6* LINEAR OPTIMAL CONTROL THEORY FOR DISCRETE-TIME SYSTEMS

6.1 INTRODUCTION

In the first five chapters of this book, we treated in considerable detail linear control theory for continuous-time systems. In this chapter we give a condensed review of the same theory for discrete-time systems. Since the theory of linear discrete-time systems very closely parallels the theory of linear continuous-time systems, many of the results are similar. For this reason the comments in the text are brief, except in those cases where the results for discrete-time systems deviate markedly from the continuous-time situation. For the same reason many proofs are omitted.

Discrete-time systems can be classified into two types:

1. Inherently discrete-time systems, such as digital computers, digital filters, monetary systems, and inventory systems. In such systems it makes sense to consider the system at discrete instants of time only, and what happens in between is irrelevant.
2. Discrete-time systems that result from considering continuous-time systems at discrete instants of time only. This may be done for reasons of convenience (e.g., when analyzing a continuous-time system on a digital computer), or may arise naturally when the continuous-time system is interconnected with inherently discrete-time systems (such as digital controllers or digital process control computers).

Discrete-time linear optimal control theory is of great interest because of its application in computer control.

6.2 THEORY OF LINEAR DISCRETE-TIME SYSTEMS

6.2.1 Introduction

In this section the theory of linear discrete-time systems is briefly reviewed. The section is organized along the lines of Chapter 1. Many of the results stated in this section are more extensively discussed by Freeman (1965).

6.2.2 State Description of Linear Discrete-Time Systems

It sometimes happens that when dealing with a physical system it is relevant not to observe the system behavior at all instants of time t but only at a sequence of instants t_i, $i = 0, 1, 2, \cdots$. Often in such cases it is possible to characterize the system behavior by quantities defined at those instants only. For such systems the natural equivalent of the state differential equation is the *state difference equation*

$$x(i + 1) = f[x(i), u(i), i], \tag{6-1}$$

where $x(i)$ is the state and $u(i)$ the input at time t_i. Similarly, we assume that the output at time t_i is given by the *output equation*

$$y(i) = g[x(i), u(i), i]. \tag{6-2}$$

Linear discrete-time systems are described by state difference equations of the form

$$x(i + 1) = A(i)x(i) + B(i)u(i), \tag{6-3}$$

where $A(i)$ and $B(i)$ are matrices of appropriate dimensions. The corresponding output equation is

$$y(i) = C(i)x(i) + D(i)u(i). \tag{6-4}$$

If the matrices A, B, C, and D are independent of i, the system is *time-invariant*.

Example 6.1. *Savings bank account*

Let the scalar quantity $x(n)$ be the balance of a savings bank account at the beginning of the n-th month, and let α be the monthly interest rate. Also, let the scalar quantity $u(n)$ be the total of deposits and withdrawals during the n-th month. Assuming that the interest is computed monthly on the basis of the balance at the beginning of the month, the sequence $x(n)$, $n = 0, 1, 2, \cdots$, satisfies the linear difference equation

$$\begin{aligned} x(n + 1) &= (1 + \alpha)x(n) + u(n), \qquad n = 0, 1, 2, \cdots, \\ x(0) &= x_0, \end{aligned} \tag{6-5}$$

where x_0 is the initial balance. These equations describe a linear time-invariant discrete-time system.

6.2.3 Interconnections of Discrete-Time and Continuous-Time Systems

Systems that consist of an interconnection of a discrete-time system and a continuous-time system are frequently encountered. An example of particular interest occurs when a digital computer is used to control a continuous-time plant. Whenever such interconnections exist, there must be some type of *interface system* that takes care of the communication between the discrete-time and continuous-time systems. We consider two particularly simple types

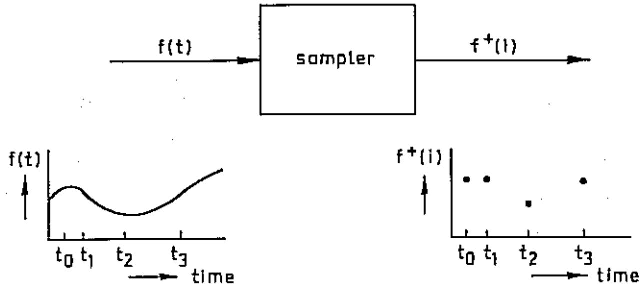

Fig. 6.1. Continuous-to-discrete-time conversion.

of interface systems, namely, *continuous-to-discrete-time* (*C-to-D*) *converters* and *discrete-to-continuous-time* (*D-to-C*) *converters*.

A C-to-D converter, also called a *sampler* (see Fig. 6.1), is a device with a continuous-time function $f(t)$, $t \geq t_0$, as input, and the sequence of real numbers $f^+(i)$, $i = 0, 1, 2, \cdots$, at times t_i, $i = 0, 1, 2, \cdots$, as output, where the following relation holds:

$$f^+(i) = f(t_i), \qquad i = 0, 1, 2, \cdots. \qquad \textbf{6-6}$$

The sequence of time instants t_i, $i = 0, 1, 2, \cdots$, with $t_0 < t_1 < t_2 < \cdots$, is given. In the present section we use the superscript $+$ to distinguish sequences from the corresponding continuous-time functions.

