

Semiparametric Estimation of Heterogeneous Firm-level Production Functions*

Jordi Jaumandreu[†]

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Abstract

Many exercises need to estimate firm-level production functions. The doubts start with what production function, i.e. which functional form and what heterogeneity. This paper develops a semiparametric specification for variable factors that only assumes common scale and firm's cost minimization. It is easy to apply and use for testing restrictions. When size of the firm is not correlated with the heterogeneity of factor proportions, is equivalent to a loglinear specification with firm and time-varying elasticities. It nests the commonly used CES and homogeneous Translog production functions. It admits Hicks-neutral and biased productivity and non-competitive outcomes in the input markets.

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[†]Boston University, Department of Economics. Email: jordij@bu.edu.

1 Introduction

Production functions at the firm-level are estimated to predict output, cost, input use (e.g. employment), emissions, and so on, as well as its variations when firms are subject to exogenous changes. The theory of production makes everything depend on a few characteristics: the output elasticities of the inputs and its sum (scale), as well as the elasticity of substitution among inputs. One leading difficulty is that firms' equilibria and outcomes are distorted by how production functions are impacted by an array of heterogeneous unobservables, mainly productivity (Hicks-neutral and input-biased), market power in the product markets, and wedges on input prices resulting from imperfect competition in the input markets. Ideally, researchers want to estimate the production function controlling that these unobservables do not affect the result, i.e. they want to be robust. And sometimes, the objective is reversed. What researchers look for is just to use the presumable distortions to derive the distribution of values of the unobservables (e.g. productivity, markups, markdowns). In any case, doubts tend to start with the decision of which particular production function to use and what heterogeneity to include explicitly and how.

The contribution of this note is to derive a way to estimate semiparametrically an unknown production function that is more general than the production functions usually estimated, and nests commonly used alternatives such as the CES and the homogeneous Translog. The specification needs minimum assumptions that can be accepted in many contexts: common differentiable technology, with a scale represented by an arbitrary degree of homogeneity in variable factors, and cost minimization. Semiparametric refers to the fact that identification is reached by replacing the elasticities and its variation by the observed shares in variable cost according to the assumption of cost minimization and estimating the scale. A sequence of estimators can be used to decide to impose specific restrictions, in particular on the elasticity of substitution of the inputs, or to leave the specification as it is.

Our semiparametric specification makes clear that the complexity of the estimation depends on how the heterogeneity of the relative input use is related to the size of the output. If the size of the firms is not related to the impact of the ratio of inputs, estimation can be carried out with a loglinear form with the input elasticities varying firm to firm and over time. Semiparametrics can deal with this situation very easily.

The semiparametric specification admits, of course, neutral productivity and input-biased productivity. It also admits the presence of imperfect competition in input markets and the recovering of the wedges that it raises

across firms without the need to assume particular behavior.¹

The substitution of the shares for the elasticities may imply the use of noisy proxies in the sense that the shares reflect shadow costs of adjustment whose values are not reflected in the observed prices. The correct way to deal with this situation is an important empirical aspect of estimation. See, for example, the approach to correct the shares used in Doraszelski and Jaumandreu (2019). The procedure is likely to be very specific to the exercise at hand, and we will not discuss this topic further in this note.

Azzam, Jaumandreu and Lopez (2025) used successfully this type of semi-parametric estimation to assess market power in two input markets of the U.S. meatpacking industry (cattle and labor).² Dobbelaere, Jaumandreu, and Mairesse (2026) use this semiparametric estimation to develop how to estimate wage wedges and product markups under behavior-unconstrained market power (they recover the entire distribution), with Compustat data on U.S. manufacturing firms.

Nonparametric and semiparametric estimation

Solow (1957) showed that, differentiating an unknown production function with respect to time, the proportional changes in output could be expressed in terms of the proportional input changes multiplied by the corresponding elasticities (plus the shift due to technical change). And, assuming competition and constant returns to scale, replaced elasticities by shares in revenue. This inaugurated a nonparametric estimation of the production function, for assessing “total factor productivity” changes, that has been massively used.

Many economists asked themselves how to apply a similar method when there are productive units to be compared in the cross-section. The units present differences that are not amenable to differential calculus, and the problem remained elusive. However, Christensen, Jorgenson and Lau (1973) presented then the (homogeneous) Translog production functions as “...functions that are quadratic in the logarithms of the quantities of inputs and outputs. These functions provide a second-order approximation to any production frontier.” And Diewert (1976), using a Quadratic Identity Lemma (QUIL),³ showed that the difference in the translog value in two input-output vectors could be written using the change in input multiplied by the

¹Having measured marginal cost that is impacted by productivity and the marginal prices determined in the input market, it is also possible to deduce the value of the markup and hence the market power hold by the firm.

²Syverson (2004), in the robustness checks of his nonparametric exercises of measurement of productivity in the concrete sector, used the shares-weighted sum of inputs as a regressor, as we do in the linear part of the equations.

³The Quadratic Identity Lemma shows that the difference between two values of a

average of the values of each elasticity in the two points. That is, using only one input x for simplicity of notation, $\Delta f(x) = \frac{1}{2}(\beta_0 + \beta_1)\Delta x$, where the function and variable are in logs, 0 and 1 represent the two points, and β is the elasticity.

Using this result, Caves, Christensen and Diewert (1982a, 1982b) developed comparisons between the productivity of two units (bilateral) or of a unit with the geometric mean of the units (multilateral). These comparisons are usually applied employing nonparametric estimation of the production function.⁴ If we have $\Delta f(x) = \Delta\omega + \frac{1}{2}(\beta_0 + \beta_1)\Delta x$, the productivity difference can be estimated as $\Delta\omega = \Delta f(x) - \frac{1}{2}(S_0 + S_1)\Delta x$ where the elasticities have been replaced by the corresponding shares of the input in revenue (assuming that the scale is unitary to make the elasticities fully observable).

Of course, these results can be used in estimation in a more general way. The elasticities can be replaced by the elasticity of scale times the share in cost, making the estimation robust to market power (only cost minimization is then implied), just by estimating the elasticity of scale. We will use in fact this type of semiparametric specification. However, the differences form of the relationship implied by the QUIL complicates the expressions in the cross-section and in panel data. In addition, the results depend on the assumption that the true form of the production function is quadratic.

