, then

$$\begin{aligned}
IC(p, p'; F) &= \chi(p) - p' \left(\frac{\chi(p) - 2x}{p} - \frac{1}{f_i(p)} \int_p^{p'} \frac{\chi(\hat{p})}{p^2} f_i(\hat{p}) d\hat{p} \right) \\
&= \chi(p) \frac{p - p'}{p} + 2x \frac{p'}{p} + \frac{p'}{f_i(p)} \int_p^{p'} \frac{\chi(\hat{p})}{p^2} f_i(\hat{p}) d\hat{p} \\
&\geq \hat{\pi} \frac{\hat{\pi} - 1}{\hat{\pi}} + 2x + \frac{p'}{f_i(p)} \int_p^{p'} \frac{\chi(\hat{p})}{p^2} f_i(\hat{p}) d\hat{p} > 0,
\end{aligned}$$

where the weak inequality uses $p \geq \hat{\pi}$, $p' \in (p, 1]$, and $\chi(p) \leq \hat{\pi}$, and the strict inequality follows as $\hat{\pi} > 2x$ and hence $\hat{\pi} - 1 + 2x > 4x - 1 > 0$ (since $x \geq 1/3$).

A.8 Proof of Lemma 5

For each $p, q \in [0, 1]^2$ and $p' < p$, define

$$\phi(p, q, p') = p - p' + x - \frac{p}{2} (\mathbf{1}_{q \leq p} + \mathbf{1}_{q < p}) + p' \mathbf{1}_{q < p'}.$$

Intuitively, $\phi(p, q, p')$ is the payoff loss incurred by a bidder who deviates from p to p' when the opponent bids q . Note that

$$\begin{aligned}
IC(p, p'; F) &= \int_q \phi(p, q, p') dF_j(q|p) \quad \forall p > p', \\
G(p, p; \Lambda, \mu) &= - \int_{p' \leq p} \phi(p, p, p') d\Lambda(p'|p) + \mu - p \quad \forall p, \quad \text{and} \\
G(p, q; \Lambda, \mu) &= - \int_{p' \leq p} \phi(p, q, p') d\Lambda(p'|p) - \int_{p' \leq q} \phi(q, p, p') d\Lambda(p'|q) + 2(\mu - q) \quad \forall p > q.
\end{aligned}$$

Now let F be feasible for \mathbf{P} , and let (Λ, μ) be feasible for \mathbf{D} . Note that

$$\begin{aligned}
0 &\leq \int_{p,q \leq p} G(p, q; \Lambda, \mu) dF(p, q) \\
&= - \int_{p,q \leq p} \left(\int_{p' \leq p} \phi(p, q, p') d\Lambda(p'|p) + \mathbf{1}_{q < p} \int_{p' \leq q} \phi(q, p, p') d\Lambda(p'|q) \right) dF(p, q) \\
&\quad + \int_{p,q \leq p} (2(\mu - q) - \mathbf{1}_{q=p}(\mu - p)) dF(p, q) \\
&= - \int_{p,q \leq p} \left(\int_{p' \leq p} \phi(p, q, p') d\Lambda(p'|p) + \mathbf{1}_{q < p} \int_{p' \leq q} \phi(q, p, p') d\Lambda(p'|q) \right) dF(p, q) + \mu - \Pi(F),
\end{aligned} \tag{24}$$

where the last equality uses $\int_{p,q \leq p} (2 - \mathbf{1}_{q=p}) dF(p, q) = 1$ (by symmetry of F) and $\int_{p,q \leq p} (2q - \mathbf{1}_{q=p}p) dF(p, q) = \Pi(F)$.

Since F is symmetric,

$$\begin{aligned}
\int_{p,q \leq p} \mathbf{1}_{q < p} \int_{p' \leq q} \phi(q, p, p') d\Lambda(p'|q) dF(p, q) &= \int_{p,q < p} \int_{p' \leq q} \phi(q, p, p') d\Lambda(p'|q) dF(p, q) \\
&= \int_{p,q > p} \int_{p' \leq p} \phi(p, q, p') d\Lambda(p'|p) dF(p, q).
\end{aligned}$$

Using this in (24), we get

$$\begin{aligned}
0 &\leq - \int_{p,q} \int_{p' \leq p} \phi(p, q, p') d\Lambda(p'|p) dF(p, q) + \mu - \Pi(F) \\
&= - \int_p \int_q \int_{p' \leq p} \phi(p, q, p') d\Lambda(p'|p) dF(q|p) dF_i(p) + \mu - \Pi(F) \\
&= - \int_p \int_{p' \leq p} \int_q \phi(p, q, p') dF(q|p) d\Lambda(p'|p) dF_i(p) + \mu - \Pi(F) \\
&= - \int_p \int_{p' \leq p} IC(p, p'; F) d\Lambda(p'|p) dF_i(p) + \mu - \Pi(F) \\
&\leq \mu - \Pi(F),
\end{aligned} \tag{25}$$

where the first two equalities follow from Fubini's theorem, which applies as ϕ , Λ , and F are bounded and measurable. This establishes part (i).

Suppose next that F and (Λ, μ) satisfy the conditions in part (ii). Then, the inequalities in (24) and (25) hold with equality, and so $\Pi(F) = \mu$. By part (i), it follows that F is

optimal for \mathbf{P} , and (Λ, μ) is optimal for \mathbf{D} .

