Durable Goods Monopoly with Stochastic Costs

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Abstract

I study the problem of a durable goods monopolist who lacks commitment power and whose marginal cost of production varies stochastically over time. I show that a monopolist with stochastic costs usually serves the different types of consumers at different times and charges them different prices. When the distribution of consumer valuations is discrete, the monopolist exercises market power and there is inefficient delay. When there is a continuum of types, the monopolist cannot extract rents and the market outcome is efficient.

Keywords: durable goods, Coase conjecture, stochastic costs, dynamic games.

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1 Introduction

Consider a monopolist who produces a durable good and who cannot commit to a path of prices. For settings in which production costs do not change over time, Coase (1972) argued that this producer would not be able to sell at the static monopoly price. After selling the initial quantity, the monopolist has the temptation to reduce prices to reach consumers with lower valuations. This temptation leads the monopolist to continue cutting prices after each sale. Forward-looking consumers expect prices to fall, so they are unwilling to pay a high price. Coase conjectured that these forces would lead the monopolist to post an opening price arbitrarily close to marginal cost. The monopolist would then serve the entire market "in the twinkling of an eye" and the market outcome would be efficient.

The classic papers on durable goods monopoly (i.e., Stokey (1981); Fudenberg et al. (1985); Gul et al. (1985)) provide formal proofs of the Coase conjecture: in any stationary equilibrium, as the period length goes to zero the monopolist's opening price converges to the lowest consumer valuation. In the limit all buyers trade immediately, the market outcome is efficient and the monopolist is unable to extract rents from buyers with higher valuations: she obtains the same profits she would have earned if she were selling to a market in which all consumers had the lowest valuation.

The goal of this paper is to study the problem of a durable goods monopolist who lacks commitment power and whose cost of production varies stochastically over time. The assumption that costs are subject to stochastic shocks is natural in many markets. Stochastic costs may arise due to changes in input prices. For instance, high-tech firms face uncertain and time-varying costs, partly because the prices of their key inputs tend to fall over time and partly due to fluctuations in the prices of the raw materials that they use. Changes in exchange rates also lead to stochastic costs if the monopolist sells an imported good or if she uses imported inputs. The results in this paper show how changes in costs affect the dynamics of prices, the timing of sales and the seller's profits in durable goods markets.

The model is set up in continuous-time and the monopolist's marginal cost evolves as a geometric Brownian motion.[1] Costs are publicly observable and at each moment the monopolist can produce any quantity at the current marginal cost. Continuous-time methods are especially suitable to perform the option value calculations that arise with time-varying costs, allowing me to obtain a tractable characterization of the equilibrium. The model delivers simple expressions for equilibrium prices, allowing for the computation of profit margins as

[1] Section 2 considers a simple case in which costs fall deterministically over time; Section 8 discusses how the results in the paper extend to a setting in which costs evolve as a Markov chain.

a function of costs and the level of market penetration.

When the monopolist's costs are time-varying, serving the entire market immediately is in general not efficient. The reason for this is that changes in costs introduce an option value of delaying trade. The efficient outcome in this setting is for the monopolist to serve consumers with valuation v the first time costs fall below a threshold z_v which is increasing in v. The efficient outcome can be implemented by choosing a stochastically decreasing path of prices that induces consumers to buy at the efficient time; for instance, consumers will buy at the efficient time if the seller prices at marginal cost at all times.

This observation suggests the following generalization of the Coase conjecture for markets with time-varying costs. Given a distribution of consumer valuations, say that an outcome is *Coasian* if (i) it is Pareto efficient, and (ii) the monopolist earns the same profit as she would earn if she were selling to a market in which all consumers had the lowest valuation. Note that these conditions do not require the monopolist to sell to all consumers at the same time or at the same price; in fact, doing so would be inefficient.

In this paper, I show that a durable goods monopolist with stochastic costs typically serves the different types of consumers at different points in time and charges them different prices. The market outcome is Coasian only when the set of consumer valuations is a continuum. Indeed, when types are discrete the monopolist is able to extract rents from consumers with higher valuations, and the equilibrium may feature inefficient delays. Consistent with empirical evidence (e.g., Zhao (2008); Conlon (2012)), the monopolist's profit margins are initially large and decrease over time together with costs. These results contrast with those obtained when costs don't change over time, where the Coase conjecture holds for any value distribution.

To see why the market outcome is in general not Coasian, consider a setting with two types of consumers with valuations $v_2 > v_1 > 0$. Suppose first that costs decrease monotonically, and note that efficiency requires serving high types when costs fall below a threshold z_2 and serving low types when costs fall below $z_1 < z_2$. After selling to all high types, the monopolist finds it optimal to sell to low types when costs fall to z_1, charging them a price of v_1. This allows the monopolist to extract rents from high types when costs are above z_1. Indeed, high types expect prices to remain high until costs fall to z_1, and are therefore willing to pay a high price. Moreover, when costs are Brownian the equilibrium outcome is not efficient. Intuitively, if the outcome were efficient the monopolist would serve high and low types immediately when costs are initially below z_1. The seller would thus extract rents from high types when costs are above z_1, but would not extract any rents from them when costs are below z_1. This introduces an option value to delay trade: a seller with cost close to z_1 finds it profitable to wait and speculate with an increase in costs. In equilibrium the monopolist doesn't completely delay trade; instead, she sells to high types gradually at a high price and attains the same profits as if she did delay.

Consider next a market with a continuum of consumer types. In contrast to the discrete case, in this setting the monopolist has an incentive to cut her price immediately after each sale to reach consumers with lower valuations. This erodes her ability to extract rents from high type buyers: the profit margin she earns on them is equal to the expected discounted profit margin she earns from consumers with the lowest valuation. Moreover, the market outcome is efficient regardless of whether costs fall monotonically or not. Indeed, the incentive to inefficiently delay trade disappears when types are a continuum, since the seller is unable to extract rents from higher types even when costs are high.

Coase's original arguments illustrate how commitment problems may prevent a monopolist producer of a durable good from exercising market power. The results in this paper show that these forces are more general than what Coase described. In particular, these forces do not rely on serving the entire market immediately, nor on serving every buyer at the same price. In markets with time-varying costs, to attain efficiency and zero rent extraction it is enough that the monopolist cannot credibly commit to delay trade from one sale to the next.

A natural interpretation of a model with discrete types is that it represents a market in which there is a clear segmentation among consumers. With this interpretation, my results imply that a dynamic monopolist with time-varying costs will be able to obtain more profits in segmented markets, possibly at the expense of efficiency. For intermediary durable goods, market segmentation arises naturally when the monopolist sells to firms in different industries. For consumer durable goods, market segmentation may arise when buyers can make investments prior to participating in the market. For instance, when the seller's good is complementary to some other good, the value distribution can be approximated by a two-type distribution (i.e., those who own the complementary good and those who don't).

The literature on durable goods monopoly has identified different ways in which a dynamic monopolist can exercise market power. For instance, a durable goods monopolist can ameliorate her lack of commitment power by renting her good rather than selling it (Bulow (1982)), or by introducing best-price provisions (Butz (1990)). The Coase conjecture also fails when the monopolist faces capacity constraints (Bulow (1982); Kahn (1986); McAfee and Wiseman (2008)), when consumers use non-stationary strategies (Ausubel and Deneckere (1989)), or when buyers have an outside option (Board and Pycia (2014)).[2] The current paper studies

[2]A dynamic monopolist can also extract rents when there is a deadline of trade; see, for instance, Güth and Ritzberger (1998) and Hörner and Samuelson (2011).

the problem of a durable goods monopolist with time-varying costs and identifies a new setting in which such a seller can exercise market power: when costs are time-varying and types are discrete a dynamic monopolist can extract rents from buyers with higher valuations.[3]

This paper relates to Biehl (2001), who studies a durable goods monopoly model in which the valuations of the buyers are subject to idiosyncratic stochastic shocks (see also Deb (2011)). Biehl (2001) shows that the monopolist may charge a constant high price in this setting. Intuitively, idiosyncratic changes in valuations lead to a renewal of high type consumers, allowing the seller to truthfully commit to a constant high price. In contrast, time-varying costs produce aggregate changes in the net valuations of the buyers (i.e., valuations minus marginal cost), leading to a different equilibrium dynamics.

This paper also shares some features with models of bargaining with one-sided incomplete information. Deneckere and Liang (2006) study a bargaining game in which the valuation of the buyer is correlated with the cost of the seller (see also Evans (1989); Vincent (1989)). They show that there are recurring bursts of trade in equilibrium, with short periods of high probability of agreement followed by long periods of delay. In the current paper there are also recurring bursts of trade when types are discrete. For instance, with two types of buyers the monopolist first sells to all high types when costs are initially large, and then sells to low types when costs fall below some given threshold.

Fuchs and Skrzypacz (2010) study a one-sided incomplete information bargaining game in which a new trader may arrive according to a Poisson process. The payoffs that the seller and the buyer get upon an arrival depend on the buyer's valuation for the seller's good; for instance, upon arrival the seller may run a second price auction between the original buyer and the new trader. Fuchs and Skrzypacz (2010) show that the seller is unable to extract rents in this setting: her inability to commit to a path of offers drives her profits down to her outside option of waiting until the arrival of a new buyer. Moreover, the possibility of arrivals leads to inefficient delays, with the seller slowly screening out high type buyers. In the current paper, the monopolist is also unable to extract rents when there is a continuum of types. However, the equilibrium outcome is efficient in this setting, with the seller serving the different buyers at the point in time that maximizes total surplus.[4]

[3] Other papers study dynamic monopoly models in non-stationary environments. Stokey (1979) solves the full commitment path of prices of a durable good monopolist when costs fall deterministically over time. Sobel (1991) studies the problem of a dynamic monopolist in a setting in which new consumers enter the market each period. Board (2008) characterizes the full commitment strategy of a durable good monopolist when incoming demand varies over time. Garrett (2012) solves the full commitment strategy of a durable good monopolist in a setting in which buyers arrive over time and their valuations are time-varying.

[4] In Fuchs and Skrzypacz (2010), achieving an efficient outcome requires the seller to post an initial price weakly below the buyer's lowest valuation. This can never be an equilibrium outcome in their model, since

The possibility of arrivals in Fuchs and Skrzypacz (2010) introduces interdependencies in the net valuations of the buyer and the seller, making their setting similar to the model in Deneckere and Liang (2006). In a different paper, Fuchs and Skrzypacz (2013) consider the model in Deneckere and Liang (2006) with a continuum of types. They show that the equilibrium of this model converges to the outcome in Fuchs and Skrzypacz (2010) as the gap between the seller's lowest cost and the buyer's lowest valuation converges to zero. The seller is therefore unable to extract rents from the buyer in this gapless limit. In the current paper, the monopolist also loses the ability to extract rents as valuations become a continuum. The difference, however, is that this result holds for any model with a continuum of types, regardless of the lowest consumer valuation, i.e., regardless of the size of the gap.

Finally, Bagnoli et al. (1989), von der Fehr and Kühn (1995) and Montez (2013) show that the Coase conjecture fails when the monopolist faces a discrete number of buyers: in this setting there are inefficient equilibria under which the monopolist exercises market power. In contrast, the model in the current paper considers a monopolist with time-varying costs who faces a continuum of consumers, each of whom is individually infinitesimal. In this setting, whether the market outcome is efficient or not depends on the distribution of consumer valuations (i.e., on whether types are discrete or a continuum).[5]

2 A simple example

This section illustrates some of the main results in the paper through a simple example in which the seller's costs fall deterministically over time. For conciseness, I keep the exposition at an informal level and focus on one equilibrium of the game.[6]

Suppose the monopolists' marginal cost at time $t \geq 0$ is $x_t = x_0 e^{\mu t}$, with $x_0 > 0$ and $\mu < 0$. The monopolist faces a unit mass of buyers indexed by $i \in [0, 1]$. Each consumer is in the market to buy a single unit of the seller's good. Consumer i's valuation is $f(i) \in [\underline{v}, \overline{v}]$, with $\overline{v} > \underline{v} > 0$. All players share the same discount rate $r > 0$. Costs are publicly observable and it is common knowledge among buyers and seller that costs fall at rate μ.

the seller's profits from posting this low price would be strictly below what she would earn by waiting until a new buyer arrives. As a result, any equilibrium must necessarily involve inefficiencies.

[5] More broadly, this paper also relates to the growing literature that uses continuous-time methods to analyze strategic interactions. For instance, continuous-time methods have been used to study the provision of incentives in dynamic settings (Sannikov (2007, 2008)), political campaigns and political bargaining (Gul and Pesendorfer (2012); Ortner (2015)) and dynamic markets for lemons (Daley and Green (2012)).

[6] Arguments similar to those in Sections 3-7 can be used to show that the equilibrium I present is the unique stationary equilibrium satisfying certain "natural" properties.

First-best. The first-best outcome is to serve buyers with value v at the time that solves

$$V_v(x_0) = \max_{t\geq 0} e^{-rt}(v - x_t).$$

The solution to this problem is to sell the first time costs fall below $z_v = \frac{r}{r-\mu}v$; i.e., to sell at time $t_v(x_0) = \max\{0, \frac{1}{\mu}\ln(\frac{z_v}{x_0})\}$. Note for all $x > z_v$, $V_v(x) = \frac{-\mu}{r-\mu}v\left(\frac{z_v}{x}\right)^{\frac{-r}{\mu}}$.

Markets with two types of buyers. Consider a market with two types of consumers: high types with valuation $v_2 > 0$ and low types with valuation $v_1 \in (0, v_2)$. Let $\alpha \in (0,1)$ be the fraction of high type consumers in the market.

As is standard in durable goods monopoly models, consumers with high valuation buy earlier, since delaying trade is costlier for them. Suppose that the game reaches a point at which all high types have bought and left the market, and let x be the monopolist's cost at that point. Since all remaining buyers have valuation v_1, the seller can charge them price v_1 and extract all their surplus. Therefore, the seller's continuation profits at this point are

$$\Pi(x) = \max_{t\geq 0} e^{-rt}(1-\alpha)(v_1 - xe^{\mu t}) = (1-\alpha)V_{v_1}(x).$$

The monopolist finds it optimal to sell to low types at the efficient time $t_{v_1}(x) = \max\{0, \frac{1}{\mu}\ln(\frac{z_{v_1}}{x})\}$.

Consider next a point in time at which high type buyers are still in the market. High types know that, after they buy, the monopolist will sell to low types when costs fall to z_{v_1}. When costs are x, high types are willing to pay $P(x, v_2) = v_2 - e^{-rt_{v_1}(x)}(v_2 - v_1)$, which is the price that leaves them indifferent between buying now or waiting and buying together with low types. High type consumers are willing to pay a price strictly larger than v_1 when costs are above z_{v_1}, since they know that the seller won't serve low types until costs fall to z_{v_1}. Thus, unlike the standard setting with time-invariant costs, the monopolist is able to extract additional profits from high types.[7] The rent that a v_2-consumer gets from buying at price $P(x, v_2)$ is $e^{-rt_{v_1}(x)}(v_2 - v_1)$.

Given that high types are willing to pay $P(x, v_2)$, the monopolist sells to them at the

[7] Note that for $x \in (z_{v_1}, z_{v_2}]$, the profit margin from selling at price $P(x, v_2)$ is strictly larger than the discounted profit margin from selling at price v_1 when costs fall to z_{v_1}; indeed $P(x, v_2) - x - e^{-rt_{v_1}(x)}(v_1 - xe^{\mu t_{v_1}(x)}) = v_2 - x - e^{-rt_{v_1}(x)}(v_2 - xe^{\mu t_{v_1}(x)}) > 0$.

time that solves

$$\max_{t\geq 0} e^{-rt}\left[\alpha(P(x_t,v_2)-x_t)+\Pi(x_t)|x_0=x\right]$$
$$=\max_{t\geq 0} e^{-rt}\left[(\alpha(v_2-x_t-e^{-rt_{v_1}(x_t)}(v_2-v_1))+(1-\alpha)e^{-rt_{v_1}(x_t)}(v_1-x_{t_{v_1}(x_t)})|x_0=x\right]$$

The solution to this problem is $t_{v_2}(x)=\max\{0,\frac{1}{\mu}\ln(\frac{z_{v_2}}{x})\}$; that is, the monopolist finds it optimal to serve high types at the efficient time. Indeed, since the monopolist sells to low types when costs reach z_{v_1}, the discounted payoff that she gets from serving high types at time t is equal to the total surplus $e^{-rt}(v_2-xe^{\mu t})$ minus the rent $e^{-rt_{v_1}(x)}(v_2-v_1)$. Therefore, it is optimal for the seller to serve high types at the efficient time.

More generally, in markets with a finite number of consumer types, the monopolist sells to the different consumers at the efficient time. For any valuation v_k, the price that v_k-consumers are willing to pay is $P(x,v_k)=v_k-e^{-rt_{v_{k-1}}(x)}(v_k-P(x_{t_{v_{k-1}}(x)},v_{k-1}))$. By an induction argument, $P(x,v_k)=v_k-\sum_{m=1}^{k-1}e^{-rt_{v_m}(x)}(v_{m+1}-v_m)$. Note that the rent that a consumer with valuation v_k gets in this market is $\sum_{m=1}^{k-1}e^{-rt_{v_m}(x)}(v_{m+1}-v_m)$.

Markets with a continuum of types. Consider next a market with a continuum of consumer valuations $[\underline{v},\overline{v}]$. This setting can be approximated by a sequence of discrete models with valuations $\mathcal{V}_j=\{v_1^j,...,v_{n_j}^j\}\subset[\underline{v},\overline{v}]$ such that $\mathcal{V}_j$ becomes dense as $j\to\infty$.

Fix a valuation $v\in[\underline{v},\overline{v}]$ and a sequence $v_{k_j}^j\in\mathcal{V}_j$ such that $\lim_{j\to\infty}v_{k_j}^j=v$. For any model j in the sequence and for all $x>z_{v_{k_j}^j}$, the price that consumers with valuation $v_{k_j}^j$ are willing to pay is

$$P^j(x,v_{k_j}^j)=v_{k_j}^j-\sum_{m=1}^{k_j-1}\left(\frac{z_{v_m^j}}{x}\right)^{\frac{-r}{\mu}}(v_{m+1}^j-v_m^j)\to v-\int_{\underline{v}}^{v}\left(\frac{z_{\tilde{v}}}{x}\right)^{\frac{-r}{\mu}}d\tilde{v}\text{ as }j\to\infty,$$

where I used $e^{-rt_{v_m}(x)}=\left(\frac{z_{v_m}}{x}\right)^{\frac{-r}{\mu}}$. If costs are initially larger than z_v, the monopolist will sell to buyers with valuation v at the time costs reach $z_v=\frac{rv}{r-\mu}$. The price at which such a consumer buys is

$$v-\int_{\underline{v}}^{v}\left(\frac{z_{\tilde{v}}}{z_v}\right)^{\frac{-r}{\mu}}d\tilde{v}=z_v+V_{\underline{v}}(z_v).$$

Note that the seller losses her ability to extract rents when she faces a continuum of types: the profit margin she obtains from buyers with valuation $v>\underline{v}$ is exactly equal to $V_{\underline{v}}(z_v)$, which is the expected discounted profits she gets from a buyer with value $\underline{v}$. Intuitively, when there is a continuum of types the monopolist has an incentive to reduce her price immediately after each sale. Forward looking buyers anticipate this, so they are not willing to pay a high price.

Why is the seller's profit margin from serving consumers with valuation $v > \underline{v}$ exactly equal to $V_{\underline{v}}(z_v)$? Since the outcome is efficient, the total surplus from selling to a buyer with valuation v is $V_v(z_v)$. By the Envelope Theorem $\frac{\partial V_v(x)}{\partial v} = e^{-rt_v(x)} = (z_v/x)^{\frac{-r}{\mu}}$, so $V_v(z_v) = V_{\underline{v}}(z_v) + \int_{\underline{v}}^{v} (z_v/x)^{\frac{-r}{\mu}} d\tilde{v}$. Of this surplus, a buyer with value v gets rents $\int_{\underline{v}}^{v} (z_v/x)^{\frac{-r}{\mu}} d\tilde{v}$, and the seller gets $V_{\underline{v}}(z_v)$.

This deterministic example illustrates how changes in costs can affect the equilibrium dynamics in durable goods markets. To summarize, the main takeaways of the example are: (i) the monopolist is able to extract rents when types are discrete; (ii) this ability to extract rents disappears when the set of consumer valuations is a continuum; and (iii) the market outcome is always efficient.

The rest of the paper considers an environment in which the monopolist's marginal cost evolves as a geometric Brownian motion. Some of the results in this section carry through when costs are stochastic; in particular, the monopolist can only extract rents when types are discrete. Importantly, stochastic costs also lead to new results. Indeed, in contrast to this deterministic environment, there is inefficient delay when costs are stochastic.

3 Model

A monopolist faces a continuum of consumers indexed by $i \in [0, 1]$. Consumers are in the market to buy one unit of the monopolist's good. Time is continuous and consumers can make their purchase at any time $t \in [0, \infty)$. The valuation of consumer $i \in [0, 1]$ is given by $f(i)$, where $f : [0, 1] \to \mathbb{R}_+$ is a non-increasing and left-continuous function that is right-continuous at 0. All players are risk-neutral expected utility maximizers and discount future payoffs at rate $r > 0$. I assume that f is a step function taking n values $v_1, ..., v_n$, with $0 < v_1 < ... < v_n$. Section 7 studies the case in which f approximates a continuous function.

Let $B = \{B_t, \mathcal{F}_t : 0 \leq t < \infty\}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. The monopolist's marginal cost x_t evolves as a geometric Brownian motion,

$$dx_t = \mu x_t dt + \sigma x_t dB_t, \tag{1}$$

with $\sigma > 0$ and $\mu < r$. The constants μ and σ measure the expected rate of change and the percentage volatility of x_t, respectively. Let $x_0 > 0$ be the (degenerate) initial level of cost. At any time t the monopolist can produce any quantity at marginal cost x_t. I assume that the cost of maintaining an inventory is sufficiently large that the monopolist always finds it optimal to produce on demand.[8] The process x_t is publicly observable and its structure is common knowledge: seller and buyers commonly know that x_t evolves as (1).

A (stationary) strategy for consumer $i \in [0,1]$ is a function $p : \mathbb{R}_+ \to \mathbb{R}_+$ that describes the cutoff price for i given the current level of costs. Suppose consumer i is still in the market at time t. Under strategy $p(\cdot)$ consumer i purchases the good at time t if and only if the price that the monopolist charges is weakly lower than $p(x_t)$.[9]

Let $\mathbf{P} = P(x,i)$ be a strategy profile for the consumers, with $P(\cdot,i)$ denoting the strategy of consumer $i \in [0,1]$. Optimal behavior by the buyers implies that $P(x,i)$ must satisfy the *skimming property*: for all $i < j$, $P(x,i) \geq P(x,j)$ for all x. That is, buyers with higher valuations are willing to pay higher prices. The reason for this is that it is costlier for buyers with higher valuation to delay their purchase: if a buyer with valuation v finds it weakly optimal to buy at some time t given a future path of prices, then buyers with valuation $v' > v$ find it strictly optimal to buy at t. For technical reasons, I restrict attention to strategy profiles such that $P(x,i)$ is left-continuous in i and continuous in x (this restriction guarantees that payoffs are well defined).

The skimming property implies that at any time t there exists a cutoff $q_t \in [0,1]$ such that consumers $i \leq q_t$ have already left the market, while consumers $i > q_t$ are still in the market. The cutoff q_t describes the level of market penetration at time t. At each time t, the level of market penetration and the monopolist's marginal cost describe the payoff relevant state of the game.

The monopolist chooses an $\mathcal{F}_t$-progressively measurable and right-continuous path of prices $\{p_t\}$ to maximize her profits. Given a strategy profile $\mathbf{P}$ of the consumers and a path of prices $\{p_t\}$, at each time $s \geq 0$ all consumers remaining in the market whose cutoff is above p_s make their purchase; that is, the set of consumers that buy at time s are those consumers $i \in [0,1]$ with $P(x_s, i) \geq p_s$ who have not bought before s.

Since $\mathbf{P}$ satisfies the skimming property and since consumers use stationary strategies, for any path of prices $\{p_t\}$ there is a unique process $\{q_t\}$ which describes the induced evolution of

[8]When the production technology is "reversible", so that the monopolist can transform at each time t a unit of his good into x_t dollars, the assumption that $\mu < r$ guarantees that she would always find it optimal to produce on demand.

[9]These stationary strategies are the natural counterpart to the strategies that buyers use along the equilibrium path in a stationary equilibrium of the standard durable goods monopoly model with constant costs; see, for instance, Gul et al. (1985).

market penetration.[10] Moreover, the monopolist will find it optimal to charge price $P(x_t, q)$ if consumer q is the marginal buyer at time t. Thus, I can alternatively specify the monopolist's problem as choosing a non-decreasing process $\{q_t\}$ with $q_{0^-} = 0$ and $q_t \leq 1$ for all t, describing the level of market penetration at each time t. With this specification, under strategy $\{q_t\}$ the seller charges $P(x_t, q_t)$ at every time t, and at this price all buyers $i \leq q_t$ who are still in the market buy. Since the monopolist sells to all consumers whose cutoff is above the current price, the process $\{q_t\}$ must be such that, for all t, $P(x_t, i) \geq P(x_t, q_t)$ implies $q_t \geq i$.

Monopolist's problem. Given a strategy profile $\mathbf{P}$ of the consumers, a strategy for the seller is an $\mathcal{F}_t$-progressively measurable process $\{q_t\}$ satisfying the conditions above such that $q_{0^-} = 0$, q_t is non-decreasing with $q_t \leq 1$ for all t, and $\{q_t\}$ is right-continuous with left-hand limits.[11] Let $\mathcal{A}^{\mathbf{P}}$ denote the set of all such processes. Given a strategy profile $\mathbf{P}$ of the consumers and a strategy $\{q_t\} \in \mathcal{A}^{\mathbf{P}}$, the monopolist's profits are

$$\Pi = E\left[\int_0^\infty e^{-rt}\left(P(x_t, q_t) - x_t\right) dq_t\right]. \tag{2}$$

Let $\Pi(x, q)$ denote the monopolist's future discounted profits conditional on the current state being (x, q), and let $\mathcal{A}^{\mathbf{P}}_{q,t}$ denote the set of processes $\{q_t\} \in \mathcal{A}^{\mathbf{P}}$ such that $q_{t^-} = q$. The monopolist's payoffs conditional on the state at time t^- being equal to (x, q) are[12]

$$\Pi(x, q) = \sup_{\{q_t\} \in \mathcal{A}^{\mathbf{P}}_{q,t}} E\left[\int_t^\infty e^{-r(s-t)}\left(P(x_s, q_s) - x_s\right) dq_s \,\middle|\, \mathcal{F}_t\right]. \tag{3}$$

Consumer's problem. Given a strategy $\{q_t\}$ of the monopolist and a strategy profile $\mathbf{P}$ of the consumers, the path of prices is $\{P(x_t, q_t)\}$. The strategy $P(x, i)$ of each consumer i must be optimal given the path of prices $\{P(x_t, q_t)\}$: the payoff that consumer i gets from buying at the time strategy $P(x, i)$ tells her to buy must be weakly larger than what she would get from purchasing at any other point in time. Formally, for any time t before consumer i buys, it must be that $f(i) - p \leq \sup_\tau E[e^{-r(\tau-t)}(f(i) - P(x_\tau, q_\tau))|\mathcal{F}_t]$ for all $p > P(x, i)$, and $f(i) - p \geq \sup_\tau E[e^{-r(\tau-t)}(f(i) - P(x_\tau, q_\tau))|\mathcal{F}_t]$ for all $p \leq P(x, i)$.

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[10] Indeed, for any strategy profile $\mathbf{P}$ of the consumers and any path of prices $\{p_t\}$, the process $\{q_t\}$ is described by: $q_{0^-} = 0$ and, for all $t \geq 0$, $q_t = \max\{\max\{q_s\}_{s<t}, \sup\{i \in [0,1] : P(x_t, i) \geq p_t\}\}$.

[11] These continuity requirements on $\{q_t\}$ together with the continuity requirements on $P(x, i)$ guarantee that the integrals in (2) and (3) are well-defined.

[12] Note that the profits $\Pi(x, q)$ are conditional on the state at time t^-. The reason for this is to preserve the right-continuity of $\{q_t\}$, since this process may jump at time t.