Trying to avoid the limitations introduced by the QUIL by estimating directly the second order approximation to any production function is not better. This type of estimation hopes that the estimated first and second derivatives are going to be a good approximation to the elasticities and curvature of the arbitrary unknown function, but this is in general not true because of the correlation of the remainder with the included variables. White (1980, 1981) showed how the estimation of the “truncated” production function only produces the least squares best approximation (best linear predictor if linear) to the unknown function.

This paper shows that it is possible to write an exact second order expansion of an homogeneous unknown production function in terms of the elasticity of scale, the shares of the inputs in variable cost, and the elasticity of substitution. We rely on this result to show how to estimate the parameter of scale and the elasticity of substitution (under different assumptions), and to treat the main kind of unobservables that can affect the production func-

quadratic function with constant parameters can be written in terms of the first derivatives at the two values. It is a multivariate generalization of the fact that, if $f(x)$ is quadratic with constant parameters, $\Delta f(x) = \frac{1}{2}(\frac{\partial f(x_0)}{\partial X_0} + \frac{\partial f(x_1)}{\partial X_1})\Delta x$.

⁴They called it comparisons by means of superlative indices, where the word superlative refers to the fact that they use quantity indices that are exact for an underlying translog production function.

tion (neutral and biased productivities, wedges in input prices determined by noncompetitive behavior). Therefore, it constitutes a semiparametric estimation of an unknown production function characterized by heterogeneity.

Another type of estimate is relatively close to our proposal. Gottardo (2026), drawing on Gandhi, Navarro, and Rivers (2020), proposes to estimate nonparametrically the elasticities in terms of the inputs from the shares in (variable) cost. Consisting of only variable factors (Gottardo, 2026, assumes that K is variable), the production function can then be “integrated” up to an arbitrary constant from the sum of the integrals of the estimated elasticities. Although this is possible, it has several drawbacks. First, it is doubtful why one needs to take the effort to estimate expressions for the elasticities. Second, it is not clear how to ensure that the estimated expressions generally meet the conditions for “integrability” (see Hurwicz and Uzawa, 1971). Third, the procedure has not been adapted to treat scales different from one, biased productivity, and lack of competition in input markets. The only advantage may be that the estimation of the elasticities can purge the observed shares of errors, something that we have obviated in our proposal (but can be done as proposed or done also with other procedures).

2 A semiparametric specification

Assume a cross-section population of firms with an unknown (log) production function $f(\cdot)$ differentiable and homogeneous of degree ν . We follow the convention of writing logs in lowercase letters and levels in uppercase. We use subindex j to denote a unit, to stress without ambiguity everything that is possibly varying across units. The outputs are

$$q_j = f(x_j) + \varepsilon_j, \tag{1}$$

where x_j is a vector of variable inputs $1 \times p$ and ε_j is an error that is not correlated with the information available at the moment. The context can be straightforwardly extended to the case of a panel population with small t . If there are fixed inputs, the function can be assumed to produce an aggregate intermediate input and nested in a total output production function.⁵

We call $\beta(x_j)$ the vector of $p \times 1$ derivatives $\frac{\partial f(x_j)}{\partial x_{kj}}$, $k = 1 \dots p$, or the output elasticities. Denote by $D\beta(x_j)$ the $p \times p$ Hessian representing the slopes of the elasticities. We will find the following useful.

⁵So the total output production function will be weakly separable in variable inputs (Shephard, 1953; Blackorby, Primont, and Russell, 1978).

LEMMA 1.

If $f(\cdot)$ shows weak input essentiality, $f(0) = 0$, the production function can be written as $f(x_j) = x_j\beta(x_j) - \frac{1}{2}x_jD\beta(t_jx_j)x'_j$, where t_j is a scalar such that $0 \leq t_j \leq 1$.

Proof. See Appendix.

Next, we introduce some notation and an important implication of cost minimization. Let us develop, for the sake of simplicity, the particular case of a production function with two inputs x_1 and x_2 .

The substitution elasticity, which characterizes the curvature of the isoquants, is $\sigma(x_j) = \frac{\frac{X_{hj}}{X_{kj}} \frac{\partial \frac{X_{kj}}{X_{hj}}}{\partial |MRS_j|}}{\frac{1}{|MRS_j|}}$, where $MRS_j = -\frac{\frac{\partial F(X_j)}{\partial X_{hj}}}{\frac{\partial F(X_j)}{\partial X_{kj}}}$, with $k, h = 1, 2$. Cost minimization implies that the input choices are made according to the FOCs, $MC_j(x_j) \frac{\partial F(X_j)}{\partial X_{kj}} = W_{kj}$, where W_{kj} is the rental price of the input k for firm j , for $k = 1, 2$. Notice that $MC_j(x_j) \sum_k \frac{X_{kj}}{F(X_j)} \frac{\partial F(X_j)}{\partial X_{kj}} = \sum_k \frac{W_{kj} X_{kj}}{F(X_j)}$ and hence $\frac{AVC_j(x_j)}{MC_j(x_j)} = \sum_k \beta_k(x_j) = \nu$ or elasticity of scale.

Multiplying both sides of the FOCs by $\frac{X_{kj}}{F(X_j)}$, and multiplying and dividing the right hand side by $\frac{AVC_j(x_j)}{MC_j(x_j)}$, we can write $\beta_k(x_j) = \nu S_{kj}(x_j)$, with $S_{kj}(x_j) = \frac{W_{kj} X_{kj}}{W_{kj} X_{kj} + W_{hj} X_{hj}}$ or share of the input k cost in (variable) cost. Notice that $S_{hj}(x_j) = 1 - S_{kj}(x_j)$ for $k, h = 1, 2$.⁶

That is, the production elasticities of the cost minimizing firm can be written as the elasticity of scale times the cost share of the input. It follows that, for a cost minimizing firm, we can write the production function in terms, in addition to the inputs, of the shares of input costs in variable cost, the elasticity of scale parameter ν , and the elasticity of substitution $\sigma(x_j)$. The next theorem shows this.