A.9 Proof of Lemma 6

To show that $\Lambda \in \mathcal{M}$, it suffices to show that Λ is bounded and measurable: given this, $\Lambda(\cdot|p)$ is a Borel measure on $\mathcal{B}([0, 1])$ by construction. Boundedness follows because, for each $p \in [0, 1]$, we have

$$\begin{aligned} \Lambda([0, 1]|p) &\leq \int_{\chi(p)}^{\hat{\pi}} \frac{2}{q} dq + \int_{\hat{\pi}}^p \frac{2(\chi(q) - x)(-\chi'(q))}{x\chi(q)} dq \\ &\leq \int_{\chi(p)}^{\hat{\pi}} \frac{2}{q} dq + \int_{\hat{\pi}}^p \frac{2(-\chi'(q))}{x} dq \\ &\leq 2(\log(\hat{\pi}) - \log(2x)) + \frac{2}{x}(\hat{\pi} - 2x), \end{aligned}$$

where the first inequality is by definition of λ , the second follows since $\chi(q) \geq x$, and the third follows since $\chi(p) \geq 2x$ and $\chi(\hat{\pi}) - \chi(p) \leq \hat{\pi} - 2x$. Measurability follows because $\Lambda([p', p'']|p)$ is continuous on $[0, 1]^3$ in the weak topology, which suffices for continuity on $[0, 1] \times \mathcal{B}([0, 1])$.

For the proof that G satisfies the desired properties, we ease notation by writing G as a function of p, q only, suppressing the arguments $\Lambda, \hat{\pi}$. We use the following equations, which follow immediately from the definitions of G and Λ .

1. For $p \geq \hat{\pi}$,

$$G(p, p) = \int_{\chi(p)}^p \left(-\frac{p}{2} - x + p'\right) \lambda(p'|p) dp' + \hat{\pi} - p. \quad (26)$$

2. For $p \geq q \geq \hat{\pi}$,

$$G(p, q) = \int_{\chi(p)}^p (-x + p' \mathbf{1}_{\{p' \leq q\}}) \lambda(p'|p) dp' + \int_{\chi(q)}^q (-q - x + q') \lambda(q'|q) dq' + 2(\hat{\pi} - q). \quad (27)$$

3. For $p \geq \hat{\pi} \geq q \geq \chi(p)$,

$$G(p, q) = \int_{\chi(p)}^p (-x + p' \mathbf{1}_{\{p' \leq q\}}) \lambda(p'|p) dp' + 2(\hat{\pi} - q). \quad (28)$$

4. For $p \geq \hat{\pi} \geq \chi(p) \geq q$,

$$G(p, q) = 2(\hat{\pi} - q). \quad (29)$$

5. For $p \geq \hat{\pi}$ and $q \in [\hat{\pi}, p)$,

$$\lambda(q|p) = \frac{(\chi(q) - x)(-\chi'(q))\lambda(\chi(q)|p)}{x}. \quad (30)$$

We prove the lemma through a series of claims.

Claim 1. For any sequence $(p_n, q_n) \rightarrow (\hat{\pi}, \hat{\pi})$ such that $p_n \geq \hat{\pi} \geq q_n$ for all n , we have $\lim_{n \rightarrow \infty} G(p_n, q_n) = 0$.

Proof. This follows from two observations. First, $G(\hat{\pi}, \hat{\pi}) = 0$, by (26) and $\chi(\hat{\pi}) = \hat{\pi}$. Second, for any $\pi \in [0, 1]$ and any sequence $(p_n, q_n) \rightarrow (\pi, \pi)$ such that $p_n \geq \pi \geq q_n$ and $p_n \neq q_n$ for all n , we have $\lim_{n \rightarrow \infty} G(p_n, q_n) = 2G(\pi, \pi)$.

Claim 2. $G(p, q) = 0$ for all p, q such that $1 > p \geq \hat{\pi} \geq q \geq \chi(p)$.

Proof. By Claim 1, it suffices to show that $\partial G(p, q)/\partial p = \partial G(p, q)/\partial q = 0$ whenever $p > q$ and $1 > p \geq \hat{\pi} \geq q \geq \chi(p)$. By (28),

$$\begin{aligned} \frac{\partial G(p, q)}{\partial p} &= -x\lambda(p|p) + (\chi(p) - x)(-\chi'(p))\lambda(\chi(p)|p), \quad \text{and} \\ \frac{\partial G(p, q)}{\partial q} &= q\lambda(q|p) - 2. \end{aligned}$$

By (11) and (30), both of these derivatives equal 0.

Claim 3. $G(p, q) = 0$ for all p, q such that $1 > p \geq q \geq \hat{\pi}$.

Proof. Suppose first that $p > q$. By Claim 1, it suffices to show that $\partial G(p, q)/\partial p = \partial G(p, q)/\partial q = 0$ whenever $1 > p > q \geq \hat{\pi}$. By (27),

$$\begin{aligned} \frac{\partial G(p, q)}{\partial p} &= -x\lambda(p|p) + (\chi(p) - x)(-\chi'(p))\lambda(\chi(p)|p), \quad \text{and} \\ \frac{\partial G(p, q)}{\partial q} &= q\lambda(q|p) - \int_{\chi(q)}^q \lambda(q'|q) dq' - x\lambda(q|q) + (\chi(q) - x + q)(-\chi'(q))\lambda(\chi(q)|q) - 2. \end{aligned}$$