I impose two additional conditions on the strategies of the consumers. The first condition says that v_1-consumers pay their valuation:

$$\forall i \text{ such that } f(i) = v_1,\ P(x,i) = v_1 \text{ for all } x. \tag{4}$$

The second condition says that, for $k > 1$, the incentive compatibility constraint of v_k-consumers is tight: the price at which the last consumer with valuation v_k buys leaves this consumer indifferent between paying that price or waiting and buying when the monopolist sells to v_{k-1}-consumers. Formally, fix a strategy profile $(\mathbf{P}, \{q_t\})$. For $k = 1, ..., n$, let $\alpha_k = \max\{i \in [0,1] : f(i) = v_k\}$ and let $\tau_k = \inf\{t : q_t > \alpha_{k+1}\}$ be the (random) time at which the monopolist starts selling to consumers with valuation v_k. Then, for $k = 2, ..., n$,

$$v_k - P(x, \alpha_k) = E\left[e^{-r(\tau_{k-1}-t)} \left(v_k - P\left(x_{\tau_{k-1}}, q_{\tau_{k-1}}\right)\right) \middle| \mathcal{F}_t \right], \tag{5}$$

whenever the state at t is (x, α_k).

Definition 1 *A strategy profile* $(\mathbf{P}, \{q_t\})$ *is an equilibrium if:*

(i) $\{q_t\}$ *is optimal for all states* $(x,q) \in \mathbb{R}_+ \times [0,1]$ *given* $\mathbf{P}$ *(i.e.,* $\{q_t\}$ *satisfies* (3)*),*

(ii) for each $i \in [0,1]$*,* $P(x,i)$ *is optimal given* $\{q_t\}$ *and* $\mathbf{P}$*, and*

(iii) $\mathbf{P}$ *satisfies conditions* (4) *and* (5) *given* $\{q_t\}$*.*

Conditions (i) and (ii) in Definition 1 require that the strategies of the seller and the buyers be optimal. Condition (iii), on the other hand, imposes additional restrictions on the buyers' strategies. These restrictions determine the rents that the monopolist extracts from different types of consumers. Consider first a time at which all consumers remaining in the market have valuation v_1. Since there is a single type of consumer left and the monopolist has all the bargaining power (i.e., she is posting the prices), she should be able to extract all the rents from the remaining buyers. This is what equation (4) requires. Similarly, when all consumers with valuation $v \geq v_{k+1}$ have left the market, the monopolist knows that the highest remaining types have valuation v_k. These buyers have the option of mimicking v_{k-1}-consumers, getting a rent of $E\left[e^{-r(\tau_{k-1}-t)} \left(v_k - P\left(x_{\tau_{k-1}}, q_{\tau_{k-1}}\right)\right) \middle| \mathcal{F}_t \right]$. Since the monopolist has all the bargaining power, she should be able to leave v_k-consumers with only this much rent; this is what equation (5) requires.

The Online Appendix shows that the restrictions that condition (iii) imposes on the buyers' strategies necessarily hold in any subgame perfect equilibrium (SPE) of a discrete time version of this model. However, without imposing condition (iii), the continuous-time game would have equilibria in which (4) or (5) are violated. The following example illustrates.

Example 1 Assume there are two types of buyers, high types with valuation $v_2 > 0$ and low types with valuation $v_1 \in (0, v_2)$. I construct a strategy profile under which consumers are never willing to pay more than the monopolist's marginal cost; i.e., such that $P(x,i) \leq x$ for all $x \in \mathbb{R}_+$ and all $i \in [0,1]$. This strategy profile satisfies conditions (i) and (ii) in Definition 1, but doesn't satisfy (iii).

The strategy profile I construct is as follows. For each $i \in [0,1]$ with $f(i) = v_k$,

$$P(x,i) = v_k - \sup_{\tau \in T} E[e^{-r\tau}(v_k - x_\tau)|x_0 = x],$$

where T is the set of stopping times. The solution to $\sup_{\tau\in T} E[e^{-r\tau}(v_k - x_\tau)|x_0 = x]$ is of the form $\tau_k = \inf\{t : x_t \leq z_k\}$ for some cutoff $z_k \in (0, v_k)$ such that $z_1 < z_2$ (see Lemma 1 below). Therefore, for all $i \in [0,1]$ with $f(i) = v_k$,

$$P(x,i) = \begin{cases} v_k - E[e^{-r\tau_k}(v_k - x_\tau)|x_0 = x] & \text{if } x > z_k, \\ x & \text{if } x \leq z_k. \end{cases}$$

Note that $P(x,i) - x = v_k - x - \sup_{\tau\in T} E[e^{-r\tau}(v_k - x_\tau)|x_0 = x] < 0$ for all $x > z_k$, so under this strategy profile buyers are never willing to pay more than marginal cost. Therefore, an optimal strategy for the monopolist is to set prices always equal to marginal cost (i.e., choose $\{p_t\} = \{x_t\}$), and sell to all consumers with valuation v_k at time $\tau_k = \inf\{t : x_t \leq z_k\}$. Moreover, each individual buyer finds it optimal to buy the first time costs fall below z_k (and pay a price equal to marginal cost) when the monopolist follows this strategy. Thus, this strategy profile satisfies conditions (i) and (ii) in Definition 1. However, this strategy profile doesn't satisfy condition (iii). First, when $x < v_1$, consumers with valuation v_1 are willing to pay a price strictly lower than v_1. Moreover, for all $x_t \in (z_1, z_2]$ and for all i with $f(i) = v_2$,

$$v_2 - P(x_t, i) = v_2 - x_t > E[e^{-r(\tau_1 - t)}(v_2 - x_{\tau_1})|\mathcal{F}_t] = E[e^{-r(\tau_1 - t)}(v_2 - P(x_{\tau_1}, q_{\tau_1}))|\mathcal{F}_t],$$

where the inequality follows since $\tau_2 = \inf\{t : x_t \leq z_2\}$ solves $\sup_\tau E[e^{-r\tau}(v_2 - x_\tau)|x_0 = x]$.

In a discrete-time version of this game, the strategy profile that buyers use in Example 1 can never be part of a SPE. To see this, suppose the game is in discrete-time and let $\Delta > 0$ be the time period. Consider a buyer with valuation v_k who expects to buy in the next period at a price equal to marginal cost. If current costs are x, today this consumer is willing to pay any price p such that $v_k - p > E[e^{-r\Delta}(v_k - x_{t+\Delta})|x_0 = x]$. Note that $v_k - x > E[e^{-r\Delta}(v_k - x_{t+\Delta})|x_0 = x]$ for x small. Therefore, when x is small this consumer finds it strictly optimal to buy at a price strictly larger than marginal cost.[13]

Intuitively, in a discrete-time game buyers incur a fixed cost of delay if they choose not to buy at the current price, since they must wait one time period for the seller to post a new price. This cost of delay limits the rents that buyers can get in an SPE. Indeed, low type buyers get a payoff of zero in any SPE of the discrete-time game (Lemma OA10 in the Online Appendix). Similarly, the rents that high type buyers get in discrete-time are equal to the payoff they get from mimicking low types (Lemma OA11 in the Online Appendix). In contrast, buyers do not face a fixed cost of delay when the game is in continuous-time. As a result, without imposing condition (iii) there would be equilibria in which the monopolist is completely unable to extract any rents from consumers.[14]

4 First-best outcome

To compute the first-best outcome, consider first the problem of choosing the surplus maximizing time at which to serve a homogeneous group of buyers with valuation v_k:

$$V_k(x) = \sup_{\tau \in T} E\left[e^{-r\tau}(v_k - x_\tau)\right| x_0 = x\right]. \tag{6}$$

Let λ_N be the negative root of $\frac{1}{2}\sigma^2\lambda(\lambda-1) + \mu\lambda = r$. Note that for any $y < x$, the expected discounted time until x_t reaches y when $x_0 = x$ is $(x/y)^{\lambda_N}$. Let $z_k = \frac{-\lambda_N}{1-\lambda_N} v_k$.

Lemma 1 *The stopping time $\tau_k = \inf\{t : x_t \leq z_k\}$ solves (6). Moreover,*

$$V_k(x) = \begin{cases} (v_k - z_k)\left(\frac{x}{z_k}\right)^{\lambda_N} & \text{if } x > z_k, \\ v_k - x & \text{if } x \leq z_k. \end{cases} \tag{7}$$

Proof. Appendix B.1. ■

Lemma 1 captures the option value that arises when costs vary stochastically over time.

[13] This also implies that, in an SPE, the monopolist will not charge price equal to marginal cost the next period, and so the price that consumers with valuation v_k are willing to pay is even higher.

[14] The fact that, without additional restrictions, the continuous-time game has equilibria that can never arise as limits of discrete-time SPE is related to the multiplicity of equilibria that arises in continuous-time bilateral bargaining games in which there are no restrictions on the timing of offers and counteroffers; see, for instance, Bergin and MacLeod (1993) and the discussion in Perry and Reny (1993), pp. 66-67.

The total surplus from serving a group of buyers with valuation v_k is maximized by waiting until costs fall below the threshold z_k. The threshold z_k is increasing in μ and decreasing in σ, so it is optimal to wait longer when costs fall faster or when they are more volatile. By Lemma 1, under the first-best outcome the monopolist serves consumers with valuation v_k at time τ_k. Proposition 1 summarizes this result.

Proposition 1 *Under the first-best outcome, the monopolist serves buyers with valuation v_k at time $\tau_k = \inf\{t : x_t \leq z_k\}$.*

Proposition 1 characterizes the first-best outcome when costs follow a geometric Brownian motion. Note that $\lim_{\sigma \to 0} z_k = \frac{r}{r-\mu} v_k$ whenever $\mu < 0$; that is, when $\mu < 0$ and $\sigma \to 0$ the outcome in Proposition 1 converges to the first-best outcome in Section 2. Finally, note that the first-best outcome can be implemented by choosing a stochastically decreasing path of prices that induces buyers to purchase at the optimal time. For example, consumers will buy at the efficient time if the seller prices at marginal cost at all points in time.

5 Markets with two types of consumers

In this section I study markets with two types of buyers. Section 5.1 characterizes the equilibrium. Section 5.2 discusses the most salient features of the equilibrium.

5.1 Equilibrium

Consider a market with two types of buyers, with values $v_2 > v_1 > 0$. Let $\alpha \in (0,1)$ be the fraction of high type buyers, so that $f(i) = v_2$ for all $i \in [0, \alpha]$ and $f(i) = v_1$ for all $i \in (\alpha, 1]$. By equation (4), consumers with valuation v_1 are willing to pay a price equal to v_1.

For any $q \in [\alpha, 1]$, let $\Pi(x, q)$ denote the monopolist's profits when the level of market penetration is q and costs are x. Note that at such a state only consumers with valuation v_1 remain in the market. Since all v_1-consumers are willing to pay v_1, at any state (x, q) with $q \geq \alpha$ the monopolist's problem is to optimally choose the time at which to sell to all remaining consumers; that is, $\Pi(x, q) = (1-q) \sup_\tau E[e^{-r\tau}(v_1 - x_\tau)| x_0 = x]$. By Lemma 1, the solution to this problem is $\tau_1 = \inf\{t : x_t \leq z_1\}$. Note then that, for all $q \in [\alpha, 1]$,

$$\Pi(x, q) = (1-q) V_1(x) = \begin{cases} (1-q)(v_1 - z_1) \left(\frac{x}{z_1}\right)^{\lambda_N} & \text{if } x > z_1, \\ (1-q)(v_1 - x) & \text{if } x \leq z_1. \end{cases}$$

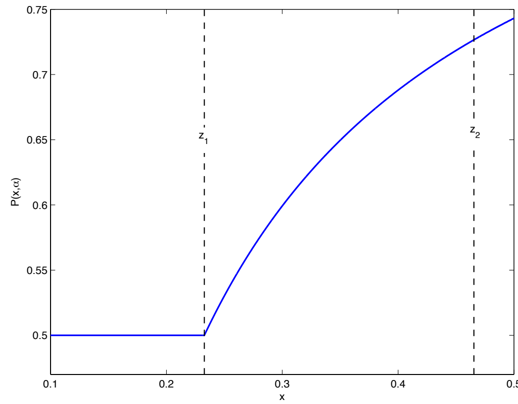

Figure 1: Strategy $P(x, \alpha)$ of consumer α;
Parameters: $v_1 = \frac{1}{2}$, $v_2 = 1$, $\mu = -0.02$, $\sigma = 0.2$ and $r = 0.05$.

Consider next a level of market penetration $q \in [0, \alpha)$. Recall that $P(x, \alpha)$ is the strategy of consumer α, the last consumer with valuation v_2. After consumer α buys, the monopolist sells to low types when costs fall below z_1 at price v_1. By equation (5), $P(x, \alpha)$ satisfies

$$P(x, \alpha) = v_2 - E\left[e^{-r\tau_1}(v_2 - v_1)\middle| x_0 = x\right]. \tag{8}$$

Figure 1 plots $P(x, \alpha)$. When $x_t > z_1$, consumer α knows that the monopolist won't lower her price to v_1 until costs fall to z_1, so she is willing to pay strictly more than v_1. On the other hand, when $x_t \leq z_1$ consumer α knows that the monopolist will serve types immediately after she buys, so she is only willing to pay v_1.

For future reference, note that $P(x, \alpha)$ has a kink at z_1. This kink is related to the seller's incentives to delay trade with low types. For levels of costs below z_1, the seller always sells to low types immediately after high types leave the market. As a result, in this range of costs, small changes in x don't affect the price that high types are willing to pay. In contrast, for costs weakly above z_1 an increase in costs strictly increases the expected waiting time until the monopolist serves low types. Therefore, $P(x, \alpha)$ is strictly increasing in this range.

Lemma 2 *$P(x, \alpha) - x > V_1(x)$ for all $x \in (z_1, z_2]$. Moreover,*

$$P(x, \alpha) = \begin{cases} v_2 - (v_2 - v_1)\left(\frac{x}{z_1}\right)^{\lambda_N} & x > z_1, \\ v_1 & x \leq z_1, \end{cases} \tag{9}$$

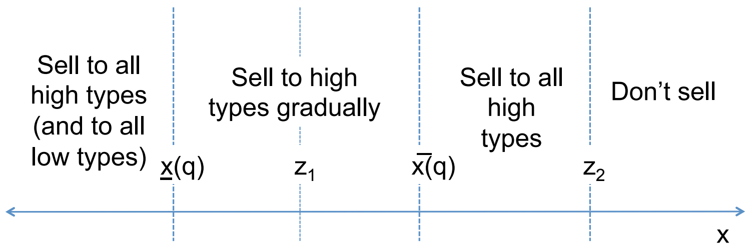

Figure 2: Description of seller's equilibrium strategies.

where λ_N *is the negative root of* $\frac{1}{2}\sigma^2\lambda(\lambda-1)+\mu\lambda=r$*.*

Proof. Appendix B.1. ■

Since the strategy profile of the buyers satisfies the skimming property, $P(x,i)\geq P(x,\alpha)$ for all x and all $i<\alpha$. This implies that for $q<\alpha$, the monopolist has the following strategy available: choose optimally when to sell to all remaining high type consumers at price $P(x_t,\alpha)$, and then play the continuation game optimally. Therefore, for all states (x,q) with $q\in[0,\alpha)$, the monopolist's profits are bounded below by

$$L(x,q)=\sup_{\tau\in T}E\left[e^{-r\tau}\left[(\alpha-q)\left(P(x_\tau,\alpha)-x_\tau\right)+\Pi(x_\tau,\alpha)\right]\Big|\,x_0=x\right]. \tag{10}$$

Theorem 1 *There exists a unique equilibrium. In equilibrium, at every state* (x,q) *with* $q\in[0,\alpha)$ *the monopolist's profits are* $L(x,q)$*. Moreover, for all* $t\geq 0$ *with* $q_{t^-}<\alpha$*, there exists* $\underline{x}(q_{t^-})<z_1$ *and* $\overline{x}(q_{t^-})\in(z_1,z_2)$ *such that*

(i) if $x_t>z_2$*, the monopolist doesn't sell;*

(ii) if $x_t\in[\overline{x}(q_{t^-}),z_2]$*, the monopolist sells to all remaining high type consumers at price* $P(x_t,\alpha)$*, so* $dq_t=\alpha-q_{t^-}$*;*

(iii) if $x_t\leq\underline{x}(q_{t^-})$*, the monopolist sells to all remaining consumers (high and low types) at price* v_1*, so* $dq_t=1-q_{t^-}$*;*

(iv) if $x_t\in(\underline{x}(q_{t^-}),\overline{x}(q_{t^-}))$*, the monopolist sells gradually to high type consumers at price*

$$P(x_t,q_t)=x_t-L_q(x_t,q_t), \tag{11}$$

and q_t evolves according to

$$\frac{dq_t}{dt} = \frac{r\left(v_2 - x_t\right) + \mu x_t}{L_{qq}\left(x_t, q_t\right)} > 0. \tag{12}$$

Proof. Appendix A.1. ■

Theorem 1 shows that the monopolist's equilibrium profits are exactly equal to the lower bound $L(x,q)$. Figure 2 summarizes the monopolist's equilibrium strategies.[15]

The force that drives the monopolist's profits down to the lower bound $L(x,q)$ is her inability to commit to future prices. As in the standard model with time-invariant costs, the monopolist has the temptation to accelerate trade whenever the prices that the buyers are willing to pay are too high. To avoid this temptation, the price that the marginal high type buyer is willing to pay when costs are in $\left(\underline{x}\left(q\right), \overline{x}\left(q\right)\right)$ is such that the monopolist is indifferent between selling to high type buyers at any rate. The prices in equation (11) are determined by this indifference condition of the seller.

At the same time, in equilibrium all high type buyers must obtain the same expected payoff. Since high types buy at different points in time when costs are in $\left(\underline{x}\left(q\right), \overline{x}\left(q\right)\right)$, prices must fall in expectation at a rate that leaves these consumers indifferent between buying now and waiting. This indifference condition determines the rate (12) at which the monopolist sells to high types when costs are in this region.

The equilibrium is efficient when costs are initially large. The intuition for this is the same as in the example in Section 2. The profit margin the seller gets from serving all high types at cost x is $P(x,\alpha) - x = v_2 - x - E[e^{-r\tau_1}(v_2 - v_1)|x_0 = x]$. When x_0 is large, the rents $E[e^{-r\tau_1}(v_2 - v_1)|x_0 = x]$ that high types get are independent of the time at which the seller serves them. As a result, it is optimal for the seller to serve high types at the efficient time.

On the other hand, the equilibrium is inefficient when $x \in (\underline{x}(q), \overline{x}(q))$. This feature of the equilibrium can be best understood by studying the properties of the solution to the optimal stopping problem in (10). Lemma B2 in the appendix shows that the solution to this optimal stopping problem consists of delaying trade with high types when costs are in $(\underline{x}(q), \overline{x}(q))$; that is, for all x in $(\underline{x}(q), \overline{x}(q))$, the monopolist prefers to completely delay trade with high types than to sell to all of them immediately.

[15] There are two ways of interpreting the gradual sales when $x_t \in \left(\underline{x}\left(q_{t^-}\right), \overline{x}\left(q_{t^-}\right)\right)$. The first one is to assume, as I do, that different consumers with valuation v_2 use different strategies, and therefore buy at different points in time. An alternative interpretation is that buyers with valuation v_2 mix between buying and not buying when $x_t \in \left(\underline{x}\left(q_{t^-}\right), \overline{x}\left(q_{t^-}\right)\right)$.

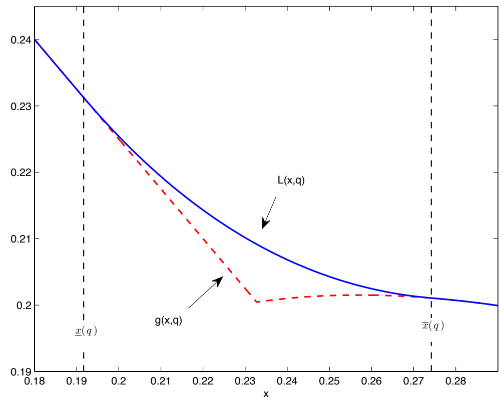

Figure 3: Equilibrium profits $L(x, q)$;
Parameters: $v_1 = \frac{1}{2}$, $v_2 = 1$, $\alpha = 0.7$, $\mu = -0.02$, $\sigma = 0.25$ and $r = 0.05$.

To see why, for all $q < \alpha$ and all x let $g\,(x, q)$ be the seller's profits from selling to all high types immediately at state (x, q); i.e., $g\,(x, q) = (\alpha - q)(P\,(x, \alpha) - x) + \Pi(x, \alpha)$. Note that $L\,(x, q) = \sup_{\tau \in T} E[e^{-r\tau} g\,(x_\tau, q)|\, x_0 = x]$. Since $P\,(x, \alpha)$ has a convex kink at z_1, $g\,(x, q)$ also has a convex kink at z_1. As a result, when x is close to z_1 the monopolist can obtain larger profits by delaying trade with high types than by serving all of them immediately. Intuitively, the seller benefits from delaying trade when costs are close to z_1, since an increase in costs allows her to extract more rents from high types. However, delaying trade with high types is also costly for the seller, since this means that she must also delay trade with low types. The cutoffs $\underline{x}(q)$ and $\overline{x}(q)$ are the points at which the benefits from delaying trade by an instant are equal to the costs. Figure 3 plots $L(x, q)$ and $g(x, q)$ and illustrates the gains the seller gets by waiting when $x \in (\underline{x}\,(q)\,, \overline{x}\,(q))$.

Since $L(x, q) > g(x, q)$ for all $x \in (\underline{x}(q), \overline{x}(q))$, the monopolist does not sell to all remaining high types immediately when costs are in this range. However, instead of delaying trade completely with high types, the monopolist sells to them gradually and attains the same profits as if she delayed. Indeed, not making any sales when $x \in (\underline{x}(q), \overline{x}(q))$ cannot be equilibrium behavior. To see why, suppose that the monopolist doesn't make any sales when costs are in $(\underline{x}(q), \overline{x}(q))$. Since this delay is inefficient, there exist a price at which consumers and monopolist would strictly prefer to trade immediately than to wait; offering such a price constitutes a profitable deviation for the monopolist.

I end this section by providing a brief sketch of the arguments I use to show that the seller's profits are equal to $L(x, q)$. To establish this, I start by showing that when $q < \alpha$, the monopolist makes sales at a positive rate if and only if her costs are below z_2 (Lemma B8). If the monopolist sells to all remaining high types, her profits are $(\alpha-q)(P(x,\alpha)-x)+\Pi(x,\alpha) \leq L(x,q)$. In this case the monopolist earns exactly $L(x,q)$, since this is a lower bound to profits. If instead the monopolist sells to some remaining high types, the price that she charges them cannot be too high; otherwise, the monopolist would have a temptation to accelerate trade. Indeed, the proof of Theorem 1 shows that, when the monopolist sells to a fraction of the remaining high types, the price that she charges must leave her indifferent between making these sales or waiting and selling to all remaining high-type consumers at a future date τ (Lemmas B12-B14). Letting $\Pi(x,q)$ be the seller's equilibrium payoff at state (x,q), this implies that $\Pi(x,q) = E[e^{-r\tau}((\alpha-q)(P(x_\tau,\alpha)-x_\tau)+\Pi(x_\tau,\alpha))] \leq L(x,q)$. Since equilibrium profits are bounded below by $L(x,q)$, it must be that $\Pi(x,q) = L(x,q)$.

5.2 Features of the equilibrium

In this section I present the most salient features of the equilibrium. I start by discussing how the equilibrium outcome relates to the results on the Coase conjecture. In markets with time-invariant costs the Coase conjecture predicts that the monopolist will post an opening price equal to the lowest valuation. All buyers trade immediately at this price, the market outcome is efficient, and the seller earns the same profits she would have earned if all buyers in the market had the lowest valuation.

With time-varying costs, selling to all consumers immediately is in general not efficient. By Proposition 1, efficiency requires the monopolist to serve the different types of consumers sequentially as costs decrease. On the other hand, the monopolist's profits would be $V_1(x) = \sup_\tau E[e^{-r\tau}(v_1 - x_\tau)|\, x_0 = x]$ if all consumers had the lowest valuation, since in this case she would sell to all buyers at a price equal to v_1. This observation suggests the following generalization of the Coase conjecture for markets with time-varying costs:

Definition 2 *Say that an outcome is* Coasian *if (i) it is efficient, and (ii) the monopolist's profits are equal to $V_1(x_0)$.*

Under a Coasian outcome the monopolist is unable to extract more rents from buyers with higher valuations than what she extracts from buyers with valuation v_1. Note that $V_1(x_0) \to 0$ as $v_1 \to 0$: under a Coasian outcome the seller's profits converge to zero as the lowest valuation goes to zero.

The equilibrium outcome is not Coasian when there are two types of buyers in the market. First, the monopolist extracts rents from high type buyers: by Lemma 2, $P(x,\alpha)-x > V_1(x)$

for all $x \in (z_1, z_2]$, so $L(x, 0) > V_1(x)$ for all $x \in (z_1, z_2]$. Second, the equilibrium outcome is inefficient when $x_0 \in (\underline{x}(0), \overline{x}(0))$. The following result summarizes this discussion.

Corollary 1 *With two types of consumers, the equilibrium outcome is not Coasian.*

A way to measure the size of the rents that the monopolist extracts from high type buyers is to compare her profits $L(x, 0)$ to the profits $\Pi^{FC}(x)$ she would earn if she could commit to a path of prices. In the Online Appendix I show that $\Pi^{FC}(x) = E[e^{-r\tau_2}\alpha(v_2 - x_\tau)| x_0 = x]$ when $\alpha v_2 > v_1$. That is, under full commitment the monopolist finds it optimal to sell only to high types when the share of high types is large. The following result shows that a monopolist with time-varying costs may get profits close to full-commitment profits.

Proposition 2 *For all $x > 0$, equilibrium profits $L(x, 0)$ converge to full commitment profits $\Pi^{FC}(x)$ as $v_1 \to 0$.*

Proof. Appendix B.4. ∎

Intuitively, the monopolist effectively commits not to serve low types when v_1 goes to zero. Indeed, when costs evolve as a geometric Brownian motion, the optimal time to sell to v_1 consumers becomes unboundedly large as v_1 goes to zero. As a result, the monopolist can extract all the rents from high types. While Proposition 2 depends crucially on the geometric Brownian motion specification for costs, the idea that a monopolist with time-varying costs may extract more rents from high types as v_1 decreases is robust to other cost specifications.

Another salient feature of the equilibrium is that the price that the monopolist charges at time $t > 0$ may depend upon the history of costs. To see this, suppose that $x_0 \in (\underline{x}(0), \overline{x}(0))$ and let $\tau = \inf\{t : x_t \notin (\underline{x}(q_t), \overline{x}(q_t))\}$. By equation (12), the rate at which the monopolist sells at time $s \in [0, \tau)$ depends on the current cost x_s and on the current level of market penetration q_s. Therefore, for all $t \in [0, \tau)$ the level of market penetration $q_t = \int_0^t dq_s$ depends upon the path of costs from time zero to t; and so the price $P(x_t, q_t)$ that the seller charges at time t also depends upon the history of costs. Figure 4 plots the path of prices and the evolution of market penetration for a path of costs with $x_0 \in (\underline{x}(0), \overline{x}(0))$ (the dashed lines in the top-left figure are the quantities $\underline{x}(q_t)$ and $\overline{x}(q_t)$).[16]

The next result characterizes the evolution of prices in this market.

[16] Under the parameters in Figure 4 the efficient cutoffs are $z_1 \approx 0.23$ and $z_2 \approx 0.47$. Therefore, under the cost path in Figure 4 the efficient outcome would be to sell to all high types immediately.

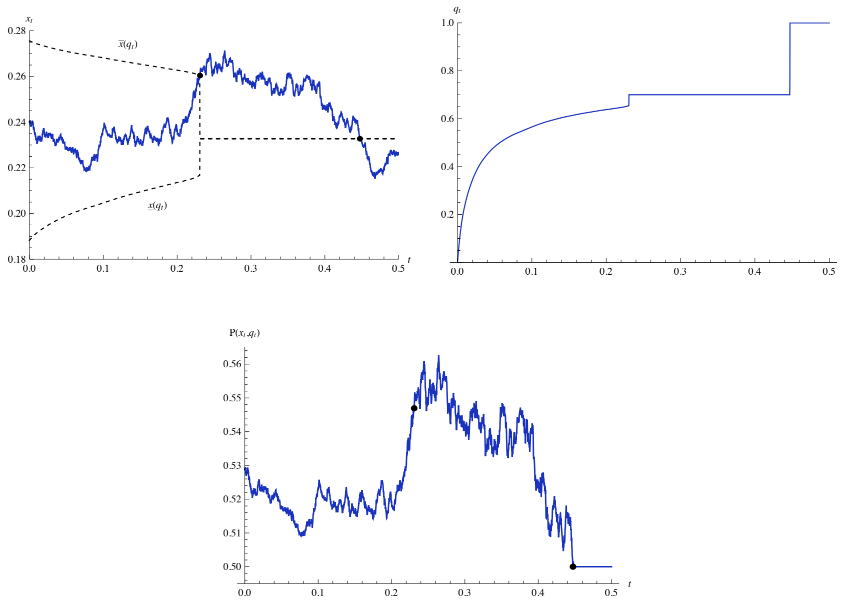

Figure 4: Sample path of costs x_t (top, left), and its associated paths of market penetration q_t (top, right) and of prices $P(x_t, q_t)$ (bottom).
Parameters: $v_1 = 0.5$, $v_2 = 1$, $\alpha = 0.7$, $\mu = -0.02$, $\sigma = 0.2$ and $r = 0.05$.