A D-to-C converter is a device that accepts a sequence of numbers $f^+(i)$, $i = 0, 1, 2, \cdots$, at given instants $t_i, i = 0, 1, 2, \cdots$, with $t_0 < t_1 < t_2 < \cdots$, and produces a continuous-time function $f(t)$, $t \geq t_0$, according to a well-defined prescription. We consider only a very simple type of D-to-C converter known as a *zero-order hold*. Other converters are described in the literature (see, e.g., Saucedo and Schiring, 1968). A zero-order hold (see Fig. 6.2) is described by the relation

$$f(t) = f^+(i), \qquad t_i \leq t < t_{i+1}, \qquad i = 0, 1, 2, \cdots. \qquad \textbf{6-7}$$

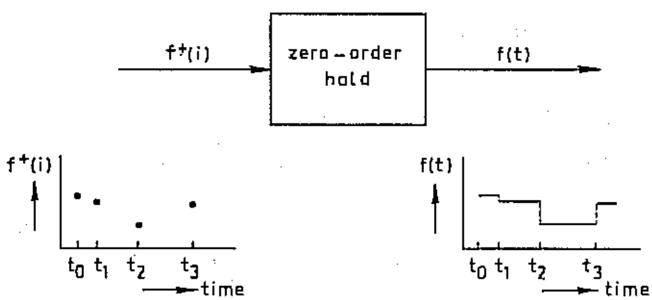

Fig. 6.2. Discrete-to-continuous-time conversion.

Figure 6.3 illustrates a typical example of an interconnection of discrete-time and continuous-time systems. In order to analyze such a system, it is often convenient to represent the continuous-time system together with the D-to-C converter and the C-to-D converter by an *equivalent discrete-time system*. To see how this equivalent discrete-time system can be found in a specific case, suppose that the D-to-C converter is a zero-order hold and that the C-to-D converter is a sampler. We furthermore assume that the continuous-time system of Fig. 6.3 is a linear system with state differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{6-8}$$

and output equation

$$y(t) = C(t)x(t) + D(t)u(t). \tag{6-9}$$

Since we use a zero-order hold,

$$u(t) = u(t_i), \qquad t_i \leq t < t_{i+1}, \qquad i = 0, 1, 2, \cdots. \tag{6-10}$$

Then from **1-61** we can write for the state of the system at time t_{i+1}

$$x(t_{i+1}) = \Phi(t_{i+1}, t_i)x(t_i) + \left[\int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, \tau)B(\tau)\, d\tau\right]u(t_i), \tag{6-11}$$

where $\Phi(t, t_0)$ is the transition matrix of the system **6-8**. This is a linear state difference equation of the type **6-3**. In deriving the corresponding output equation, we allow the possibility that the instants at which the output is sampled do not coincide with the instants at which the input is adjusted. Thus we consider the *output associated with the i-th sampling interval*, which is given by

$$y(t_i'), \tag{6-12}$$

where

$$t_i \leq t_i' < t_{i+1}, \tag{6-13}$$

for $i = 0, 1, 2, \cdots$. Then we write

$$y(t_i') = C(t_i')\Phi(t_i', t_i)x(t_i) + \left[C(t_i')\int_{t_i}^{t_i'} \Phi(t_i', \tau)B(\tau)\, d\tau\right]u(t_i) + D(t_i')u(t_i). \tag{6-14}$$

Now replacing $x(t_i)$ by $x^+(i)$, $u(t_i)$ by $u^+(i)$, and $y(t_i')$ by $y^+(i)$, we write the system equations in the form

$$\begin{aligned} x^+(i+1) &= A_d(i)x^+(i) + B_d(i)u^+(i), \\ y^+(i) &= C_d(i)x^+(i) + D_d(i)u^+(i), \qquad i = 0, 1, 2, \cdots, \end{aligned} \tag{6-15}$$

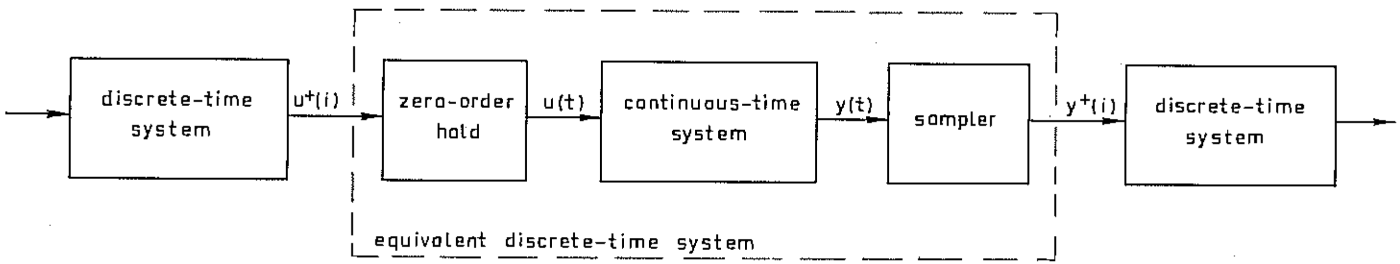

Fig. 6.3. Interconnection of discrete-time and continuous-time systems.

where

$$\begin{aligned} A_d(i) &= \Phi(t_{i+1}, t_i), \\ B_d(i) &= \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, \tau)B(\tau)\, d\tau, \\ C_d(i) &= C(t_i')\Phi(t_i', t_i), \\ D_d(i) &= C(t_i')\int_{t_i}^{t_i'} \Phi(t_i', \tau)B(\tau)\, d\tau + D(t_i'). \end{aligned} \qquad \text{6-16}$$

We note that the discrete-time system defined by **6-15** has a direct link even if the continuous-time system does not have one because $D_d(i)$ can be different from zero even when $D(t_i')$ is zero. The direct link is absent, however, if $D(t) \equiv 0$ and the instants t_i' coincide with the instants t_i, that is, $t_i' = t_i$, $i = 0, 1, 2, \cdots$.