⁶A comment on notation is in order. We write $AVC_j(x_j)$ and $MC_j(x_j)$, with j -specific subindex, because AVC and MC are functions of the prices in addition of the inputs, and prices are firm-specific. We choose to write MRS_j instead of $MRS(x_j)$ for simplicity, but the first notation is a shorthand for the second. We write $\beta_k(x_j)$ because the elasticities of any production function are simply a function of the inputs and it would be redundant to double the first subindex. Writing as we do $S_{kj}(x_j)$ implies some redundancy, because shares under cost minimization turn out to be a function of the inputs ratio. But it stresses the underlying heterogeneity when we use this expression reflecting the observed shares.

THEOREM 1.

Let $\sigma(x_j)$ be the elasticity of substitution and ν the elasticity of scale, using Lemma 1 the production function of a cost minimizing firm can be written as

$$f(x_j) = \nu(S_{1j}(x_j)x_{1j} + S_{2j}(x_j)x_{2j}) + \frac{1}{2}\nu\frac{1-\sigma(x_j)}{\sigma(x_j)}S_{1j}(x_j)S_{2j}(x_j)(x_{2j} - x_{1j})^2. \quad (2)$$

Proof. See Appendix.

We call (2) a semiparametric specification in the sense that some objects (output elasticities), which usually are the object of estimation, have been replaced by observable variables. According to the conditions assumed to hold (cost minimization), the observable shares reveal the value of the elasticities. We have justified this for any production function, but we can apply it to particular specifications.

Consider the particular functional forms CES, $f(x_j) = -\frac{\nu\sigma}{1-\sigma} \ln[x_{1j}^{-\frac{1-\sigma}{\sigma}} + x_{2j}^{-\frac{1-\sigma}{\sigma}}]$ (see a recent example of use in Demirer, 2025), and the homogeneous Translog of degree $\nu = \alpha_1 + \alpha_2$, $f(x_j) = \alpha_0 + \alpha_1x_{1j} + \alpha_2x_{2j} + \frac{1}{2}\alpha(x_{2j} - x_{1j})^2$ (recently used by Doraszelski and Jaumandreu, 2019).

COROLLARY. The semiparametric specification (2) has as especial cases the CES and the homogeneous Translog production functions.

Proof. See Appendix.

In the CES case, compared to the general semiparametric form, the substitution elasticity $\sigma(x_j)$ is restricted to be a parameter σ . In the case of Translog, $\nu_j\frac{1-\sigma(x_j)}{\sigma(x_j)}S_{1j}(x_j)S_{2j}(x_j)$ is restricted to be equal to the constant α .

3 Estimation

Under cost minimization, we have derived a semiparametric specification for an unknown (variable inputs) production function, based on the replacement of the elasticities and their derivatives by the observed cost shares in variable cost, together with the parameter of scale, and the elasticity of substitution. We have also shown how this semiparametric specification encompasses the CES and homogeneous Translog production functions, functional forms that

are often chosen to estimate the production function. Now, we detail how these results can be used in estimation.

Our next proposition shows the consistency of the cross-section (or short-panel) estimation of the elasticity of scale parameter ν (the sum of output elasticities or proportional increase of output when all inputs are changed simultaneously). Moreover, if the heterogeneity of input proportions is uncorrelated with the size of the firm, the semiparametric specification can be simplified to a loglinear expression in terms of the (varying) elasticities. In this case, the CES and the Translog will tend to produce the same estimation of the scale parameter, although there can be some differences, because the CES and Translog impose restrictions on the substitution elasticity that the most general model does not impose. In contrast, if the input ratio heterogeneity does vary in a way that is correlated with the size of the firm, the different assumptions about the elasticity of substitution embodied in each of the specifications are very relevant, and the estimated scale parameters will differ.

Let us use the notation $z_{1j} = S_{1j}(x_j)x_{1j} + S_{2j}(x_j)x_{2j}$ and note that $\exp(z_{1j}) = X_{1j}^{S_{1j}} X_{2j}^{S_{2j}}$ can be taken as a (constant return Cobb-Douglas) index of the size of the output, so z_{1j} can be seen as a positive monotonic transformation of an index of size of the firm. Define also the regressor $z_{2j} = S_{1j}(x_j)S_{2j}(x_j)(x_{2j} - x_{1j})^2$. Given Theorem 1 (expression (2)), the output of the firms can be written⁷

$$q_j = \beta_0 + \nu z_{1j} + \frac{1}{2} \nu \frac{1 - \sigma(x_j)}{\sigma(x_j)} z_{2j} + \varepsilon_j. \quad (3)$$

Let us establish a couple of results that we will need to prove the next proposition: both the shares in cost and the elasticity of substitution are functions of the ratio of inputs .

LEMMA 2.

For a cost minimizing firm, the shares in variable cost can be written as a function of the ratio of inputs. That is, we can write $S_{kj}(x_j) = S(x_{hj} - x_{kj})$ for $k, h = 1, 2$.

Proof. See Appendix.

⁷We include systematically a constant without loss of generality. Some models have no constant, but it is always convenient to estimate with a constant that the data will detect, if the model is valid, as not significantly different from zero.

LEMMA 3.

The elasticity of substitution is a function of the ratio of inputs. That is, we can write $\sigma(x_j) = \sigma(x_{2j} - x_{1j})$.

Proof. See Appendix.

The variation in $\sigma(x_{2j} - x_{1j})$ can take different forms. The literature has rarely discussed what to expect from this variation. In one leading case, we can have an input that becomes systematically more difficult to substitute for the other as its relative amount grows larger. Suppose, for example, that substitution of labor for some material becomes more difficult when the possibilities of saving the material have already been intensively explored and exploited.

However, we can also have non-monotonic cases in which the elasticity of substitution decreases (or increases?) when we leave the vicinity of a proportion favored by the available technology. For example, the elasticity of substitution of the homogeneous Translog decreases with the inequality of the cost shares, $\frac{\partial \sigma(x_j)}{\partial |S_{2j} - S_{1j}|} < 0$.

PROPOSITION 1.

