Semiparametric Estimation of Firm-level Production Functions with Heterogeneity

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Introduction

- An unknown differentiable production function (PF) of variable inputs $Q = F(X)$ homogeneus of arbitrary degree v.

- Under cost minimization it is well known (e.g. Chambers, 1988) that

$$\beta_k = vS_k(X),\ k = 1\ldots p,$$

where $v = \sum_k \beta_k$ and $S_k(\bullet)$ is the share in cost.

- Semiparametric estimation: Exploring estimation replacing the elasticities by the shares multiplied by the elasticity of scale.

- Estimation of what? v, input elasticities, and the elasticity of substitution, but also productivities, and input and output market power.

Main results (p=2 for simplicity):

1) The (log) function can be written exactly as

$$q = \beta_0 + vz + g(r) + \varepsilon,$$

where

$$z = S_1 x_1 + S_2 x_2,$$
$$r = x_2 - x_1,$$

and $g(\bullet)$ is an unknown function.

Functional form coincides with the econometric model of Robinson (1988).

2) Any differentiable PF of two variable inputs, homogeneous of degree v, is nested in semiparametric form, in the sense that only impose restrictions on the function $g(\bullet)$.

In particular the semiparametric CES and homogeneous Translog (TL) are nested.

- The elasticity of substitution $\sigma(X)$ of the general specification is unrestricted.

- CES imposes $\sigma(X) = \sigma$.

- Translog restricts $\sigma(X)$ to a particular variation (decreases with $|r|$).

3) The production function can have unobserved Hicksian productivity.

4) The production function can have, at the same time, unobserved input-augmenting productivity.

5) The model can be used to identify nonparametrically the degree of imperfect competition in the settng of one of the inputs.

Azzam, Jaumandreu and Lopez (2022): four meatpacking firms are accused of exploiting farmers and workers and charging high prices to consumers.

Interest

- Researchers may want ν or the distributions of β_k or σ across firms (e. g. to compute markups, or firm-level labor substitution effects).

- May also want the distributions of Hicks-neutral ω_j and/or Labor-augmenting productivity ω_{Lj}, or degree of imperfection in the labor market τ_j.

- Semiparametric estimation allows the more general results, with an unrestricted σ, at the cost of some computing. It can also happen that elasticities can be computed with an "extended CD."

- One may be willing to restrict σ. It can be done, e.g., with semiparametric CES and TL. The framework allows the restrictions to be tested.

Semiparametric specification

• Cross-section population of firms with production function $F(X)$. Will use subindex j to underline what differs from firm to firm.

• X_j is a vector of $1 \times p$ variable inputs.

• $F(X_j)$: Unknown production function, differentiable, short-run, homogeneous of arbitrary degree v.

• Writing in logs: $f(x_j) = \ln F(\exp(x_{1j}), \ldots, \exp(x_{pj}))$. We observe

$$q_j = f(x_j) + \varepsilon_j,$$

with ε_j uncorrelated with the available information.

• Call $\beta(x_j) = \frac{\partial f(x_j)}{\partial x_j}$ the $p \times 1$ vector of derivatives, and $D\beta(x_j)$ the $p \times p$ Hessian with the slopes of the derivatives.

• Predict $0 = f(0)$ from the actual use of inputs x_j by means of a Taylor (second order) expansion. Reordering:

$$f(x_j) = x_j\beta(x_j) - \frac{1}{2}x_j D\beta(t_j x_j)x_j', \quad \text{with } t_j \in [0,1]. \tag{1}$$

• Exact expression for the unknown production function, in terms of the elasticities and their derivatives, and the values t_j that happen in Taylor's second (and last) term.

• Assume that firms minimize cost in competitive input markets. For each input k

$$MC_j \frac{\partial F(X_j)}{\partial X_{kj}} = W_{kj},$$

where W_{kj} is the price of input k for firm j. It follows

$$\beta_k(x_j) = \nu S_{kj}(x_j), \tag{2}$$

with S_{kj} is the share of input k in variable cost.