Proposition 3 *For all t such that $q_t < \alpha$ and $x_t \in (\underline{x}(q_t), \overline{x}(q_t))$, there exists γ_t such that*

$$dP(x_t, q_t) = -r(v_2 - P(x_t, q_t))dt - \gamma_t dB_t. \tag{13}$$

Proof. Appendix B.4. ■

The expected rate at which prices fall in (13) is determined by the indifference condition of high type buyers: when prices fall at rate $-r(v_2 - P(x, q))$, high types remain indifferent between buying now or delaying trade by an instant. Note that equation (13) can be used to estimate the valuation of high types from observed evolution of prices. Moreover, since $P(x, q)$ is increasing in x, by equation (13) prices fall faster when costs are smaller. Lastly, equation (13) implies that $d(P(x_t, q_t) - x_t) = (-r(v_2 - P(x_t, q_t)) - \mu x_t)dt + \beta_t dB_t$ for some β_t. Since $v_2 > P(x_t, q_t)$, the seller's profit margin falls over time when the rate at which costs fall is not too large. This is consistent with the evidence in Zhao (2008) and Conlon (2012), who show that prices fall faster than costs in high-tech markets.

The next result studies settings in which costs fall deterministically over time by analyzing the limiting properties of the equilibrium when $\mu < 0$ and $\sigma \to 0$. Recall from Sections 2 and 4 that the efficient outcome in this case is for the monopolist to serve consumers with valuation v_k the first time costs fall below $\frac{r}{r-\mu}v_k$.

Proposition 4 *Suppose $\mu < 0$. Then, in the limit as $\sigma \to 0$ the market outcome becomes efficient: the monopolist sells to all buyers with valuation v_2 the first time costs fall below $\frac{r}{r-\mu}v_2$ and sells to all buyers with valuation v_1 the first time costs fall below $\frac{r}{r-\mu}v_1$.*

Proof. Appendix B.4. ■

The intuition behind Proposition 4 is as follows. When $\mu < 0$ and $\sigma > 0$ there is positive probability that costs will go up if the monopolist delays trade with high type buyers. Since the price $P(x, \alpha)$ is increasing in x, this increase in costs would allow the monopolist to extract more rents from high types. This gives rise to an option value of delaying trade. However, the probability that costs will increase becomes negligible when $\mu < 0$ and $\sigma \to 0$. As a result, inefficiently delaying trade with high types is no longer profitable in this limiting case. These results suggest that we should expect to see delays and inefficiencies in markets in which costs are subject to significant stochastic shocks. One such example is firms that act as intermediaries of durable commodities, like steel, who face large and rapid variations in costs, and who therefore have strong incentives to engage in pricing speculation.[17]

The last result of this section studies the limiting properties of the equilibrium as the drift and volatility of costs converge to zero. For any $x \in [0, v_2]$, let $p(x)$ denote the lowest consumer valuation that is larger than x: $p(x) = v_1$ if $x \leq v_1$ and $p(x) = v_2$ if $x \in (v_1, v_2]$.

Proposition 5 *Suppose $x_0 \leq v_2$. Then, as $(\sigma, \mu) \to (0, 0)$ the monopolist sells at $t = 0$ to all consumers with valuation larger than x_0 at a price $p(x_0)$.*

Proof. Appendix B.4. ■

Proposition 5 shows that the market outcome converges to the standard Coase conjecture outcome when costs become time-invariant: if costs are initially below v_1 the monopolist's opening price converges to v_1 as $(\sigma, \mu) \to (0, 0)$ and all buyers trade immediately.

6 Markets with $n > 2$ types of consumers

This section generalizes the results in Section 5 to settings in which the function $f : [0, 1] \to [\underline{v}, \overline{v}]$ describing the valuations of the consumers takes $n > 2$ values $v_1 < ... < v_n$. For $k = 1, ..., n$, let $\alpha_k = \max\{i \in [0, 1] : f(i) = v_k\}$ be the last v_k-consumer. Let $\alpha_{n+1} = 0$.

[17] See Hall and Rust (2000) for high-frequency cost and pricing data of an intermediary seller of steel.

As a first step, note that at states (x,q) with $q \in [\alpha_3, 1)$ there are one or two types of buyers in the market: buyers with valuation v_1 and, if $q < \alpha_2$, buyers with valuation v_2. Thus, for states (x,q) with $q \in [\alpha_3, 1)$ the equilibrium is the one derived in Section 5.

Consider next states (x,q) with $q \in [\alpha_4, \alpha_3)$, at which there are $\alpha_3 - q$ buyers with valuation v_3 remaining in the market. By equation (5), it must be that $P(x, \alpha_3) = v_3 - E[e^{-r\tau_2}(v_3 - P_2(x_{\tau_2}, q_{\tau_2}))| x_0 = x]$, where $\tau_2 = \inf\{t : x_t \leq z_2\}$ is the time at which the monopolist starts selling to consumers with valuation v_2 when all consumers with valuation v_3 have left the market. The skimming property implies that $P(x,i) \geq P(x,\alpha_3)$ for all $i \leq \alpha_3$, so the monopolist can sell to all buyers with valuation v_3 at price $P(x, \alpha_3)$. Therefore, at states (x,q) with $q \in [\alpha_4, \alpha_3)$ the seller's profits are bounded below by

$$L(x,q) = \sup_{\tau \in T} E\left[e^{-r\tau}\left((\alpha_3 - q)(P(x_\tau, \alpha_3) - x_\tau) + L(x_\tau, \alpha_3)\right)\middle| x_0 = x\right], \tag{14}$$

where $L(x, \alpha_3)$ are the seller's profits at state (x, α_3).

Following the same steps as in the derivation of (14), I can extend $L(x,q)$ to all $q \in [0,1]$ in a way such that, for $k = 2, ..., n$ and all $q \in [\alpha_{k+1}, \alpha_k)$,

$$L(x,q) = \sup_{\tau \in T} E\left[e^{-r\tau}\left((\alpha_k - q)(P(x_\tau, \alpha_k) - x_\tau) + L(x_\tau, \alpha_k)\right)\middle| x_0 = x\right], \tag{15}$$

where $P(x, \alpha_k)$ is the price at which the monopolist can sell to all consumers with valuation v_k, and $L(x, \alpha_k)$ is the lower bound to the monopolist's profits at state (x, α_k). For $q \in [\alpha_2, 1]$, let $L(x,q) = (1-q)V_1(x)$.

Theorem 2 *Suppose f is a step function taking $n > 2$ values. Then, there exists a unique equilibrium. In equilibrium, the monopolist's profits are equal to $L(x,q)$ at every state (x,q).*

Proof. Online Appendix. ■

Figure 5 plots $L(x,q)$ for an environment with three types of consumers. For all $q < \alpha_3$, let $g(x,q)$ be the profits that the monopolist earns from selling to all consumers with valuation v_3 immediately when costs are x and market penetration is q:

$$\begin{aligned} g(x,q) &= (\alpha_3 - q)(P(x,\alpha_3) - x) + L(x,\alpha_3) \\ \Rightarrow L(x,q) &= \sup_{\tau} E[e^{-r\tau} g(x_\tau, q) | x_0 = x]. \end{aligned}$$

The solution to this stopping problem involves delaying when x_t lies in $(\underline{x}(q,1), \overline{x}(q,1)) \cup$

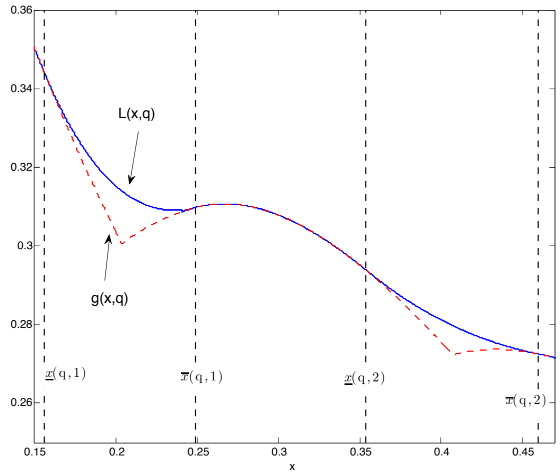

Figure 5: Lower bound to profits $L(x, q)$;
Parameters: $v_1 = \frac{1}{2}$, $v_2 = 1$, $v_3 = \frac{3}{2}$, $\alpha_2 = \frac{17}{20}$, $\alpha_3 = \frac{1}{2}$, $\mu = -0.02$, $\sigma = 0.25$ and $r = 0.05$.

$(\underline{x}(q,2), \overline{x}(q,2))$ or when $x_t > z_3$, and stopping otherwise.[18] In equilibrium, when x is in the delay region $(\underline{x}(q,1), \overline{x}(q,1)) \cup (\underline{x}(q,2), \overline{x}(q,2))$ the monopolist sells gradually to buyers with valuation v_3 at price $P(x,q) = x - L_q(x,q)$. If $x > z_3$, the seller waits until costs fall to z_3, and at this point sells to all buyers with valuation v_3 at price $P(x, \alpha_3)$. Finally, if x lies in the stopping region, the seller serves all remaining buyers with valuation v_3 at price $P(x, \alpha_3)$.

As in the model with two types, in this setting the monopolist is also able to extract rents from buyers with higher valuations. Indeed, arguments similar to those in Lemma 2 imply that, for all $k \geq 2$, $P(x, \alpha_k) - x > V_1(x)$ for all $x \in (z_1, z_k]$. Since the monopolist can sell to all consumers with valuation v_k and higher at a price of $P(x, \alpha_k)$, it follows that $L(x,q) > (1-q)V_1(x)$ for all $x \in (z_1, z_k]$ and $q < \alpha_2$. Moreover, the equilibrium outcome is inefficient: when $x_0 < z_n$ lies in the delay region of (15) the efficient outcome is to serve all buyers with valuation v_n immediately, but the monopolist serves them gradually.

7 Markets with a continuum of types

In this section I study markets in which the buyers' valuations are described by a continuous and decreasing function $h : [0,1] \to \mathbb{R}_+$, with $h(0) = \overline{v} > \underline{v} = h(1) > 0$. I study such markets by considering a sequence of discrete models $\{f^j\} \to h$, where $f^j : [0,1] \to [\underline{v}, \overline{v}]$ takes finitely

[18]In Figure 5, the quantities of $z_1 \in (\underline{x}(q,1), \overline{x}(q,1))$ and $z_2 \in (\underline{x}(q,2), \overline{x}(q,2))$ are the values of x at which $g(x,q)$ has kinks.

many values for each $j = 1, 2, ...$ For all j, f^j satisfies the assumptions in Section 3.

Given such a sequence $\{f^j\}$, for each $j = 1, 2, ...$ let $L^j(x, q)$ denote the monopolist's profits at state (x, q) in an environment in which the valuations of the consumers are described by f^j. For each $v \in [\underline{v}, \overline{v}]$, let $V_v(x) = \sup_\tau E[e^{-r\tau}(v - x_\tau)| x_0 = x]$, $z_v = \frac{-\lambda_N}{1-\lambda_N}v$ and $\tau_v = \inf\{t : x_t \leq z_v\}$. Note that the seller's profits would be $V_{\underline{v}}(x)$ if all buyers had value $\underline{v}$.

Theorem 3 *Fix a sequence of step functions $\{f^j\}$ such that $\{f^j\} \to h$. Then, the market outcome becomes Coasian as $j \to \infty$: the monopolist's profits converge to $V_{\underline{v}}(x)$ and the limiting market outcome is efficient.*

Proof. Appendix A.2. ■

To see intuition behind Theorem 3, consider first a setting with two types of buyers: high types with valuation $\overline{v}$, and low types with valuation $\underline{v}$. After high types buy, the monopolist can commit to keep high prices until costs fall below $z_{\underline{v}} = \frac{-\lambda_N}{1-\lambda_N}\underline{v}$. High type buyers know that prices won't fall to $\underline{v}$ until x_t falls below $z_{\underline{v}}$, so they are willing to pay higher prices when costs are above $z_{\underline{v}}$. Consider next a setting with three types of buyers, with valuations $\overline{v}$, $(\overline{v}+\underline{v})/2$ and $\underline{v}$. In this setting, after all consumers with valuation $\overline{v}$ buy, the monopolist can only commit not to cut prices until costs fall below $\frac{-\lambda_N}{1-\lambda_N}\frac{\overline{v}+\underline{v}}{2}$, since at this point it becomes optimal for her to sell to buyers with intermediate valuation. This puts a limit to the price buyers with valuation $\overline{v}$ are willing to pay, since they can now wait for costs to fall to $\frac{-\lambda_N}{1-\lambda_N}\frac{\overline{v}+\underline{v}}{2}$ and get the good at a lower price.

More generally, the proof of Theorem 3 shows that the price consumers are willing to pay decreases as the set of values becomes dense. In the limit, the rents that the monopolist extracts from high type buyers is the same as the expected discounted rent that the monopolist obtains from consumers with the lowest valuation $\underline{v}$. As a result of this, the monopolist no longer profits from inefficiently delaying trade, and the market outcome becomes efficient.

Why do the monopolist's profits converge to $V_{\underline{v}}(x)$? The reason is the same as in the example in Section 2. Since the market outcome is efficient, the total surplus from selling to a buyer with valuation v is $V_v(x)$. By the Envelope Theorem, $\frac{\partial V_v(x)}{\partial v} = E[e^{-r\tau_v}|x_0 = x]$, and so $V_v(x) = V_{\underline{v}}(x) + \int_{\underline{v}}^{v} E[e^{-r\tau_{\tilde{v}}}|x_0 = x]d\tilde{v}$. Out of this total surplus, a consumer with valuation v gets rents equal to $\int_{\underline{v}}^{v} E[e^{-r\tau_{\tilde{v}}}|x_0 = x]d\tilde{v}$ and the seller gets $V_{\underline{v}}(x)$; i.e., when costs are x, she charges price $V_{\underline{v}}(x) + x$ and earns a profit margin of $V_{\underline{v}}(x)$.[19]

[19] Unlike the setting with discrete types, in this case the monopolist's profit margin increases in expectation over time. Indeed, for all $x_t > z_{\underline{v}}$, $dV_{\underline{v}}(x_t) = (\mu x_t V'_{\underline{v}}(x_t) + \frac{1}{2}\sigma^2 x_t^2 V''_{\underline{v}}(x_t))dt + \sigma x_t V'_{\underline{v}}(x_t)dB_t = rV_{\underline{v}}(x_t)dt + \sigma x_t V'_{\underline{v}}(x_t)dB_t$, where I used $rV_{\underline{v}}(x) = \mu x V'_{\underline{v}}(x) + \frac{1}{2}\sigma^2 x^2 V''_{\underline{v}}(x)$ for all $x > z_{\underline{v}}$ (see proof of Lemma 1).

When costs are time-invariant, the literature on the Coase conjecture refers to the difference between the lowest consumer valuation and the monopolist's cost as the gap. When costs don't change over time, the market outcome becomes competitive as the gap goes to zero: the monopolist sets a price equal to marginal cost and earns zero profits. In this paper's setting the lowest valuation $\underline{v}$ measures the gap. Note that $V_{\underline{v}}(x) \to 0$ as $\underline{v} \to 0$. Therefore, with a continuum of types the market outcome converges to the perfectly competitive outcome as the gap goes to zero: in the limit as $\underline{v} \to 0$ the monopolist charges marginal cost and earns zero profits. The following corollary summarizes this discussion.

Corollary 2 *As $\underline{v} \to 0$, the monopolist sells at marginal cost and earns zero profits.*

I now turn to study settings in which costs fall deterministically over time. I do this by analyzing the limiting properties of the market outcome when $\mu < 0$ and $\sigma \to 0$. Since the market outcome is efficient with a continuum of types, in the limit as $\sigma \to 0$ the monopolist sells to consumers with valuation v the first time costs fall below $z_v^* = \lim_{\sigma\to 0} z_v = \frac{r}{r-\mu} v$ and the seller's profits converges to $V_{\underline{v}}^*(x) = \lim_{\sigma\to 0} V_{\underline{v}}(x) = (\underline{v} - z_{\underline{v}}^*)\left(x/z_{\underline{v}}^*\right)^{\frac{r}{\mu}}$; i.e., the equilibrium outcome converges to the outcome in Section 2. The next corollary summarizes these results.

Corollary 3 *Suppose $\mu < 0$ and $x_0 > \underline{v}$. Then, as $\sigma \to 0$ the monopolist serves consumers with valuation v the first time costs fall below $\frac{r}{r-\mu} v$ and the seller's profit converges to $V_{\underline{v}}^*(x_0)$.*

Remark 1 The assumption that the function $h : [0, 1] \to [\underline{v}, \overline{v}]$ is continuous implies that the distribution of consumer valuations has convex support. This assumption is crucial for Theorem 3. Indeed, suppose that there is a gap in the support of the distribution of valuations; for example, suppose $h(i) = 1 - \gamma i - \beta \mathbf{1}_{i>\hat{q}}$ for some $\hat{q} \in (0, 1)$, where $\gamma \in (0, 1)$ and $\beta \in (0, 1-\gamma)$. In such a setting, after selling to all consumers $i \leq \hat{q}$ the monopolist won't reduce her price until costs fall below $\frac{-\lambda_N}{1-\lambda_N} h(\hat{q}^+)$. This allows the seller to extract rents from buyers with valuation larger than $h(\hat{q})$ when costs are above $\frac{-\lambda_N}{1-\lambda_N} h(\hat{q}^+)$. Moreover, this ability of the monopolist to extract rents creates a wedge between profit maximization and efficiency, just as in the two types case of Section 5. As a result, the market outcome fails to be efficient in this setting. Finally, note that the seller's ability to extract rents depends on the size of the discontinuity at $\hat{q}$; i.e., depends on how large β is. As $\beta \to 0$ the gap in the set of valuations vanishes and the monopolist again losses her ability to extract rents. On the other hand, note that the model of Section 5 can be approximated by letting $\gamma \to 0$.

Remark 2 This section studies markets with a continuum of consumer types by taking limits of discrete type models. It is worth noting that the limiting equilibrium that obtains is also an equilibrium of the game with a continuum of types. Indeed, under this equilibrium the monopolist charges price $p_t = V_{\underline{v}}(x_t) + x_t$ at each point in time. Given this path of prices, buyers find it optimal to purchase at the efficient time.[20] Theorem 3 shows that this equilibrium of the game with a continuum of types is the unique limiting equilibrium of "close-by" games with discrete types.

8 Other cost processes

This section discusses how the results in the paper extend to settings in which costs follow a (continuous-time) Markov chain. For concision, I consider the case in which the Markov chain takes values $x_L \geq 0$ and $x_H > x_L$. I show that the main results of the paper continue to hold under this cost process: (i) with discrete types the monopolist earns rents and there are inefficiencies if costs are likely increase; (ii) with a continuum of types the monopolist does not earn rents, and the market outcome is efficient.

First-best. Let v^* be the value that solves $v^* - x_H = E[e^{-r\tau_L}(v^* - x_L)|x_0 = x_H]$, where $\tau_L = \inf\{t : x_t = x_L\}$. Note that the first-best outcome in this setting is to serve buyers with value $v \geq v^*$ when costs are x_H, and to serve buyers with value $v \leq v^*$ when costs are x_L.

Markets with two types of buyers. Suppose that there are two types of buyers, with valuations $v_H > v^*$ and $v_L \in (x_L, v^*)$. If costs are initially x_H, in continuous-time the monopolist serves all high type buyers immediately, and then sells to low types at time $\tau_L = \inf\{t : x_t = x_L\}$ at price v_L. The price the seller charges high types when $x = x_H$ is $P_H = v_H - E[e^{-r\tau_L}(v_H - v_L)|x_0 = x_H]$, which leaves high types indifferent between buying when costs are x_H or waiting and buying together with low types. The seller extracts rents from high types when costs are x_H, since she can commit not to cut prices until costs fall.[21]

Suppose next that the initial level of costs is x_L. If the monopolist sells to all high types immediately she has to charge them a price equal to v_L, since high types know that

[20] Formally, consumers use the following strategy profile under this equilibrium. Consumer $i \in [0,1]$ with valuation $h(i) > \underline{v}$ uses strategy $P(x,i) = h(i) - \sup_\tau E[e^{-r\tau}(h(i) - V_{\underline{v}}(x_\tau) + x_\tau)|x_0 = x]$, and consumer $i \in [0,1]$ with valuation $h(i) = \underline{v}$ uses strategy $P(x,i) = \underline{v}$. It can be shown that the solution to $\sup_\tau E[e^{-r\tau}(h(i) - V_{\underline{v}}(x_\tau) + x_\tau)|x_0 = x]$ is $\tau_{h(i)} = \inf\{t : x_t \leq z_{h(i)}\}$, so consumers find it optimal to buy at the efficient time. Note that $P(x,i) \leq V_{\underline{v}}(x) + x$ for all x, with strict inequality for all $x > z_{h(i)}$. Therefore, given this strategy profile the monopolist finds it optimal to charge price $p_t = V_{\underline{v}}(x_t) + x_t$.

[21] Indeed, the profit margin the monopolist gets from high types when costs are x_H is strictly larger than the expected discounted profit margin she gets from low types: $P_H - x_H - E[e^{-r\tau_L}(v_L - x_L)|x_0 = x_H] = v_H - x_H - E[e^{-r\tau_L}(v_H - x_L)|x_0 = x_H] > 0$.

the monopolist would then sell to low types immediately after they all buy. However, the monopolist has the option of waiting until costs go up to x_H and then selling to all high types at price P_H. If the fraction of high types remaining in the market is large enough, if the expected waiting time until costs increase to x_H is short and if the price P_H is large enough compared to v_L, the monopolist would find it more profitable to delay trade with high types when costs are x_L than to sell to all of them at price v_L. In this case, in equilibrium the monopolist sells to high types gradually over time when costs are initially low, and the equilibrium outcome is inefficient.

Markets with a continuum of types. Consider now a setting with a continuum of types $[\underline{v}, \overline{v}]$, with $v^* \in (\underline{v}, \overline{v})$ and $\underline{v} > x_L$. When costs are initially x_H, in continuous-time the monopolist sells to all buyers with valuation $v \in [v^*, \overline{v}]$ immediately at $t = 0$, and then sells to the rest of the buyers at time τ_L (charging them price $\underline{v}$). The price that the monopolist charges when costs are x_H is equal to $P_{v^*} = v^* - E[e^{-r\tau_L}(v^* - \underline{v})|x_0 = x_H]$, which leaves the marginal consumer v^* indifferent between buying now or getting the good at time τ_L at price $\underline{v}$. The profit margin that the monopolist earns on consumers with valuation above v^* when costs are x_H is $P_{v^*} - x_H = v^* - x_H - E[e^{-r\tau_L}(v^* - \underline{v})|x_0 = x_H] = E[e^{-r\tau_L}(\underline{v} - x_L)|x_0 = x_H]$, where the last equality follows since the valuation v^* satisfies $v^* - x_H = E[e^{-r\tau_L}(v^* - x_L)|x_0 = x_H]$. That is, the profit margin that the seller earns from buyers with type higher than v^* is equal to the expected discounted profit margin she earns from buyers with valuation $\underline{v}$.

Finally, with a continuum of types the market outcome is efficient. Indeed, by the previous paragraph the market outcome is efficient when costs are initially x_H. On the other hand, when costs are initially x_L, in continuous-time the monopolist sells to all consumers immediately at price $\underline{v}$ and the market closes. Indeed, in this setting the monopolist does not have an incentive to delay trade with higher types until costs are x_H, since she is not able to extract rents from them when costs are high.[22]

9 Conclusion

This paper studies the problem of a durable goods monopolist who lacks commitment power and who faces uncertain and time-varying costs. I show that a durable goods monopolist

[22]The profit margin that the monopolist earns from selling to consumers with valuations above v^* immediately when costs are x_L is $\underline{v} - x_L$. This profit margin is strictly larger than the expected discounted margin that the monopolist would get by delaying trade with these consumers until costs are x_H, since the margin she gets on buyers with type above v^* when costs are x_H is $E[e^{-r\tau_L}(\underline{v} - x_L)|x_0 = x_H] < \underline{v} - x_L$.

with time-varying costs usually serves the different types of buyers at different times and charges them different prices. When the distribution of valuations has non-convex support, the monopolist extracts rents from buyers with higher valuations and there is inefficient delay. When the set of types is a continuum, the monopolist is unable to extract rents and the market outcome is efficient.

The model assumes that the seller's costs are publicly observable. This assumption approximates situations in which the seller's production cost is to a large extent determined by one or more commodities whose prices are publicly observable (e.g, minerals, oil or cement). Another example in which the evolution of costs is publicly observable is that of a monopolist who sells an imported good or uses imported intermediary goods, and who is therefore exposed to exchange rate risk. An interesting avenue for future research is to extend the current analysis to settings in which costs are only observed by the firm, and in which buyers can only observe the history of prices and of past sales. In such a model, one would expect there to be equilibria in which the seller can signal the evolution of costs through her choice of prices. While solving this model is beyond the scope of the current paper, it is worth noting that privately observed costs would introduce additional incentive constraints on the sellers' side, leading to a different equilibrium dynamic. Indeed, we already know from the literature on bargaining with two-sided asymmetric information (e.g. Cho (1990); Ausubel and Deneckere (1992)) that, with time-invariant costs, the standard Coasian dynamics may not arise when the monopolist has private information about her production cost.

A Appendix

A.1 Proof of Theorem 1

Proof of Theorem 1. Appendix B.3 shows that, in any equilibrium, the monopolist's profits are equal to $L(x,q)$ for all (x,q) with $q<\alpha$. Here, I complete the proof of Theorem 1 by constructing the unique equilibrium. In order for the monopolist to obtain profits equal to $L(x,q)$, she must sell to all high types at price $P(x_t,\alpha)$ when $x_t \in [0,\underline{x}(q_{t^-})] \cup [\overline{x}(q_{t^-}), z_2]$ (and also to low types when $x_t \leq \underline{x}(q_{t^-})$). Moreover, by Lemma B8 in Appendix B.3, the monopolist sells at a positive rate when $x_t \in (\underline{x}(q_t), \overline{x}(q_t))$. I now determine the price that the monopolist charges and the rate at which she sells when $x_t \in (\underline{x}(q_t), \overline{x}(q_t))$.