In the special case in which the sampling instants are equally spaced:

$$t_{i+1} - t_i = \Delta, \qquad \text{6-17}$$

and

$$t_i' - t_i = \Delta', \qquad \text{6-18}$$

while the system **6-8**, **6-9** is time-invariant, the discrete-time system **6-15** is also time-invariant, and

$$\begin{aligned} A_d &= e^{A\Delta}, & B_d &= \left(\int_0^{\Delta} e^{A\tau}\, d\tau\right)B, \\ C_d &= Ce^{A\Delta'}, & D_d &= C\left(\int_0^{\Delta'} e^{A\tau}\, d\tau\right)B + D. \end{aligned} \qquad \text{6-19}$$

We call Δ the *sampling period* and $1/\Delta$ the *sampling rate.*

Once we have obtained the discrete-time equations that represent the continuous-time system together with the converters, we are in a position to study the interconnection of the system with other discrete-time systems.

Example 6.2. *Digital positioning system*

Consider the continuous-time positioning system of Example 2.4 (Section 2.3) which is described by the state differential equation

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t). \qquad \text{6-20}$$

Suppose that this system is part of a control system that is commanded by a digital computer (Fig. 6.4). The zero-order hold produces a piecewise constant input $\mu(t)$ that changes value at equidistant instants of time separated by

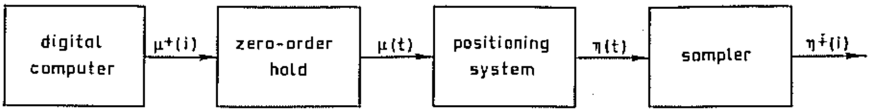

Fig. 6.4. A digital positioning system.

intervals of length Δ. The transition matrix of the system **6-20** is

$$\Phi(t, t_0) = \begin{pmatrix} 1 & \dfrac{1}{\alpha}[1 - e^{-\alpha(t-t_0)}] \\ 0 & e^{-\alpha(t-t_0)} \end{pmatrix}. \qquad \textbf{6-21}$$

From this it is easily found that the discrete-time description of the positioning system is given by

$$x^+(i+1) = Ax^+(i) + b\mu^+(i), \qquad \textbf{6-22}$$

where

$$A = \begin{pmatrix} 1 & \dfrac{1}{\alpha}(1 - e^{-\alpha\Delta}) \\ 0 & e^{-\alpha\Delta} \end{pmatrix} \qquad \textbf{6-23}$$

and

$$b = \begin{pmatrix} \dfrac{\kappa}{\alpha}\left(\Delta - \dfrac{1}{\alpha} + \dfrac{1}{\alpha}e^{-\alpha\Delta}\right) \\ \dfrac{\kappa}{\alpha}(1 - e^{-\alpha\Delta}) \end{pmatrix}. \qquad \textbf{6-24}$$

Note that we have replaced $x(t_i)$ by $x^+(i)$ and $\mu(t_i)$ by $\mu^+(i)$.

With the numerical values

$$\begin{aligned} \alpha &= 4.6\ \mathrm{s}^{-1}, \\ \kappa &= 0.787\ \mathrm{rad/(V\ s^2)}, \\ \Delta &= 0.1\ \mathrm{s}, \end{aligned} \qquad \textbf{6-25}$$

we obtain for the state difference equation

$$x^+(i+1) = \begin{pmatrix} 1 & 0.08015 \\ 0 & 0.6313 \end{pmatrix} x^+(i) + \begin{pmatrix} 0.003396 \\ 0.06308 \end{pmatrix} \mu^+(i). \qquad \textbf{6-26}$$

Let us suppose that the output variable $\eta(t)$ of the continuous-time system, where

$$\eta(t) = (1, \quad 0)x(t), \qquad \textbf{6-27}$$

is sampled at the instants t_i, $i = 0, 1, 2, \cdots$. Then the output equation for the discrete-time system clearly is

$$\eta^+(i) = (1, \quad 0)x^+(i), \tag{6-28}$$

where we have replaced $\eta(t_i)$ with $\eta^+(i)$.

Example 6.3. *Stirred tank*

Consider the stirred tank of Example 1.2 (Section 1.2.3) and suppose that it forms part of a process commanded by a process control computer. As a result, the valve settings change at discrete instants only and remain constant in between. It is assumed that these instants are separated by time intervals of constant length Δ. The continuous-time system is described by the state differential equation

$$\dot{x}(t) = \begin{pmatrix} -\dfrac{1}{2\theta} & 0 \\ 0 & -\dfrac{1}{\theta} \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ \dfrac{c_1 - c_0}{V_0} & \dfrac{c_2 - c_0}{V_0} \end{pmatrix} u(t). \tag{6-29}$$

It is easily found that the discrete-time description is

$$x^+(i+1) = Ax^+(i) + Bu^+(i),$$

where

$$A = \begin{pmatrix} e^{-\Delta/(2\theta)} & 0 \\ 0 & e^{-\Delta/\theta} \end{pmatrix},$$

$$B = \begin{pmatrix} 2\theta(1 - e^{-\Delta/(2\theta)}) & 2\theta(1 - e^{-\Delta/(2\theta)}) \\ \dfrac{\theta(c_1 - c_0)}{V_0}(1 - e^{-\Delta/\theta}) & \dfrac{\theta(c_2 - c_0)}{V_0}(1 - e^{-\Delta/\theta}) \end{pmatrix}. \tag{6-30}$$