• Plug (2) into (1). For $p = 2$, e. g., we get

$$
\begin{aligned}
f(x_j) = {} & \nu(S_{1j}(x_j)x_{1j} + S_{2j}(x_j)x_{2j}) \\
& + \frac{1}{2}\nu\frac{1-\sigma(t_jx_j)}{\sigma(t_jx_j)}S_{1j}(t_jx_j)S_{2j}(t_jx_j)(x_{2j}-x_{1j})^2,
\end{aligned}
$$

where $\sigma(\bullet)$ is the elasticity of substitution between the two inputs.

• A ($p = 2$) unknonw differentiable homogeneous production function, if cost is minimized, admits an exact representation in terms of the scale parameter, the input shares in variable cost, the elasticity of substitution, and Taylor's t_j values.

• The first part of the expression is what Syverson (2004) used in a robustness check.

Econometric model and estimation

• Proposition ($p = 2$): The elasticity of substitution $\sigma(x_j)$, the shares in variable cost $S_k(x_j)$, and the $t_j^{\prime}s$, are functions of the ratio of inputs $r_j = x_2 - x_1$.

It follows that the PF can be written as

$$q_j = \beta_0 + \nu z_j + g(r_j) + \varepsilon_j, \tag{3}$$

where

$$z_j = S_{1j}x_{1j} + S_{2j}x_{2j},$$

a weighted sum of inputs, and $g(\bullet)$ is an unknown function.

• Econometrically this is the "semiparametric" model of Robinson (1988). Estimation of ν controlling for $g(r_j)$ is consistent and asymptotically normal.

• Robinson (1988) proposed projecting q_j and z_j on r_j by means of kernels or sieves. Pagan and Ullah (1999) show how it is equivalent to use a sieve for $g(r_j)$.

Consequences for PF estimation

• It is perfectly legitimate to assume that the "true" production function has the form

$$q_j = \beta_0 + \sum_k \beta_{kj} x_{kj} + \varepsilon_j,$$

("extended Cobb-Douglas") and use the FOCs to replace the elasticities.

• Unfortunately the data can reject the assumption (if we test).

• However, with $p = 2$, there is one case in which full model (3) is perfectly compatible with using the extended CD: $Cov(z_j, g(r_j)) = 0$ or the "scale" of the firm is uncorrelated with the function $g(r_j)$.

• The proposition shows (for $p = 2$) that PFs meeting the assumptions should be writable in terms of the term z_j and a function of the (log) ratio of inputs r_j.

• This is the case for the CES,

$$f(x_j) = -\frac{\nu\sigma}{1-\sigma} \ln[\exp(-\frac{1-\sigma}{\sigma} x_{1j}) + \exp(-\frac{1-\sigma}{\sigma} x_{2j})].$$

Using one FOC can be written

$$q_j = \beta_0 + \nu x_{kj} + \nu \frac{\sigma}{1-\sigma} \ln S_{kj} + \varepsilon_j$$

or

$$q_j = \beta_0 + \nu z_j + \nu \frac{\sigma}{1-\sigma} \ln S_{1j} - \nu S_{2j} r_j + \varepsilon_j$$

and using the two FOCs

$$q_j = \beta_0 + \nu z_j + \frac{\nu}{2}(\frac{\sigma}{1-\sigma}(\ln S_{1j} + \ln S_{2j}) - (S_{2j} - S_{1j}) r_j) + \varepsilon_j.$$

- And is the case of the homogeneous Translog,

$$f(x_j) = \alpha_0 + \alpha_1 x_{1j} + \alpha_2 x_{2j} - \frac{1}{2}\alpha(x_{2j} - x_{1j})^2, \ \nu = a_1 + a_2,$$

that can be written

$$q_j = \beta_0 + \nu z_j + \frac{1}{2}\alpha r_j^2 + \varepsilon_j.$$

- In contrast with (3), CES restricts $\sigma(\bullet)$ to be a constant, TL restricts $\sigma(\bullet)$ to take the form $\sigma_j = \frac{\nu S_{1j} S_{2j}}{\alpha + \nu S_{1j} S_{2j}}$. Notice that $\frac{\partial \sigma_j}{\partial |S_{2j} - S_{1j}|} < 0.$

- Zhang (2019) and Ziyao (2025) use the CES model, Doraszelski and Jaumandreu (2019) the TL model. All include neutral and biased productivity.