Suppose $x_t \in (\underline{x}(q_t), \overline{x}(q_t))$ and let $\tau = \inf\{s>t : x_s \notin (\underline{x}(q_s), \overline{x}(q_s))\}$. At any $s \in [t,\tau)$, the seller's expected discounted profits (which are equal to $L(x_s, q_s)$) are given by

$$L(x_s, q_s) = E\left[\int_s^\tau e^{-r(u-s)} (P(x_u, q_u) - x_u)\, dq_u + e^{-r(\tau - s)} L(x_\tau, q_\tau) \middle| \mathcal{F}_s\right].$$

By the Law of Iterated Expectations, the process

$$\begin{aligned} Y_s &= \int_0^s e^{-ru} (P(x_u, q_u) - x_u)\, dq_u + e^{-rs} L(x_s, q_s) \\ &= E\left[\int_0^\tau e^{-ru} (P(x_u, q_u) - x_u)\, dq_u + e^{-r\tau} L(x_\tau, q_\tau) \middle| \mathcal{F}_s\right], \end{aligned} \tag{A.1}$$

is a continuous martingale for all $s \in [t, \tau)$. By the Martingale Representation Theorem (Karatzas and Shreve (1998), p. 182), there exists a progressively measurable and square integrable process β such that $dY_s = e^{-rs}\beta_s dB_s$. Differentiating (A.1) with respect to s gives

$$\begin{aligned} dY_s &= e^{-rs} (P(x_s, q_s) - x_s)\, dq_s - re^{-rs} L(x_s, q_s)\, ds + e^{-rs} dL(x_s, q_s) \\ \Rightarrow dL(x_s, q_s) &= rL(x_s, q_s)\, ds - (P(x_s, q_s) - x_s)\, dq_s + \beta_s dB_s. \end{aligned}$$

Since $L(x, q) \in C^2$ for all $x \in (\underline{x}(q), \overline{x}(q))$ and all $q \in [0, \alpha)$ (Lemma B6 in Appendix B.2), we can apply Ito's Lemma to get

$$dL(x_s, q_s) = \left(\mu x_s L_x(x_s, q_s) + \frac{\sigma^2 x_s^2}{2} L_{xx}(x_s, q_s)\right) ds + L_q(x_s, q_s)\, dq_s + \sigma x L_x(x_s, q_s)\, dB_s.$$

From these two equations, it follows that the seller's profit function must satisfy

$$rL(x_s, q_s)\, ds = (P(x_s, q_s) - x_s + L_q(x_s, q_s))\, dq_s + \mu x_s L_x(x_s, q_s)\, ds + \frac{\sigma^2 x_s^2}{2} L_{xx}(x_s, q_s)\, ds,$$

at all states (x_s, q_s) with $x_s \in (\underline{x}(q_s), \overline{x}(q_s))$. On the other hand, the proof of Lemma B2 shows that $L(x, q)$ satisfies

$$rL(x, q) = \mu x L_x(x, q) + \frac{1}{2}\sigma^2 x^2 L_{xx}(x, q),$$

for all $x \in (\underline{x}(q), \overline{x}(q))$ (Lemma B6 in Appendix B.2). Comparing these two equations it follows that $P(x_s, q_s) - x_s = -L_q(x_s, q_s)$ for all (x_s, q_s) with $x_s \in (\underline{x}(q_s), \overline{x}(q_s))$. That is, the profit margin $P(x_s, q_s) - x_s$ that the monopolist earns from selling to high types must be equal to the cost $-L_q(x_s, q_s)$ that she incurs in terms of a lower continuation payoff. This expression pins down the prices that the seller charges when $x_s \in (\underline{x}(q_s), \overline{x}(q_s))$ (and hence the buyers' strategy profile in this region). It is worth noting that, since $L(x,q)$ is the solution solves the stopping (10), $P(x,q) - x = -L_q(x,q) = E[e^{-r\tau(q)}(P(x_{\tau(q)}, \alpha) - x_{\tau(q)})|x_0 = x]$ for all $x \in (\underline{x}(q), \overline{x}(q))$.

Finally, I pin down the rate dq_t at which the monopolist sells when $x_t \in (\underline{x}(q_t), \overline{x}(q_t))$. Note that in equilibrium all high types must get the same payoff; otherwise, a buyer getting a lower payoff would find it strictly optimal to deviate and mimic the strategy of one who is getting a larger payoff. Therefore, prices must evolve in such a way that high types are indifferent between buying at any $s \in [t, \tau)$ (where $\tau = \inf\{s > t : x_s \notin (\underline{x}(q_s), \overline{x}(q_s))\}$). That is,

$$e^{-rs}(v_2 - P(x_s, q_s)) = E\left[e^{-ru}(v_2 - P(x_u, q_u))\middle|\mathcal{F}_s\right], \tag{A.2}$$

for any $s, u \in [t, \tau)$, $s < u$. By the Law of Iterated Expectations, the process $M_s = E\left[e^{-ru}(v_2 - P(x_u, q_u))\middle|\mathcal{F}_s\right]$ is a continuous martingale. By the Martingale Representation Theorem, there exists a progressively measurable process γ such that $dM_s = e^{-rs}\gamma_s dB_s$. Differentiating (A.2) with respect to s gives

$$\begin{aligned} dM_s &= -re^{-rs}(v_2 - P(x_s, q_s))\,ds - e^{-rs}dP(x_s, q_s) \\ \Rightarrow dP(x_s, q_s) &= -r(v_2 - P(x_s, q_s))\,ds - \gamma_s dB_s. \end{aligned} \tag{A.3}$$

Equation (A.3) shows that, in expectation, prices must fall at rate $-r(v_2 - P(x_s, q_s))$ in order to keep high types indifferent between buying at any time $s \in [t, \tau)$. By the arguments above, $P(x_s, q_s) = x_s - L_q(x_s, q_s)$ for all $s \in [t, \tau)$. The proof of Lemma B2 shows that $L(x,q) \in C^2$ for all $x \in (\underline{x}(q), \overline{x}(q))$, so $P(x,q) \in C^{2,1}$ for all $x \in (\underline{x}(q), \overline{x}(q))$. Ito's Lemma then gives

$$dP(x_s, q_s) = \left(\mu x_s P_x(x_s, q_s) + \frac{\sigma^2 x_s^2}{2} P_{xx}(x_s, q_s)\right) ds + P_q(x_s, q_s)\,dq_s + P_x(x_s, q_s)\,\sigma x_s dB_s,$$

for all $s \in [t, \tau)$. This equation and (A.3) give two expressions for $dP(x_s, q_s)$. Since these expressions must be equal, it follows that

$$\frac{dq_s}{ds} = \frac{-r(v_2 - P(x_s, q_s)) - \mu x_s P_x(x_s, q_s) - \frac{1}{2}\sigma^2 x_s^2 P_{xx}(x_s, q_s)}{P_q(x_s, q_s)}.$$

The proof of Lemma B2 shows that $L_q(x,q)$ solves $rL_q(x,q) = \mu x L_{qx}(x,q) + \frac{\sigma^2 x^2}{2} L_{qxx}(x,q)$ for all $x \in (\underline{x}(q), \overline{x}(q))$ (footnote 24 in Appendix B.2). Using this with the fact that $P(x_s, q_s) -$

$x_s = -L_q(x_s, q_s)$ gives $\frac{dq_s}{ds} = -\frac{r(v_2 - x_s) + \mu x_s}{P_q(x_s, q_s)} = \frac{r(v_2 - x_s) + \mu x_s}{L_{qq}(x_s, q_s)} > 0$, where the equality follows since $P_q(x, q) = -L_{qq}(x, q)$, and the inequality follows since $L_{qq}(x, q) > 0$ for all $x \in (\underline{x}(q), \overline{x}(q))$ (Lemma B7 in Appendix B.2) and since $r(v_2 - x) + \mu x > 0$ for all $x < z_2$. ■

A.2 Proof of Theorem 3

Fix a sequence $\{f^j\} \to h$, with $f^j : [0, 1] \to [\underline{v}, \overline{v}]$ taking finitely many values for all j. For $j = 1, 2, ...,$ let $v_1^j < v_2^j < ... < v_{n_j}^j$ be the set of possible valuations under f^j. For $k = 1, ..., n_j$, let $z_k^j = \frac{-\lambda_N}{1-\lambda_N} v_k^j$. For each j, define the function $P^j(x)$ as follows. For $x \leq z_1^j$, $P^j(x) = v_1^k$. For $k = 2, ..., n_j$ and $x \in (z_{k-1}^j, z_k^j]$, $P^j(x) = P^j(x, \alpha_k^j)$. Equation (5) then implies that, for $k = 2, ..., n_j$ and $x \in (z_{k-1}^j, z_k^j]$, $P^j(x) = v_k^j - E[e^{-r\tau_{k-1}^j}(v_k^j - P^j(z_{k-1}^j))|x_0 = x]$, where for $k = 1, ..., n_j$, $\tau_k^j = \inf\{t : x_t \leq z_k^j\}$ is the time at which the monopolist starts selling to buyers with valuation v_k^j when v_k^j is the highest valuation remaining in the market.

Lemma A1 *For $k = 2, ..., n_j$ and $x \in (z_{k-1}^j, z_k^j]$, $P^j(x) = v_k^j - \sum_{m=1}^{k-1}(v_{m+1}^j - v_m^j)(x/z_m^j)^{\lambda_N}$.*

Proof. The proof is by induction. By equation (9) in the main text, $P^j(x) = v_2^j - (v_2^j - v_1^j)(x/z_1^j)^{\lambda_N}$ for $x \in (z_1^j, z_2^j]$, so the statement is true for $k = 2$. Suppose the statement is true for $l = 2, .., k-1$. By Corollary B1 in Appendix B.1, $P^j(x) = v_k^j - (v_k^j - P^j(z_{k-1}^j))(x/z_{k-1}^j)^{\lambda_N}$ for all $x \in (z_{k-1}^j, z_k^j]$. The induction hypothesis then implies that $P^j(x) = v_k^j - (v_k^j - P^j(z_{k-1}^j))(x/z_{k-1}^j)^{\lambda_N} = v_k^j - \sum_{m=1}^{k-1}(v_{m+1}^j - v_m^j)(x/z_m^j)^{\lambda_N}$ for $x \in (z_{k-1}^j, z_k^j]$. ■

Let $\overline{z} = \frac{-\lambda_N}{1-\lambda_N}\overline{v}$ and $\underline{z} = \frac{-\lambda_N}{1-\lambda_N}\underline{v}$, and let $V_{\underline{v}}(x) = \sup_\tau E[e^{-r\tau}(\underline{v} - x_\tau)|\, x_0 = x]$. By Lemma 1, $V_{\underline{v}}(x) = E[e^{-r\underline{\tau}}(\underline{v} - x_{\underline{\tau}})|\, x_0 = x]$, where $\underline{\tau} = \inf\{t : x_t \leq \underline{z}\}$.

Lemma A2 *$P^j(x) - x \to V_{\underline{v}}(x)$ uniformly on $[0, \overline{z}]$ as $j \to \infty$.*

Proof. I first show that $\lim_{j\to\infty} P^j(x) = V_{\underline{v}}(x) + x$ for all $x \in [0, \overline{z}]$. Note first that, for all $x \leq \underline{z}$, $\lim_{j\to\infty} P^j(x) = \lim_{j\to\infty} v_1^j = \underline{v} = V_{\underline{v}}(x) + x$. Next, fix $x \in (\underline{z}, \overline{z}]$ and for $j = 1, 2, ...,$ let k_j be such that $x \in (z_{k_j-1}^j, z_{k_j}^j]$. Let $v(x) = \frac{1-\lambda_N}{-\lambda_N}x$. Lemma A1 and the fact that $x/z_m^j = v(x)/v_m^j$ imply that $P^j(x) = v_{k_j}^j - \sum_{m=1}^{k_j-1}(v_{m+1}^j - v_m^j)(v(x)/v_m^j)^{\lambda_N}$. Since $x \in (z_{k_j-1}^j, z_{k_j}^j]$ for all j and since $\lim_{j\to\infty} z_{k_j-1}^j - z_{k_j}^j = 0$, it follows that $z_{k_j}^j = \frac{-\lambda_N}{1-\lambda_N}v_{k_j}^j \to x$ as $j \to \infty$. Hence, $\lim_{j\to\infty} v_{k_j}^j = \frac{1-\lambda_N}{-\lambda_N}x = v(x)$. Since $(v(x)/v)^{\lambda_N}$ is Riemann integrable, $\lim_{j\to\infty} P^j(x) = v(x) - \int_{\underline{v}}^{v(x)}(v(x)/v)^{\lambda_N}\, dv = x + (\underline{v} - \underline{z})(x/\underline{z})^{\lambda_N} = x + V_{\underline{v}}(x)$. Finally, since $P^j(x)$ is increasing in x for all j and since $\lim_{j\to\infty} P^j(x) = V_{\underline{v}}(x) + x$ for all $x \in [0, \overline{z}]$, it follows that $P^j(x) \to V_{\underline{v}}(x) + x$ uniformly on $[0, \overline{z}]$ as $j \to \infty$. Thus, $P^j(x) - x \to V_{\underline{v}}(x)$ uniformly on $[0, \overline{z}]$ as $j \to \infty$. ■

Proof of Theorem 3. I first show that, for all x, $L^j(x,0) \to V_{\underline{v}}(x)$ as $j \to \infty$. Note first that $L^j(x,0) \geq V_{\underline{v}}(x)$ for all x, since at any state $(x,0)$ the monopolist can wait until time τ_1^j and sell to all buyers at price $v_1^j \geq \underline{v}$, obtaining a profit of $E[e^{-r\tau_1^j}(v_1^j - x_{\tau_1^j}) \mid x_0 = x] \geq V_{\underline{v}}(x)$.

Consider the case in which $x_0 = x \geq \overline{z}$. In this case, for any model j, in equilibrium the monopolist sells to consumers with valuation v_k^j at time $\tau_k^j = \inf\{t : x_t \leq z_k^j\}$ (for $k = 1, ..., n_j$) at a price $P(z_k^j, \alpha_k^j) = P^j(z_k^j)$ (see Corollary C3 in Appendix C.1). Let $\alpha_{n_j+1}^j = 0$. Then, the seller's profits are $L^j(x,0) = \sum_{k=1}^{n_j} E[e^{-r\tau_k^j}(P^j(z_k^j) - z_k^j)|x_0 = x](\alpha_k^j - \alpha_{k+1}^j)$. Since $P^j(x) - x \to V_{\underline{v}}(x)$ uniformly on $[0, \overline{z}]$ as $j \to \infty$, for every $\eta > 0$ there exists N such that $P^j(x) - x - V_{\underline{v}}(x) < \eta$ for all $j > N$ and all $x \in [0, \overline{z}]$. Thus, for $j > N$, $L^j(x,0) < \sum_{k=1}^{n_j} d\alpha_k^j E[e^{-r\tau_k^j} V_{\underline{v}}(x_{\tau_k^j})|x_0 = x] + \eta$, where $d\alpha_k^j = \alpha_k^j - \alpha_{k+1}^j$ (so $\sum_{k=1}^{n_j} d\alpha_k^j = 1$). Note further that for $x \geq \overline{z}$ and $k = 1, 2, .., n_j$,

$$E[e^{-r\tau_k^j} V_{\underline{v}}(x_{\tau_k^j})|x_0 = x] = E[e^{-r\tau_k^j}[e^{-r(\tau_{\underline{v}} - \tau_k^j)}(\underline{v} - x_{\tau_{\underline{v}}})|x_{\tau_k^j}]|x_0 = x] = V_{\underline{v}}(x).$$

Using this and the fact $\sum_{k=1}^{n_j} d\alpha_k^j = 1$, it follows that $V_{\underline{v}}(x) \leq L^j(x,0) < \underline{V}(x) + \eta$ for all $j > N$. Therefore, $\lim_{j\to\infty} L^j(x,0) = V_{\underline{v}}(x)$ for all $x \geq \overline{z}$.

Consider next the case with $x_0 = x < \overline{z}$, and suppose by contradiction that $L^j(x,0) \nrightarrow V_{\underline{v}}(x)$ as $j \to \infty$. Since $L^j(x,0) \geq V_{\underline{v}}(x)$ for all j, there exists a subsequence $\{j_r\}$, N and $\gamma > 0$ such that $L^{j_r}(x,0) > V_{\underline{v}}(x) + \gamma$ for all $j_r > N$. Fix $y \geq \overline{z}$ and let $\tau_x = \inf\{t : x_t \leq x\}$. Since the seller can delay trade until time τ_x, it must be that $L^{j_r}(y,0) \geq E[e^{-r\tau_x} L^{j_r}(x_{\tau_x}, 0)|x_0 = y]$. Thus, for all $j_r > N$, $L^{j_r}(y,0) > E[e^{-r\tau_x} V_{\underline{v}}(x_{\tau_x})|x_0 = y] + E[e^{-r\tau_x}\gamma|x_0 = y]$. But this contradicts $\lim_{j_k\to\infty} L^{j_k}(y,0) = V_{\underline{v}}(y)$, since $E[e^{-r\tau_x} V_{\underline{v}}(x_{\tau_x})|x_0 = y] = V_{\underline{v}}(y)$ and since $E[e^{-r\tau_x}\gamma|x_0 = y] > 0$. Thus, $\lim_{j\to\infty} L^j(x,0) = V_{\underline{v}}(x)$.

Finally, I show that the limiting equilibrium outcome is efficient. Note that Lemma A2 implies that, for all $i \in [0,1]$, the price $P^j(x,i)$ that consumer i is willing to pay converges to $V_{\underline{v}}(x) + x$ for all $x \leq \frac{-\lambda_N}{1-\lambda_N} h(i)$ as $j \to \infty$. This in turn implies that, in the limit as $j \to \infty$, the monopolist always sells at price $V_{\underline{v}}(x_t) + x_t$. By the same arguments used in the proof of Lemma 1, for all $v \in [\underline{v}, \overline{v}]$, the solution to $\sup_\tau E\left[e^{-r\tau}(v - V_{\underline{v}}(x_\tau) - x_\tau)\right| x_0 = x]$ is $\tau_v = \inf\{t : x_t \leq \frac{-\lambda_N v}{1-\lambda_N}\}$. Since the seller always charges $V_{\underline{v}}(x_t) + x_t$ as $j \to \infty$ and since buyers buy at the time that maximizes their surplus, in the limit a buyer with valuation v buys at the efficient time τ_v. ■

B Omitted proofs

B.1 Proofs of Lemmas 1 and 2

Fix $y_2 > y_1 > 0$ and let $\tau_y = \inf\{t : x_t \notin (y_1, y_2)\}$ and $\tau_{y_1} = \inf\{t : x_t \leq y_1\}$. Note that τ_y and τ_{y_1} are random variables, whose distributions depend on the initial level of costs x_0.

Lemma B1 *Let g be a bounded function and let W be the solution to*

$$rW(x) = \mu x W'(x) + \frac{1}{2}\sigma^2 x^2 W''(x), \tag{B.1}$$

with $W(y_i) = g(y_i)$ for $i = 1, 2$. Then, $W(x) = E[e^{-r\tau_y} g(x_{\tau_y}) | x_0 = x]$ for all $x \in (y_1, y_2)$.

Proof. Let W satisfy (B.1) with $W(y_1) = g(y_1)$ and $W(y_2) = g(y_2)$. The general solution to (B.1) is $W(x) = Ax^{\lambda_N} + Bx^{\lambda_P}$, where $\lambda_N < 0$ and $\lambda_P > 1$ are the roots of $\frac{1}{2}\sigma^2\lambda(\lambda - 1) + \mu\lambda = r$, and where A and B are constants determined by the boundary conditions:

$$A = \frac{g(y_2)\, y_1^{\lambda_P} - g(y_1)\, y_2^{\lambda_P}}{y_1^{\lambda_P} y_2^{\lambda_N} - y_1^{\lambda_N} y_2^{\lambda_P}} \text{ and } B = -\frac{g(y_2)\, y_1^{\lambda_N} - g(y_1)\, y_2^{\lambda_N}}{y_1^{\lambda_P} y_2^{\lambda_N} - y_1^{\lambda_N} y_2^{\lambda_P}} \tag{B.2}$$

Let $f(x,t) = e^{-rt}W(x)$. By Ito's Lemma, for all $x_t \in (y_1, y_2)$

$$\begin{aligned} df(x_t, t) &= e^{-rt}\left(-rW(x_t) + \mu x W'(x_t) + \frac{1}{2}\sigma^2 x^2 W''(x_t)\right) dt + e^{-rt}\sigma x W'(x_t)\, dB_t \\ &= e^{-rt}\sigma x W'(x_t)\, dB_t, \end{aligned}$$

where the second equality follows from the fact that W solves (B.1). Then,

$$\begin{aligned} E\left[e^{-r\tau_y} g(x_{\tau_y}) \middle| x_0 = x\right] &= E\left[f(x_{\tau_y}, \tau_y) \middle| x_0 = x\right] = f(x, 0) + E\left[\int_0^{\tau_y} df(x_t, t) \middle| x_0 = x\right] \\ &= W(x) + E\left[\int_0^{\tau_y} e^{-rt}\sigma x W'(x_t)\, dB_t \middle| x_0 = x\right] = W(x), \end{aligned}$$

since $\int_0^{\tau_y} e^{-rt}\sigma x W'(x_t)\, dB_t$ is a Martingale with expectation zero. ■

Corollary B1 *Let g be a bounded function and let w be a solution to (B.1) with $w(y_1) = g(y_1)$ and $\lim_{x\to\infty} w(x) = 0$. Then, $w(x) = E[e^{-r\tau_{y_1}} g(x_{\tau_{y_1}}) | x_0 = x]$ for all $x > y_1$. Moreover, $w(x) = g(y_1)(x/y_1)^{\lambda_N}$ for all $x > y_1$.*

Proof. Since w solves (B.1), it follows that $w(x) = Cx^{\lambda_N} + Dx^{\lambda_P}$. The conditions $w(y_1) =$

$g(y_1)$ and $\lim_{x\to\infty} w(x) = 0$ imply $D = 0$ and $C = g(y_1)(1/y_1)^{\lambda_N}$, so $w(x) = g(y_1)(x/y_1)^{\lambda_N}$. Next, note that for all $x_0 > y_1$, $\tau_y \to \tau_{y_1}$ as $y_2 \to \infty$. By Dominated convergence, $W(x) = E\left[e^{-r\tau_y} g\left(x_{\tau_y}\right)\middle| x_0 = x\right] \to E\left[e^{-r\tau_{y_1}} g\left(x_{\tau_{y_1}}\right)\middle| x_0 = x\right]$ as $y_2 \to \infty$. By Lemma B1, $W(x) = Ax^{\lambda_N} + Bx^{\lambda_P}$ for $x \in (y_1, y_2)$, with A and B satisfying (B.2). Since $\lim_{y_2\to\infty} B = 0$ and $\lim_{y_2\to\infty} A = g(y_1)/y_1^{\lambda_N}$, it follows that $E[e^{-r\tau_{y_1}} g(x_{\tau_{y_1}})| x_0 = x] = \lim_{y_2\to\infty} W(x) = g(y_1)(x/y_1)^{\lambda_N} = w(x)$ for all $x > y_1$. ■

Proof of Lemma 1. Let $V_k(\cdot)$ be as in the statement of the Lemma. Note that V_k is twice differentiable with a continuous first derivative. One can show that $V_k(x) > v_k - x$ for $x > z_k$, so $V_k(x) \geq v_k - x$ for all $x \geq 0$. Note also that $V_k(\cdot)$ solves (B.1) for all $x > z_k$, with $V_k(z_k) = v_k - z_k$ and $\lim_{x\to\infty} V_k(x) = 0$. By Corollary B1, $V_k(x) = E[e^{-r\tau_k}(v_k - x_{\tau_k})| x_0 = x]$. Moreover, $r(v_k - x) = rV_k(x) > \mu x V_k'(x) + \frac{1}{2}\sigma^2 x^2 V_k''(x) = -\mu x$, for all $x \leq z_k$.

By the previous paragraph, V_k is twice differentiable with a continuous first derivative and satisfies

$$-rV_k(x) + \mu x V_k'(x) + \frac{1}{2}\sigma^2 x^2 V_k''(x) \leq 0, \text{ with equality on } (z_k, \infty).$$

Then, by standard verification theorems (Theorem 3.17 in Shiryaev (2008)) V_k solves (6). ■

Remark B1 Since V_k is the solution to the optimal stopping problem (6), then $e^{-rt}V_k(x_t)$ is superharmonic; i.e., $V_k(x) \geq E[e^{-r\tau}V_k(x_\tau)| x_0 = x]$ for any stopping time τ (e.g., Theorem 10.1.9 in Oksendal (2007)). I will use this property of V_k in the proof of Lemma B2 below.

Proof of Lemma 2. By Corollary B1, $E[e^{-r\tau_1}(v_2 - v_1)|x_0 = x] = (v_2 - v_1)(x/z_1)^{\lambda_N}$ for all $x > z_1$. This gives equation (9). Moreover, for all $x \in (z_1, z_2]$,

$$\begin{aligned} P(x,\alpha) - x - V_1(x) &= v_2 - x - E\left[e^{-r\tau_1}(v_2 - v_1)\middle| x_0 = x\right] - E\left[e^{-r\tau_1}(v_1 - x_{\tau_1})\middle| x_0 = x\right] \\ &= v_2 - x - E[e^{-r\tau_1}(v_2 - x_{\tau_1})| x_0 = x] > 0 \end{aligned}$$

since by Lemma 1, $v_2 - x = V_2(x) > E[e^{-r\tau_1}(v_2 - x_{\tau_1})| x_0 = x]$ for all $x \in (z_1, z_2]$. ■

B.2 Solution to (10)

The following Lemma characterizes the solution to (10).

Lemma B2 *For every $q \in [0, \alpha)$, there exists $\underline{x}(q) \in (0, z_1)$ and $\overline{x}(q) \in (z_1, z_2)$ such that $\tau(q) = \inf\{t : x_t \in [0, \underline{x}(q)] \cup [\overline{x}(q), z_2]\}$ solves (10). Moreover, $\underline{x}(\cdot)$ and $\overline{x}(\cdot)$ are continuous, with $\lim_{q\to\alpha} \underline{x}(q) = \lim_{q\to\alpha} \overline{x}(q) = z_1$.*

The proof of Lemma B2 is organized as follows. Lemmas B3 and B4 give properties of solutions to equation (B.1). Lemma B5 uses these properties to characterize the solution to the optimal stopping problem (10). Finally, Lemmas B6 and B7 prove properties of the solution to (10).

Lemma B3 *Let U and $\widetilde{U}$ be two solutions to (B.1). If $\widetilde{U}(y) \geq U(y)$ and $\widetilde{U}'(y) > U'(y)$ for some $y > 0$, then $\widetilde{U}'(x) > U'(x)$ for all $x > y$, and so $\widetilde{U}(x) > U(x)$ for all $x > y$. Similarly, if $\widetilde{U}(y) \leq U(y)$ and $\widetilde{U}'(y) > U'(y)$ for some $y > 0$, then $\widetilde{U}'(x) > U'(x)$ for all $x < y$, and so $\widetilde{U}(x) < U(x)$ for all $x < y$.*

Proof. I prove the first statement of the Lemma. The proof of the second statement is symmetric and omitted. Suppose the claim is not true, and let $y_1 > y$ be the smallest point with $U'(y_1) = \widetilde{U}'(y_1)$. Therefore, $\widetilde{U}'(x) > U'(x)$ for all $x \in [y, y_1)$, so $\widetilde{U}(y_1) > U(y_1)$. Since U and $\widetilde{U}$ solve (B.1), then $\widetilde{U}''(y_1) = \frac{2(r\widetilde{U}(y_1) - \mu y_1 \widetilde{U}'(y_1))}{\sigma^2 y_1^2} > \frac{2(rU(y_1) - \mu y_1 U'(y_1))}{\sigma^2 y_1^2} = U''(y_1)$. But this implies that $U'(y_1 - \varepsilon) > \widetilde{U}'(y_1 - \varepsilon)$ for $\varepsilon > 0$ small, a contradiction. ■

Lemma B4 *Fix $q \in [0, \alpha)$ and $y \in (0, z_1)$, and let $U_y(x)$ be the solution to (B.1) with $U_y(y) = (1-q)(v_1 - y)$ and $U_y'(y) = -(1-q)$. Then, $U_y(x)$ is strictly convex for all $x > 0$. Moreover, if $y < y' < z_1$, then $U_y(x) > U_{y'}(x)$ for all $x \geq y'$.*

Proof. Since $U_y(\cdot)$ solves (B.1), it follows that $U_y(x) = Ax^{\lambda_N} + Bx^{\lambda_P}$. The constants A and B are determined by the conditions $U_y(y) = (1-q)(v_1 - y)$ and $U_y'(y) = -(1-q)$:

$$A = y^{-\lambda_N}(1-q)\frac{\lambda_P(v_1 - y) + y}{\lambda_P - \lambda_N} > 0 \text{ and } B = y^{-\lambda_P}(1-q)\frac{-(v_1 - y)\lambda_N - y}{\lambda_P - \lambda_N} > 0,$$

where $B > 0$ follows since $y < z_1 = -v_1\lambda_N/(1-\lambda_N)$. Thus, $U_y''(x) = \lambda_N(\lambda_N - 1)Ax^{\lambda_N - 2} + \lambda_P(\lambda_P - 1)Bx^{\lambda_P - 2} > 0$ for all $x > 0$ (since $\lambda_N < 0$ and $\lambda_P > 1$). Finally, let $y < y' < z_1$. Since $U_y(\cdot)$ is strictly convex, it follows that $U_y(y') > (1-q)(v_1 - y') = U_{y'}(y')$ and $U_y'(y') > -(1-q) = U_{y'}'(y')$. Hence, by Lemma B3 $U_y(x) > U_{y'}(x)$ for all $x \geq y'$. ■

For $q \in [0, \alpha)$ and $x > 0$, let $g(x, q) = (\alpha - q)(P(x, \alpha) - x) + \Pi(x, \alpha)$, so that $L(x, q) = \sup_\tau E[e^{-r\tau} g(x_\tau, q) | x_0 = x]$. Note that $g(x, q) = (1-q)(v_1 - y)$ for all $x \leq z_1$.