The parameter ν of expression (3) can be consistently estimated in the cross-section, controlling the effects of the term in z_{2j} by an unknown function of the ratio of inputs, estimated nonparametrically. If the unknown function is uncorrelated with the size of the firm, a consistent estimation can be carried out dropping the control term. In this case, the CES and Translog estimators will be closer to the semiparametric estimator.

Proof.

The regressor z_{2j} is a function of $x_{2j} - x_{1j}$ because it is a composition of functions of the ratio of the inputs (Lemma 2). The expression $\frac{1 - \sigma(x_j)}{\sigma(x_j)}$ is also, as shown in Lemma 3, a function of $x_{2j} - x_{1j}$. It follows that the replacement of the entire third term of (3) by an unknown function of the input ratio is an appropriate control. That is, the model $q_j = \beta_0 + \nu z_{1j} + g(x_{2j} - x_{1j}) + \varepsilon_j$, where $g(\cdot)$ is an unknown function, can be used to estimate consistently ν . This is in fact the model of Robinson (1988) to estimate consistently a partially linear model controlling nonparametrically by an observed variable. Robinson (1988) showed that the estimator is (root-N) consistent and asymptotically normal.

Notice that $g(x_{2j} - x_{1j})$ depends only on the input ratio, is presumably quite nonlinear, and not necessarily monotonic. The variable $x_{2j} - x_{1j}$ is not independent of z_{1j} , but z_{1j} can be uncorrelated with $g(\cdot)$. If variable z_{1j}

is not correlated with $g(\cdot)$, $Cov(z_{1j}g(x_{2j} - x_{1j})) = 0$, consistent estimation may be carried out dropping the term in $g(\cdot)$. Multiplying the model by $\tilde{z}_{1j} = z_{1j} - E(z_{1j})$ and solving for ν gives $\nu = \frac{Cov(z_{1j}, q_j)}{Var(z_{1j})}$. The parameter is identified because it can be written in terms of population moments (see e. g. Wooldridge, 2010).

It can be consistently estimated from sample analogs of the population moments $Cov(z_{1j}, q_j)$ and $Var(z_{1j})$. In this case, the specifications corresponding to the CES and Translog functions should tend to produce the same ν estimate. Not necessarily the same because they are imposing restrictions on the elasticity of substitution that may not be true. ■

We have been left in practice with three options to estimate ν . Models are

$$q_j = \beta_0 + \nu z_{1j} + g(x_{2j} - x_{1j}) + \varepsilon_j, \quad (4)$$

$$q_j = \beta_0 + \nu z_{1j} + \frac{1}{2}\nu \frac{1 - \sigma}{\sigma} z_{2j} + \varepsilon_j, \quad (5)$$

$$q_j = \beta_0 + \nu z_{1j} + \frac{1}{2}\alpha(x_{2j} - x_{1j})^2 + \varepsilon_j, \quad (6)$$

The first is the general semiparametric specification, the second the semiparametric specification of the CES, and the third the semiparametric specification of the Translog. In each case, in addition to the estimate $\hat{\nu}$, we get semiparametric estimates of the elasticities β_{1j} and β_{2j} , as $\hat{\beta}_{1j} = \hat{\nu}S_{1j}$ and $\hat{\beta}_{2j} = \hat{\nu}S_{2j}$. Moreover, the CES model computes directly an estimate $\hat{\sigma}$, the general semiparametric estimator allows us to estimate the distribution of the substitution elasticity as $\hat{\sigma}_j = \frac{\hat{\nu}z_{2j}}{2\hat{g}_j + \hat{\nu}z_{2j}}$, and the Translog's restricted distribution can be estimated as $\hat{\sigma}_j = \frac{\hat{\nu}S_{1j}S_{2j}}{\hat{\alpha} + \hat{\nu}S_{1j}S_{2j}}$.

The estimation of model (4) can be done by projecting q_j and z_{1j} on $x_{2j} - x_{1j}$, obtaining $E(q_j|x_{2j} - x_{1j})$ and $E(z_{1j}|x_{2j} - x_{1j})$. Since $E(q_j|x_{2j} - x_{1j}) = \beta_0 + \nu E(z_{1j}|x_{2j} - x_{1j}) + g(x_{2j} - x_{1j})$, we have that $q_j - E(q_j|x_{2j} - x_{1j}) = \nu(z_{1j} - E(z_{1j}|x_{2j} - x_{1j})) + \varepsilon_j$ and ν is identified by the expression of population moments $\nu = \frac{E[(z_{1j} - E(z_{1j}|x_{2j} - x_{1j}))(q_j - E(q_j|x_{2j} - x_{1j}))]}{E[(z_{1j} - E(z_{1j}|x_{2j} - x_{1j}))^2]}$.

Nonparametric projections can be carried out using kernel-based estimators (e.g. a Normal Nadaraya-Watson estimator, as Robinson, 1988, proposed) or sieves based on polynomials of $x_{2j} - x_{1j}$ (see, for example, the discussion of the application of both methods in Pagan and Ullah, 1999).

Model (5) constitutes a simple multivariate linear regression with two parameters linked by a cross restriction. Model (6) is a multivariate regression with two parameters.

A possible way to start is to check if $g(\cdot)$ matters by estimating the model including it and not including it alternatively. This is enough to decide to adopt, if accepted as identical, the loglinear model. However, the general semiparametric specification can also be tested against more specific alternatives.

4 Hicks-neutral productivity

Until now, for simplicity, we have not considered any type of productivity. Let us now introduce the Hicks-neutral productivity ω_j .