Including productivity (neutral and biased)

- With neutral, model becomes

$$q_j = \beta_0 + v z_j + g(r_j) + \omega_j + \varepsilon_j,$$

where will assume $\omega_j = \rho\omega_{j-1} + \xi_j$.

- Since the ratio of inputs is a function of the input prices only (and biased productivity, see later), we can assume that $E(\omega_j | w_{2j} - w_{1j}) = 0$. This ensures $E(\omega_j | r_j) = 0$ and hence identification.

- Estimation can be based on the following generalized dynamic panel method (DP):

$$q_j = \rho q_{j-1} + v(z_j - \rho z_{j-1}) + g(r_j) - \rho g(r_{j-1}) + \xi_j + \varepsilon_j - \rho\varepsilon_{j-1}.$$

- We estimate $\widehat{v}$ and $\widehat{\omega}_j$. We can separate ω_j from ε_j with state-space models (e.g. Ziyao, 2025).

- Biased productivity: one input is only partially observed, say $x_1^* = x_1 + \omega_1$.

- Dropping neutral for simplicity, model becomes

$$\begin{aligned} f(x_j) &= v(S_{1j}(x_j)x_{1j} + S_{2j}(x_j)x_{2j}) + vS_1(x_j)\omega_{1j} \\ &\quad + \frac{1}{2}v\frac{1-\sigma(t_jx_j)}{\sigma(t_jx_j)}S_{1j}(t_jx_j)S_{2j}(t_jx_j)(x_{2j} - x_{1j} - \omega_{1j})^2, \end{aligned}$$

where the $S's$ and σ do not change because they are the invariant to writing in efficiency or real terms.

- Using the ratio of FOCs, $\omega_{1j} = x_{2j} - x_{1j} - \frac{\sigma(x_j)}{1-\sigma(x_j)} \ln \frac{S_{1j}(x_j)}{S_{2j}(x_j)}$, model is

$$q_j = \beta_0 + vx_{2j} + \widetilde{g}(r_j) + \varepsilon_j.$$

- Semiparametric CES and TL confirm this surprising simplicity:

$$q_j = \beta_0 + vx_{2j} + v\frac{\sigma}{1-\sigma}\ln S_{2j} + \varepsilon_j,$$

$$q_j = \beta_0 + vx_{2j} - \frac{v^2}{\alpha}S_{1j}^2 + \varepsilon_j.$$

Identifying nonparametrically input market imperfections

• Most challenging situation: input-augmenting productivity and the input is decided in non-competition (e.g. monpsony or rent sharing). Drop neutral ω.

• Consider the FOC for input X_1 is

$$MC_j \frac{\partial F_j(X_1)}{\partial(X_{1j}\exp(\omega_{1j}))}\exp(\omega_{1j}) = W_{1j}(1+\tau_j),$$

$$\text{or} \quad \beta_1(x_j) = \nu^*(1+\tau_j)S_{1j}(x_j), \quad \text{where } \nu^* = \frac{\nu}{1+S_{1j}\tau_j}.$$

• Two new things affect the relationship between $\beta_1(x_j)$ and $S_{1j}(x_j)$:

- Elasticity shows a gap τ_j over the share because competition.
- The ratio $\frac{MC_j}{AVC_j} = \frac{1+S_{1j}\tau_j}{\nu}$ is (with $\tau_j > 0$) greater than technology.

- Result: We can identify nonparametrically ω_{1j} and τ_j.