Lemma B5 *For all $q \in [0, \alpha)$, there exists $\underline{x}(q) \in (0, z_1)$ and $\overline{x}(q) \in (z_1, z_2)$ such that $\tau(q) = \inf\{t : x_t \in [0, \underline{x}(q)] \cup [\overline{x}(q), z_2]\}$ solves (10). Moreover,*

(i) for all $x \in (\underline{x}(q), \overline{x}(q)) \cup (z_2, \infty)$, $L(x, q)$ solves (B.1), with $\lim_{x\to\infty} L(x, q) = 0$.

(ii) for all $x \leq \underline{x}(q)$ and all $x \in [\overline{x}(q), z_2]$, $L(x,q) = g(x,q)$.

(iii) the cutoffs $\underline{x}(q)$ and $\overline{x}(q)$ are such that

$$L(\underline{x}(q), q) = g(\underline{x}(q), q), L(\overline{x}(q), q) = g(\overline{x}(q), q), \tag{VM}$$

$$L_x(\underline{x}(q), q) = g_x(\underline{x}(q), q), L_x(\overline{x}(q), q) = g_x(\overline{x}(q), q). \tag{SP}$$

Proof. First I show that there exists a function $G(x,q)$ satisfying (i)-(iii). Then I show that $G(x,q) = \sup_\tau E[e^{-r\tau} g(x_\tau, q) | x_0 = x] = L(x,q)$. Let $W(x)$ be the solution to (B.1) with $\lim_{x\to\infty} W(x) = 0$ and $W(z_2) = g(z_2, q)$. By Corollary B1, $W(x) = g(z_2,q)(x/z_2)^{\lambda_N}$. Note that $W'(z_2) = \frac{\lambda_N}{z_2} g(z_2, q)$, and that, for all $x > z_1$

$$\begin{aligned}
\left.\frac{\partial}{\partial x} g(x,q)\right|_{x=z_2} &= \frac{\lambda_N}{z_2}\left[(\alpha - q)\left(-(v_2 - v_1)\left(\frac{z_2}{z_1}\right)^{\lambda_N} - \frac{z_2}{\lambda_N}\right) + (1-\alpha)(v_1 - z_1)\left(\frac{z_2}{z_1}\right)^{\lambda_N}\right] \\
&= \frac{\lambda_N}{z_2}\left[(\alpha - q)\left(v_2 - (v_2 - v_1)\left(\frac{z_2}{z_1}\right)^{\lambda_N} - z_2\right) + (1-\alpha)(v_1 - z_1)\left(\frac{z_2}{z_1}\right)^{\lambda_N}\right] \\
&= \frac{\lambda_N}{z_2} g(z_2, q),
\end{aligned}$$

where the second equality follows since $v_2 - z_2 = \frac{v_2}{1-\lambda_N} = -\frac{z_2}{\lambda_N}$. It then follows that $W'(z_2) = g_x(z_2,q)$. Moreover, one can check that $W(x) > g(x,q)$ for all $x > z_2$ and that $g(x,q) < W(x)$ for all $x < z_2$. For all $x \geq z_2$, let $G(x,q) = W(x)$.

Next, I show that there exists a function $G(x,q)$ and unique cutoffs $\underline{x}(q) < z_1$ and $\overline{x}(q) \in (z_1, z_2)$ such that $G(x,q)$ solves (B.1) on $(\underline{x}(q), \overline{x}(q))$ and satisfies (iii). For each $y < z_1$, let U_y be the solution to (B.1) with $U_y(y) = g(y,q) = (1-q)(v_1 - y)$ and $U_y'(y) = g_x(y,q) = -(1-q)$. Since solutions to (B.1) are continuous in initial conditions, then the solutions I'm considering are continuous in y. If y is small enough, then $U_y(x)$ will remain above $g(x,q)$ for all $x > y$. On the other hand, if y is close to z_1 then U_y will cross $g(x,q)$ at some $\widetilde{x} > z_1$ (see solutions I-IV in Figure B1). Since $U_{y''}(x) > U_{y'}(x)$ for all $y'' < y'$ and all $x \geq y'$ (Lemma B4), the point at which U_y crosses $g(x,q)$ moves to the right as y decreases. Let $\underline{x}(q) = \inf\{y : U_y(x) = g(x,q) \text{ for some } x > z_1\}$, and let $\overline{x}(q) > z_1$ be such that $U_{\underline{x}(q)}(\overline{x}(q)) = g(\overline{x}(q), q)$. Since a solution with $y < \underline{x}(q)$ never reaches $g(x,q)$, it follows that $U_{\underline{x}(q)}(x) \geq g(x,q)$ for all x. Thus, $U_{\underline{x}(q)}(x)$ is tangent to $g(x,q)$ at $\overline{x}(q)$, so $U'_{\underline{x}(q)}(\overline{x}(q)) = g_x(\overline{x}(q), q)$ (see solution III in Figure B1). Let $G(x,q) = U_{\underline{x}(q)}(x)$ for $x \in [\underline{x}(q), \overline{x}(q)]$.

Note that by construction it must be that $\underline{x}(q) \in (0, z_1)$ and that $\overline{x}(q) > z_1$. I now show

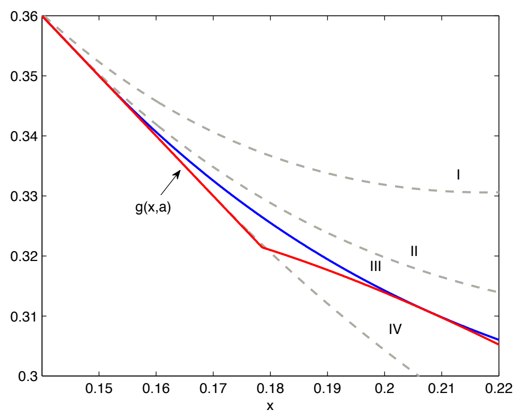

Figure B1: Solutions U_y to ODE (B.1).

that $\overline{x}(q) < z_2$. To see this, suppose by contradiction that $\overline{x}(q) \geq z_2$. Recall that $W(x)$ is the solution to (B.1) with $\lim_{x\to\infty} W(x) = 0$ and $W(z_2) = g(z_2, q)$. Recall further that $W'(z_2) = g_x(z_2, q)$. If $\overline{x}(q) = z_2$, then $U_{\underline{x}(q)}(x)$ and $W(x)$ both solve (B.1), with $U_{\underline{x}(q)}(z_2) = W(z_2) = g(z_2, q)$ and $U'_{\underline{x}(q)}(z_2) = W'(z_2) = g_x(z_2, q)$. This implies that $W(x) = U_{\underline{x}(q)}(x)$ for all x, which cannot be since $U_{\underline{x}(q)}(x) = Ax^{\lambda_N} + Bx^{\lambda_P}$ for some $B > 0$ (see proof of Lemma B5) and $W(x) = g(z_2, q)(x/z_2)^{\lambda_N}$. Suppose next that $\overline{x}(q) > z_2$. Since $W(x) > g(x, q)$ for all $x > z_2$ and since $U_{\underline{x}(q)}(x) > g(x, q)$ for all $x \in (\underline{x}(q), \overline{x}(q))$, then $W(\overline{x}(q)) > U_{\underline{x}(q)}(\overline{x}(q)) = g(\overline{x}(q), q)$ and $W(z_2) = g(z_2, q) < U_{\underline{x}(q)}(z_2)$. Moreover, since $W(x) > g(x, q)$ for all $x < z_2$, it follows that $U_{\underline{x}(q)}(\underline{x}(q)) < W(\underline{x}(q))$. By the intermediate value theorem, there exists $y_1 \in (\underline{x}(q), z_2)$ and $y_2 \in (z_2, \overline{x}(q))$ with $U_{\underline{x}(q)}(y_i) = W(y_i)$ for $i = 1, 2$. This implies that $U_{\underline{x}(q)}(x) = W(x)$ for all x, since $U_{\underline{x}(q)}$ and W both solve (B.1) on (y_1, y_2) with $U_{\underline{x}(q)}(y_i) = W(y_i)$ for $i = 1, 2$. But this cannot be, since $W(x) = g(z_2, q)(x/z_2)^{\lambda_N}$ and since $U_{\underline{x}(q)}(x) = Ax^{\lambda_N} + Bx^{\lambda_P}$ for some $B > 0$. Hence, it must be that $\overline{x}(q) < z_2$.

Finally, for all $x \leq \underline{x}(q)$ and for all $x \in [\overline{x}(q), z_2]$ let $G(x, q) = g(x, q)$. Note that, by construction, $G(x, q)$ satisfies (i)-(iii).

I now show that $G(x, q) = L(x, q) = \sup_\tau E[e^{-r\tau} g(x_\tau, q) | x_0 = x]$. By construction, $G(x, q) \geq g(x, q)$ for all $x \geq 0$. Moreover, $G(x, q)$ is twice differentiable in x, with a continuous first derivative. Finally, the function $G(x, q)$ satisfies:

$$-rG(x, q) + \mu x G_x(x, q) + \frac{1}{2}\sigma^2 x^2 G_{xx}(x, q) \leq 0, \text{ with equality on } (\underline{x}(q), \overline{x}(q)) \cup (z_2, \infty). \tag{B.3}$$

Indeed, $G(x, q)$ satisfies (B.3) with equality on $(\underline{x}(q), \overline{x}(q)) \cup (z_2, \infty)$ since it solves (B.1) in this region. One can also check that $rG(x,q) \geq \mu x G_x(x,q) + \frac{1}{2}\sigma^2 x^2 G_{xx}(x,q)$ for all $x \in [0, \underline{x}(q)] \cup [\overline{x}(q), z_2]$.[23] Therefore, by standard verification theorems (e.g., Theorem 3.17 in Shiryaev, 2008), $G(x,q) = \sup_\tau E[e^{-r\tau} g(x_\tau, q) | x_0 = x] = L(x,q)$. Finally, note that Lemma B1 and Corollary B1 imply that $L(x,q) = E[e^{-r\tau(q)} g\left(x_{\tau(q)}, q\right) \big| x_0 = x]$, so $\tau(q)$ solves (10). ■

Lemma B6 *$L(x,q) \in C^2$ for all $x \in (\underline{x}(q), \overline{x}(q))$ and all $q \in [0,\alpha)$. Moreover, $\underline{x}(q)$ and $\overline{x}(q)$ are C^2, with $\lim_{q\to\alpha} \underline{x}(q) = \lim_{q\to\alpha} \overline{x}(q) = z_1$.*

Proof. By Lemma B5, $L(x,q) = A(q) x^{\lambda_N} + B(q) x^{\lambda_P}$ for all $x \in (\underline{x}(q), \overline{x}(q))$, where $A(q)$, $B(q)$, $\underline{x}(q)$ and $\overline{x}(q)$ are determined by the system of equations (VM) + (SP). Denote this system of equations by $F(\underline{x}(q), \overline{x}(q), A(q), B(q)) = 0$. One can check that $F \in C^2$ and its Jacobian at $(\underline{x}(q), \overline{x}(q), A(q), B(q))$ has a non-zero determinant. By the Implicit Function Theorem, the functions $A(q)$, $B(q)$, $\underline{x}(q)$ and $\overline{x}(q)$ are all C^2 with respect to q. Since $L(x,q) = A(q) x^{\lambda_N} + B(q) x^{\lambda_P}$ for all $x \in (\underline{x}(q), \overline{x}(q))$, it follows that $L(x,q) \in C^2$ for all $x \in (\underline{x}(q), \overline{x}(q))$.[24]

Next, I show that $\lim_{q\to\alpha} \underline{x}(q) = \lim_{q\to\alpha} \overline{x}(q) = z_1$. Let $\underline{x} = \lim_{q\to\alpha} \underline{x}(q)$ and $\overline{x} = \lim_{q\to\alpha} \overline{x}(q)$. Since $\underline{x}(q) < z_1$ and $\overline{x}(q) > z_1$ for all $q < \alpha$ (Lemma B5), it follows that $\underline{x} \leq z_1 \leq \overline{x}$. Let $\widehat{\tau} = \inf\{t : x_t \in [0, \underline{x}] \cup [\overline{x}, z_2]\}$, so $\tau(q_n) \to \widehat{\tau}$ for every sequence $\{q_n\} \to \alpha$. Note that $L(x,q) \geq g(x,q) \geq \Pi(x,\alpha) = (1-\alpha) V_1(x)$ for all $q \leq \alpha$, so $\lim_{q\to\alpha} L(x,q) \geq (1-\alpha) V_1(x)$.

Fix a sequence $\{q_n\} \to \alpha$. Since $\lim_{q\to\alpha} g(x,q) = (1-\alpha) V_1(x)$, by Dominated Convergence, $L(x,q_n) = E\left[e^{-r\tau(q_n)} g\left(x_{\tau(q_n)}, q_n\right) \big| x_0 = x\right] \to E\left[e^{-r\widehat{\tau}} (1-\alpha) V_1(x) \big| x_0 = x\right]$ as $n \to \infty$. Suppose by contradiction that $\underline{x} < z_1$. Then, for $x \in (\underline{x}, z_1)$, $E\left[e^{-r\widehat{\tau}} V_1(x) \big| x_0 = x\right] < V_1(x)$, where the inequality follows from Lemma 1. But this implies that $\lim_{q\to\alpha} L(x,q) = (1-\alpha) E[e^{-r\widehat{\tau}} V_1(x) | x_0 = x] < (1-\alpha) V_1(x)$ for all $x \in (\underline{x}, z_1)$, which contradicts $L(x,q) \geq (1-\alpha) V_1(x)$ for all $q < \alpha$. Thus, it must be that $\underline{x} = z_1$.

[23] Indeed, for all $x \leq \underline{x}(q)$, $rG(x,q) = r(1-q)(v_1 - x) > \mu x G_x(x,q) + \frac{1}{2}\sigma^2 x^2 G_{xx}(x,q) = -\mu x(1-q)$. For all $x \in [\overline{x}(q), z_2]$,

$$G(x,q) = (\alpha - q)(v_2 - x) - (\alpha - q)(v_2 - z_1) \left(\frac{x}{z_1}\right)^{\lambda_N} + (1-\alpha) V_1(x).$$

Note that $r(v_2 - x) = rV_2(x) \geq \mu x V_2'(x) + \frac{\sigma^2 x^2}{2} V_2''(x) = -\mu x$ for all $x \leq z_2$. Since $r(\alpha - q)(v_2 - z_1)\left(\frac{x}{z_1}\right)^{\lambda_N}$ and $r(1-\alpha) V_1(x)$ both solve (B.1) for all $x \in [\overline{x}(q), z_2]$, it follows that $rG(x,q) \geq \mu x G_x(x,q) + \frac{1}{2}\sigma^2 x^2 G_{xx}(x,q)$ for all $x \in [\overline{x}(q), z_2]$.

[24] Note that this implies that $L_q(x,q) = A'(q) x^{\lambda_N} + B'(q) x^{\lambda_P}$ for all $x \in (\underline{x}(q), \overline{x}(q))$, so $L_q(x,q)$ also solves (B.1) in this range.

Suppose next that $\overline{x} > z_1 = \underline{x}$. Let $U(x) = E[e^{-r\widehat{\tau}}(P(x_{\widehat{\tau}}, \alpha) - x_{\widehat{\tau}}) | x_0 = x]$ and $Y_t = e^{-rt}(P(x_t, \alpha) - x_t)$. By Ito's Lemma, for all $x_t \in (z_1, \overline{x})$,

$$\begin{aligned} dY_t &= e^{-rt}\left(\left(-r(P(x_t,\alpha) - x_t) + \mu x_t (P_x(x_t,\alpha) - 1) + \frac{\sigma^2 x_t^2}{2} P_{xx}(x_t,\alpha)\right) dt + \sigma x_t P_x(x_t,\alpha)\, dB_t\right) \\ &= e^{-rt}(-r(v_2 - x_t) - \mu x_t)\, dt + e^{-rt}\sigma x_t P_x(x_t,\alpha)\, dB_t, \end{aligned}$$

where the second equality follows since equation (9) implies that $rP(x,\alpha) = rv_2 + \mu x P_x(x,\alpha) + \frac{\sigma^2 x^2}{2} P_{xx}(x,\alpha)$ for all $x > z_1$. Therefore, for $x \in (z_1, \overline{x})$,

$$U(x) = E[Y_{\widehat{\tau}} | x_0 = x] = Y_0 + E\left[\int_0^{\widehat{\tau}} e^{-rt}(-r(v_2 - x_t) - \mu x_t)\, dt \middle| x_0 = x\right].$$

Note that $-r(v_2 - x) < \mu x$ for all $x < \overline{x} \leq z_2$, so $U(x) < Y_0 = P(x,\alpha) - x$ for all such x.

For each $q \in [0, \alpha)$, let $W(x,q) = E[e^{-r\tau(q)}(P(x_{\tau(q)}, \alpha) - x_{\tau(q)}) | x_0 = x]$. Pick a sequence $\{q_n\} \to \alpha$, so that $\tau(q_n) \to \widehat{\tau}$ as $n \to \infty$. By dominated Convergence, $W(x, q_n) \to U(x)$ as $n \to \infty$. Fix $x \in (z_1, \overline{x})$. Since $U(x) < P(x,\alpha) - x$, there exists N such that $W(x, q_n) < P(x,\alpha) - x$ for all $n > N$. On the other hand, $E[e^{-r\tau(q_n)} V_1(x_{\tau(q_n)}) | x_0 = x] \leq V_1(x)$ for all x and all n (see Remark B1). Therefore, for $n > N$

$$\begin{aligned} L(x, q_n) &= E\left[e^{-r\tau(q_n)}\left((\alpha - q_n)(P(x_{\tau(q_n)}, \alpha) - x_{\tau(q_n)}) + (1-\alpha) V_1(x_{\tau(q_n)})\right) \middle| x_0 = x\right] \\ &< (\alpha - q_n)(P(x,\alpha) - x) + (1-\alpha) V_1(x) = g(x, q_n), \end{aligned}$$

which contradicts the fact that $L(x, q_n) = \sup_\tau E[e^{-r\tau} g(x_\tau, q_n) | x_0 = x]$. Thus, $\overline{x} = z_1$. ■

Proof of Lemma B2. Follows from Lemmas B5 and B6. ■

Lemma B7 *$L(x,q)$ is strictly convex in q for all $x \in (\underline{x}(q), \overline{x}(q))$.*

Proof. I first show that, for all $q' < q < \alpha$, either $\underline{x}(q) \neq \underline{x}(q')$ and/or $\overline{x}(q) \neq \overline{x}(q')$. For $\hat{q} \in [0, \alpha)$, let $W(x, \hat{q}) = E[e^{-r\tau(\hat{q})}(P(x_{\tau(\hat{q})}, \alpha) - x_{\tau(\hat{q})}) | x_0 = x]$ and $U(x, \hat{q}) = E[e^{-r\tau(\hat{q})} V_1(x_{\tau(\hat{q})}) | x_0 = x]$, where $\tau(\hat{q})$ is the solution to (10). Note that $L(x, \hat{q}) = (\alpha - \hat{q}) W(x, \hat{q}) + (1-\alpha) U(x, \hat{q})$.

Fix $q' < q < \alpha$ and suppose by contradiction that $\underline{x}(q) = \underline{x}(q')$ and $\overline{x}(q) = \overline{x}(q')$. Note that this implies that $\tau(q) = \tau(q')$, and so $W(x,q) = W(x,q')$ and $U(x,q) = U(x,q')$. By Lemma B5, it must be that $L_x(\underline{x}(\hat{q}), \hat{q}) = g_x(\underline{x}(\hat{q}), \hat{q}) = -(1-\hat{q})$ for all $\hat{q} < \alpha$. Since $L(x, \hat{q}) = (\alpha - \hat{q}) W(x, \hat{q}) + (1-\alpha) U(x, \hat{q})$ for all $\hat{q} < \alpha$ (and hence $L_x(x, \hat{q}) = (\alpha - \hat{q}) W_x(x, \hat{q}) +$

$(1-\alpha) U_x(x,\hat{q}))$, it follows that,

$$\begin{aligned} -(1-q') &= L_x(\underline{x}(q'),q') = (\alpha-q')W_x(\underline{x}(q'),q') + (1-\alpha)U_x(\underline{x}(q'),q') \\ &= (\alpha-q')W_x(\underline{x}(q),q) + (1-\alpha)U_x(\underline{x}(q),q) \\ &= (\alpha-q)W_x(\underline{x}(q),q) + (1-\alpha)U_x(\underline{x}(q),q) + (q-q')W_x(\underline{x}(q),q) \\ &= -(1-q) + (q-q')W_x(\underline{x}(q),q), \end{aligned}$$

where the second line follows since $W(x,q)=W(x,q')$, $U(x,q)=U(x,q')$ and $\underline{x}(q)=\underline{x}(q')$, and the last equality follows since $L_x(\underline{x}(q),q) = (\alpha-q)W_x(\underline{x}(q),q) + (1-\alpha)U_x(\underline{x}(q),q) = -(1-q)$. The equalities above imply that $W_x(\underline{x}(q),q) = -1$. Since $-(1-q) = (\alpha-q)W_x(\underline{x}(q),q) + (1-\alpha)U_x(\underline{x}(q),q)$, this in turn implies that $U_x(\underline{x}(q),q) = -1$.

Note next that by Lemma B1, $U(x,q) = E[e^{-r\tau(q)}V_1(x_{\tau(q)}) | x_0 = x]$ solves (B.1) for all $x \in (\underline{x}(q), \overline{x}(q))$ with $U(\underline{x}(q),q) = v_1 - \underline{x}(q) = V_1(\underline{x}(q))$ and $U(\overline{x}(q),q) = V_1(\overline{x}(q))$. Since $U_x(\underline{x}(q),q) = -1$, it follows from Lemma B4 that $U(x,q)$ is strictly convex for all $x \in [\underline{x}(q), \overline{x}(q)]$. This implies that $U_x(x,q) > -1$ and $U(x,q) > v_1 - x$ for all $x > \underline{x}(q)$; in particular, $U_x(z_1,q) > -1 = V_1'(z_1)$ and $U(z_1,q) > v_1 - z_1 = V_1(z_1)$. Since both $U(x,q)$ and $V_1(x)$ solve equation (B.1) for all $x \in [z_1, \overline{x}(q)]$, it follows from Lemma B3 that $U(x,q) > V_1(x)$ for all $x \in [z_1, \overline{x}(q)]$, a contradiction to the fact that $U(\overline{x}(q),q) = V_1(\overline{x}(q))$. Hence, it must be that either $\underline{x}(q) \neq \underline{x}(q')$ or $\overline{x}(q) \neq \overline{x}(q')$.

Finally, I show that $L(x,q)$ is strictly convex in q for all $x \in (\underline{x}(q), \overline{x}(q))$. Take $q' < q < \alpha$, and for each $\gamma \in (0,1)$ let $q_\gamma = \gamma q + (1-\gamma)q'$. By the previous paragraphs, the pairs $\{\underline{x}(q), \overline{x}(q)\}$, $\{\underline{x}(q'), \overline{x}(q')\}$ and $\{\underline{x}(q_\gamma), \overline{x}(q_\gamma)\}$ are all different, which in turn implies that the stopping times $\tau(q)$, $\tau(q')$ and $\tau(q_\gamma)$ are all different. Note that $g(x,q_\gamma) = (\alpha - q_\gamma)(P(x,\alpha) - x) + (1-\alpha)V_1(x) = \gamma g(x,q) + (1-\gamma)g(x,q')$. Therefore,

$$\begin{aligned} L(x,q_\gamma) &= E\left[e^{-r\tau(q_\gamma)} g(x_{\tau(q_\gamma)}, q_\gamma) \middle| x_0 = x\right] \\ &= \gamma E\left[e^{-r\tau(q_\gamma)} g(x_{\tau(q_\gamma)}, q) \middle| x_0 = x\right] + (1-\gamma) E\left[e^{-r\tau(q_\gamma)} g(x_{\tau(q_\gamma)}, q') \middle| x_0 = x\right] \\ &< \gamma L(x,q) + (1-\gamma) L(x,q'), \end{aligned}$$

for all $x \in (\underline{x}(q), \overline{x}(q))$, so $L(x,q)$ is strictly convex in q on $(\underline{x}(q), \overline{x}(q))$. ■

B.3 Supplement to proof of Theorem 1

This appendix shows that, with two types of buyers, in any equilibrium the seller's profits at state (x,q) with $q < \alpha$ are equal to $L(x,q)$. The proof is divided into a series of lemmas.

Lemma B8 *Let* $(\{q_t\}, \mathbf{P})$ *be an equilibrium. Then,*

(i) if $x_t \leq z_2$ *and* $q_{t^-} < \alpha$*, then* $q_s > q_{t^-}$ *for all* $s > t$ *(i.e., the monopolist makes positive sales between* t *and* $s > t$*),*

(ii) if $x_t > z_2$ *and* $q_{t^-} < \alpha$*, then* $q_s = q_{t^-}$ *for all* $s \in (t, \tau_2)$ *(i.e., the monopolist doesn't make sales until costs reach* z_2*).*

Proof. (i) Suppose that $x_t \leq z_2$, and that there exists $s > t$ such that $q_s = q_{t^-}$. Let $\tau = \inf\{u > t : q_u > q_t\}$, so $\tau > t$. By payoff maximization, the price that the marginal buyer $q_t^+ = \lim_{\varepsilon \downarrow 0} q_t + \varepsilon$ is willing to pay at time t satisfies $P\left(x_t, q_t^+\right) = v_2 - E_t[e^{-r(\tau-t)}(v_2 - P(x_\tau, q_\tau)]$, where $E_t[\cdot] = E[\cdot|\mathcal{F}_t]$. The monopolist gets a profit margin of $E_t[e^{-r(\tau-t)}(P(x_\tau, q_\tau) - x_\tau)]$ from selling to consumer q_t^+ at time τ, while she would get $P\left(x_t, q_t^+\right) - x_t = v_2 - x_t - E_t[e^{-r(\tau-t)}(v_2 - P(x_\tau, q_\tau)]$ from selling to consumer q_t^+ at time t. Note that

$$P\left(x_t, q_t^+\right) - x_t - E_t[e^{-r(\tau-t)}(P(x_\tau, q_\tau) - x_\tau)] = v_2 - x_t - E_t[e^{-r(\tau-t)}(v_2 - x_\tau)] > 0,$$

where the inequality follows since $v_2 - x_t > E_t[e^{-r(\tau-t)}(v_2 - x_\tau)]$ for all $\tau > t$ when $x_t \leq z_2$ (Lemma 1). Thus, the seller is better off selling to q_t^+ at t. Finally, since $\lim_{i \downarrow q_t} P(x, i) = P\left(x, q_t^+\right)$, there exists $\varepsilon > 0$ such that the monopolist is better off selling to all $i \in (q_t^+, q_t^+ + \varepsilon]$ at time t than after time τ, so $(\{q_t\}, \mathbf{P})$ cannot be an equilibrium.

(ii) Suppose that the monopolist makes sales while $x_t > z_2$. Let τ_α denote the time at which consumer α buys and recall that $\tau_2 = \inf\{t : x_t \leq z_2\}$. Let $\tau = \min\{\tau_\alpha, \tau_2\}$. I first show that the price at which the monopolist sells at any $s \in [t, \tau_2]$ satisfies $P(x_s, q_s) = v_2 - E_s[e^{-r(\tau_2-s)}(v_2 - P(x_{\tau_2}, q_{\tau_2}))]$. To see this, note that all high types must get the same payoff in equilibrium. Hence, for all $s < \tau$ and $u \in [s, \tau]$, the price at which the monopolist sells at s satisfies $P(x_s, q_s) = v_2 - E_s[e^{-r(u-s)}(v_2 - P(x_u, q_u))]$. Thus, if $\tau_\alpha \geq \tau_2$, then $P(x_s, q_s) = v_2 - E_s[e^{-r(\tau_2-s)}(v_2 - P(x_{\tau_2}, q_{\tau_2}))]$ for all $s \in [t, \tau_2]$. On the other hand, if $\tau_\alpha < \tau_2$, then $P(x_s, q_s) = v_2 - E_s[e^{-r(\tau_\alpha-s)}(v_2 - P(x_{\tau_\alpha}, \alpha))]$ for all $s \in [t, \tau_\alpha]$. By equation (8),

$$P(x_{\tau_\alpha}, \alpha) = v_2 - E_{\tau_\alpha}\left[e^{-r(\tau_1-\tau_\alpha)}(v_2 - v_1)\right] = v_2 - E_{\tau_\alpha}\left[e^{-r(\tau_2-\tau_\alpha)}(v_2 - P(x_{\tau_2}, \alpha))\right],$$

where the second equality follows since $P(x_{\tau_2}, \alpha) = v_2 - E\left[e^{-r(\tau_1-\tau_2)}(v_2 - v_1)\middle| \mathcal{F}_{\tau_2}\right]$. The law of iterated expectations and the fact that $q_{\tau_2} = \alpha$ whenever $\tau_\alpha < \tau_2$ imply that $P(x_s, q_s) = v_2 - E_s[e^{-r(\tau_\alpha-s)}(v_2 - P(x_{\tau_\alpha}, \alpha))] = v_2 - E_s[e^{-r(\tau_2-s)}(v_2 - P(x_{\tau_2}, q_{\tau_2}))]$.