Assume that we have a minimum of two time observations per firm, so we can assume that neutral productivity follows the Markovian process $\omega_{jt} = \rho\omega_{jt-1} + \xi_{jt}$. We will assume that neutral productivity is mean independent of the ratio of prices, $E(\omega_{jt}|w_{2jt} - w_{1jt}) = 0$. This ensures, because of the law of iterated expectations, that $E(\omega_{jt}|x_{2jt} - x_{1jt}) = 0$.⁸

Let us first discuss the case of the general semiparametric specification. With neutral productivity, the model becomes:

$$q_{jt} = \beta_0 + \nu z_{1jt} + g(x_{2jt} - x_{1jt}) + \omega_{jt} + \varepsilon_{jt},$$

Since $E(\omega_{jt}|x_{2jt} - x_{1jt}) = 0$, taking expectations we have $E(q_{jt}|x_{2jt} - x_{1jt}) = \beta_0 + \nu E(z_{1jt}|x_{2jt} - x_{1jt}) + g(x_{2jt} - x_{1jt})$. Subtracting from the original equation, we get $q_{jt} - E(q_{jt}|x_{2jt} - x_{1jt}) = \nu(z_{1jt} - E(z_{1jt}|x_{2jt} - x_{1jt})) + \omega_{jt} + \varepsilon_{jt}$. Estimation of the parameter of scale can then be based on the pseudo-differentiation of this equation (dynamic panel method)

$$\begin{aligned} q_{jt} - E(q_{jt}|x_{2jt} - x_{1jt}) &= \rho[q_{jt-1} - E(q_{jt-1}|x_{2jt-1} - x_{1jt-1})] \\ &\quad + \nu[E(z_{1jt}|x_{2jt} - x_{1jt}) - \rho E(z_{1jt-1}|x_{2jt-1} - x_{1jt-1})] \\ &\quad + \xi_{jt} + \varepsilon_{jt} - \rho\varepsilon_{jt-1}. \end{aligned}$$

We have two parameters to estimate, ρ and ν . The variable q_{jt-1} is correlated with ε_{jt-1} , and z_{1jt} is likely to be correlated with ω_{jt} and hence with ξ_{jt} , but not the projection of z_{1jt} on $x_{2jt} - x_{1jt}$. Variables x_{1t-1} and x_{2t-1} are also valid instruments to which k_{t-1} , lagged input prices and/or a lagged share of cost may be added. Hence, we can estimate with several overidentification restrictions.

⁸Neutral productivity multiplies the FOCs of the two inputs, $MC_{jt}(x_{jt}) \frac{\partial F_{jt}(X_{jt})}{\partial X_{kjt}} \exp(\omega_{jt}) = W_{kt}$, where W_{kt} is the rental price of the input k , for $k = 1, 2$. It follows that the log-ratio $x_{2jt} - x_{1jt}$ depends only on the ratio of input prices.

With the semiparametric specification of the CES and Translog, models are

$$\begin{aligned} q_{jt} &= \beta_0 + \nu z_{1jt} + \frac{1}{2} \nu \frac{1-\sigma}{\sigma} z_{2jt} + \omega_{jt} + \varepsilon_{jt}, \\ q_{jt} &= \beta_0 + \nu z_{1jt} + \frac{1}{2} \alpha (x_{2j} - x_{1j})^2 + \omega_{jt} + \varepsilon_{jt}, \end{aligned}$$

respectively. For having an estimable model, we only need pseudodifferentiation, and the same instruments as before are valid. In comparison with the fully semiparametric form, we do not require neutral productivity to be mean independent of the ratio of input prices.

However, the last two semiparametric forms suggest that there might be an advantage in imposing restrictions on the elasticity of substitution if one keeps the assumption of mean independence of the ratio of input prices. With enough variation of the ratios, using $E(\omega_j | x_2 - x_1) = 0$, the estimation of the parameter ν could also be based on the models

$$\begin{aligned} E(q_j | x_{2j} - x_{1j}) &= \nu [E(z_{1j} | x_{2j} - x_{1j})] + \frac{1}{2} \nu \frac{1-\sigma}{\sigma} E(z_{2j} | x_{2j} - x_{1j}) + \varepsilon_j, \\ E(q_j | x_{2j} - x_{1j}) &= \nu [E(z_1 | x_{2j} - x_{1j})] + \frac{1}{2} \alpha E(z_{2j} | x_{2j} - x_{1j}) + \varepsilon_j, \end{aligned}$$

that do not need any dynamic assumption on the productivity process or time observations.⁹

Using an Olley and Pakes (1996)/ Levinsohn and Petrin (2003) method (OP/LP method) to control for productivity does not seem to make much sense in our context. The OP/LP method exploits the FOCs, and the FOCs have already been exploited in the specification of the production function. We are obviously assuming that they are met, and then the inclusion of the “proxies” implied by an OP/LP method must be redundant (see Jaumandreu, 2026). However, there is one sense in which this can have meaning, and it is just to test the specification (checking if the FOCs are met).

5 Non-neutral productivity

Now we have an input that is not fully observable (we do not observe its log-efficiency ω_{kj}): $x_{kj}^* = x_{kj} + \omega_{kj}$. Let us drop momentarily neutral-Hicks productivity for the sake of simplicity. We consider it again later in the section.

⁹Although still remains the task to separate the estimated ω_j and ε_j .

Assume, without loss of generality, that biased productivity affects input x_1 . The expression (3) for the production function can be written as

$$q_j = \beta_0 + \nu z_{1j} + \nu S_{1j}(x_j) \omega_{1j} + \frac{1}{2} \nu \frac{1 - \sigma(x_j)}{\sigma(x_j)} S_{1j}(x_j) S_{2j}(x_j) (x_{2j} - x_{1j} - \omega_{1j})^2 + \varepsilon_j.$$

We will replace the unobservable by observables (as in Doraszelski and Jaumandreu, 2019 or Demirer, 2025). Log-linearizing the ratio of FOCs we get $x_{2j} - x_{1j} = \sigma(x_j)(w_{1j} - w_{2j}) + (1 - \sigma(x_j))\omega_{1j}$, which can be solved for unobservable productivity as $\omega_{1j} = x_{2j} - x_{1j} - \frac{\sigma(x_j)}{1 - \sigma(x_j)} \ln \frac{S_{1j}(x_j)}{1 - S_{1j}(x_j)}$. Substituting this expression for the unobservable and rearranging we have

$$q_j = \beta_0 + \nu x_{2j} - \nu \frac{\sigma(x_j)}{1 - \sigma(x_j)} S_{1j}(x_j) \ln \frac{S_{1j}(x_j)}{1 - S_{1j}(x_j)} \left(1 - \frac{1}{2} S_{2j}(x_j) \ln \frac{S_{1j}(x_j)}{1 - S_{1j}(x_j)}\right) + \varepsilon_j.$$