With the numerical data of Example 1.2, we find

$$A = \begin{pmatrix} 0.9512 & 0 \\ 0 & 0.9048 \end{pmatrix}, \quad B = \begin{pmatrix} 4.877 & 4.877 \\ -1.1895 & 3.569 \end{pmatrix}, \tag{6-31}$$

where we have chosen

$$\Delta = 5 \text{ s}. \tag{6-32}$$

Example 6.4. *Stirred tank with time delay*

As an example of a system with a time delay, we again consider the stirred tank but with a slightly different arrangement, as indicated in Fig. 6.5. Here

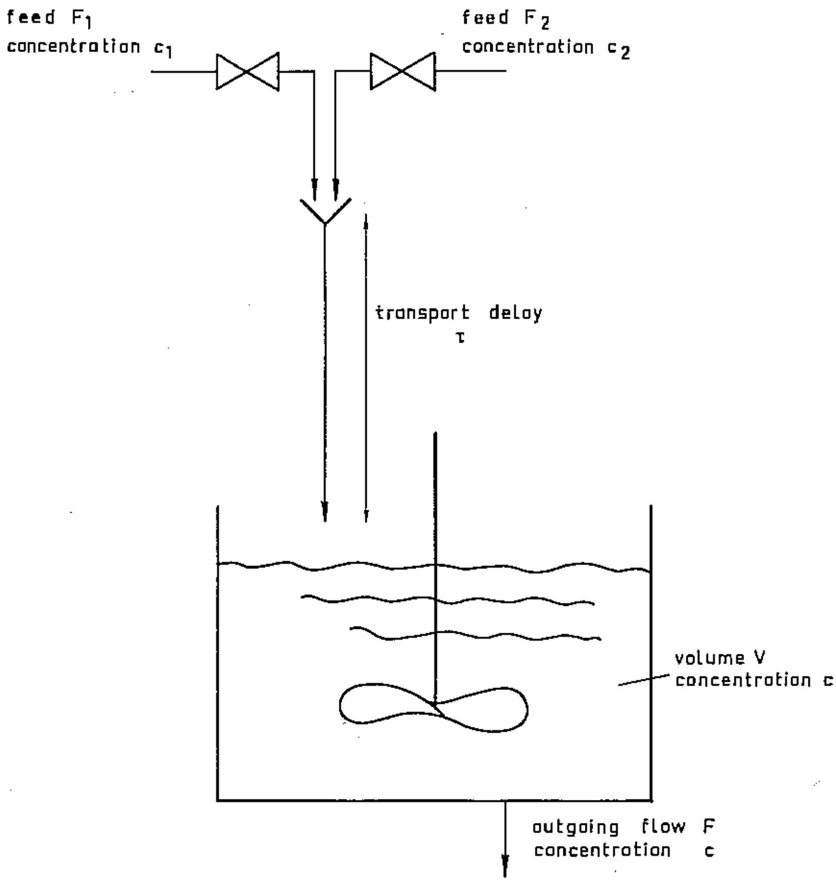

Fig. 6.5. Stirred tank with modified configuration.

the feeds are mixed *before* they flow into the tank. This would not make any difference in the dynamic behavior of the system if it were not for a transport delay τ that occurs in the common section of the pipe. Rewriting the mass balances and repeating the linearization, we find that the system equations now are

$$\dot{\xi}_1(t) = -\frac{1}{2\theta}\,\xi_1(t) + \mu_1(t) + \mu_2(t),$$

$$\dot{\xi}_2(t) = -\frac{1}{\theta}\,\xi_2(t) + \frac{c_1 - c_0}{V_0}\,\mu_1(t-\tau) + \frac{c_2 - c_0}{V_0}\,\mu_2(t-\tau), \qquad \text{6-33}$$

where the symbols have the same meanings as in Example 1.2 (Section

1.2.3). In vector form we write

$$\dot{x}(t) = \begin{pmatrix} -\dfrac{1}{2\theta} & 0 \\ 0 & -\dfrac{1}{\theta} \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 & 0 \\ \dfrac{c_1 - c_0}{V_0} & \dfrac{c_2 - c_0}{V_0} \end{pmatrix} u(t - \tau). \qquad \textbf{6-34}$$

Note that changes in the feeds have an immediate effect on the volume but a delayed effect on the concentration.

We now suppose that the tank is part of a computer controlled process so that the valve settings change only at fixed instants separated by intervals of length Δ. For convenience we assume that the delay time τ is an exact multiple $k\Delta$ of the sampling period Δ. This means that the state difference equation of the resulting discrete-time system is of the form

$$x^+(i + 1) = Ax^+(i) + B_1u^+(i) + B_2u^+(i - k). \qquad \textbf{6-35}$$

It can be found that with the numerical data of Example 1.2 and a sampling period

$$\Delta = 5 \text{ s}, \qquad \textbf{6-36}$$

A is as given by **6-31**, while

$$B_1 = \begin{pmatrix} 4.877 & 4.877 \\ 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 \\ -1.1895 & 3.569 \end{pmatrix}. \qquad \textbf{6-37}$$