The profits that the monopolist gets from selling to high valuation consumers between time t and τ_2 are $E_t[e^{-r(s-t)}\int_t^{\tau_2}(P(x_s, q_s) - x_s)dq_s]$. If instead the monopolist waits until time

τ_2 and sells to all consumers $i \in [q_{t^-}, q_{\tau_2}]$ at that instant at price $P(x_{\tau_2}, q_{\tau_2})$, her profits are $E_t[e^{-r(\tau_2-t)}(P(x_{\tau_2}, q_{\tau_2}) - x_{\tau_2})(q_{\tau_2} - q_{t^-})]$. Since $P(x_s, q_s) = v_2 - E_s[e^{-r(\tau_2-s)}(v_2 - P(x_{\tau_2}, q_{\tau_2}))]$ for all $s \in [t, \tau_2)$,

$$P(x_s, q_s) - x_s - E_s\left[e^{-r(\tau_2-s)}(P(x_{\tau_2}, q_{\tau_2}) - x_{\tau_2})\right] = v_2 - x_s - E_s\left[e^{-r(\tau_2-s)}(v_2 - x_{\tau_2})\right] < 0,$$

since, by Lemma 1, $v_2 - x_s < E_s[e^{-r(\tau_2-s)}(v_2 - x_{\tau_2})]$ whenever $x_s > z_2$. Hence, the seller is better off by delaying sales until τ_2, so $(\{q_t\}, \mathbf{P})$ cannot be an equilibrium. ■

Lemma B9 *Let $(\{q_s\}, \mathbf{P})$ be an equilibrium. Then, for all $x \in (0, z_2]$, $P(x, \cdot)$ is continuous on $[0, \alpha)$.*

Proof. Suppose that there exists $x' \in (0, z_2]$ such that $P(x', \cdot)$ is discontinuous at $j \in [0, \alpha)$, with $P(x, j) > P(x, j^+)$. Since $P(\cdot, i)$ is assumed to be continuous for all i, there must exist $x_a > x' > x_b$ such that $P(x, \cdot)$ is discontinuous at j for all $x \in (x_a, x_b)$. By Lemma B8, the monopolist always makes sales when the level of market penetration is below α and costs are below z_2. This implies that, in equilibrium, when $q = j$ and $x \in (x_a, x_b)$, prices must jump down immediately after consumer j buys. But this cannot happen in equilibrium, since consumer j would be strictly better-off delaying trade than buying at price $P(x, j)$. Hence, $P(x, \cdot)$ is continuous on $i \in [0, \alpha)$ for all $x \in (0, z_2]$. ■

Lemma B10 *Let $(\{q_s\}, \mathbf{P})$ be an equilibrium and let $t \in [0, \infty)$ be such that $q_{t^-} < \alpha$. If $\{q_s\}$ is continuous and strictly increasing in $[t, \tau)$ for some $\tau > t$, then there exists $u > t$ such that $P(x_t, q_t) - x_t = E_t[e^{-r(s-t)}(P(x_s, q_t) - x_s)]$ for all $s \in [t, u]$.*

Proof. Note first that, for all times t, $s \geq t$ and $i < \alpha$, $E_t[e^{-r(s-t)}(P(x_s, i) - x_s)]$ is continuous in s (this follows since $P(x, i) - x$ is a continuous function of x, and since the distribution of x_s conditional on x_t is continuous in s).

For all $s \geq t$ and i, define $\Delta(s, i) := P(x_t, i) - x_t - E_t[e^{-r(s-t)}(P(x_s, i) - x_s)]$. By the previous paragraph, $\Delta(\cdot, i)$ is continuous for all $s \geq t$. Moreover, by Lemma B9, $\Delta(s, \cdot)$ is continuous on $[0, \alpha)$ for all s close enough to t.[25] Suppose that the result is not true, so that for all $u > t$ there exists $s \in (t, u]$ with $\Delta(s, q_t) \neq 0$. Since $\Delta(\cdot, q_t)$ is continuous, there exist $u > 0$ such that either (i) $\Delta(s, q_t) > 0$ for all $s \in (t, u]$ or (ii) $\Delta(s, q_t) < 0$ for all $s \in (t, u]$.

Consider case (i) first, so that $\Delta(s, q_t)$ is strictly increasing in s at $s = t$. Since $\Delta(s, i)$ is continuous in i for all s close to t, there must exist $i^* > q_t$ such that $\Delta(s, i)$ is increasing in s at $s = t$ for all $i \in [q_t, i^*]$ (i.e., $\Delta(s, i) > 0$ for all $i \in [q_t, i^*]$ and all s close enough to t).

[25] Indeed, by Lemma B8, if q_t is strictly increasing in $[t, \tau)$ for some $\tau > t$, it must be that $x_t < z_2$.

I now show that, in this case, the monopolist has a profitable deviation. For each $i \in [0,1]$, let $\tau(i) = \inf\{s : q_s \geq i\}$ be the time at which consumer i buys. Let $\Pi(x_t, q_t)$ be the seller's continuation payoff at time t, and note that for all $q > q_t$

$$\Pi(x_t, q_t) = E_t\left[\int_t^{\tau(q)} e^{-r(s-t)}\left(P\left(x_s, q_s\right) - x_s\right) dq_s + e^{-r(\tau(q)-t)}\Pi\left(x_{\tau(q)}, q\right)\right]. \tag{B.4}$$

Note that for any $q > q_t$, at time t the monopolist can get profits arbitrarily close to $\int_{q_t}^{q}(P(x_t, i) - x_t)di + E_t[e^{-r(\tau(q)-t)}\Pi(x_{\tau(q)}, q)]$ by selling to consumers in $[q_t, q]$ immediately and then not making any sale until time $\tau(q)$. Since $\Delta(s, i) > 0$ for all $i \in [q_t, i^*]$ and all s close enough to t, there exists $q \in (q_t, i^*)$ such that $\int_{q_t}^{q}(P(x_t, i) - x_t)di + E_t[e^{-r(\tau(q)-t)}\Pi(x_{\tau(q)}, q)]$ is strictly larger than the monopolist's profits in (B.4). But this contradicts the fact that $(\{q_s\}, \mathbf{P})$ is an equilibrium, so $\Delta(s, q_t)$ must be weakly decreasing at $s = t$.

Consider next case (ii), so that $\Delta(s, q_t)$ is strictly decreasing in s at $s = t$. Since $\Delta(s, i)$ is continuous in i and s, there must exist $i^* > q_t$ such that $\Delta(s, i)$ is decreasing in $s = t$ for all $i \in [q_t, i^*]$ (i.e., $\Delta(s, i) < 0$ for all $i \in [q_t, i^*]$ and all s close enough to t). I now show that, if this were the case, the monopolist would also have a profitable deviation.

Note that for any $q > q_t$, at time t the monopolist can get profits arbitrarily close to $E_t[e^{-r(\tau(q)-t)}(\int_{q_t}^{q}(P(x_{\tau(q)}, i) - x_{\tau(q)})di + \Pi(x_{\tau(q)}, q))]$ by not making any sales until time $\tau(q)$, selling to all consumers $i \in [q_t, q]$ arbitrarily fast at time $\tau(q)$, and then continuing playing the equilibrium. Since $\Delta(s, i) < 0$ for all $i \in [q_t, i^*]$ and all s close enough to t, there exists a $q \in (q_t, i^*)$ such that $E_t[e^{-r(\tau(q)-t)}(\int_{q_t}^{q}(P(x_{\tau(q)}, i) - x_{\tau(q)})di + \Pi(x_{\tau(q)}, q))]$ is larger than the monopolist's profits in (B.4), which cannot be since $(\{q_s\}, \mathbf{P})$ is an equilibrium. Hence, $\Delta(s, q_t)$ must be weakly increasing at $s = t$. Together with the arguments above, there must exist $u > t$ such that $\Delta(s, q_t) = 0$ for all $s \in [t, u]$. ■

Lemma B11 *Let $(\{q_s\}, \mathbf{P})$ be an equilibrium and let $\Pi(x, q)$ be the seller's profits. Let $t \in [0, \infty)$ be such that $q_{t^-} < \alpha$ and such that $\{q_s\}$ is continuous in $[t, \tau)$ for some $\tau > t$. Then there exists $u \in (t, \tau]$ such that $\Pi(x_t, q_t) = E_t[e^{-r(s-t)}\Pi(x_s, q_t)]$ for all $s \in [t, u]$.*

Proof. Suppose first that $\{q_s\}$ is constant on $[t, \tau)$. Then, $\Pi(x_t, q_t) = E_t[e^{-r(u-t)}\Pi(x_u, q_t)]$ for all $u \in [t, \tau)$, since the monopolist doesn't make any sales on $[t, \tau)$.

Suppose next that $\{q_s\}$ is continuous and increasing in $[t, \tau)$. Note first that $\Pi(x_t, q_t) \geq E_t\left[e^{-r(u-t)}\Pi(x_u, q_t)\right]$ for all $u > t$, since the monopolist can always choose to make no sales between t and $u > t$. Suppose by contradiction that the statement in the Lemma is not true.

Hence, for every $u > t$ there exists $s \in (t, u]$ such that

$$\Pi(x_t, q_t) = E_t \left[\int_t^s e^{-r(\tilde{s}-t)} (P(x_{\tilde{s}}, q_{\tilde{s}}) - x_{\tilde{s}}) dq_{\tilde{s}} + e^{-r(s-t)} \Pi(x_s, q_s) \right] > E_t \left[e^{-r(s-t)} \Pi(x_s, q_t) \right].$$

Note that $\Pi(x_s, q_t) \geq \int_{q_t}^{q_s} (P(x_u, i) - x_u) di + \Pi(x_s, q_s)$, since at state (x_s, q_t) the monopolist can get profits arbitrarily close to $\int_{q_t}^{q_s} (P(x_u, i) - x_u) di + \Pi(x_s, q_s)$ by selling to all buyers $i \in [q_t, q_s]$ arbitrarily fast. Combining this with the equation above, it follows that for all $u > t$ there exists $s \in (t, u]$ such that

$$E_t \left[\int_t^s e^{-r(\tilde{s}-t)} (P(x_{\tilde{s}}, q_{\tilde{s}}) - x_{\tilde{s}}) dq_{\tilde{s}} \right] > E_t \left[e^{-r(s-t)} \int_{q_t}^{q_s} (P(x_s, i) - x_s) di \right]. \tag{B.5}$$

Equation (B.5) in turn implies that, for all $u \in (t, \tau]$ there exists $s \in (t, u]$ and $i \in [q_t, q_s)$ such that $P(x_{\tau(i)}, i) - x_{\tau(i)} > E_{\tau(i)} \left[e^{-r(s-\tau(i))} (P(x_s, i) - x_s) \right]$ (where $\tau(i) = \inf\{t : q_t \geq i\}$ is the time at which consumer i buys), which contradicts Lemma B10. ■

Lemma B12 *Let $(\{q_t\}, \mathbf{P})$ be an equilibrium and let $\Pi(x, q)$ be the seller's profits. Suppose $\{q_s\}$ is continuous in $[t, \tau)$ for some $\tau > t$. Then, there exists $\hat{\tau} > t$ such that $\{q_s\}$ is discontinuous at state $(x_{\hat{\tau}}, q_{t^-})$; i.e., $\{q_s\}$ jumps up at this state. Moreover,*

$$\Pi(x_t, q_{t^-}) = E_t \left[e^{-r(\hat{\tau}-t)} ((P(x_{\hat{\tau}}, q_t + dq_{\hat{\tau}}) - x_{\hat{\tau}}) dq_{\hat{\tau}} + \Pi(x_{\hat{\tau}}, q_t + dq_{\hat{\tau}})) \right],$$

where $dq_{\hat{\tau}}$ denotes the jump of $\{q_s\}$ at state $(x_{\hat{\tau}}, q_{t^-})$.

Proof. Let $\hat{\tau} = \sup\{u > t : \Pi(x_t, q_t) = E_t[e^{-r(s-t)} \Pi(x_s, q_t)]$ for all $s \in [t, u]\}$. By Lemma B11, $\hat{\tau} > 0$. I now show that $\{q_t\}$ jumps at state $(x_{\hat{\tau}}, q_t)$. Suppose not. Then, by Lemma B11 there exists $u' > \hat{\tau}$ such that $\Pi(x_{\hat{\tau}}, q_t) = E_t[e^{-r(s-\hat{\tau})} \Pi(x_s, q_t)]$ for all $s \in [\hat{\tau}, u']$. By the Law of Iterated Expectations, $\Pi(x_t, q_t) = E_t[e^{-r(\hat{\tau}-t)} \Pi(x_{\hat{\tau}}, q_t)] = E_t[e^{-r(u'-t)} \Pi(x_{u'}, q_t)]$, which contradicts the definition of $\hat{\tau}$. Hence, it must be that $\{q_t\}$ jumps at state $(x_{\hat{\tau}}, q_t)$, with $\Pi(x_{\hat{\tau}}, q_t) = (P(x_{\hat{\tau}}, q_t + dq_{\hat{\tau}}) - x_{\hat{\tau}}) dq_{\hat{\tau}} + \Pi(x_{\hat{\tau}}, q_t + dq_{\hat{\tau}})$. Therefore,

$$\Pi(x_t, q_t) = E_t \left[e^{-r(\hat{\tau}-t)} ((P(x_{\hat{\tau}}, q_t + dq_{\hat{\tau}}) - x_{\hat{\tau}}) dq_{\hat{\tau}} + \Pi(x_{\hat{\tau}}, q_t + dq_{\hat{\tau}})) \right].$$

■

Lemma B13 *Let $(\{q_t\}, \mathbf{P})$ be an equilibrium and let $\Pi(x, q)$ be the seller's profits. If $\{q_s\}$ is continuous and strictly increasing in $[t, \tau)$ for some $\tau > t$, then $-\Pi_q(x_s, q_s) = P(x_s, q_s) - x_s$ for all $s \in [t, \tau)$.*

Proof. Note first that for all $\varepsilon > 0$ and for all $s \in [t, \tau)$, it must be that

$$\Pi(x_s, q_s) \geq (P(x_s, q_s + \varepsilon) - x_s)\varepsilon + \Pi(x_s, q_s + \varepsilon), \tag{B.6}$$

since the monopolist can always choose at time s to sell to all buyers $i \in [q_s, q_s + \varepsilon]$ at price $P(x_s, q_s + \varepsilon)$. Next, I show that for all $s \in [t, \tau)$ it must also be that

$$\Pi(x_s, q_s) \leq (P(x_s, q_s) - x_s)\varepsilon + \Pi(x_s, q_s + \varepsilon), \tag{B.7}$$

for all $\varepsilon > 0$ small enough. To see this, let $\tau_\varepsilon = \inf\{u : q_u \geq q_s + \varepsilon\}$, so that

$$\Pi(x_s, q_s) = E_s\left[\int_s^{\tau_\varepsilon} e^{-r(u-s)}(P(x_u, q_u) - x_u)\,dq_u + e^{-r(\tau_\varepsilon - s)}\Pi(x_{\tau_\varepsilon}, q_s + \varepsilon)\right].$$

Note that $\Pi(x_s, q_s + \varepsilon) \geq E_s[e^{-r(\tau_\varepsilon - s)}\Pi(x_{\tau_\varepsilon}, q_s + \varepsilon)]$, since at state $(x_s, q_s + \varepsilon)$ the monopolist can always achieve profits $E_s[e^{-r(\tau_\varepsilon - s)}\Pi(x_{\tau_\varepsilon}, q_s + \varepsilon)]$ by not making any sales until time τ_ε. Combining this with the equation above, it follows that

$$\begin{aligned} & \Pi(x_s, q_s) - (P(x_s, q_s) - x_s)\varepsilon - \Pi(x_s, q_s + \varepsilon) \\ \leq\ & E_s\left[\int_s^{\tau_\varepsilon} e^{-r(u-s)}(P(x_u, q_u) - x_u)\,dq_u\right] - (P(x_s, q_s) - x_s)\varepsilon \end{aligned} \tag{B.8}$$

To establish (B.7) it suffices to show that the right-hand side of (B.8) is less than zero for $\varepsilon > 0$ small enough. By Lemma B10, there exists $\hat{u} > s$ such that, for all $u \in [s, \hat{u}]$, $P(x_s, q_s) - x_s = E_s\left[e^{-r(u-s)}(P(x_u, q_s) - x_u)\right] \geq E_s\left[e^{-r(u-s)}(P(x_u, q_u) - x_u)\right]$, where the inequality follows since $q_u > q_s$ and since $P(x, i)$ is decreasing in i. Hence, for $\varepsilon > 0$ small the right-hand side of (B.8) is less than zero, and so (B.7) holds.

Finally, by (B.6) and (B.7), for $\varepsilon > 0$ small, $P(x_s, q_s + \varepsilon) - x_s \leq -\frac{\Pi(x_s, q_s + \varepsilon) - \Pi(x_s, q_s)}{\varepsilon} \leq P(x_s, q_s) - x_s$. Since $P(x, q)$ is continuous in q at (x_s, q_s) (Lemma B9), $-\Pi_q(x_s, q_s) = P(x_s, q_s) - x_s$. ■

Corollary B2 *Let $(\{q_t\}, \mathbf{P})$ be an equilibrium and let $\Pi(x, q)$ be the seller's profits. For all states (x, q) with $x \leq z_2$ and $q \in [0, \alpha)$, $-\Pi_q(x, q) = P(x, q) - x$.*

Proof. Fix a state (x, q) with $x \leq z_2$ and $q \in [0, \alpha)$ and note that by Lemma B8 $\{q_t\}$ is strictly increasing at time s if $(x_s, q_{s^-}) = (x, q)$. There are two possibilities: (i) $\{q_t\}$ is continuous and increasing at time s if $(x_s, q_{s^-}) = (x, q)$, or (ii) $\{q_t\}$ jumps at time s if $(x_s, q_{s^-}) = (x, q)$. In case (i), the Corollary follows from Lemma B13. In case (ii), $\Pi(x, q) =$

$(P(x,q+dq)-x)dq+\Pi(x,q+dq)$, where $dq>0$ is the amount by which $\{q_t\}$ jumps at time s if $(x_s,q_{s^-})=(x,q)$. Hence, in this case $-\Pi_q(x,q)=P(x,q+dq)-x$. Finally, note that in this latter case it must be that $P(x,i)=P(x,q+dq)$ for all $i\in[q,q+dq]$: all buyers in this range know that the monopolist will charge price $P(x,q+dq)$ at state (x,q), so in equilibrium they are not willing to pay a higher price. It then follows that $-\Pi_q(x,q)=P(x,q)-x$. ■

Lemma B14 *Let* $(\{q_t\},\mathbf{P})$ *be an equilibrium and let* $\Pi(x,q)$ *denote the monopolist's profits. Then,* $\Pi(x,q)=L(x,q)$ *for all states* (x,q) *with* $q<\alpha$.

Proof. By the arguments in the main text, $\Pi(x,q)\geq L(x,q)$ for all states (x,q) with $q<\alpha$. I now show that $\Pi(x,q)\leq L(x,q)$ for all states (x,q) with $q<\alpha$.

Fix a state (x,q) with $q<\alpha$, and suppose $(x_t,q_{t^-})=(x,q)$. Note that at this state either $\{q_u\}$ jumps at t (i.e., $dq_t=q_t-q_{t^-}>0$) or $\{q_u\}$ is continuous on $[t,s)$ for some $s>t$. In the first case, $\Pi(x_t,q_{t^-})=(P(x_t,q_{t^-}+dq_t)-x_t)dq_t+\Pi(x_t,q_{t^-}+dq_t)$. In the second case, by Lemma B12 there exists $\hat{\tau}>t$ and $dq_{\hat{\tau}}>0$ such that $\Pi(x_t,q_{t^-})=E_t[e^{-r(\hat{\tau}-t)}((P(x_{\hat{\tau}},q_{t^-}+dq_{\hat{\tau}})-x_{\hat{\tau}})dq_{\hat{\tau}}+\Pi(x_{\hat{\tau}},q_{t^-}+dq_{\hat{\tau}}))]$. Let

$$\tilde{\tau}=\sup\{\tau\geq t:\Pi(x_t,q_{t^-})=E_t[e^{-r(\tau-t)}((P(x_\tau,q_{t^-}+dq_\tau)-x_\tau)dq_\tau+\Pi(x_\tau,q_{t^-}+dq_\tau))]\}.$$

Note that if $dq_{\tilde{\tau}}\geq\alpha-q_{t^-}$, then

$$\begin{aligned}&(P(x_{\tilde{\tau}},q_{t^-}+dq_{\tilde{\tau}})-x_{\tilde{\tau}})dq_{\tilde{\tau}}+\Pi(x_{\tilde{\tau}},q_t+dq_{\tilde{\tau}})\\ \leq&(P(x_{\tilde{\tau}},\alpha)-x_{\tilde{\tau}})(\alpha-q_{t^-})+(P(x_{\tilde{\tau}},q_{t^-}+dq_{\tilde{\tau}})-x_{\tilde{\tau}})(dq_{\tilde{\tau}}-\alpha+q_{t^-})+\Pi(x_{\tilde{\tau}},q_{t^-}+dq_{\tilde{\tau}})\\ \leq&(P(x_{\tilde{\tau}},\alpha)-x_{\tilde{\tau}})(\alpha-q_{t^-})+(1-\alpha)V_1(x_{\tilde{\tau}})=g(x_{\tilde{\tau}},q_{t^-}),\end{aligned}$$

where the first inequality follows since $P(x_{\tilde{\tau}},q_{t^-}+dq_{\tilde{\tau}})\leq P(x_{\tilde{\tau}},\alpha)$ and the second inequality follows since $(1-\alpha)V_1(x_{\tilde{\tau}})\geq(P(x_{\tilde{\tau}},q_{t^-}+dq_{\tilde{\tau}})-x_{\tilde{\tau}})(dq_{\tilde{\tau}}-\alpha+q_{t^-})+\Pi(x_{\tilde{\tau}},q_{t^-}+dq_{\tilde{\tau}})$ when $dq_{\tilde{\tau}}\geq\alpha-q_{t^-}$. This implies that $\Pi(x_t,q_{t^-})\leq E_t[e^{-r(\tilde{\tau}-t)}g(x_{\tilde{\tau}},q_{t^-})]\leq L(x_t,q_{t^-})=\sup_\tau E[e^{-r\tau}g(x_\tau,q_{t^-})]$, and so $\Pi(x_t,q_{t^-})=L(x_t,q_{t^-})$. The rest of the proof establishes that, indeed, $dq_{\tilde{\tau}}\geq\alpha-q_{t^-}$.

Towards a contradiction, suppose that $dq_{\tilde{\tau}}=q_{\tilde{\tau}}-q_{t^-}<\alpha-q_{t^-}$, so that $\Pi(x_{\tilde{\tau}},q_{t^-})=(P(x_{\tilde{\tau}},q_{\tilde{\tau}})-x_{\tilde{\tau}})(q_{\tilde{\tau}}-q_{t^-})+\Pi(x_{\tilde{\tau}},q_{\tilde{\tau}})$. By Lemma B12, there exists $\tau'>\tilde{\tau}$ such that $\Pi(x_{\tilde{\tau}},q_{\tilde{\tau}})=E_{\tilde{\tau}}[e^{-r(\tau'-\tilde{\tau})}((P(x_{\tau'},q_{\tilde{\tau}}+dq_{\tau'})-x_{\tau'})dq_{\tau'}+\Pi(x_{\tau'},q_{\tilde{\tau}}+dq_{\tau'}))]$, where $dq_{\tau'}$ denotes the jump of $\{q_t\}$ at state $(x_{\tau'},q_{\tilde{\tau}})$. This implies that $-\Pi_q(x_{\tilde{\tau}},q_{\tilde{\tau}})=E_{\tilde{\tau}}[e^{-r(\tau'-\tilde{\tau})}(P(x_{\tau'},q_{\tilde{\tau}}+dq_{\tau'})-x_{\tau'})]$. On the other hand, by Corollary B2 it must be that $P(x_{\tilde{\tau}},q_{\tilde{\tau}})-x_{\tilde{\tau}}=-\Pi_q(x_{\tilde{\tau}},q_{\tilde{\tau}})$,[26]

[26] Indeed, by Lemma B8 the monopolist only sells to high types when $x\leq z_2$, so it must be that $x_{\tilde{\tau}}\leq z_2$.

and so $P\left(x_{\tilde{\tau}}, q_{\tilde{\tau}}\right) - x_{\tilde{\tau}} = E_{\tilde{\tau}}[e^{-r(\tau'-\tilde{\tau})}(P(x_{\tau'}, q_{\tilde{\tau}} + dq_{\tau'}) - x_{\tau'})]$. Since $\Pi\left(x_{\tilde{\tau}}, q_{t^-}\right) = (P(x_{\tilde{\tau}}, q_{\tilde{\tau}}) - x_{\tilde{\tau}})(q_{\tilde{\tau}} - q_{t^-}) + \Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}})$ and since $\Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}}) = E_{\tilde{\tau}}[e^{-r(\tau'-\tilde{\tau})}((P(x_{\tau'}, q_{\tilde{\tau}} + dq_{\tau'}) - x_{\tau'})dq_{\tau'} + \Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}} + dq_{\tau'}))]$, it follows that

$$\Pi\left(x_{\tilde{\tau}}, q_{t^-}\right) = E_{\tilde{\tau}}\left[e^{-r(\tau'-\tilde{\tau})}\left(\left(P\left(x_{\tau'}, q_{\tau'}\right) - x_{\tau'}\right)\left(dq_{\tau'} + q_{\tilde{\tau}} - q_{t^-}\right) + \Pi\left(x_{\tau'}, q_{\tau'}\right)\right)\right].$$

By the Law of Iterated Expectations,

$$\begin{aligned}\Pi\left(x_t, q_{t^-}\right) &= E_t[e^{-r(\tilde{\tau}-t)}((P\left(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}\right) - x_{\tilde{\tau}})\, dq_{\tilde{\tau}} + \Pi\left(x_{\tilde{\tau}}, q_{\tilde{\tau}}\right))] \\ &= E_t\left[e^{-r(\tau'-t)}\left(\left(P\left(x_{\tau'}, q_{\tau'}\right) - x_{\tau'}\right)\left(dq_{\tau'} + dq_{\tilde{\tau}}\right) + \Pi\left(x_{\tau'}, q_{\tau'}\right)\right)\right],\end{aligned}$$

which contradicts the definition of $\tilde{\tau}$. Hence, it must be that that $dq_{\tilde{\tau}} \geq \alpha - q_{t^-}$. ■

B.4 Proofs of Section 5.2

Proof of Proposition 2. Note first that

$$\lim_{v_1 \to 0} P(x, \alpha) = \lim_{v_1 \to 0} v_2 - (v_2 - v_1)(x/z_1)^{\lambda_N} = v_2,$$

where the last equality follows since $\lambda_N < 0$ and since $z_1 = \frac{-\lambda_N v_1}{1-\lambda_N} \to 0$ as $v_1 \to 0$.