By (11) and (30), $\partial G(p, q)/\partial p = 0$. To show that $\partial G(p, q)/\partial q = 0$, note that $G(q, \chi(q)) = 0$

by Claim 2, so (28) gives

$$\begin{aligned} G(q, \chi(q)) &= \int_{\chi(q)}^q (-x + q') \lambda(q'|q) dq' + 2(\hat{\pi} - q) = 0 \quad \Longleftrightarrow \\ \int_{\chi(q)}^q \lambda(q'|q) dq' &= 2 \frac{\hat{\pi} - \chi(q)}{x}. \end{aligned} \quad (31)$$

Note also that

$$-x\lambda(q|q) + (\chi(q) - x)(-\chi'(q))\lambda(\chi(q)|q) = 0, \quad (32)$$

since $\partial G(q, q')/\partial q = 0$ for all $q > q' \geq \chi(q)$. In total, we have

$$\begin{aligned} \frac{\partial G(p, q)}{\partial q} &= q\lambda(q|p) - \int_{\chi(q)}^q \lambda(q'|q) dq' - x\lambda(q|q) + (\chi(q) - q - x)(-\chi'(q))\lambda(\chi(q)|q) - 2 \\ &= q\lambda(q|p) - 2 \frac{\hat{\pi} - \chi(q)}{x} - x\lambda(q|q) + (\chi(q) - q - x)(-\chi'(q))\lambda(\chi(q)|q) - 2 \\ &= -2 \frac{\hat{\pi} - \chi(q)}{x} + q(\lambda(q|p) + \chi'(q)\lambda(\chi(q)|q)) - 2 \\ &= -2 \frac{\hat{\pi} - \chi(q)}{x} + q \left(\frac{2(\chi(q) - x)(-\chi'(q))}{x\chi(q)} + \chi'(q) \frac{2}{\chi(q)} \right) - 2 \\ &= 2 \left(-\frac{\hat{\pi} - \chi(q)}{x} - \frac{\chi(q) - 2x}{x} q \frac{\chi'(q)}{\chi(q)} - 1 \right) \\ &= 2 \left(-\frac{\hat{\pi} - \chi(q)}{x} - \frac{\chi(q) - 2x}{x} \frac{\chi(q) - (\hat{\pi} + x)}{\chi(q) - 2x} - 1 \right) = 0, \end{aligned}$$

where the second line is by (31), the third line is by (32), the fourth line uses $\lambda(q|p) = \frac{2(\chi(q)-x)(-\chi'(q))}{x\chi(q)}$ and $\lambda(\chi(q)|q) = \frac{2}{\chi(q)}$ (by (11)), the fifth line collects terms, and the sixth line is by (7). Finally, the claim also holds when $p = q$, as $\lim_{q \uparrow p} G(p, q) = 2G(p, p)$ for all p .

Claim 4. $G(p, q) > 0$ if either $p \in (\hat{\pi}, 1)$, $q < \chi(p)$ or $p < \hat{\pi}$, $q \leq p$.

Proof. If $p \in (\hat{\pi}, 1)$ and $q < \chi(p)$, then $\partial G(p, q)/\partial q = -2 < 0$ by (29). Since $G(p, \chi(p)) = 0$ by Claim 2, this implies that $G(p, q) > 0$.

If $p < \hat{\pi}$ and $q \leq p$, then $d\Lambda(p'|p) = d\Lambda(q'|q) = 0$ for all $p' < p, q' < q$. Hence, $G(p, q) = 2(\hat{\pi} - q) > 0$.

Claims 1–4 establish the desired conclusion when $p < 1$. The $p = 1$ case follows by continuity, since $\chi(p)$ and $G(p, q)$ are continuous in p for all $q < 1$.

A.10 Proof of Proposition 7

For any positive integer N , let \mathcal{F}^N denote the set of all symmetric cdfs on $\mathbf{N} = \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\}^2$. Let \mathbf{P}^N denote the primal program with \mathcal{F}^N in place of \mathcal{F} . Note that in \mathbf{P}^N firms can still contemplate deviations to arbitrary prices, so \mathbf{P}^N is a strictly more constrained program than \mathbf{P} . Let π^N and π denote the values of \mathbf{P}^N and \mathbf{P} , respectively.

Lemma 7 *For all $\varepsilon > 0$, there exists \bar{N} such that $\pi^N > \pi - \varepsilon$ for all $N > \bar{N}$.*

We prove Lemma 7 through a series of claims. We write $\mathbf{P}(x)$ and $\mathbf{P}^N(x)$ to make the dependence on x explicit. Let $\pi(x)$ and $\pi^N(x)$ denote the values for $\mathbf{P}(x)$ and $\mathbf{P}^N(x)$. Let $\mathcal{F}_{IC}(x) \subset \mathcal{F}$ and $\mathcal{F}_{IC}^N(x) \subset \mathcal{F}^N$ denote the set of symmetric cdfs that satisfy incentive compatibility in $\mathbf{P}(x)$ and $\mathbf{P}^N(x)$. Note that $\mathcal{F}_{IC}^N(x) = \mathcal{F}^N \cap \mathcal{F}_{IC}(x)$.

We first bound $\pi(x)$ by the values of $\pi^N(x)$ with slightly different values for x .

Claim 1 $\pi^N(x) \leq \pi(x) \leq \pi^N(x + \frac{1}{N}) + \frac{1}{N}$.

Proof. The first inequality is immediate as $\mathcal{F}_{IC}^N(x) \subset \mathcal{F}_{IC}(x)$.