It is not difficult to bring the difference equation **6-35** into standard state difference equation form. We illustrate this for the case $k = 1$. This means that the effect of changes in the valve settings are delayed by one sampling interval. To evaluate the effect of valve setting changes, we must therefore remember the settings of one interval ago. Thus we define an augmented state vector

$$x'(i) = \begin{pmatrix} \xi_1^+(i) \\ \xi_2^+(i) \\ \mu_1^+(i - 1) \\ \mu_2^+(i - 1) \end{pmatrix}. \qquad \textbf{6-38}$$

By using this definition it is easily found that in terms of the augmented state the system is described by the state difference equation

$$x'(i + 1) = A'x'(i) + B'u^+(i), \qquad \textbf{6-39}$$

where

$$A' = \begin{pmatrix} 0.9512 & 0 & 0 & 0 \\ 0 & 0.9048 & -1.1895 & 3.569 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$B' = \begin{pmatrix} 4.877 & 4.877 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad \textbf{6-40}$$

We point out that the matrix A' has two characteristic values equal to zero. Discrete-time systems obtained by operating finite-dimensional time-invariant linear differential systems with a piecewise constant input never have zero characteristic values, since for such systems $A_d = \exp(A\Delta)$, which is always a nonsingular matrix.

6.2.4 Solution of State Difference Equations

For the solution of state difference equations, we have the following theorem, completely analogous to Theorems 1.1 and 1.3 (Section 1.3).

Theorem 6.1. *Consider the state difference equation*

$$x(i + 1) = A(i)x(i) + B(i)u(i). \qquad \textbf{6-41}$$

The solution of this equation can be expressed as

$$x(i) = \Phi(i, i_0)x(i_0) + \sum_{j=i_0}^{i-1} \Phi(i, j + 1)B(j)u(j), \qquad i \geq i_0 + 1, \qquad \textbf{6-42}$$

where $\Phi(i, i_0)$, $i \geq i_0$, *is the matrix*

$$\Phi(i, i_0) = \begin{cases} A(i-1)A(i-2)\cdots A(i_0) & \text{for } i \geq i_0 + 1, \\ I & \text{for } i = i_0. \end{cases} \qquad \textbf{6-43}$$

The ***transition matrix*** $\Phi(i, i_0)$ *is the solution of the difference equation*

$$\begin{aligned} \Phi(i + 1, i_0) &= A(i)\Phi(i, i_0), \qquad i \geq i_0, \\ \Phi(i_0, i_0) &= I. \end{aligned} \qquad \textbf{6-44}$$

If $A(i)$ *does not depend upon* i,

$$\Phi(i, i_0) = A^{i-i_0}. \qquad \textbf{6-45}$$

Suppose that the system has an output

$$y(i) = C(i)x(i). \tag{6-46}$$

If the initial state is zero, that is, $x(i_0) = 0$, we can write with the aid of **6-42**:

$$y(i) = \sum_{j=i_0}^{i} K(i, j)u(j), \qquad i \geq i_0. \tag{6-47}$$

Here

$$K(i, j) = \begin{cases} C(i)\Phi(i, j+1)B(j), & j \leq i-1, \\ 0, & j = i, \end{cases} \tag{6-48}$$

will be termed the *pulse response matrix* of the system. Note that for time-invariant systems K depends upon $i - j$ only. If the system has a *direct link*, that is, the output is given by

$$y(i) = C(i)x(i) + D(i)u(i), \tag{6-49}$$

the output can be represented in the form

$$y(i) = \sum_{j=i_0}^{i} K(i, j)u(j), \qquad i \geq i_0, \tag{6-50}$$

where

$$K(i, j) = \begin{cases} C(i)\Phi(i, j+1)B(j) & \text{for } j \leq i-1, \\ D(i) & \text{for } j = i. \end{cases} \tag{6-51}$$

Also in the case of time-invariant discrete-time linear systems, diagonalization of the matrix A is sometimes useful. We summarize the facts.

Theorem 6.2. *Consider the time-invariant state difference equation*

$$x(i+1) = Ax(i). \tag{6-52}$$

Suppose that the matrix A has n distinct characteristic values $\lambda_1, \lambda_2, \cdots, \lambda_n$ with corresponding characteristic vectors $e_1, e_2, \cdots, e_n$. Define the $n \times n$ matrices

$$\begin{aligned} T &= (e_1, e_2, \cdots, e_n), \\ \Lambda &= \text{diag}\,(\lambda_1, \lambda_2, \cdots, \lambda_n). \end{aligned} \tag{6-53}$$

Then the transition matrix of the state difference equation **6-41** *can be written as*

$$\Phi(i, i_0) = A^{i-i_0} = T\Lambda^{i-i_0}T^{-1}. \tag{6-54}$$

Suppose that the inverse matrix T^{-1} is represented as

$$T^{-1} = \begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix}, \qquad \text{6-55}$$

where $f_1, f_2, \cdots, f_n$ are row vectors. Then the solution of the difference equation **6-52** *can be expressed as*

$$x(i) = \sum_{j=1}^{n} \lambda_j^{i-i_0} e_j f_j x_0, \qquad \text{6-56}$$

where $x_0 = x(i_0)$.

Expression **6-56** shows that the behavior of the system can be described as a composition of expanding (for $|\lambda_j| > 1$), sustained (for $|\lambda_j| = 1$), or contracting (for $|\lambda_j| < 1$) motions along the characteristic vectors $e_1, e_2, \cdots, e_n$ of the matrix A.