The seller's full commitment profits are $\Pi^{FC}(x) = E[e^{-r\tau_2}\alpha\,(v_2 - x_\tau)|\, x_0 = x]$ when $v_1 < \alpha v_2$ (see Appendix C.3). On the other hand, the seller's equilibrium profits $L(x, 0)$ are larger than $E[e^{-r\tau_2}\alpha(P(x_{\tau_2}, \alpha) - x_{\tau_2})|x_0 = x]$. Therefore, when $v_1 < \alpha v_2$

$$\Pi^{FC}(x) = E[\,e^{-r\tau_2}\alpha\,(v_2 - x_\tau)\big|\, x_0 = x] \geq L(x, 0) \geq E[\,e^{-r\tau_2}\alpha\,(P(x_{\tau_2}, \alpha) - x_{\tau_2})\big|\, x_0 = x].$$

Since $\lim_{v_1 \to 0} P(x, \alpha) = v_2$, it follows that the seller's equilibrium profits converge to her full commitment profits as $v_1 \to 0$. ■

Proof of Proposition 3. The result follows from equation (A.3) in Appendix A.1. ■

Proof of Proposition 4. Let $\mu < 0$ and fix a sequence $\sigma_n \to 0$. For each n, let λ_N^n be the negative root of $\frac{1}{2}\sigma_n^2\lambda\,(\lambda - 1) + \mu\lambda = r$. For $k = 1, 2$, let $z_k^n = \frac{-\lambda_N^n}{1-\lambda_N^n}v_k$. Let $P^n(x, \alpha)$ be buyer α's strategy when $\sigma = \sigma_n$. By Theorem 1, for each n there exists $\underline{x}^n(0) < z_1^n$ and $\overline{x}^n(0) \in (z_1^n, z_2^n)$ such that the monopolist sells at $t = 0$ to all high types at price $P^n(x, \alpha)$ when $x_0 \in [\overline{x}^n(0), z_2^n]$, and sells at $t = 0$ to all buyers at price v_1 when $x_0 \leq \underline{x}^n(0)$. Let $\underline{x}^* = \lim_{n\to\infty} \underline{x}^n(0)$ and $\overline{x}^* = \lim_{n\to\infty} \overline{x}^n(0)$. Since $\lim_{n\to\infty} z_k^n = z_k^* := \frac{r}{r-\mu}v_k$ for $k = 1, 2$, and since $\underline{x}^n(0) < z_1^n$ and $\overline{x}^n(0) \in (z_1^n, z_2^n)$ for all n, it follows that $\underline{x}^* \leq z_1^*$ and $\overline{x}^* \geq z_1^*$. To prove Proposition 4, it suffices to show that $\underline{x}^* = \overline{x}^* = z_1^*$.

Suppose by contradiction that $\underline{x}^* < \underline{x}^*$. Let $L^n(x_0, 0) = E^n[e^{-r\tau_n(0)}(\alpha(P^n(x_{\tau_n(0)}, \alpha) - x_{\tau_n(0)}) + \Pi^n(x_{\tau_n(0)}, \alpha))]$ be the seller's profits when $\sigma = \sigma_n$, where $\tau_n(0) = \inf\{t : x_t \in [0, \underline{x}^n(0)] \cup [\overline{x}^n(0), z_2^n]\}$ and where $E^n[\cdot]$ and $\Pi^n(x, \alpha)$ denote, respectively, the expectation operator and the seller's profits at state (x, α) when $\sigma = \sigma_n$. Let $\hat{\tau} = \inf\{t : x_t \leq \underline{x}^*\}$ and note that $\tau_n(0) \to \hat{\tau}$ as $n \to \infty$ when $x_0 \in (\underline{x}^*, \overline{x}^*)$: as $\sigma \to 0$ the probability that x_t reaches $\overline{x}^*$ before it reaches $\underline{x}^*$ approaches zero if $x_0 \in (\underline{x}^*, \overline{x}^*)$ and $\mu < 0$. Moreover, note that $P^n(\underline{x}^*, \alpha) \to v_1$ and $\Pi^n(\underline{x}^*, \alpha) \to (1-\alpha)(v_1 - \underline{x}^*)$ as $n \to \infty$. Thus, $\lim_{n\to\infty} L^n(x, 0) = \lim_{n\to\infty} E^n[e^{-r\hat{\tau}}(v_1 - \underline{x}^*)|x_0 = x]$ for $x \in (\underline{x}^*, \overline{x}^*)$. There are two cases to consider: (i) $\underline{x}^* < z_1^*$, or (ii) $\overline{x}^* > z_1^*$. In case (i), for n large enough the seller's profits $L_n(x, 0) \approx E^n[e^{-r\hat{\tau}}(v_1 - \underline{x}^*)|x_0 = x]$ are strictly lower than $v_1 - x_0$ when $x_0 \in (\underline{x}^*, z_1^*)$, since, by Lemma 1, $v_1 - x > E^n[e^{-r\tau}(v_1 - x_\tau)|x_0 = x]$ for all τ and $x < z_1^n$. This contradicts the fact that $L^n(x, 0) \geq v_1 - x$ for all $x \leq z_1^n$, so $\underline{x}^* = z_1^*$. In case (ii), $L^n(x, 0) \approx E^n[e^{-r\hat{\tau}}(v_1 - \underline{x}^*)|x_0 = x] < (P^n(x, \alpha) - x)\alpha + \Pi^n(x, \alpha)$ for all $x \in (z_1^*, \overline{x})$ and for n large enough, since by Lemma 2, $P^n(x, \alpha) - x > \sup_\tau E^n[e^{-r\tau}(v_1 - x_\tau)|x_0 = x]$ for all $x \in (z_1^n, z_2^n)$ and since $\Pi^n(x, \alpha) \geq (1 - \alpha)E^n[e^{-r\hat{\tau}}(v_1 - \underline{x}^*)|x_0 = x]$. But this cannot be, since $L^n(x, 0) \geq (P^n(x, \alpha) - x)\alpha + \Pi^n(x, \alpha)$. Hence, $\overline{x}^* = z_1^*$. ■

Proof of Proposition 5. Fix a sequence $(\sigma_n, \mu_n) \to 0$. For each n, let λ_N^n be the negative root of $\frac{1}{2}\sigma_n^2\lambda(\lambda - 1) + \mu_n\lambda = r$. For $i = 1, 2$, let $z_i^n = \frac{-\lambda_N^n}{1-\lambda_N^n}v_i$. Note that $\lim_{n\to\infty} \lambda_N^n = -\infty$ and $\lim_{n\to\infty} z_i^n = v_i$. Let $P^n(x, \alpha)$ be buyer α's strategy when $(\sigma, \mu) = (\sigma_n, \mu_n)$. Note that $\lim_{n\to\infty} P^n(x, \alpha) = v_1$ for $x \leq v_1$ and $\lim_{n\to\infty} P^n(x, \alpha) = v_2$ for $x > v_1$. By Theorem 1, for each n there exists $\underline{x}^n(0) < z_1^n$ and $\overline{x}^n(0) \in (z_1^n, z_2^n)$ such that the monopolist sells at $t = 0$ to all high types at price $P^n(x, \alpha)$ when $x_0 \in [\overline{x}^n(0), z_2^n]$, and sells at $t = 0$ to all buyers at price v_1 when $x_0 \leq \underline{x}^n(0)$. To prove the Proposition it suffices to show that $\underline{x}^* = \lim_{n\to\infty} \underline{x}^n(0) = v_1$ and $\overline{x}^* = \lim_{n\to\infty} \overline{x}^n(0) = v_1$.

Since $\underline{x}^n(0) < z_1^n$ and $\overline{x}^n(0) \in (z_1^n, z_2^n)$ for all n and since $\lim_{n\to\infty} z_1^n = v_1$, $\underline{x}^* \leq v_1$ and $\overline{x}^* \geq v_1$. Suppose by contradiction that $\underline{x}^* \neq v_1$ or $\overline{x}^* \neq v_1$, so that $\underline{x}^* < \overline{x}^*$. Thus, there exists N and $\underline{y} < \overline{y}$ such that $\underline{x}^n(0) \leq \underline{y}$ and $\overline{x}^n(0) \geq \overline{y}$ for all $n \geq N$. Let $L^n(x, 0)$ be the monopolist's profits at state $(x, 0)$ when $(\sigma, \mu) = (\sigma_n, \mu_n)$. By Theorem 1, $L^n(x_0, 0) = E^n[e^{-r\tau_n(0)}(\alpha(P^n(x_{\tau_n(0)}, \alpha) - x_{\tau_n(0)}) + \Pi^n(x_{\tau_n(0)}, \alpha))]$, where $\tau_n(0) = \inf\{t : x_t \in [0, \underline{x}^n(0)] \cup [\overline{x}^n(0), z_2^n]\}$ and where $E^n[\cdot]$ and $\Pi^n(x, \alpha)$ denote, respectively, the expectation operator and the seller's profits at state (x, α) when $(\sigma, \mu) = (\sigma_n, \mu_n)$. Let $\widehat{\tau} = \inf\{t : x_t \notin (\underline{y}, \overline{y})\}$ and note that for all $n \geq N$, $\tau_n(0) \geq \widehat{\tau}$ whenever $x_0 \in [\underline{y}, \overline{y}]$. Fix $x \in (\underline{y}, \overline{y})$. Since $P^n(x_{\tau_n(0)}, \alpha) < v_2$

and $\Pi^n(x_{\tau_n(0)}, \alpha) < (1-\alpha)\, v_2$ for all n, $L^n(x, 0) < v_2 E^n\left[e^{-r\hat{\tau}}\right]$ for all $n \geq N$. Finally, note that $\lim_{n\to\infty} E^n\left[e^{-r\hat{\tau}}\right] = 0$ when $x_0 \in (\underline{y}, \overline{y})$: as $(\sigma, \mu) \to 0$ it takes arbitrarily long until costs leave the interval $(\underline{y}, \overline{y})$. This implies that $L^n(x, 0) \to 0$, which cannot be since $L^n(x, 0) \geq \alpha(P^n(x, \alpha) - x) + (1-\alpha)\Pi^n(x, \alpha) > 0$ for all $x < z_2^n$. Thus, $\underline{x}^* = \overline{x}^* = v_1$. ■

C Online Appendix

C.1 Proof of Theorem 2

The proof of Theorem 2 is organized as follows. First, I show that the lower bound $L(x, q)$ (i.e., the value function of the optimal stopping problem (15)) is well defined for all $q \in [0, 1]$. I then show that the monopolist's equilibrium profits are equal to $L(x, q)$ for all states (x, q).

I begin by showing that the lower bound $L(x, q)$ is well defined for all $q \in [0, 1]$. I use the following result in optimal stopping problems.

Lemma OA1 *Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function that is bounded on any compact subset of* $[0, \infty)$. *Then,*

$$W(x) = \sup_{\tau} E[e^{-r\tau} h(x_\tau) | x_0 = x], \tag{C.1}$$

is continuous in x. Moreover, the stopping time $\tau(S) = \inf\{t : x_t \in S\}$ solves (C.1), *where* $S = \{x \in [0, \infty) : W(x) = h(x)\}$.

The proof of Lemma OA1 can be found in Dayanik and Karatzas (2003).

Lemma OA1 can be used to show that $L(x, q)$ is well defined for all $q \in [0, 1]$ and all x. Indeed, by Lemma B2, $L(x, q)$ is well-defined for all $q \in [\alpha_3, 1]$. Consider next $q \in [\alpha_4, \alpha_3)$, and let

$$\begin{aligned} g(x, q) &= (\alpha_3 - q)(P(x, \alpha_3) - x) + L(x, \alpha_3) \\ \Rightarrow L(x, q) &= \sup_{\tau} E[e^{-r\tau} g(x_\tau, q) | x_0 = x]. \end{aligned} \tag{C.2}$$

Note that $g(x, q)$ is continuous and bounded on any compact subset of $[0, \infty)$ (since $P(x, \alpha_3) - x$ and $L(x, \alpha_3)$ satisfy these conditions). Therefore, by Lemma OA1, $L(x, q)$ is continuous in x for all $q \in [\alpha_4, \alpha_3)$. Moreover, the stopping time $\tau(q) = \inf\{t : x_t \in S(q)\}$ solves (C.2), where $S(q) = \{x \in (0, \infty) : L(x, q) = g(x, q)\}$. Repeating this argument inductively establishes that, for all k, $L(x, q)$ is continuous in x for all $q \in [\alpha_{k+1}, \alpha_k)$.

Note that since $L(x,q)$ and $g(x,q)$ are continuous, the optimal stopping region $S(q)$ is a union of intervals. Fix $x \notin S(q)$. When $x_0 = x$, the stopping time $\tau(q)$ is equal to the first time at which x_t reaches either $\overline{z}(x) = \inf\{y \in S(q), y > x\}$ or $\underline{z}(x) = \sup\{y \in S(q), y < x\}$ (if the first set is empty, set $\overline{z}(x) = \infty$; if the second set is empty, set $\underline{z}(x) = 0$). Let $\tau_x = \inf\{t : x_t \notin (\underline{z}(x), \overline{z}(x))\}$, and note that $L(y,q) = E[e^{-r\tau_x} g(x_{\tau_x}, q)|x_0 = y]$ for all $y \in (\underline{z}(x), \overline{z}(x))$. By Lemma B1, for all $y \in (\underline{z}(x), \overline{z}(x))$, $L(y,q)$ solves[27]

$$rL(y,q) = \mu y L_y(y,q) + \frac{\sigma^2 y^2}{2} L_{yy}(y,q). \tag{C.3}$$

The following results are the counterparts of Lemmas B8-B13 to the current setting with $n > 2$ types of consumers. Their proofs are the same as the proofs of Lemmas B8-B13, and hence omitted for conciseness.

Lemma OA2 *Let $(\{q_t\}, \mathbf{P})$ be an equilibrium.*

(i) if $x_t \leq z_k$ and $q_{t^-} \in [\alpha_{k+1}, \alpha_k)$, $q_s > q_{t^-}$ for all $s > t$ (i.e., the monopolist makes positive sales between t and $s > t$),

(ii) if $x_t > z_k$ and $q_{t^-} \in [\alpha_{k+1}, \alpha_k)$, $q_s = q_{t^-}$ for all $s \in (t, \tau_k)$ (i.e., the monopolist doesn't make sales until costs reach z_k).

Lemma OA3 *Let $(\{q_s\}, \mathbf{P})$ be an equilibrium. Then, for all k and all $x \in (0, z_k]$, $P(x, \cdot)$ is continuous on $[\alpha_{k+1}, \alpha_k)$.*

Lemma OA4 *Let $(\{q_s\}, \mathbf{P})$ be an equilibrium and let $t \in [0, \infty)$ be such that $q_{t^-} < \alpha_2$. If $\{q_s\}$ is continuous and increasing in $[t, \tau)$ for some $\tau > t$, then there exists $u > t$ such that $P(x_t, q_t) - x_t = E_t[e^{-r(s-t)}(P(x_s, q_t) - x_s)]$ for all $s \in [t, u]$.*

Lemma OA5 *Let $(\{q_s\}, \mathbf{P})$ be an equilibrium and let $\Pi(x,q)$ be the seller's profits. Let $t \in [0, \infty)$ be such that $q_{t^-} < \alpha_2$ and such that $\{q_s\}$ is continuous in $[t, \tau)$ for some $\tau > t$. Then there exists $u > t$ such that $\Pi(x_t, q_t) = E_t[e^{-r(s-t)}\Pi(x_s, q_t)]$ for all $s \in [t, u]$.*

Lemma OA6 *Let $(\{q_t\}, \mathbf{P})$ be an equilibrium and let $\Pi(x,q)$ be the seller's profits. Let $t \in [0, \infty)$ be such that $q_{t^-} < \alpha_2$ and such that $\{q_s\}$ is continuous in $[t, \tau)$ for some $\tau > t$.*

[27] The boundary condition at $\overline{z}(x)$ depends on whether $\overline{z}(x) < \infty$ or $\overline{z}(x) = \infty$. In the first case, $L(\overline{z}(x), q) = g(\overline{z}(x), q)$; in the second case, $\lim_{x\to\infty} L(x,q) = 0$. Similarly, the boundary condition at $\underline{z}(x)$ depends on whether $\underline{z}(x) > 0$ or $\underline{z}(x) = 0$. In the first case, $L(\underline{z}(x), q) = g(\underline{z}(x), q)$; in the second case, $\lim_{x\to 0} L(x,q) = 0$.

Then, there exists $\hat{\tau} > t$ such that $\{q_t\}$ is discontinuous at state $(x_{\hat{\tau}}, q_t)$; i.e., $\{q_t\}$ jumps up at this state. Moreover,

$$\Pi(x_t, q_t) = E_t\left[e^{-r(\hat{\tau}-t)}\left(\left(P\left(x_{\hat{\tau}}, q_t + dq_{\hat{\tau}}\right) - x_{\hat{\tau}}\right) dq_{\hat{\tau}} + \Pi\left(x_{\hat{\tau}}, q_t + dq_{\hat{\tau}}\right)\right)\right],$$

where $dq_{\hat{\tau}}$ denotes the jump of $\{q_t\}$ at state $(x_{\hat{\tau}}, q_t)$.

Lemma OA7 *Let $(\{q_t\}, \mathbf{P})$ be an equilibrium and let $\Pi(x, q)$ be the seller's profits. Let $t \in [0, \infty)$ be such that $q_{t^-} < \alpha_2$ and such that $\{q_s\}$ is continuous in $[t, \tau)$ for some $\tau > t$. Then, $-\Pi_q(x_s, q_s) = P(x_s, q_s) - x_s$ for all $s \in [t, \tau)$.*

The following result, which is the counterpart of Lemma B14 to the current setting, uses Lemmas OA2-OA7 to establish that in any equilibrium the monopolist's profits are equal to $L(x, q)$ for all states (x, q).[28]

Lemma OA8 *Let $(\{q_t\}, \mathbf{P})$ be an equilibrium and let $\Pi(x, q)$ denote the monopolist's profits. Then, $\Pi(x, q) = L(x, q)$ for all states (x, q) with $q \in [0, 1]$.*

Proof. By the arguments in the main text, $\Pi(x, q) \geq L(x, q)$ for all states (x, q) with $q \in [\alpha_{k+1}, \alpha_k)$. I now show that $\Pi(x, q) \leq L(x, q)$ for all such states.

The proof is by induction on k. From Theorem 1, we know that the result is true for all states (x, q) with $q \geq \alpha_3$. Suppose next that the result holds for all states (x, q) with $q \in [\alpha_{\tilde{k}+1}, \alpha_{\tilde{k}})$ with $\tilde{k} \leq k - 1$. I now show that this implies that the result also holds for all states (x, q) with $q \in [\alpha_{k+1}, \alpha_k)$.

Fix a state (x, q) with $q \in [\alpha_{k+1}, \alpha_k)$, and suppose $(x_t, q_{t^-}) = (x, q)$. Note that at this state either $\{q_u\}$ jumps at t (i.e., $dq_t = q_t - q_{t^-} > 0$) or $\{q_u\}$ is continuous on $[t, s)$ for some $s > t$. In the first case, $\Pi(x_t, q_{t^-}) = (P(x_t, q_{t^-} + dq_t) - x_t)dq_t + \Pi(x_t, q_{t^-} + dq_t)$. In the second case, by Lemma OA6 there exists $\hat{\tau} > t$ and $dq_{\hat{\tau}} > 0$ such that $\Pi(x_t, q_{t^-}) = E_t[e^{-r(\hat{\tau}-t)}((P(x_{\hat{\tau}}, q_{t^-} + dq_{\hat{\tau}}) - x_{\hat{\tau}}) dq_{\hat{\tau}} + \Pi(x_{\hat{\tau}}, q_{t^-} + dq_{\hat{\tau}}))]$. Let

$$\tilde{\tau} = \sup\{\tau \geq t : \Pi(x_t, q_{t^-}) = E_t[e^{-r(\tau-t)}((P(x_\tau, q_{t^-} + dq_\tau) - x_\tau) dq_\tau + \Pi(x_\tau, q_{t^-} + dq_\tau))]\}.$$

[28] I include the proof of Lemma OA8, since it uses an induction argument that is not present in the proof of Lemma B14.

Note that if $dq_{\tilde{\tau}} \geq \alpha_k - q_{t^-}$, then

$$\begin{aligned}
&(P(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}) - x_{\tilde{\tau}})dq_{\tilde{\tau}} + \Pi(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}) \\
\leq &(P(x_{\tilde{\tau}}, \alpha_k) - x_{\tilde{\tau}})(\alpha_k - q_{t^-}) + (P(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}) - x_{\tilde{\tau}})(dq_{\tilde{\tau}} - \alpha_k + q_{t^-}) + \Pi(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}) \\
\leq &(P(x_{\tilde{\tau}}, \alpha_k) - x_{\tilde{\tau}})(\alpha_k - q_{t^-}) + L(x_{\tilde{\tau}}, \alpha_k) = g(x_{\tilde{\tau}}, q_{t^-}),
\end{aligned}$$

where the first inequality follows since $P(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}) \leq P(x_{\tilde{\tau}}, \alpha_k)$ and the second inequality follows since, by the induction hypothesis, $L(x_{\tilde{\tau}}, \alpha_k) \geq (P(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}) - x_{\tilde{\tau}})(dq_{\tilde{\tau}} - \alpha_k + q_{t^-}) + \Pi(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}})$ when $dq_{\tilde{\tau}} \geq \alpha_k - q_{t^-}$. This implies that $\Pi(x_t, q_{t^-}) \leq E_t[e^{-r(\tilde{\tau}-t)} g(x_{\tilde{\tau}}, q_{t^-})] \leq L(x_t, q_{t^-}) = \sup_\tau E[e^{-r\tau} g(x_\tau, q_{t^-})]$, and so $\Pi(x_t, q_{t^-}) = L(x_t, q_{t^-})$. The rest of the proof establishes that, indeed, $dq_{\tilde{\tau}} \geq \alpha_k - q_{t^-}$.

Towards a contradiction, suppose that $dq_{\tilde{\tau}} = q_{\tilde{\tau}} - q_{t^-} < \alpha_k - q_{t^-}$, so that $\Pi(x_{\tilde{\tau}}, q_{t^-}) = (P(x_{\tilde{\tau}}, q_{\tilde{\tau}}) - x_{\tilde{\tau}})(q_{\tilde{\tau}} - q_{t^-}) + \Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}})$. By Lemma OA6, there exists $\tau' > \tilde{\tau}$ such that $\Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}}) = E_{\tilde{\tau}}[e^{-r(\tau'-\tilde{\tau})}((P(x_{\tau'}, q_{\tilde{\tau}} + dq_{\tau'}) - x_{\tau'})dq_{\tau'} + \Pi(x_{\tau'}, q_{\tilde{\tau}} + dq_{\tau'}))]$, where $dq_{\tau'}$ denotes the jump of $\{q_t\}$ at state $(x_{\tau'}, q_{\tilde{\tau}})$. This implies that $-\Pi_q(x_{\tilde{\tau}}, q_{\tilde{\tau}}) = E_{\tilde{\tau}}[e^{-r(\tau'-\tilde{\tau})}(P(x_{\tau'}, q_{\tilde{\tau}} + dq_{\tau'}) - x_{\tau'})]$. On the other hand, by Lemma OA7 it must be that $P(x_{\tilde{\tau}}, q_{\tilde{\tau}}) - x_{\tilde{\tau}} = -\Pi_q(x_{\tilde{\tau}}, q_{\tilde{\tau}})$, and so $P(x_{\tilde{\tau}}, q_{\tilde{\tau}}) - x_{\tilde{\tau}} = E_{\tilde{\tau}}[e^{-r(\tau'-\tilde{\tau})}(P(x_{\tau'}, q_{\tilde{\tau}} + dq_{\tau'}) - x_{\tau'})]$. Since $\Pi(x_{\tilde{\tau}}, q_{t^-}) = (P(x_{\tilde{\tau}}, q_{\tilde{\tau}}) - x_{\tilde{\tau}})(q_{\tilde{\tau}} - q_{t^-}) + \Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}})$ and since $\Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}}) = E_{\tilde{\tau}}[e^{-r(\tau'-\tilde{\tau})}((P(x_{\tau'}, q_{\tilde{\tau}} + dq_{\tau'}) - x_{\tau'})dq_{\tau'} + \Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}} + dq_{\tau'}))]$, it follows that

$$\Pi(x_{\tilde{\tau}}, q_{t^-}) = E_{\tilde{\tau}}\left[e^{-r(\tau'-\tilde{\tau})}\left((P(x_{\tau'}, q_{\tau'}) - x_{\tau'})(dq_{\tau'} + q_{\tilde{\tau}} - q_{t^-}) + \Pi(x_{\tau'}, q_{\tau'})\right)\right].$$

By the Law of Iterated Expectations,

$$\begin{aligned}
\Pi(x_t, q_{t^-}) &= E_t[e^{-r(\tilde{\tau}-t)}((P(x_{\tilde{\tau}}, q_{t^-} + dq_{\tilde{\tau}}) - x_{\tilde{\tau}})\, dq_{\tilde{\tau}} + \Pi(x_{\tilde{\tau}}, q_{\tilde{\tau}}))] \\
&= E_t\left[e^{-r(\tau'-t)}\left((P(x_{\tau'}, q_{\tau'}) - x_{\tau'})(dq_{\tau'} + dq_{\tilde{\tau}}) + \Pi(x_{\tau'}, q_{\tau'})\right)\right],
\end{aligned}$$

which contradicts the definition of $\tilde{\tau}$. Hence, it must be that that $dq_{\tilde{\tau}} \geq \alpha_k - q_{t^-}$. ■

By Lemma OA8, in any equilibrium the monopolist's profits are equal to the lower bound $L(x, q)$. Using this, the equilibrium strategies can be constructed as in the proof of Theorem 1. For any $q \in [\alpha_{k+1}, \alpha_k)$ and any $x \in S(q)$, the monopolist sells to all consumers with valuation v_k at price $P(x, \alpha_k)$.[29] For all $x \notin S(q)$, $x > z_k$, the monopolist does not make sales. Finally, for all $x \notin S(q)$, $x \leq z_k$, the monopolist sells gradually to consumers with

[29]For all $x \in S(q)$, the equilibrium strategy of all buyers $i \in [q, \alpha_k]$ is $P(x, i) = P(x, \alpha_k)$.

valuation v_k. By the same arguments as in the proof of Theorem 1, for all $x_s \notin S(q_s)$, $x_s \leq z_k$

$$rL(x_s, q_s)ds = (P(x_s, q_s) - x_s + L_q(x_s, q_s))dq_s + \mu x_s L_x(x_s, q_s)ds + \frac{\sigma^2 x_s^2}{2} L_{xx}(x_s, q_s)ds.$$

Comparing this equation with (C.3), it follows that $P(x, q) - x = -L_q(x, q)$ for all $x \notin S(q), x \leq z_k$. This pins down the prices that consumers are willing to pay for all $x \notin S(q), x \leq z_k$. Note that, for all $q \in [\alpha_{k+1}, \alpha_k)$ and all $x \notin S(q)$,

$$\begin{aligned} L(x, q) &= E[e^{-r\tau(q)}[(\alpha_k - q)(P(x_{\tau_x}, \alpha_k) - x_{\tau(q)}) + L(x_{\tau(q)}, \alpha_k)]|x_0 = x] \\ \Rightarrow -L_q(x, q) &= P(x, q) - x = E[e^{-r\tau(q)}(P(x_{\tau(q)}, \alpha_k) - x_{\tau(q)})|x_0 = x]. \end{aligned} \tag{C.4}$$

Lastly, the rate at which the monopolist sells to consumers with valuation v_k when $x_s \notin S(q_s)$, $x_s \leq z_k$ is determined by the indifference condition of v_k-consumers, as in the proof of Theorem 1:

$$\frac{dq_s}{ds} = \frac{-r\left(v_k - P\left(x_s, q_s\right)\right) - \mu x_s P_x\left(x_s, q_s\right) - \frac{1}{2}\sigma^2 x_s^2 P_{xx}\left(x_s, q_s\right)}{P_q\left(x_s, q_s\right)}.$$

I end this appendix by showing that, for all $q \in [\alpha_{k+1}, \alpha_k)$ and all $x_0 > z_k$, the solution to (C.2) is to sell to all consumers with valuation v_k the first time costs reach z_k. Recall that, for any $q \in [0, 1]$, the stopping time $\tau(q) = \inf\{t : x_t \in S(q)\}$ solves (C.2), where $S(q) = \{x : L(x, q) = g(x, q)\}$. For any $q \in [0, 1]$, let $z(q) := \sup\{x \in S(q)\}$. Since $L(x, q)$ and $g(x, q)$ are continuous, $z(q) \in S(q)$. The following result shows that $z(q) = z_k$ for all $q \in [\alpha_{k+1}, \alpha_k)$.

Lemma OA9 *For all integers $k \in \{2, ..., , n\}$ and all $q \in [\alpha_{k+1}, \alpha_k)$, $z(q) = z_k$.*

Before proceeding to its proof, note that Lemma OA9 implies that the monopolist sells to the different types of consumers at the efficient time when $x_0 > z_n$: for any integer $k \in \{1, ..., n\}$, the monopolist sells to all consumers with valuation v_k the first time costs reach z_k at price $P(z_k, \alpha_k)$. Moreover, for $k \in \{2, ..., , n\}$ and for all $x \geq z_{k-1}$, $P(z_k, \alpha_k) = v_k - E[e^{-r\tau_{k-1}}(v_k - P(x_{\tau_{k-1}}, \alpha_{k-1}))|x_0 = z_k]$. The following corollary summarizes this.