Using the same reasoning as in Proposition 1 we can write the third term as an unknown function of the ratio of inputs

$$q_j = \beta_0 + \nu x_{2j} + \bar{g}(x_{2j} - x_{1j}) + \varepsilon_j. \quad (7)$$

The general semiparametric specification changes in the linear part, where parameter ν shows now as the coefficient of the input not affected by biased productivity (identification needs one unaffected), while the new nonparametric unknown function controls for the ratio and for the partially unobserved input (via ratio). We obviously expect the ratio now to be highly correlated with ω_{1j} . After estimating ν , the substitution elasticity can be recovered as $\hat{\sigma}_j = \frac{\hat{g}_j}{\hat{g}_j + \hat{\nu} S_{1j} \ln \frac{S_{1j}}{1 - S_{1j}} (1 + \frac{1}{2} S_{2j} \ln \frac{S_{1j}}{1 - S_{1j}})}$.

The cases of CES and Translog can be treated by applying the corresponding restrictions on the general semiparametric specification. If the production function is a CES, then the elasticity of scale becomes a parameter and we can use the regressor $z_{3j} = S_{1j}(x_j) \ln \frac{S_{1j}(x_j)}{1 - S_{1j}(x_j)} (1 - \frac{1}{2} S_{2j}(x_j) \ln \frac{S_{1j}(x_j)}{1 - S_{1j}(x_j)})$:

$$q_j = \beta_0 + \nu x_{2j} - \nu \frac{\sigma}{1 - \sigma} z_{3j} + \varepsilon_j. \quad (8)$$

If the production function is a homogeneous Translog, we should solve the ratio of FOCs specific to this form. It gives $\omega_{1j} = x_{2j} - x_{1j} + \frac{\alpha_L}{\alpha} - \frac{\nu}{\alpha} S_{1j}(x_j)$. Plugging this expression and collecting terms, one gets

$$q_j = \beta_0 + \nu x_{2j} - \frac{1}{2} \frac{\nu^2}{\alpha} [S_{1j}(x_j)]^2 + \varepsilon_j, \quad (9)$$

with $\beta_0 = \frac{1}{2} \frac{\alpha_L^2}{\alpha}$, that is, expression (11) of Doraszelski and Jaumandreu (2019).

Given formula (7) it becomes clear that Hicks-neutral productivity ω_j can be re-introduced and treated in the way that we have described in Section 4. The semiparametric specification can separate both productivities. To recover input-augmenting productivity, we will use $\widehat{\omega}_{1j} = x_{2j} - x_{1j} - \frac{\widehat{\sigma}_j}{1-\widehat{\sigma}_j} \ln \frac{S_{1j}(x_j)}{1-S_{1j}(x_j)}$.

The semiparametric forms of the CES and Translog ((8) and (9) respectively) also admit the reintroduction of the specification of Hicks-neutral productivity. All forms can produce estimates of both productivities.

6 Imperfect competition in the determination of an input

Assume that the input x_k is not hired in a perfectly competitive market. It can be the result of determination with monopsony power, a rent sharing model, or any other unspecified type of imperfection in competition. For simplicity, let us leave aside both productivities; we consider them simultaneously later in this section. The FOC will look as $MC_j(x_j) \frac{\partial F_j(X_j)}{\partial X_{kj}} \exp(\omega_j) = W_k(1 + \tau_j)$, where τ_j is a proportional wedge arising from the specific market imperfection. Under monopsony, we expect τ_j to be positive (“markdown”), but in other circumstances (may be less frequent) can be negative.

Now the problem is that the relationship between $MC_j(x_j)$ and $AVC_j(x_j)$ is a little more complex: $\frac{AVC_j(x_j)}{MC_j(x_j)} = \frac{\nu}{1+S_{kj}(x_j)\tau_j} \equiv \nu^*(S_{kj}, \tau_j)$. This implies that $\beta_{kj} = (1 + \tau)\nu^*(S_{kj}, \tau_j)S_{kj}(x_j)$ and $\beta_{hj} = \nu^*(S_{kj}, \tau_j)S_{hj}(x_j)$.

Assume, without loss of generality, that x_1 is the input which is not set noncompetitively.

On the one hand, $x_j\beta(x_j) = \beta_{1j}(x_j)x_{1j} + \beta_{2j}(x_j)x_2 = \nu^*(S_{1j}, \tau_j)(S_{1j}(x_j)x_{1j} + S_{2j}(x_j)x_2 + \tau_j S_{1j}(x_j)x_{1j}) = \nu^*(S_{1j}, \tau_j)z_{1j} + \nu^*(S_{1j}, \tau_j)\tau_j S_{1j}(x_j)x_{1j}$.

On the other hand, the quadratic term of Lemma 1 becomes

$\frac{1}{2}\nu^*(S_{kj}, \tau_j) \frac{\sigma(x_j)}{1-\sigma(x_j)} S_{1j}(x_j)S_{2j}(x_j)[(x_2 - x_1)^2 - \tau_j x_1(x_2 - x_1)]$, which contains the unobservable τ_j twice.

Developing now the log-linearization of the ratio of FOCs, we get that $\ln(1 + \tau_j) = \frac{1-\sigma(x_j)}{\sigma(x_j)}(x_{2j} - x_{1j}) - \ln \frac{S_{1j}(x_j)}{1-S_{1j}(x_j)}$, again is a function of the ratio of inputs (if τ_j is not expected to be very large, we can use $\ln(1 + \tau_j) \simeq \tau_j$). The control becomes now the unknown function $\bar{g}(x_{1j}, x_{2j} - x_{1j})$ and a general semiparametric form, denoting $z_{4j} = S_{1j}(x_j)x_{1j}$, can be set as

$$q_j = \beta_0 + \nu^*(S_{1j}, \tau_j)z_{1j} + \nu^*(S_{1j}, \tau_j)\tau_j z_{4j} + \bar{g}(x_{1j}, x_{2j} - x_{1j}) + \varepsilon_j. \quad (10)$$

Notice that equation (10) implies a qualitative jump over previous equations. The part that was linear is not linear anymore, because it depends nonlinearly on τ_j . On the other hand, the unknown function is now slightly more complicated and its relevance harder to test. The difficulty seems to be that τ_j acts as a price that is not directly reflected in the cost shares, and hence its identification is quite difficult without restrictions on the elasticity of substitution.