For the second inequality, consider any $F \in \mathcal{F}_{IC}(x)$. Let

$$f^N(p, q) = \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q} \in [p, p + \frac{1}{N})} dF(\hat{p}, \hat{q}) \quad \text{for all } (p, q) \in \mathbf{N},$$

and let F^N denote the cdf on \mathbf{N} with probability mass function $f^N(p, q)$. That is, $F^N(p, q)$ is the cdf that results when bids are drawn according to F and then rounded down to the nearest element of \mathbf{N} . Note that $F^N \in \mathcal{F}^N$ and, moreover, $\Pi(F^N) \geq \Pi(F) - \frac{1}{N}$. We will show that $F^N \in \mathcal{F}_{IC}^N(x + \frac{1}{N})$: that is,

$$p \left(\frac{1}{2} f^N(p, p) + \sum_{q > p: (p, q) \in \mathbf{N}} f^N(p, q) \right) - p' \left(\sum_{q > p': (p', q) \in \mathbf{N}} f^N(p', q) + \sum_{q: (p', q) \in \mathbf{N}} f^N(p', q) \left(x + \frac{1}{N} \right) \right) \geq 0 \quad (33)$$

for all $(p, p') \in \mathbf{N}$. (This inequality says that it is unprofitable for a firm that is recommended bid p to deviate to a bid just below p' . Clearly, if all such deviations are unprofitable, then so are deviations to any bids in $[0, 1]$.) Since this holds for any choice of $F \in \mathcal{F}_{IC}(x)$, we have $\pi^N(x + \frac{1}{N}) + \frac{1}{N} \geq \Pi(F)$ for all $F \in \mathcal{F}_{IC}(x)$, and hence $\pi^N(x + \frac{1}{N}) + \frac{1}{N} \geq \pi(x)$.

To see why (33) holds, first note that

$$\frac{1}{2}f^N(p, p) + \sum_{q > p: (p, q) \in \mathbf{N}} f^N(p, q) = \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q}} \left(\frac{1}{2} \times \mathbf{1}\{\hat{p} = \hat{q}\} + \mathbf{1}\{\hat{p} < \hat{q}\} \right) dF(\hat{p}, \hat{q}). \quad (34)$$

This follows because

$$\begin{aligned} \frac{1}{2}f^N(p, p) &= \frac{1}{2} \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q}} \mathbf{1}\{\hat{p} = \hat{q}\} dF(\hat{p}, \hat{q}) \\ &\quad + \frac{1}{2} \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q} \in [p, p + \frac{1}{N})} \mathbf{1}\{\hat{p} < \hat{q}\} dF(\hat{p}, \hat{q}) + \frac{1}{2} \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q} \in [p, p + \frac{1}{N})} \mathbf{1}\{\hat{q} > \hat{p}\} dF(\hat{p}, \hat{q}) \\ &= \frac{1}{2} \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q}} \mathbf{1}\{\hat{p} = \hat{q}\} dF(\hat{p}, \hat{q}) + \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q} \in [p, p + \frac{1}{N})} \mathbf{1}\{\hat{p} < \hat{q}\} dF(\hat{p}, \hat{q}), \end{aligned}$$

by symmetry, and for $(p, q) \in \mathbf{N}$ with $q > p$ (and hence $q \geq p + \frac{1}{N}$),

$$f^N(p, q) = \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q} \in [q, q + \frac{1}{N})} dF(\hat{p}, \hat{q}) = \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q} \in [q, q + \frac{1}{N})} \mathbf{1}\{\hat{p} < \hat{q}\} dF(\hat{p}, \hat{q}).$$

Similarly,

$$\sum_{q \geq p': (p, q) \in \mathbf{N}} f^N(p, q) = \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q}} \mathbf{1}\{p' \leq \hat{q}\} dF(\hat{p}, \hat{q}) \quad \text{and} \quad \sum_{q, (p, q) \in \mathbf{N}} f^N(p, q) = \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q}} dF(\hat{p}, \hat{q}).$$

Therefore,

$$\begin{aligned}
& p \left(\frac{1}{2} f^N(p, p) + \sum_{q > p: (p, q) \in \mathbf{N}} f^N(p, q) \right) - p' \sum_{q \geq p': (p, q) \in \mathbf{N}} f^N(p, q) + \sum_{q: (p, q) \in \mathbf{N}} f^N(p, q) \left(x + \frac{1}{N} \right) \\
&= p \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q}} \left(\frac{1}{2} \times \mathbf{1}\{\hat{p} = \hat{q}\} + \mathbf{1}\{\hat{p} < \hat{q}\} \right) dF(\hat{p}, \hat{q}) - p' \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q}} \mathbf{1}\{p' \leq \hat{q}\} dF(\hat{p}, \hat{q}) \\
&\quad + \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q}} dF(\hat{p}, \hat{q}) \left(x + \frac{1}{N} \right) \\
&\geq \left(p + \frac{1}{N} \right) \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q}} \left(\frac{1}{2} \times \mathbf{1}\{\hat{p} = \hat{q}\} + \mathbf{1}\{\hat{p} < \hat{q}\} \right) dF(\hat{p}, \hat{q}) - p' \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q}} \mathbf{1}\{p' \leq \hat{q}\} dF(\hat{p}, \hat{q}) \\
&\quad + \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q}} dF(\hat{p}, \hat{q}) x \\
&\geq \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q}} \hat{p} \left(\frac{1}{2} \times \mathbf{1}\{\hat{p} = \hat{q}\} + \mathbf{1}\{\hat{p} < \hat{q}\} \right) dF(\hat{p}, \hat{q}) - p' \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q}} \mathbf{1}\{p' \leq \hat{q}\} dF(\hat{p}, \hat{q}) \\
&\quad + \int_{\hat{p} \in [p, p + \frac{1}{N}), \hat{q}} dF(\hat{p}, \hat{q}) x \\
&\geq 0,
\end{aligned}$$

where the last inequality follows from integrating $IC(\hat{p}, p'; F, x)$ over $\hat{p} \in [p, p + \frac{1}{N})$. This establishes (33) and completes the proof. ■

Next, we show $\pi(x)$ is concave.³⁴

Claim 2 For all $\alpha > 1$, we have $\pi(x) \leq \alpha \pi(x/\alpha)$.