- The quadratic part shows now an extra term,

$$-\frac{1}{2}x_j D\beta(t_j x_j)x_j' = \frac{1}{2}\nu\frac{1+\tau_j}{1+S_{1j}\tau_j}\frac{1-\sigma_j}{\sigma_j}S_{1j}S_{2j}(x_{2j}-x_{1j})^2,$$

which depends on τ_j. But can be written as an unknown function of r_j.

- The log-linear part becomes now

$$x_j\beta = \nu^* z_{1j} + \nu^*\tau_j S_{1j}x_{1j} + \nu^*(1+\tau_j)S_{1j}\omega_{1j},$$

that is nonlinear and adds a term in τ_j and another in $1+\tau_j$ and ω_{1j}.

- How to control for both unobservables?

- Dobbelaere, Jaumandreu, and Mairesse, 2025: Get τ_j and ω_{1j} from inverting two equations which include them (e.g. determinants of the ratio r_j and demand for x_{2j}).

- With only homogeneity ν, system and solutions are:

$$\begin{bmatrix} \sigma_j & (1-\sigma_j) \\ \sigma_j & -\sigma_j \end{bmatrix} \begin{bmatrix} \tau_j \\ \omega_{1j} \end{bmatrix} = \begin{bmatrix} (1-\sigma_j)(x_{2j}-x_{1j}) - \sigma_j \ln \frac{S_{1j}}{1-S_{1j}} \\ \frac{1}{S_{1j}} x_{2j} - \sigma_j (w_{1j} - w_{2j}) - \frac{1}{\nu S_{1j}} q_j^* \end{bmatrix},$$

$$\begin{aligned} \tau_j &= \frac{1-\sigma_j}{\sigma_j} \frac{1}{S_{1j}} (x_{2j} - \frac{1}{\nu} q_j^*) - \ln \frac{S_{1j}}{1-S_{1j}}, \\ \omega_{1j} &= x_{2j} - x_{1j} - \frac{1}{S_{1j}} (x_{2j} - \frac{1}{\nu} q_j^*). \end{aligned}$$

• Researchers can in fact choose: take τ_j as a constant, model τ_j according to the determinants of a given theory (e.g. Cournot monopsony), or estimate nonparametrically freely varying τ_j and ω_{1j} from the production data.

• If one is willing to accept the restrictions of the CES or TL, estimation becomes easier.

• Semiparametric CES with τ_j and ω_{1j} becomes:

$$q_j = \beta_0 + \nu^* x_{2j} + \frac{\nu^* \sigma}{1-\sigma} \ln S_2 + \frac{\nu^* \sigma}{1-\sigma} \ln(1 + S_{1j}\tau_j),$$

• Semiparametric TL with τ_j and ω_{1j} becomes:

$$q_j = \beta_0 + \nu^* z_{1j} + \nu^* S_{1j}\tau_j x_{1j} + \nu^* S_{1j}(1+\tau_j)\omega_{1j} + \frac{1}{2}\alpha r_j^2.$$

Conclusions

- A differentiable, unknown PF, with two variable inputs, homogeneous of degree ν, can be written exactly, if firms minimize costs, as ν times a weighted mean of the logs of the inpus and an unknown function of the ratio of the inputs.

- Obtention: replacing the elasticities of the unknown PF by the cost shares of the inputs times ν. Keeps implicit an unrestricted $\sigma(\bullet)$ that is a function of the ratio of the inputs (consequence of homogeneity).

- Parameter ν and the output elasticities of the inputs can be estimated consistently by controlling the unknown function of the input's ratio. Model suggests that $\sigma(\bullet)$ can be estimated as a function of the ratio of the inputs.

- Semiparametric estimation of the CES and homogeneous TL gives forms that are nested, simplifying the unknown function to known forms. They impose a constant elasticty of substitution and an elasticity that decays with the inequality of the inputs respectively. Restrictions can be tested.

- The model can be enlarged to estimate the distribution of values of neutral productivity and input biased productivity of one input.

- It can be also used to identify imperfections of unknown origin and form in the market for one input.