6.2.5 Stability

In Section 1.4 we defined the following forms of stability for continuous-time systems: stability in the sense of Lyapunov; asymptotic stability; asymptotic stability in the large; and exponential stability. All the definitions for the continuous-time case carry over to the discrete-time case if the continuous time variable t is replaced with the discrete time variable i. Time-invariant discrete-time linear systems can be tested for stability according to the following results.

Theorem 6.3. *The time-invariant linear discrete-time system*

$$x(i + 1) = Ax(i) \qquad \text{6-57}$$

is stable in the sense of Lyapunov if and only if
(a) *all the characteristic values of A have moduli not greater than 1, and*
(b) *to any characteristic value with modulus equal to 1 and multiplicity m there correspond exactly m characteristic vectors of the matrix A.*

The proof of this theorem when A has no multiple characteristic values is easily seen by inspecting **6-56.**

Theorem 6.4. *The time-invariant linear discrete-time system*

$$x(i + 1) = Ax(i) \qquad \text{6-58}$$

is asymptotically stable if and only if all of the characteristic values of A have moduli strictly less than 1.

Theorem 6.5. *The time-invariant linear discrete-time system*

$$x(i + 1) = Ax(i), \qquad \text{6-59}$$

is exponentially stable if and only if it is asymptotically stable.

We see that the role that the left-half complex plane plays in the analysis of continuous-time systems is taken by the inside of the unit circle for discrete-time systems. Similarly, the right-half plane is replaced with the outside of the unit circle and the imaginary axis by the unit circle itself.

Completely analogously to continuous-time systems, we define the stable subspace of a linear discrete-time system as follows.

Definition 6.1. *Consider the n-dimensional time-invariant linear discrete-time system*

$$x(i + 1) = Ax(i). \qquad \text{6-60}$$

Suppose that A has n distinct characteristic values. Then we define the ***stable subspace*** *of this system as the real linear subspace spanned by those characteristic vectors of A that correspond to characteristic values with moduli strictly less than 1. Similarly, the* ***unstable subspace*** *of the system is the real subspace spanned by those characteristic vectors of A that correspond to characteristic values with moduli equal to or greater than 1.*

For systems where the characteristic values of A are not all distinct, we have:

Definition 6.2. *Consider the n-dimensional time-invariant linear discrete-time system*

$$x(i + 1) = Ax(i). \qquad \text{6-61}$$

Let $\mathcal{N}_j$ be the null space of $(A - \lambda_j I)^{m_j}$, where λ_j is a characteristic value of A and m_j the multiplicity of this characteristic value in the characteristic polynomial of A. Then we define the stable subspace of the system as the real subspace of the direct sum of those null spaces $\mathcal{N}_j$ that correspond to characteristic values of A with moduli strictly less than 1. Similarly, the unstable subspace is the real subspace of the direct sum of those null spaces $\mathcal{N}_j$ that correspond to characteristic values of A with moduli greater than or equal to 1.

Example 6.5. *Digital positioning system*

It is easily found that the characteristic values of the digital positioning system of Example 6.2 (Section 6.2.3) are 1 and $\exp(-\alpha\Delta)$. As a result, the system is stable in the sense of Lyapunov but not asymptotically stable.

6.2.6 Transform Analysis of Linear Discrete-Time Systems

The natural equivalent of the Laplace transform for continuous-time variables is the z-transform for discrete-time sequences. We define the *z-transform*

$\mathbf{V}(z)$ of a sequence of vectors $v(i)$, $i = 0, 1, 2, \cdots$, as follows

$$\mathbf{V}(z) = \sum_{i=0}^{\infty} z^{-i} v(i), \qquad \textbf{6-62}$$

where z is a complex variable. This transform is defined for those values of z for which the sum converges.

To understand the application of the z-transform to the analysis of linear time-invariant discrete-time systems, consider the state difference equation

$$x(i+1) = Ax(i) + Bu(i). \qquad \textbf{6-63}$$

Multiplication of both sides of **6-63** by z^{-i} and summation over $i = 0, 1, 2, \cdots$ yields

$$z\mathbf{X}(z) - zx(0) = A\mathbf{X}(z) + B\mathbf{U}(z), \qquad \textbf{6-64}$$

where $\mathbf{X}(z)$ is the z-transform of $x(i)$, $i = 0, 1, 2, \cdots$, and $\mathbf{U}(z)$ that of $u(i)$, $i = 0, 1, 2, \cdots$. Solution for $\mathbf{X}(z)$ gives

$$\mathbf{X}(z) = (zI - A)^{-1} B\mathbf{U}(z) + (zI - A)^{-1} zx(0). \qquad \textbf{6-65}$$

In the evaluation of $(zI - A)^{-1}$, Leverrier's algoritbm (Theorem 1.18, Section 1.5.1) may be useful. Suppose that an output $y(i)$ is given by

$$y(i) = Cx(i) + Du(i). \qquad \textbf{6-66}$$

Transformation of this expression and substitution of **6-65** yields for $x(0) = 0$

$$\mathbf{Y}(z) = H(z)\mathbf{U}(z), \qquad \textbf{6-67}$$

where $\mathbf{Y}(z)$ is the z-transform of $y(i)$, $i = 0, 1, 2, \cdots$, and

$$H(z) = C(zI - A)^{-1}B + D \qquad \textbf{6-68}$$

is the *z-transfer matrix* of the system.

For the *inverse transformation* of z-transforms, there exist several methods for which we refer the reader to the literature (see, e.g., Saucedo and Schiring, 1968).