Proof. Take any $F \in \mathcal{F}^{IC}(x)$, and define $\hat{F} \in \mathcal{F}(x)$ by $\hat{F}(p, q) = F(\alpha p, \alpha q)$. Since $\alpha \Pi(\hat{F}) = \Pi(F)$, it suffices to show that $\hat{F} \in \mathcal{F}^{IC}(x/\alpha)$. This holds because, for all $p \in$

³⁴To see that Claim 2 implies that $\pi(x)$ is concave, note that $\pi(0) = 0$.

$\text{supp}(\hat{F}_i), p'$, we have

$$\begin{aligned}
& IC(p, p'; \hat{F}, x/\alpha) \\
&= \int_q p \left(\mathbf{1}\{q > p\} + \frac{1}{2} \mathbf{1}\{q = p\} \right) d\hat{F}(q|p) - \int_q p' \mathbf{1}\{q \geq p'\} d\hat{F}(q|p) + x/\alpha \\
&= \int_{\alpha q} p \left(\mathbf{1}\{\alpha q > \alpha p\} + \frac{1}{2} \mathbf{1}\{\alpha q = \alpha p\} \right) dF(\alpha q|\alpha p) - \int_{\alpha q} p' \mathbf{1}\{\alpha q \geq \alpha p'\} dF(\alpha q|\alpha p) + x/\alpha \\
&= \left(\frac{1}{\alpha} \right) \left(\int_{\alpha q} \alpha p \left(\mathbf{1}\{\alpha q > \alpha p\} + \frac{1}{2} \mathbf{1}\{\alpha q = \alpha p\} \right) dF(\alpha q|\alpha p) - \int_{\alpha q} \alpha p' \mathbf{1}\{\alpha q \geq \alpha p'\} dF(\alpha q|\alpha p) + x \right) \\
&\geq 0,
\end{aligned}$$

where the second line follows from integration by substitution, and the inequality follows by $IC(\alpha p, \alpha p'; F, x)$ (noting that $p \in \text{supp}(\hat{F}_i) \implies \alpha p \in \text{supp}(F_i)$). ■

Proof of Lemma 7. For any $\alpha > 1$ and $N > \frac{\alpha}{(\alpha-1)x}$, we have

$$\pi(x) \leq \alpha \pi\left(\frac{x}{\alpha}\right) \leq \alpha \pi^N\left(\frac{x}{\alpha} + \frac{1}{N}\right) + \frac{\alpha}{N} \leq \alpha \pi^N(x) + \frac{\alpha}{N},$$

where the first inequality follows from Claim 2, the second inequality follow from Claim 1, and the third inequality follows from $\frac{x}{\alpha} + \frac{1}{N} < x$ (as $N > \frac{\alpha}{(\alpha-1)x}$) and monotonicity of π^N . Since $\pi^N(x) \leq 1$, taking $\alpha < 1 + \varepsilon/2$ and $N > 1/2\varepsilon$ implies that $\alpha \pi^N(x) + \frac{\alpha}{N} \leq \pi^N(x) + \varepsilon$, completing the proof. ■

Given Lemma 7, it suffices to show that, for all N sufficiently large, there exists a solution F^N to \mathbf{P}^N such that the highest winning bid under F^N equals 1. We show that this is the case for all $N > 1/x$. Let f be an optimal pmf in \mathbf{P}^N . Without loss, assume that f is symmetric. Let \bar{p} denote the highest bid in the support of the marginal. Suppose toward a contradiction that $\bar{p} < 1$. Let $\beta = f(\bar{p}, \bar{p})$ and $\gamma_p = f(\bar{p}, p)$ for $p \in [0, \bar{p}]$. Let $\gamma = \sum_{p < \bar{p}} \gamma_p$ and $\tilde{\gamma}_p = \sum_{p \leq p' < \bar{p}} \gamma_{p'}$ (and let $\tilde{\gamma}_{\bar{p}} = 0$).

Lemma 8 $\beta > 0$.

Proof. Suppose that $\beta = 0$. Then \bar{p} is never a winning bid. Let $p^* < \bar{p}$ denote the highest bid that ever wins. Consider the bid distribution that everywhere replaces all bids above p^* with bids of p^* . Clearly, this variation leaves profit unchanged. We argue that it remains an

equilibrium.

First, conditional on each recommended bid, the distribution of the opponent's bid has shifted down. This makes any upward deviation weakly less profitable. So upward IC still holds.

Next, if one's recommended bid is below p^* , the payoff from following the recommendation as well as from any downward deviation are unchanged. So downward IC holds for recommendations below p^* .