- Using the estimate of v and the ditribution of τ_j, we can estimate markups in the product market.

- Limitations:

  - Model needs real output, it should be adapted if we only know revenue.

  - Variable cost shares have been implicitly assumed without error (measurement and/or adjustment costs). They should be corrected if we are suspicious of the opposite.

  - Can homogeneity be relaxed? Yes to homotheticity.

Appendix 1: Elasticities and cost shares

- Adding the FOCs duly weighted:

$$MC_j \sum_k \frac{X_{kj}}{F(X_j)} \frac{\partial F(X_j)}{\partial X_{kj}} = \frac{W_{kj} X_{kj}}{F(X_j)} \text{ and } \frac{AVC_j}{MC_j} = \sum_k \beta_k(x_j) = \nu.$$

- This allows to rewrite the FOCs. With $S_{kj}(x_j) = \dfrac{W_{kj}X_{kj}}{\sum_h W_{hj}X_{hj}}$,

$$\beta_k(x_j) = \frac{AVC_j}{MC_j} S_{kj}(x_j) = \nu S_{kj}(x_j),$$

a well known result often used to replace elasticities by shares in cost.

Appendix 2: Extended weak essenciality

- There is weak essenciality when $F(0) = 0$.

- Assume that $\widetilde{F}(0) = 0$ and $\widetilde{F}(1) = 1.$ It follows that $F(\bullet) = A\widetilde{F}(\bullet)$ has weak essenciality and $F(1) = A,$ where A is the measure of output when all inputs are set at unit values.

- $f(0) = \ln F(1) = \ln[A\widetilde{F}(\exp(0),\ldots,\exp(0))] = \ln A = a.$

- With a convenient output scaling, we can have $f(0) = 0.$ With no-scaling, we have a common constant that we obviate in theory developments and always take into account in econometric models.

Appendix 3: Taylor expansion of order 2

- Using $f(0) = a,\ a = f(x) + (0-x)f'(x) + \frac{1}{2}(0-x)f''(tx)(0-x)'$
- It follows $f(x) = a + xf'(x) - \frac{1}{2}xf''(tx)x'$, with $t \in [0,1]$.

Appendix 4: S_{kj} and σ_j as functions of r_j ($p = 2$)

$$S_{kj} = \frac{W_{kj}X_{kj}}{W_{kj}X_{kj}+W_{hj}X_{hj}} = \frac{1}{1+\frac{W_{hj}}{W_{kj}}\frac{X_{hj}}{X_{kj}}} = \frac{1}{1+\varphi(\frac{X_{hj}}{X_{kj}})\frac{X_{hj}}{X_{kj}}} = S_{kj}(\exp(x_{kj}-x_{hj})) \; = S_{kj}(r_j),$$

where the third equality follows from cost minimization under an homogeneous function. The fifth equality implies some abuse of notation.

Define σ_j as $\frac{|MRS_j|}{\frac{X_{kj}}{X_{hj}}}\frac{\partial\frac{X_{kj}}{X_{hj}}}{\partial|MRS_j|}$, where $|MRS_j|= \varphi(\frac{X_{hj}}{X_{kj}})$ by homogeneity, and $\frac{\partial\frac{X_{hj}}{X_{kj}}}{\partial|MRS_j|}$ is the same along a ray characterized by a constant $\frac{X_{hj}}{X_{kj}}$. Therefore $\sigma_j = \sigma(\frac{X_{hj}}{X_{kj}}) = \sigma(r_j)$, where the second equality again slightly abuses notation.

Appendix 5: Proving that t_j is a function of r_j ($p = 2$)

Taylor's second order term without particularizing is $-\frac{1}{2}x_j D\beta(x_j)x_j'$. In the derivatives $D\beta(x_j)$ under cost minimization they occur $S_{kj}(\frac{X_{hj}}{X_{kj}})$ and $\sigma(\frac{X_{hj}}{X_{kj}})$ that, slightly abusing notation, can be written $S_{kj}(r_j)$ and $\sigma(r_j)$ ($\frac{X_{hj}}{X_{kj}} = \exp(x_{hj} - x_{kj}) = \exp(r_j)$).