However, equation (10) gives several suggestions about how to proceed. First, it says that we can directly identify τ_j , with completely unspecified substitution patterns if we are willing to give τ_j a limited variation. For example, it can be considered constant or make it a function of some variables. Of course, this may implicitly set specific assumptions on how x_1 is determined.

Second, the log-linearization of the ratio of inputs, which obtains an expression for τ_j , also points to a way to get a more general model: plug it to replace τ_j using a constant elasticity of substitution that can be considered a local approximation (one does not want to abandon completely the general semiparametric form). It is a very nonlinear model that identifies a varying τ_j as $\hat{\tau}_j = \frac{1-\hat{\sigma}}{\hat{\sigma}}(x_{2j} - x_{1j}) - \ln \frac{S_{1j}(x_j)}{1-S_{1j}(x_j)}$.

Third, equation (10) also clarifies how imperfect competition in an input can be combined with the identification of the two productivities, Hicks-neutral and input-augmenting. Think first of the non-control part of the general semiparametric form with input-augmenting productivity. Assuming that x_1 is the input affected by biased productivity and noncompetitive determination, we have

$$\begin{aligned} x_j \beta(x_j) &= \beta_{1j}(x_j)(x_{1j} + \omega_{1j}) + \beta_{2j}(x_j)x_2 \\ &= (1 + \tau_j)\nu^*(S_{1j}, \tau_j)(S_{1j}(x_j)(x_{1j} + \omega_{1j}) + S_{2j}(x_j)x_{2j}) \\ &= \nu^*(S_{1j}, \tau_j)z_{1j} + \nu^*(S_{1j}, \tau_j)\tau_j S_{1j}(x_j)x_{1j} + \nu^*(S_{1j}, \tau_j)(1 + \tau_j)S_{1j}(x_j)\omega_{1j}. \end{aligned}$$

We have a nonlinear expression with two unobservables, τ_j and ω_{1j} . And both unobservables show also up in the log-linearization of the FOCs. Dobbelaere, Jaumandreu and Mairesse (2026) propose to take the system formed by this ratio of FOCs and the conditional demand for x_1 , solve it for the unobservables and plug the values in terms of observables. In this way, both a varying τ_j and ω_{1j} , can be identified without an assumption of how the determination of x_1 works, provided that the substitution elasticity can be taken as locally given.

Other less general arrangements can be done as well with the CES and Translog semiparametric forms.

7 Concluding remarks

Using cost minimization, we have written any differentiable production function of variable inputs in the form of a semiparametric specification which depends on the elasticity of substitution and one parameter: the elasticity of scale. If one is willing to assume, in a given population, that the scale is common, the parameter can be consistently estimated for heterogeneous firms with different alternatives concerning the modeling of the elasticity of substitution.

If the researcher has no reason for thinking that a particular restriction on the elasticity of substitution holds, the safest way is to start the estimation with the general semiparametric form. The elasticity of substitution can vary freely, and the scale parameter and the distribution of output elasticities of the inputs are estimated under the only restrictions of common scale and variable cost minimizing firms.

If the researcher is willing to impose the restriction that the elasticity of substitution is also common, or that follows the particular variation induced by the Translog model, then the semiparametric form of the CES or the Translog is less costly in computational terms. However, a significant difference in the estimates should be taken as a sign that the restrictions may not be valid, and the best estimation is likely to be in the general semiparametric form.

In contrast, if the general semiparametric estimation shows signs that the nonparametric control for the input ratios does not affect the estimation of the scale, the three models should produce closer statistic estimates except for the fact that CES and Translog restrict the elasticity of substitution, and the restrictions may not be true.

The framework can include, with no particular difficulty, both Hicks-neutral productivity and productivity biased towards an input. We have also shown how the framework can deal with the noncompetitive determination of an input, identifying varying wedges, without the need to assume any behavior. This latest thing typically needs to restrict somewhat the variation of the elasticity of substitution.

Estimation of the unobservable productivity and the marginal price of the inputs determined noncompetitively, allows to assess the markup or market power of the firm.

Appendix

Proof of Lemma 1.

We can write the value at $f(0) = 0$ by expanding the unknown function $f(\cdot)$ around the point x_j : $f(0) = f(x_j) + (0 - x_j)\beta(x_j) + \frac{1}{2}(0 - x_j)D\beta(t_j x_j)(0 - x_j)'$, where the second derivatives are evaluated at the point $(1 - t_j) \times 0 + t_j x_j$ according to the mean value theorem. It follows that $f(x_j) = x_j\beta(x_j) - \frac{1}{2}x_j D\beta(t_j x_j)x_j'$. ■

Proof of Theorem 1.

Using $\beta_k(x_j) = \nu S_{kj}(x_j)$ for $k = 1, 2$ the linear terms of the production function of Lemma 1 become $\nu(S_{1j}(x_j)x_{1j} + S_{2j}(x_j)x_{2j})$.