It remains to establish downward IC for a recommended bid of p^* . Let \Pr denote probability under the original distribution, and $\widehat{\Pr}$ denote probability under the new distribution. We must show that, for all $p < p^*$,

$$\frac{1}{2}\widehat{\Pr}(p_i = p_j = p^*)p^* \geq \widehat{\Pr}(p_i = p^* \wedge p_j \geq p)p - \widehat{\Pr}(p_i = p^*)x. \quad (35)$$

First, note that

$$\begin{aligned} & \frac{1}{2}\widehat{\Pr}(p_i = p_j = p^*) \\ &= \frac{1}{2}(\Pr(p_i = p_j = p^*) + \Pr(p_i = p^* \wedge p_j > p^*) + \Pr(p_i > p^* \wedge p_j = p^*)) \\ &= \frac{1}{2}(\Pr(p_i = p_j = p^*) + 2\Pr(p_i = p^* \wedge p_j > p^*)) \\ &= \frac{1}{2}\Pr(p_i = p_j = p^*) + \Pr(p_i = p^* \wedge p_j > p^*), \end{aligned} \quad (36)$$

where the second equality uses symmetry. Second, note that

$$\widehat{\Pr}(p_i = p^* \wedge p_j \geq p) = \Pr(p_i = p^* \wedge p_j \geq p) + \Pr(p_i > p^* \wedge p_j \geq p) \quad \text{and} \quad (37)$$

$$\widehat{\Pr}(p_i = p^*) = \Pr(p_i = p^*) + \Pr(p_i > p^*). \quad (38)$$

Third, by $IC_{p^*, p}$, we have

$$\left(\frac{1}{2}\Pr(p_i = p_j = p^*) + \Pr(p_i = p^* \wedge p_j > p^*)\right)p^* \geq \Pr(p_i = p^* \wedge p_j \geq p)p - \Pr(p_i = p^*)x,$$

and by $IC_{p',p}$ for all $p' > p^*$, we have

$$0 \geq \Pr(p_i > p^* \wedge p_j \geq p) p - \Pr(p_i > p^*) x.$$

Adding these inequalities, we have

$$\begin{aligned} & \left(\frac{1}{2} \Pr(p_i = p_j = p^*) + \Pr(p_i = p^* \wedge p_j > p^*) \right) p^* \\ & \geq (\Pr(p_i = p^* \wedge p_j \geq p) + \Pr(p_i > p^* \wedge p_j \geq p)) p - (\Pr(p_i = p^*) + \Pr(p_i > p^*)) x \end{aligned} \quad (39)$$

Finally, (36), (37), (38), and (39) imply (35). ■

We now construct another distribution \hat{f} , which we will show is an equilibrium bid distribution that yields strictly higher profit than f . This will establish the desired contradiction, completing the proof. First, note that $2(\beta + \gamma)x \geq \beta\bar{p}$, by $IC_{\bar{p},\bar{p}}$. Let $\eta = 1/N < x$, and fix $\rho \in \left(0, \min \left\{ \frac{\bar{p}}{\bar{p}+\eta}, \frac{\bar{p}+\eta}{\bar{p}+2\eta} \right\} \right)$. Define

$$\hat{f}(p, q) = \begin{cases} f(p, q) & \text{if } \max\{p, q\} < \bar{p} \\ (1 - \rho) f(p, q) & \text{if } \min\{p, q\} < \bar{p} = \max\{p, q\} \\ \rho f(\min\{p, q\}, \bar{p}) & \text{if } \min\{p, q\} < \bar{p}, \max\{p, q\} = \bar{p} + \eta \\ \left(1 - \frac{\bar{p}+2\eta}{\bar{p}+\eta} \rho\right) \beta & \text{if } p = q = \bar{p} \\ \frac{\eta\rho}{\bar{p}+\eta} \beta & \text{if } \min\{p, q\} = \bar{p}, \max\{p, q\} = \bar{p} + \eta \\ \frac{\bar{p}\rho}{\bar{p}+\eta} \beta & \text{if } p = q = \bar{p} + \eta \end{cases}.$$

Since $\beta > 0$, industry profit under \hat{f} exceeds that under f . Thus, it remains to verify incentive compatibility. Define

$$\begin{aligned} \widehat{IC}_{p,p'} &= \left(\sum_{\tilde{p} > p} \hat{f}(\tilde{p}, p) + \frac{1}{2} \hat{f}(p, p) \right) p - \left(\sum_{\tilde{p} \geq p'} \hat{f}(\tilde{p}, p) \right) p' + \sum_{\tilde{p}} \hat{f}(\tilde{p}, p) x \quad \text{and} \\ IC_{p,p'} &= \left(\sum_{\tilde{p} > p} f(\tilde{p}, p) + \frac{1}{2} f(p, p) \right) p - \left(\sum_{\tilde{p} \geq p'} f(\tilde{p}, p) \right) p' + \sum_{\tilde{p}} f(\tilde{p}, p) x \geq 0. \end{aligned}$$

Lemma 9 *Downward IC from $\bar{p} + \eta$ is satisfied: $\widehat{IC}_{\bar{p}+\eta,p} \geq 0$ for all $p \leq \bar{p} + \eta$.*

Proof. First,

$$\begin{aligned}\widehat{IC}_{\bar{p}+\eta, \bar{p}+\eta} &= \frac{1}{2} \frac{\bar{p}\rho}{\bar{p}+\eta} \beta (\bar{p}+\eta) - \frac{\bar{p}\rho}{\bar{p}+\eta} \beta (\bar{p}+\eta) + \left(\frac{\bar{p}\rho}{\bar{p}+\eta} \beta + \frac{\eta\rho}{\bar{p}+\eta} \beta + \rho\gamma \right) x \\ &= \rho \left(-\frac{1}{2} \beta \bar{p} + (\beta + \gamma) x \right) \geq 0,\end{aligned}$$

where the inequality holds because $2(\beta + \gamma)x \geq \beta\bar{p}$.