Taylor's exact term can hence be written

$$-\frac{1}{2}v\frac{1-\sigma(t_jr_j)}{\sigma(t_jr_j)}S_{1j}(t_jr_j)S_{2j}(t_jr_j)[x_{1j}x_{2j}]\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_{1j} \\ x_{2j} \end{bmatrix}$$

$$= \frac{1}{2}v\frac{1-\sigma(t_jr_j)}{\sigma(t_jr_j)}S_{1j}(t_jr_j)S_{2j}(t_jr_j)r_j^2.$$

Define $G(t_j) = \frac{1-\sigma(t_jr_j)}{\sigma(t_jr_j)}S_{1j}(t_jr_j)S_{2j}(t_jr_j)r_j^2$. For a given $r_j,\ t_j \in [0,1]$, $G(0) = \bar{r}r_j^2$, $G(1) = \theta(r_j)$, and it can be proved that $G(t_j)$ is monotonic.

Mean value theorem: $G'(t_j) = g(t_j) = \frac{G(1)-G(0)}{1-0}$. Follows $t_j = g^{-1}(\theta(r_j) - \bar{r}r_j^2)$.

Appendix 6: Estimation of σ_j

- From the estimation of $g(r_j)$ we only get $\frac{1-\sigma(t_j x_j)}{\sigma(t_j x_j)} S_{1j}(t_j x_j) S_{2j}(t_j x_j)$, so only an approximation to σ_j is possible.

- However, we can rely on the ratio of FOCs in the form

$$\ln \frac{S_{1j}}{S_{2j}} = (1 - \sigma_j(r_j))(w_1 - w_2),$$

using a nonlinear model for $\sigma_j(r_j)$.

- For example, we can model $1 - \sigma(r_j) = \frac{1}{1+\exp(P(r_j))}$ where $P(r_j)$ is a sieve in r_j.

Appendix 7: Estimating without output prices

- Write log revenue in terms of the price and PF

$$r = p + q = \ln\mu + mc + f(x) + \omega + \varepsilon.$$

- Consider $VC = VC(W, \frac{Q^*}{\exp(\omega)})$, obtain $MC = VC_2(W, F(X))\exp(-\omega)$. Write $\log MC$ as

$$mc = \overline{mc}(w, x) - \omega$$

- Deflated revenue is

$$r - \ln\mu = \overline{mc}(w, x) + f(x) + \varepsilon.$$

- Let us use an available proxy for the markup, $\ln\frac{R}{VC} = -\ln\nu + \ln\mu + \varepsilon$:

$$vc = \ln\nu + \overline{mc}(w, x) + f(x).$$

(We could have obtained the expression from $VC = \nu MCF(X)\exp(\omega)$).

- Semiparametric estimation suggests to estimate the PF parameters from

$$vc = cons + \overline{mc}(w, x) + \nu z + g(r) + \varepsilon.$$

Appendix 8: Effects of non-semiparametrics on markup estimation

- Take the estimator of De Loecker and Warzynski (2012):

$$\widehat{\mu} = \frac{\widehat{\beta}_X}{S_X^R} \exp(-\varepsilon).$$

- In terms of possible bias,

$$\widehat{\mu} = \frac{\widehat{\beta}_X}{\beta_X} \mu = \frac{\widehat{\beta}_X}{\nu S_X} \mu = \frac{\widehat{S}_X}{S_X} \mu.$$

- The bias depends on how the "implicit estimate" of the share is wrt the real share in cost.

- Use semiparametric estimation

$$\widehat{\mu} = \frac{\widehat{\beta}_X}{S_X^R} \exp(-\varepsilon) = \frac{\nu S_X}{S_X^R} \exp(-\varepsilon) = \nu \frac{R}{VC} \exp(-\varepsilon).$$

It says that unbiased estimation is Bain (1951) corrected by the scale.