It is easily proved that the z-transform transfer matrix $H(z)$ is the z-transform of the pulse response matrix of the system. More precisely, let the pulse transfer matrix of time-invariant system be given by $K(i - j)$ (with a slight inconsistency in the notation). Then

$$H(z) = \sum_{i=0}^{\infty} z^{-i} K(i). \qquad \textbf{6-69}$$

We note that $H(z)$ is generally of the form

$$H(z) = \frac{P(z)}{\det(zI - A)}, \qquad \textbf{6-70}$$

where $P(z)$ is a polynomial matrix in z. The poles of the transfer matrix $H(z)$ are clearly the characteristic values of the matrix A, unless a factor of the form $z - \lambda_j$ cancels in all entries of $H(z)$, where λ_j is a characteristic value of A.

Just as in Section 1.5.3, if $H(z)$ is a square matrix, we have

$$\det [H(z)] = \frac{\psi(z)}{\phi(z)}, \qquad \textbf{6-71}$$

where $\phi(z)$ is the characteristic polynomial $\phi(z) = \det (zI - A)$ and $\psi(z)$ is a polynomial in z. We call the roots of $\psi(z)$ the *zeroes* of the system.

The *frequency response* of discrete-time systems can conveniently be investigated with the aid of the z-transfer matrix. Suppose that we have a complex-valued input of the form

$$u(i) = u_m e^{j\theta i}, \qquad i = 0, 1, 2, \cdots, \qquad \textbf{6-72}$$

where $j = \sqrt{-1}$. We refer to the quantity θ as the *normalized angular frequency*. Let us first attempt to find a particular solution to the state difference equation **6-63** of the form

$$x_p(i) = x_m e^{j\theta i}, \qquad i = 0, 1, 2, \cdots. \qquad \textbf{6-73}$$

It is easily found that this particular solution is given by

$$x_p(i) = (e^{j\theta}I - A)^{-1}Bu_m e^{j\theta i}, \qquad i = 0, 1, 2, \cdots. \qquad \textbf{6-74}$$

The general solution of the *homogeneous* difference equation is

$$x_h(i) = A^i a, \qquad \textbf{6-75}$$

where a is an arbitrary constant vector. The general solution of the inhomogeneous state difference equation is therefore

$$x(i) = A^i a + (e^{j\theta}I - A)^{-1}Bu_m e^{j\theta i}, \qquad i = 0, 1, 2, \cdots. \qquad \textbf{6-76}$$

If the system is asymptotically stable, the first term vanishes as $i \rightarrow \infty$; then the second term corresponds to the *steady-state response* of the state to the input **6-72**. The corresponding steady-state response of the output **6-66** is given by

$$\begin{aligned} y(i) &= C(e^{j\theta}I - A)^{-1}Bu_m e^{j\theta i} + Du_m e^{j\theta i} \\ &= H(e^{j\theta})u_m e^{j\theta i}, \end{aligned} \qquad \textbf{6-77}$$

where $H(z)$ is the transfer matrix of the system.

We see that the response of the system to inputs of the type **6-72** is determined by the behavior of the z-transfer matrix for values of z on the unit circle. The steady-state response to real "sinusoidal inputs," that is, inputs of the form

$$u(i) = \alpha \cos(i\theta) + \beta \sin(i\theta), \qquad i = 0, 1, 2, \cdots, \qquad \textbf{6-78}$$

can be ascertained from the moduli and arguments of the entries of $H(e^{j\theta})$. The steady-state response of an asymptotically stable discrete-time system with z-transfer matrix $H(z)$ to a constant input

$$u(i) = u_m, \qquad i = 0, 1, 2, \cdots, \qquad \textbf{6-79}$$

is given by

$$\lim_{i\to\infty} y(i) = H(1)u_m. \qquad \textbf{6-80}$$

In the special case in which the discrete-time system is actually an equivalent description of a continuous-time system with zero-order hold and sampler, we let

$$\theta = \omega\Delta, \qquad \textbf{6-81}$$

where Δ is the sampling period. The harmonic input

$$u(i) = e^{j\theta i}u_m = e^{j\omega\Delta i}u_m, \qquad i = 0, 1, 2, \cdots, \qquad \textbf{6-82}$$

is now the discrete-time version of the continuous-time harmonic function

$$e^{j\omega t}u_m, \qquad t \geq 0, \qquad \textbf{6-83}$$

from which **6-82** is obtained by sampling at equidistant instants with sampling rate $1/\Delta$.

For sufficiently small values of the angular frequency ω, the frequency response $H(e^{j\omega\Delta})$ of the discrete-time version of the system approximates the frequency response matrix of the continuous-time system. It is noted that $H(e^{j\omega\Delta})$ is periodic in ω with period $2\pi/\Delta$. This is caused by the phenomenon of *aliasing*; because of the sampling procedure, high-frequency signals are indistinguishable from low-frequency signals.