Second, for $p \in (\bar{p}, \bar{p} + \eta)$, it is clear that $\widehat{IC}_{\bar{p}+\eta, p} \geq \widehat{IC}_{\bar{p}+\eta, \bar{p}+\eta}$, and hence $\widehat{IC}_{\bar{p}+\eta, p} \geq 0$.

Third, for $p \leq \bar{p}$, we have

$$\begin{aligned}\widehat{IC}_{\bar{p}+\eta, p} &= \frac{1}{2} \frac{\bar{p}\rho}{\bar{p}+\eta} \beta (\bar{p}+\eta) - \left(\frac{\bar{p}\rho}{\bar{p}+\eta} \beta + \frac{\eta\rho}{\bar{p}+\eta} \beta + \rho\tilde{\gamma}_p \right) p \\ &\quad + \left(\frac{\bar{p}\rho}{\bar{p}+\eta} \beta + \frac{\eta\rho}{\bar{p}+\eta} \beta + \rho\gamma \right) x \\ &= \rho \left(\frac{1}{2} \beta \bar{p} - (\beta + \tilde{\gamma}_p) p + (\gamma + \beta) x \right) \geq 0,\end{aligned}$$

where the inequality holds by $IC_{\bar{p}, p}$. ■

Lemma 10 *Downward IC from \bar{p} is satisfied: $\widehat{IC}_{\bar{p}, p} \geq 0$ for $p \leq \bar{p}$.*

Proof. By definition,

$$\begin{aligned}\widehat{IC}_{\bar{p}, p} &= \left(\frac{1}{2} \beta - \frac{1}{2} \frac{\rho\bar{p}\beta}{\bar{p}+\eta} - \frac{\rho}{\bar{p}+\eta} \beta \eta + \frac{\rho}{\bar{p}+\eta} \beta \eta \right) \bar{p} \\ &\quad - \left(\beta - \frac{\rho\bar{p}\beta}{\bar{p}+\eta} - 2 \frac{\rho}{\bar{p}+\eta} \beta \eta + \frac{\rho}{\bar{p}+\eta} \beta \eta + (1-\rho) \tilde{\gamma}_p \right) p \\ &\quad + \left((1-\rho) \gamma + \beta - \frac{\rho\bar{p}\beta}{\bar{p}+\eta} - 2 \frac{\rho}{\bar{p}+\eta} \beta \eta + \frac{\rho}{\bar{p}+\eta} \beta \eta \right) x \\ &= (1-\rho) \left(\frac{1}{2} \beta \bar{p} - (\beta + \tilde{\gamma}_p) p + (\gamma + \beta) x \right) \\ &\quad + \rho \left(\left(\frac{1}{2} \beta - \frac{1}{2} \frac{1}{\bar{p}+\eta} \beta \bar{p} \right) \bar{p} - \left(\beta - \frac{\beta\bar{p}}{\bar{p}+\eta} - \frac{1}{\bar{p}+\eta} \beta \eta \right) p + \left(\beta - \frac{\beta\bar{p}}{\bar{p}+\eta} - \frac{1}{\bar{p}+\eta} \beta \eta \right) x \right).\end{aligned}$$

Since the first line is non-negative by $IC_{\bar{p},p}$, the proof is completed by noting that

$$\begin{aligned} & \left(\frac{1}{2}\beta - \frac{1}{2}\frac{1}{\bar{p}+\eta}\beta\bar{p} \right) \bar{p} - \left(\beta - \frac{\beta\bar{p}}{\bar{p}+\eta} - \frac{1}{\bar{p}+\eta}\beta\eta \right) p + \left(\beta - \frac{\beta\bar{p}}{\bar{p}+\eta} - \frac{1}{\bar{p}+\eta}\beta\eta \right) x \\ &= \frac{1}{2}\beta\frac{\bar{p}\eta}{\bar{p}+\eta} > 0. \end{aligned}$$

■

Lemma 11 *Upward IC from $p \leq \bar{p}$ to $p' \geq \bar{p}$ is satisfied: $\widehat{IC}_{p,p'} \geq 0$ for $p < \bar{p}$ and $p' \geq \bar{p}$.*

Proof. For $p = \bar{p}$, this holds because $\eta \leq x$. So suppose that $p < \bar{p}$. Since $(1 - \rho)\gamma_p + \rho\gamma_{\bar{p}} = \gamma_p$, $\hat{f}(p) = f(p)$ for all $p < \bar{p}$. Moreover, given recommendation p , the conditional probability that the opponent's recommendation is no less than \bar{p} is

$$\frac{(1 - \rho)\gamma_p + \rho\gamma_{\bar{p}}}{\hat{f}(p)} = \frac{\gamma_{\bar{p}}}{f(p)},$$

and hence $\widehat{IC}_{p,\bar{p}}$ follows from $IC_{p,\bar{p}}$.