Corollary C3 *When $x_0 > z_n$, the monopolist sells to the different types of consumers at the efficient time.*

Proof of Lemma OA9. To prove the lemma, I use the following claim:

Claim 1 *Fix $k \in \{2, ..., n\}$ and $q \in [\alpha_{k+1}, \alpha_k)$. Then, if $x \in S(q)$ and $x \leq z_{k-1}$, it must be that $x \in S(\alpha_k)$.*

Claim 1 (whose proof can be found below) implies that, if $x \leq z_m < z_{k-1}$ for some $m < k-1$ and $x \in S(q)$ for $q \in [\alpha_{k+1}, \alpha_k)$, then $x \in S(\alpha_{\tilde{k}})$ for all integers $\tilde{k} \in \{m+1, ..., k\}$.[30]

I now prove the lemma. The proof is by induction. Note first that, by Lemma B2, the result is true for $k = 2$. Suppose next that the result is true for all $\tilde{k} = 2, .., k-1$. I now show that this implies that the result is also true for k. Fix $q \in [\alpha_{k+1}, \alpha_k)$. Note that for all $x > z_{k-1}$,

$$P(x, \alpha_k) = v_k - E[e^{-r\tau_{k-1}}(v_k - P(x_{\tau_{k-1}}, \alpha_{k-1}))|x_0 = x], \tag{C.5}$$

where the equality follows since, by the induction hypothesis, when the state is (x, α_k) with $x > z_{k-1}$ the monopolist waits until time τ_{k-1} and at this point sells to all v_{k-1}-consumers at price $P(x_{\tau_{k-1}}, \alpha_{k-1})$. Suppose first that $z(q) > z_k$. This implies that

$$\begin{aligned}
L(z(q), q) =& (\alpha_k - q)(P(z(q), \alpha_k) - z(q)) + L(z(q), \alpha_k) \\
=& (\alpha_k - q)(v_k - z(q)) - (\alpha_k - q)E[e^{-r\tau_{k-1}}(v_k - P(x_{\tau_{k-1}}, \alpha_{k-1}))|x_0 = z(q)] + L(z(q), \alpha_k) \\
<& (\alpha_k - q)E[e^{-r\tau_k}(v_k - x_{\tau_k})|x_0 = z(q)] \\
& - (\alpha_k - q)E[e^{-r\tau_{k-1}}(v_k - P(x_{\tau_{k-1}}, \alpha_{k-1}))|x_0 = z(q)] + L(z(q), \alpha_k) \\
=& E[e^{-r\tau_k}(\alpha_k - q)(P(x_{\tau_k}, \alpha_k) - x_{\tau_k})|x_0 = z(q)] + L(z(q), \alpha_k)
\end{aligned} \tag{C.6}$$

where the strict inequality follows from Lemma 1 and the last equality follows since, for all $x > z_{k-1}$ and all stopping times $\tau < \tau_{k-1}$,

$$\begin{aligned}
& E[e^{-r\tau}(P(x_\tau, \alpha_k) - x_\tau)|x_0 = x] \\
=& E[e^{-r\tau}(v_k - x_\tau - E[e^{-r(\tau_{k-1}-\tau)}(v_k - P(x_{\tau_{k-1}}, \alpha_{k-1}))|x_\tau])|x_0 = x] \\
=& E[e^{-r\tau}(v_k - x_\tau)|x_0 = x] - E[e^{-r\tau_{k-1}}(v_k - P(x_{\tau_{k-1}}, \alpha_{k-1})|x_0 = x].
\end{aligned} \tag{C.7}$$

Note next that, by the induction hypothesis, $L(x, \alpha_k) = E[e^{-r\tau_{k-1}}g(x_{\tau_{k-1}}, \alpha_k)|x_0 = x]$ for all $x > z_{k-1}$. Therefore, by the Law of Iterated Expectations, for all $x > z_{k-1}$ and all stopping

[30]Proof: the statement follows directly from Claim 1 for $\tilde{k} = k$. Suppose next that the statement is true for $\tilde{k} = m' + 1, ..., k$, with $m' \geq m + 1$. Since $x \leq z_m \leq z_{m'-1}$ and $x \in S(\alpha_{m'+1})$, Claim 1 implies that $x \in S(\alpha_{m'})$.

times $\tau < \tau_{k-1}$,

$$E[e^{-r\tau}L(x_\tau,\alpha_k)|x_0=x] = E[e^{-r\tau}E[e^{-r(\tau_{k-1}-\tau)}g(x_{\tau_{k-1}},\alpha_k)|x_\tau]|x_0=x]$$
$$= E[e^{-r\tau_{k-1}}g(x_{\tau_{k-1}},\alpha_k)|x_0=x] = L(x,\alpha_k). \tag{C.8}$$

Combining this with the inequality in (C.6), it follows that

$$L(z(q),q) < E[e^{-r\tau_k}[(\alpha_k-q)(P(x_{\tau_k},\alpha_k)-x_{\tau_k})+L(x_{\tau_k},\alpha_k)]|x_0=z(q)],$$

a contradiction. Hence, it must be that $z(q) \le z_k$.

Suppose next that $z(q) \in [z_{k-1}, z_k)$. This implies that, whenever $x_0 > z(q)$, $\tau(q) = \inf\{t : x_t \in S(q)\} = \inf\{t : x_t = z(q)\}$. Therefore,

$$\begin{aligned}
L(z_k,q) =& E[e^{-r\tau(q)}[((\alpha_k-q)(P(x_{\tau(q)},\alpha_k)-x_{\tau(q)})+L(x_{\tau(q)},\alpha_k))]|x_0=z_k] \\
=& E[e^{-r\tau(q)}(\alpha_k-q)(v_k-x_{\tau(q)})|x_0=z_k] \\
&-(\alpha_k-q)E[e^{-r\tau_{k-1}}(v_k-P(x_{\tau_{k-1}},\alpha_{k-1})))|x_0=z_k]+L(z_k,\alpha_k) \\
<& (\alpha_k-q)((v_k-z_k)-E[e^{-r\tau_{k-1}}(v_k-P(x_{\tau_{k-1}},\alpha_{k-1}))|x_0=z_k])+L(z_k,\alpha_k) \\
=& (\alpha_k-q)(P(z_k,\alpha_k)-z_k)+L(z_k,\alpha_k)=g(z_k,q),
\end{aligned}$$

where the second equality uses (C.7) and (C.8) and the strict inequality follows from Lemma 1. This cannot be, since $L(z_k,q) = \sup_\tau E[e^{-r\tau}g(x_\tau,q)|x_0=z_k]$. Hence, $z(q) \notin [z_{k-1}, z_k)$.

Finally, I show that $z(q) \notin [0, z_{k-1}]$. Letting $z_0 := 0$, suppose that $z(q) \in (z_{m-1}, z_m]$ for some $m \le k-1$ (for the case of $m = 1$, suppose $z(q) \in [0, z_1] = [z_0, z_1]$). Note that by Claim 1 and the paragraph that follows the claim, it follows that $z(q) \in S(\alpha_{\tilde{k}})$ for all integers $\tilde{k} \in \{m+1, ..., k\}$. Since $z(q) \in S(\alpha_k)$, at state $(z(q), \alpha_k)$ the monopolist sells to all consumers with valuation v_{k-1} immediately at price $P(z(q), \alpha_{k-1})$, and so $P(z(q), \alpha_k) = P(z(q), \alpha_{k-1})$. If $k-1 \ge m+1$, then $z(q) \in S(\alpha_{k-1})$. Therefore, at state $(z(q), \alpha_{k-1})$ the monopolist sells to all consumers with valuation v_{k-2} immediately at price $P(z(q), \alpha_{k-2})$, and so $P(z(q), \alpha_k) = P(z(q), \alpha_{k-1}) = P(z(q), \alpha_{k-2})$. Continuing in this way, it follows that $P(z(q), \alpha_k) = P(z(q), \alpha_m)$.

Since $z(q) \in S(\alpha_k)$, it follows that $L(z(q), \alpha_k) = (\alpha_{k-1} - \alpha_k)(P(z(q), \alpha_{k-1}) - z(q)) +$

$L(z(q), \alpha_{k-1})$. Therefore,

$$\begin{aligned}L(z(q), q) &= (\alpha_k - q)(P(z(q), \alpha_k) - z(q)) + L(z(q), \alpha_k)\\ &= (\alpha_k - q)(P(z(q), \alpha_k) - z(q)) + (\alpha_{k-1} - \alpha_k)(P(z(q), \alpha_{k-1}) - z(q)) + L(z(q), \alpha_{k-1})\\ &= (\alpha_{k-1} - q)(P(z(q), \alpha_{k-1}) - z(q)) + L(z(q), \alpha_{k-1}),\end{aligned}$$

where the last equality follows since $P(z(q), \alpha_k) = P(z(q), \alpha_{k-1}) = P(z(q), \alpha_m)$. If $k - 1 \geq m+1$, then $L(z(q), \alpha_{k-1}) = (\alpha_{k-2} - \alpha_{k-1})(P(z(q), \alpha_{k-2}) - z(q)) + L(z(q), \alpha_{k-2})$. Moreover, in this case $P(z(q), \alpha_{k-1}) = P(z(q), \alpha_{k-2})$, and so $L(z(q), q) = (\alpha_{k-2} - q)(P(z(q), \alpha_{k-2}) - z(q)) + L(z(q), \alpha_{k-2})$. Continuing in this way, it follows that $L(z(q), q) = (\alpha_m - q)(P(z(q), \alpha_m) - z(q)) + L(z(q), \alpha_m)$.

Fix $x > z(q)$, and note that $\tau(q) = \inf\{t : x_t = S(q)\} = \inf\{t : x_t = z(q)\}$ whenever $x_0 = x$. Therefore, by the arguments in the previous paragraph, for all $x > z(q)$,

$$L(x, q) = E[e^{-r\tau(q)}[((\alpha_m - q)(P(x_{\tau(q)}, \alpha_m) - x_{\tau(q)}) + L(x_{\tau(q)}, \alpha_m))]|x_0 = x].$$

Note further that, by the induction hypothesis, for all $x > z_m$,

$$\begin{aligned}P(x, \alpha_m) &= v_m - E[e^{-r\tau_{m-1}}(v_m - P(x_{\tau_{m-1}}, \alpha_{m-1}))|x_0 = x], \text{ and}\\ L(x, \alpha_m) &= E[e^{-r\tau_{m-1}} g(x_{\tau_{m-1}}, \alpha_m)|x_0 = x].\end{aligned}$$

Applying the Law of Iterated Expectations, for all $x > z(q) \geq z_{m-1}$,

$$\begin{aligned}&E[e^{-r\tau(q)}[(\alpha_m - q)(P(x_{\tau(q)}, \alpha_m) - x_{\tau(q)}) + L(x_{\tau(q)}, \alpha_m)]|x_0 = x]\\ =&E[e^{-r\tau(q)}(\alpha_m - q)(v_m - x_{\tau(q)})|x_0 = x]\\ &- E[e^{-r\tau_{m-1}}(\alpha_m - q)(v_m - P(x_{\tau_{m-1}}, \alpha_{m-1}))|x_0 = x] + L(x, \alpha_m).\end{aligned}$$

Therefore,

$$\begin{aligned}L(z_m, q) =&E[e^{-r\tau(q)}[((\alpha_m - q)(P(x_{\tau(q)}, \alpha_m) - x_{\tau(q)}) + L(x_{\tau(q)}, \alpha_m))]|x_0 = z_m]\\ =&E[e^{-r\tau(q)}(\alpha_m - q)(v_m - x_{\tau(q)})|x_0 = z_m]\\ &- E[e^{-r\tau_{m-1}}(\alpha_m - q)(v_m - P(x_{\tau_{m-1}}, \alpha_{m-1}))|x_0 = z_m] + L(z_m, \alpha_m)\\ <&(\alpha_m - q)(v_m - z_m) - (\alpha_m - q)E[e^{-r\tau_{m-1}}(v_m - P(x_{\tau_{m-1}}, \alpha_{m-1}))|x_0 = z_m] + L(z_m, \alpha_m)\\ =&(\alpha_m - q)(P(z_m, \alpha_m) - z_m) + L(z_m, \alpha_m),\end{aligned}$$

where the strict inequality follows from Lemma 1. But this is a contradiction, since $L(z_m, q) \geq (\alpha_m - q)(P(z_m, \alpha_m) - z_m) + L(z_m, \alpha_m)$.[31] Hence, $z(q) \notin (z_{m-1}, z_m]$. Combining all these arguments, it follows that $z(q) = z_k$. ■

Proof of Claim 1. Fix $x \leq z_{k-1}$ with $x \notin S(\alpha_k)$. Then, in equilibrium, at state (x, α_k) the monopolist sells to consumers with valuation v_{k-1} gradually over time. By equation (C.4),

$$P(x, \alpha_k) - x = -L_q(x, \alpha_k) = E[e^{-r\tau(\alpha_k)}(P(x_{\tau(\alpha_k)}, \alpha_{k-1}) - x_{\tau(\alpha_k)})|x_0 = x].$$

Therefore, for all $q \in [\alpha_{k+1}, \alpha_k)$ and all $x \leq z_{k-1}, x \notin S(\alpha_k)$,

$$\begin{aligned} g(x, q) &= (\alpha_k - q)(P(x, \alpha_k) - x) + L(x, \alpha_k) \\ &= E[e^{-r\tau(\alpha_k)}[(\alpha_k - q)(P(x_{\tau(\alpha_k)}, \alpha_{k-1}) - x_{\tau(\alpha_k)}) + L(x_{\tau(\alpha_k)}, \alpha_k)]|x_0 = x] \\ &\leq \sup_{\tau} E[e^{-r\tau} g(x_\tau, q)|x_0 = x]. \end{aligned}$$

Hence, stopping when $x_t = x$ is dominated by waiting and stopping at time $\tau(\alpha_k)$, and therefore $x \notin S(q)$. ■

C.2 The discrete-time game

This section studies the discrete-time version of the model in the paper. The main goal is to show that in any subgame perfect equilibrium of this game, the strategies of the buyers must satisfy condition (iii) in Definition 1 in the main text. For conciseness, I focus on the case in which there are two types of buyers, as in Section 5. I stress however that these results generalize to settings with any (finite) number of types.

As in the main text, a monopolist faces a continuum of consumers indexed by $i \in [0, 1]$. For each $i \in [0, 1]$, let $f(i)$ denote the valuation of consumer i. There are two types of buyers: high types with valuation v_2, and low types with valuation $v_1 \in (0, v_2)$. Let $\alpha \in (0, 1)$ be the fraction of high types in the market, so $f(i) = v_2$ for all $i \in [0, \alpha]$ and $f(i) = v_1$ for all $i \in (\alpha, 1]$.

[31] Indeed, note that for all $q \in [\alpha_{k+1}, \alpha_k)$,

$$\begin{aligned} L(x, q) &\geq (\alpha_k - q)(P(x, \alpha_k) - x) + L(x, \alpha_k) \\ &\geq (\alpha_k - q)(P(x, \alpha_k) - x) + (\alpha_{k-1} - \alpha_k)(P(x, \alpha_{k-1}) - x) + L(x, \alpha_{k-1}) \\ &\geq (\alpha_{k-1} - q)(P(x, \alpha_{k-1}) - x) + L(x, \alpha_{k-1}), \end{aligned}$$

where the last inequality follows since $P(x, \alpha_k) \geq P(x, \alpha_{k-1})$. Repeating this argument inductively, it follows that $L(x, q) \geq (\alpha_m - q)(P(x, \alpha_m) - x) + L(x, \alpha_m)$ for all $m \leq k - 1$.

Time is discrete. Let $T(\Delta) = 0, \Delta, 2\Delta, ...$ be the set of times at which players take actions, with Δ measuring the time period. At each time $t \in T(\Delta)$ the monopolist announces a price $p \in \mathbb{R}_+$. All consumers who have not purchased already simultaneously choose whether to buy at this price or wait. All players have perfect recall of the history of the game. Moreover, all players in the game are expected utility maximizers, and have a common discount factor $\delta = e^{-r\Delta}$. The monopolist's marginal cost of production evolves as (1), with $\mu < r$ and $\sigma > 0$. The seller's cost is publicly observable. Note that costs evolve continuously over time, but the monopolist can only announce a price and make sales at times $t \in T(\Delta)$. Therefore, as $\Delta \to 0$, costs become more persistent across periods.

A strategy for the monopolist specifies at each time $t \in T(\Delta)$ a price to charge as a function of the history. A strategy for a consumer specifies at each time the set of prices she will accept as a function of the history (provided she has not previously made a purchase). I focus on the subgame perfect equilibria (SPE) of this game.[32] [33]

Lemma OA10 *In any SPE and after any history, all buyers accept a price equal to v_1, regardless of the current level of costs.*

Proof. Fix a SPE and let $p(x)$ be the supremum of prices accepted by all consumers after any history such that current costs are x. Let $\underline{p} := \inf_{x \in \mathbb{R}_+} p(x)$. Note first that $\underline{p} \leq v_1$, since buyers with valuation v_1 never accept a price larger than their valuation. Suppose by contradiction that the Lemma is not true, so $\underline{p} < v_1$. Note that the monopolist would never charge a price lower than $\underline{p}$. Consider the offer $p = (1 - \delta)v_1 + \delta \underline{p} > \underline{p}$. Note that every buyer would accept a price of $p - \epsilon$ for any $\epsilon > 0$, since the price in the future will never be lower than $\underline{p}$. Moreover, $p - \epsilon > \underline{p}$ for ϵ small enough. This implies that there exists a cost level x such that $p - \epsilon > p(x)$, which contradicts the fact that $p(x)$ is the supremum of prices accepted by all consumers after any history such that current costs are x. Thus, $\underline{p} = v_1$. ■

An immediate Corollary of Lemma OA10 is that, in any SPE, consumers with valuation v_1 accept a price equal to v_1; that is, condition (4) in the main text holds in any SPE. The next result shows that condition (5) also holds in any SPE of the game.

Consider the optimal stopping problem $\sup_{\tau \in \overline{T}(\Delta)} E[e^{-r\tau}(v_1 - x_\tau)|x_0 = x]$, where $\overline{T}(\Delta)$ is the set of stopping times taking values on $T(\Delta)$. The solution to this problem is to stop the first time costs fall below some level z_1^Δ. For all $s \in T(\Delta)$, let $\tau_1^\Delta(s) = \inf\{t \in$

[32]As usual in durable goods monopoly games, I restrict attention to SPE in which actions are constant on histories in which prices are the same and the sets of agents accepting at each point in time differ by sets of measure zero; see Gul et al. (1985) for a discussion of this assumption.

[33]Existence of SPE can be shown by generalizing arguments in Gul et al. (1985).

$T(\Delta), t > s : x_t \leq z_1^\Delta\}$. Let $\tau_1^\Delta = \tau_1^\Delta(0)$. Note that, in any SPE, if all remaining high type consumers buy at time $s \in T(\Delta)$ and leave the market, the monopolist will then wait until time $\tau_1^\Delta(s)$ and charge a price of v_1 (which all low type buyers accept). For all $x > 0$, let $P^\Delta(x) = v_2 - E[e^{-r\tau_1^\Delta}(v_2 - v_1)|x_0 = x]$.

Lemma OA11 *In any SPE and after any history, all buyers with valuation v_2 accept a price equal to $P^\Delta(x)$ if the current cost level is x.*

Proof. Fix a SPE and let $p_2(x)$ be the supremum of the prices that all buyers $i \in [0, \alpha]$ accept after any history if current costs are x. I first show that $p_2(x) \leq P^\Delta(x)$. To see this, note that by definition of $p_2(x)$, all buyers $i \in [0, \alpha]$ that remain in the market will buy if the seller charges a price $p_2(x)$. By our discussion above, the monopolist will then sell to all low types at a price v_1 the first time costs fall below z_1^Δ. The utility that a high type buyer gets by not purchasing at price $p_2(x)$ and waiting until the monopolist serves low types is $E[e^{-r\tau_1^\Delta}(v_2 - v_1)|x_0 = x]$. Therefore, for all buyers to be willing to purchase at price $p_2(x)$, it must be that $v_2 - p_2(x) \geq E[e^{-r\tau_1^\Delta}(v_2 - v_1)|x_0 = x]$, or $p_2(x) \leq P^\Delta(x)$.

I now complete the proof by showing that $p_2(x) \geq P^\Delta(x)$. To see this, let $\underline{p}(x) = p_2(x)$ if $x > z_1^\Delta$ and $\underline{p}(x) = v_1$ if $x \leq z_1^\Delta$. Note that the monopolist will never charge a price lower than $\underline{p}(x)$ if current costs are x. Moreover, all high type consumers will accept this price. As a first step to prove the inequality, I show that $p_2(x) \geq v_2 - E[e^{-r\Delta}(v_2 - \underline{p}(x_{t+\Delta}))|x_t = x]$. To see this, suppose by contradiction that this is not true. Then, some high type consumers would reject a price of $p^\epsilon(x) = v_2 - E[e^{-r\Delta}(v_2 - \underline{p}(x_{t+\Delta}))|x_t = x] - \epsilon$ for ϵ small enough (i.e., $p^\epsilon(x) > p_2(x)$ for ϵ small enough). Note that the lowest possible price that the seller would charge next period is $\underline{p}(x_{t+\Delta})$, and that this price would be accepted by all high types. This implies that the continuation utility of high types from rejecting today's price is bounded above by $E[e^{-r\Delta}(v_2 - \underline{p}(x_{t+\Delta}))|x_t = x]$. But this in turn implies that all high type consumers should accept a price of $v_2 - E[e^{-r\Delta}(v_2 - \underline{p}(x_{t+\Delta}))|x_t = x] - \epsilon > p_2(x)$, a contradiction to the fact that $p_2(x)$ is the supremum over all prices that all high types accept when costs are equal to x. Hence, $p_2(x) \geq v_2 - E[e^{-r\Delta}(v_2 - \underline{p}(x_{t+\Delta}))|x_t = x]$.

Recall that $\tau_1^\Delta = \tau_1^\Delta(0) = \inf\{t \in T(\Delta), t > 0 : x_t \leq z_1^\Delta\}$. For all $t \in T(\Delta)$, let $F^\Delta(t, x) = \text{Prob}(\tau_1^\Delta = t | x_0 = x)$, and note that $F^\Delta(0, x) = 0$. It then follows that

$$\begin{aligned} p_2(x) &\geq v_2 - E[e^{-r\Delta}(v_2 - \underline{p}(x_\Delta))|x_0 = x] \\ &= v_2 - e^{-r\Delta}F^\Delta(\Delta, x)(v_2 - v_1) \\ &\quad -(1 - F^\Delta(\Delta, x))E[e^{-r\Delta}(v_2 - p_2(x_\Delta))|x_0 = x, \tau_1^\Delta > \Delta], \end{aligned} \tag{C.9}$$

where the equality follows since $\underline{p}(x) = v_1$ for all $x \leq z_1^\Delta$ and $\underline{p}(x) = p_2(x)$ for all $x > z_1^\Delta$. Using the fact that $p_2(x) \geq v_2 - E[e^{-r\Delta}(v_2 - \underline{p}(x_\Delta))|x_0 = x]$ repeatedly in equation (C.9), it follows that $p_2(x) \geq v_2 - \sum_{k=1}^{\infty} e^{-rk\Delta}(v_2 - v_1)F^\Delta(k\Delta, x) = v_2 - E[e^{-\tau_1^\Delta}(v_2 - v_1)|x_0 = x]$, where the last equality follows since $F^\Delta(t, x) = \text{Prob}(\tau_1^\Delta = t|x_0 = x)$ and since $F^\Delta(0, x) = 0$. ■

Lemma OA11 shows that condition (5) holds in any SPE of this discrete-time game with two types of buyers: all buyers $i \in [0, \alpha]$ accept a price that leaves them indifferent between buying at that price or waiting and buying at the time low type consumers buy; in particular, consumer α accepts such a price.

Lemmas OA10 and OA11 together establish that condition (iii) in Definition 1 holds in any SPE of this discrete-time game with two types of consumers. If there were three types of consumers, with valuations $v_3 > v_2 > v_1$, then the monopolist would only serve v_2-consumers when costs are below some cutoff z_2^Δ. Letting $P_2^\Delta(x)$ denote the price at which the monopolist first sells to consumers with valuation v_2 (when costs are x), arguments identical to those in Lemma OA11 can be used to show that, in any SPE, all consumers with valuation v_3 accept a price equal to $P_3^\Delta(x) = v_3 - E[e^{-r\tau_2^\Delta}(v_3 - P_2^\Delta(x_{\tau_2^\Delta}))|x_0 = x]$, where $\tau_2^\Delta = \inf\{t \in T(\Delta), t > 0 : x_t \leq z_2^\Delta\}$. Hence, condition (5) also holds in any SPE of this discrete-time game with three types of buyers. Repeating this argument, one can show that condition (5) holds in any SPE of this game with any finite number of consumer types. Moreover, the arguments in Lemma OA10 don't rely on there being only two types of buyers, so condition (4) would also hold in any SPE of this game with any number of consumer types.

C.3 Full commitment

In this appendix, I solve for the full commitment strategy of the monopolist when there are two types of buyers in the market. In the full commitment problem, the monopolist chooses a path of prices $\{p_t\}$ at time $t = 0$.[34] Given a path of prices $\{p_t\}$, consumer i makes her purchase at the earliest stopping time that solves $\sup_\tau E[e^{-r\tau}(f(i) - p_\tau)]$. Hence, with two types of buyers, there will be (at most) two times of sale: the (random) time $\hat{\tau}_1$ at which low types buy, and the (random) time $\hat{\tau}_2$ at which high types buy. Moreover, high types will buy weakly earlier than low types, so $\hat{\tau}_2 \leq \hat{\tau}_1$ with probability 1. Note that, by choosing the path of prices, the monopolist effectively chooses the times $\hat{\tau}_1$ and $\hat{\tau}_2$ at which the different consumers buy.

[34] $\{p_t\}$ must be an $\mathcal{F}_t$-progressively measurable process.

Note next that it is optimal for the monopolist to charge a price of v_1 to low type buyers. Given this, the highest price that the monopolist can charge high type buyers is given by $p(x_t) = v_2 - E[e^{-r(\hat{\tau}_1 - t)}(v_2 - v_1)|x_t]$. Therefore, the optimal strategy of the monopolist boils down to optimally choosing the times $\hat{\tau}_1$ and $\hat{\tau}_2$ at which the different consumers buy. That is, the monopolist's full commitment profits $\Pi^{FC}(x)$ are given by

$$\begin{aligned}
\Pi^{FC}(x) &= \sup_{\hat{\tau}_1, \hat{\tau}_2} \alpha E\left[e^{-r\hat{\tau}_2}\left(p(x_{\hat{\tau}_2}) - x_{\hat{\tau}_2}\right)|x_0 = x\right] + (1-\alpha)E\left[e^{-r\hat{\tau}_1}\left(v_1 - x_{\hat{\tau}_1}\right)|x_0 = x\right] \\
&= \sup_{\hat{\tau}_1, \hat{\tau}_2} \alpha E\left[e^{-r\hat{\tau}_2}\left(v_2 - x_{\hat{\tau}_2}\right)|x_0 = x\right] + (1-\alpha)E\left[e^{-r\hat{\tau}_1}\left(\frac{v_1 - \alpha v_2}{1-\alpha} - x_{\hat{\tau}_1}\right)|x_0 = x\right],
\end{aligned}$$

where the equality follows from using $p(x_{\hat{\tau}_2}) = v_2 - E[e^{-r(\hat{\tau}_1 - \hat{\tau}_2)}(v_2 - v_1)|x_{\hat{\tau}_2}]$. Note the solution to the problem above involves choosing $\hat{\tau}_2$ to maximize the first term, and choosing $\hat{\tau}_1$ separately to maximize the second term. Moreover, by Lemma 1, $\hat{\tau}_2 = \tau_2 = \inf\{t : x_t \leq z_2\}$. Finally, note that the second term is always negative if $v_1 \leq \alpha v_2$, so in this case the optimal strategy for the monopolist is to set $\hat{\tau}_1 = \infty$; that is, to never sell to low types. In this case, $\Pi^{FC}(x) = \sup_\tau \alpha E\left[e^{-r\tau}\left(v_2 - x_\tau\right)|x_0 = x\right]$. Otherwise, if $v_1 > \alpha v_2$, one can use arguments similar to those in the proof of Lemma 1 to show that it is optimal to set $\hat{\tau}_1 = \inf\{t : x_t \leq \frac{-\lambda_N}{1-\lambda_N}\frac{v_1 - \alpha v_2}{1-\alpha}\}$.

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