To compute $D\beta(t_j x_j)$ we can use the fact that $S_{kj}(t_j x_j) = S_{kj}(x_j)$. However, when taking the derivative of the elasticity, we should allow the effect of the change of X_{hj} on the ratio $\frac{X_{kj}}{X_{hj}}$ and hence on the equilibrium MRS_j , that equals $-\frac{W_{hj}}{W_{kj}}$. That is,

$$\begin{aligned} \frac{\partial \beta_k(t_j x_j)}{\partial x_{hj}} &= \frac{\partial \beta_k(t_j x_j)}{\partial X_{hj}} X_{hj} = \nu \frac{\partial S_{kj}(x_j)}{\partial \frac{X_{kj}}{X_{hj}}} \frac{\partial \frac{X_{kj}}{X_{hj}}}{\partial X_{hj}} X_{hj} \\ &= \nu \frac{1 - \sigma(x_j)}{\sigma(x_j)} S_{kj}(x_j) S_{hj}(x_j), \text{ with negative sign for } \frac{\partial \beta_{kj}(t_j x_j)}{\partial x_{kj}}, k = 1, 2. \end{aligned}$$

The third equality is obtained writing $S_{kj}(x_j) = \frac{1}{1 + \frac{W_{hj} X_{hj}}{W_{kj} X_{kj}}} = \frac{1}{1 - MRS_j \frac{X_{hj}}{X_{kj}}}$, where, taking into account that $\frac{\partial RMS_j}{\partial \frac{X_{kj}}{X_{hj}}} = -\frac{S_{hj}(x_j)}{S_{kj}(x_j)} \frac{1}{\sigma(x_j)}$ we have $\frac{\partial S_{kj}(x_j)}{\partial \frac{X_{kj}}{X_{hj}}} = -\frac{1 - \sigma(x_j)}{\sigma(x_j)} S_{kj}(x_j) S_{hj}(x_j) \frac{X_{hj}}{X_{kj}}$.

It follows that the quadratic term $-\frac{1}{2}x_j D\beta(t_j x_j)x_j'$ becomes

$$-\frac{1}{2} \nu \frac{1 - \sigma(x_j)}{\sigma(x_j)} S_{1j}(x_j) S_{2j}(x_j) x_j \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x_j' = \frac{1}{2} \nu \frac{1 - \sigma(x_j)}{\sigma(x_j)} S_{1j}(x_j) S_{2j}(x_j) (x_{2j} - x_{1j})^2. \blacksquare$$

Proof of the corollary.

For the CES, the elasticity of the input k is $\beta_k(x_j) = \frac{\nu_j x_{kj}^{-\frac{1-\sigma}{\sigma}}}{x_{kj}^{-\frac{1-\sigma}{\sigma}} + x_{hj}^{-\frac{1-\sigma}{\sigma}}}$, and the derivative $\frac{\partial \beta_k(x_j)}{\partial x_{hj}} = \frac{1}{\nu} \frac{1-\sigma}{\sigma} \beta_k(x_j) \beta_h(x_j)$, with negative sign for $\frac{\partial \beta_k(x_j)}{\partial x_{kj}}$, $k = 1, 2$. It follows that using $\beta_k(x_j) = \nu S_{kj}(x_j)$ for $k = 1, 2$ we get $f(x_j) = \nu(S_{1j}(x_j)x_{1j} + S_{2j}(x_j)x_{2j}) + \frac{1}{2} \nu \frac{1-\sigma}{\sigma} S_{1j}(x_j) S_{2j}(x_j) (x_{2j} - x_{1j})^2$. The

only difference of the CES with respect to the general semiparametric specification is that it restricts the substitution elasticity to be a parameter.

Since the elasticities for the Translog can be written as $\beta_1(x_j) = \alpha_1 + \alpha(x_{2j} - x_{1j})$ and $\beta_2(x_j) = \alpha_2 - \alpha(x_{2j} - x_{1j})$, with $\alpha_1 + \alpha_2 = \nu$, the function can be easily written as $f(x_j) = \alpha_0 + \beta_1(x_j)x_{1j} + \beta_2(x_j)x_{2j} + \frac{1}{2}\alpha(x_{2j} - x_{1j})^2 = \nu(S_{1j}(x_j)x_{1j} + S_{2j}(x_j)x_{2j}) + \frac{1}{2}\alpha(x_{2j} - x_{1j})^2$, and it becomes clear that the Translog restricts the expression $\nu \frac{1-\sigma(x_j)}{\sigma(x_j)} S_{j1}(x_j)S_{j2}(x_j)$ to take the constant value α . Solving this restriction for $\sigma(x_j)$, one gets the formula for sigma that is used to measure the varying substitution elasticity, $\sigma(x_j) = \frac{\nu S_{1j}(x_j)S_{2j}(x_j)}{\alpha + \nu S_{1j}(x_j)S_{2j}(x_j)}$. ■

Proof of Lemma 2.

Consider $S_{kj} = \frac{W_{kj}X_{kj}}{W_{kj}X_{kj} + W_{hj}X_{hj}} = \frac{1}{1 + \frac{W_{hj}X_{hj}}{W_{kj}X_{kj}}}$. Due to cost minimization, $\frac{\partial F(X_{kj}, X_{hj})/\partial X_{hj}}{\partial F(X_{kj}, X_{hj})/\partial X_{kj}} = \frac{X_{hj}^{\nu-1} \partial F(\frac{X_{kj}}{X_{hj}}, 1)/\partial X_{hj}}{X_{kj}^{\nu-1} \partial F(1, \frac{X_{hj}}{X_{kj}})/\partial X_{kj}} = \varphi(\frac{X_{hj}}{X_{kj}}) = \frac{W_{hj}}{W_{kj}}$, where the first equality is valid due to homogeneity ν of the production function. It follows that $S_{kj}(x_j) = \frac{1}{1 + \varphi(\frac{X_{hj}}{X_{kj}}) \frac{X_{hj}}{X_{kj}}} = S(x_{hj} - x_{kj})$. ■

Proof of Lemma 3.

Since the production function is homogeneous of degree ν , the absolute value of the marginal rate of substitution only depends on the ratio $\frac{X_{hj}}{X_{kj}}$, i. e. $|MRS_j| = \varphi(\frac{X_{hj}}{X_{kj}})$. This implies that $\frac{X_{hj}}{X_{kj}} = \varphi^{-1}(|MRS_j|)$, and the derivative $\frac{\partial \frac{X_{hj}}{X_{kj}}}{\partial |MRS_j|}$ is the same along a ray. The substitution elasticity can be written $\sigma(x_j) = \frac{|MRS_j|}{\frac{X_{kj}}{X_{hj}}} \frac{\partial \frac{X_{kj}}{X_{hj}}}{\partial |MRS_j|}$, where we have two ratios and a derivative that remain the same along a ray. It follows that if the elasticity of substitution varies, it does as a function of the input ratio, that is, $\sigma(x_j) = \sigma(x_{hj} - x_{kj})$. ■

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