For $p' \in (\bar{p}, \bar{p} + \eta)$, it is clear that $\widehat{IC}_{p,p'} \geq \widehat{IC}_{p,\bar{p}+\eta}$. Thus, it remains to show that $\widehat{IC}_{p,\bar{p}+\eta} \geq 0$. Given recommendation p , the conditional probability that the opponent's recommendation is no less than $\bar{p} + \eta$ is

$$\frac{\rho\gamma_{\bar{p}}}{\hat{f}(p)} = \rho\frac{\gamma_{\bar{p}}}{f(p)}.$$

Thus, $\widehat{IC}_{p,\bar{p}+\eta} \geq 0$ is implied by $IC_{p,\bar{p}} \geq 0$ if

$$(\bar{p} + \eta)\rho\frac{\gamma_{\bar{p}}}{f(p)} \leq \bar{p}\frac{\gamma_{\bar{p}}}{f(p)},$$

which follows from $\rho \leq \frac{\bar{p}}{\bar{p}+\eta}$. ■

Lemma 12 *IC from $p < \bar{p}$ to $p' < \bar{p}$ is satisfied: $\widehat{IC}_{p,p'} \geq 0$ for all $p < \bar{p}, p' < \bar{p}$.*

Proof. This follows from $\hat{f}(p) = f(p)$ and $\sum_{\bar{p} \geq p'} \hat{f}(p, \tilde{p}) = \sum_{\bar{p} \geq p'} f(p, \tilde{p})$ for all $p' < \bar{p}$. ■

B Asymmetric Equilibrium without Mediation

Let $n = 3$ and $x < \frac{1}{4}$. Firms 1 and 2 mix independently, with

$$p_i = \begin{cases} x & \text{with prob } \frac{2}{7} \\ \frac{3}{2}x & \text{with prob } \frac{2}{7} \\ 4x & \text{with prob } \frac{3}{7} \end{cases}$$

Firm 3 stays out (or, equivalently, prices above $4x$). There are no on-path punishments. Any off-path price is punished by x .

The winning price distribution is given by

$$\min \{p_1, p_2\} = \begin{cases} x & \text{with prob } \frac{24}{49} \\ \frac{3}{2}x & \text{with prob } \frac{16}{49} \\ 4x & \text{with prob } \frac{9}{49} \end{cases}$$

Hence, industry profit equals $\frac{24}{49}(x) + \frac{16}{49}\left(\frac{3}{2}x\right) + \frac{9}{49}(4x) = \frac{12}{7}x$. This is greater than $\frac{3}{2}x$, which by Proposition 3, is the maximum cartel profit when all firms win with positive probability.

We check that this is an equilibrium. For firm 3, pricing just below x gives $x - x = 0$, pricing just below $\frac{3}{2}x$ gives $\frac{25}{49}\left(\frac{3}{2}x\right) - x < 0$, and pricing just below $4x$ gives $\frac{9}{49}(4x) - x < 0$, so it is optimal for firm 3 to stay out. For firm 1 or 2, pricing at x , $\frac{3}{2}x$, or $4x$ all give an expected payoff of $\frac{6}{7}x$, because

$$\underbrace{\frac{6}{7}}_{\text{prob win if } p_i=x} (x) = \underbrace{\frac{4}{7}}_{\text{prob win if } p_i=\frac{3}{2}x} \left(\frac{3}{2}x\right) = \underbrace{\frac{3}{14}}_{\text{prob win if } p_i=4x} (4x).$$

Finally, for firm 1 or 2, any off-path price gives a strictly lower payoff, because pricing just below x gives $x - x = 0$, pricing just below $\frac{3}{2}x$ gives $\frac{5}{7}\left(\frac{3}{2}x\right) - x = \frac{1}{14}x$, and pricing just below $4x$ gives $\frac{3}{7}(4x) - x = \frac{5}{7}x$.

C Optimal Bid Distributions with Only Downward Incentive Constraints



Figure 7: The optimal bid distribution when $x = .1$, with only downward incentive constraints. The color scheme in Figures 7–9 is the same as in Figure 1. Bid pair $p_1 = p_2 = 1$ is recommended with probability .06.

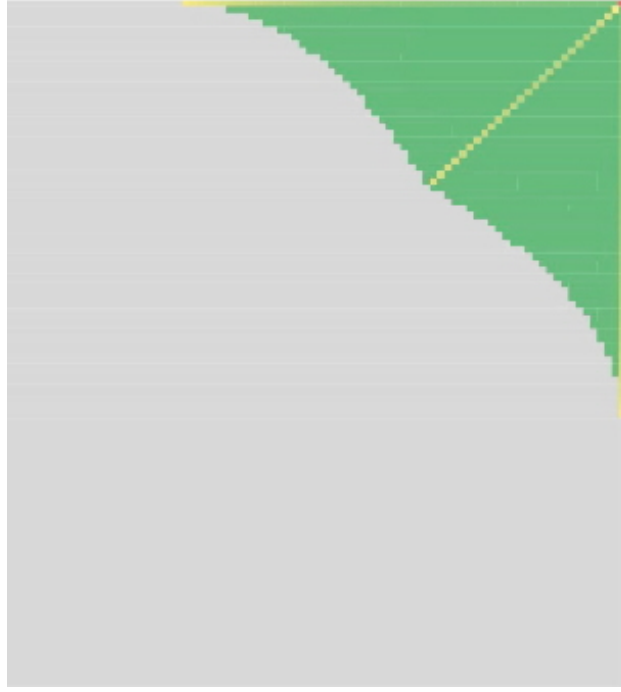


Figure 8: The optimal bid distribution when $x = .2$, with only downward incentive constraints. Bid pair $p_1 = p_2 = 1$ is recommended with probability .16.



Figure 9: The optimal bid distribution when $x = .3$, with only downward incentive constraints. Bid pair $p_1 = p_2 = 1$ is recommended with probability .31.

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