Example 6.6. *Digital positioning system*

Consider the digital positioning system of Example 6.2 (Section 6.2.3) and suppose that the position is chosen as the output:

$$y(i) = (1, \quad 0)x(i). \qquad \textbf{6-84}$$

It is easily found that the z-transfer function is given by

$$H(z) = \frac{0.003396z + 0.002912}{(z-1)(z-0.6313)}. \qquad \textbf{6-85}$$

Figure 6.6 shows a plot of the modulus and the argument of $H(e^{j\omega\Delta})$, where $\Delta = 0.1$ s. In the same figure the corresponding plots are given of the frequency response function of the original continuous-time system, which

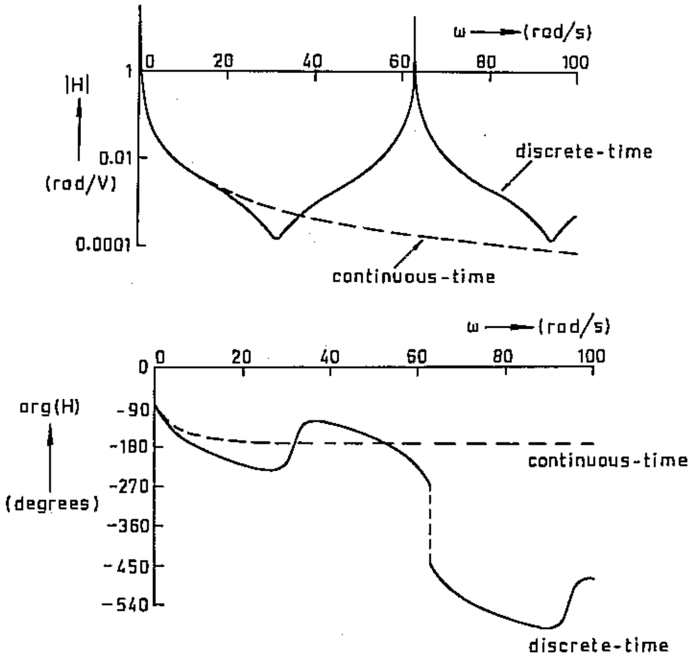

Fig. 6.6. The frequency response functions of the continuous-time and the discrete-time positioning systems.

is given by

$$\frac{0.787}{j\omega(j\omega + 4.6)}. \qquad \textbf{6-86}$$

We observe that for low frequencies (up to about 15 rad/s) the continuous-time and the discrete-time frequency response function have about the same modulus but that the discrete-time version has a larger phase shift. The plot also illustrates the aliasing phenomenon.

6.2.7 Controllability

In Section 1.6 we defined controllability for continuous-time systems. This definition carries over to the discrete-time case if the discrete-time variable i is substituted for the continuous-time variable t. For the controllability of time-invariant linear discrete-time systems, we have the following result which is surprisingly similar to the continuous-time equivalent.

Theorem 6.6. *The n-dimensional linear time-invariant discrete-time system with state difference equation*

$$x(i + 1) = Ax(i) + Bu(i) \qquad \textbf{6-87}$$

is completely controllable if and only if the column vectors of the ***controllability matrix***

$$P = (B, AB, A^2B, \cdots, A^{n-1}B) \qquad \textbf{6-88}$$

span the n-dimensional space.

For a proof we refer the reader to, for example, Kalman, Falb, and Arbib (1969). At this point, the following comment is in order. Frequently, complete controllability is defined as the property that any initial state can be reduced to the zero state in a finite number of steps (or in a finite length of time in the continuous-time case). According to this definition, the system with the state difference equation

$$x(i + 1) = 0 \qquad \textbf{6-89}$$

is completely controllable, although obviously it is not controllable in any intuitive sense. This is why we have chosen to define controllability by the requirement that the system can be brought from the zero state to any nonzero state in a finite time. In the continuous-time case it makes little difference which definition is used, but in the discrete-time it does. The reason is that in the latter case the transition matrix $\Phi(i, i_0)$, as given by **6-43**, can be singular, caused by the fact that one or more of the matrices $A(j)$ can be singular (see, e.g., the system of Example 6.4, Section 6.2.3).

The complete controllability of time-varying linear discrete-time systems can be tested as follows.

Theorem 6.7. *The linear discrete-time system*

$$x(i + 1) = A(i)x(i) + B(i)u(i) \qquad \textbf{6-90}$$

is completely controllable if and only if for every i_0 there exists an $i_1 \geq i_0 + 1$ such that the symmetric nonnegative-definite matrix

$$W(i_0, i_1) = \sum_{i=i_0}^{i_1-1} \Phi(i_1, i + 1)B(i)B^T(i)\Phi^T(i_1, i + 1) \qquad \textbf{6-91}$$

is nonsingular. Here $\Phi(i, i_0)$ is the transition matrix of the system.

Uniform controllability is defined as follows.

Definition 6.3. *The time-varying system* **6-90** *is* ***uniformly completely controllable*** *if there exist an integer $k \geq 1$ and positive constants α_0, α_1, β_0, and β_1 such that*

(a) $W(i_0, i_0 + k) > 0$ *for all i_0*; **6-92**

(b) $\alpha_0 I \leq W^{-1}(i_0, i_0 + k) \leq \alpha_1 I$ *for all i_0*; **6-93**

(c) $\beta_0 I \leq \Phi^T(i_0 + k, i_0)W^{-1}(i_0, i_0 + k)\Phi(i_0 + k, i_0) \leq \beta_1 I$ *for all i_0*. **6-94**

Here $W(i_0, i_1)$ is the matrix **6-91**, *and $\Phi(i, i_0)$ is the transition matrix of the system.*

It is noted that this definition is slightly different from the corresponding continuous-time definition. This is caused by the fact that in the discrete-time case we have avoided defining the transition matrix $\Phi(i, i_0)$ for $i < i_0$. This would involve the inverses of the matrices $A(j)$, which do not necessarily exist.

For time-invariant systems we have:

Theorem 6.8. *The time-invariant linear discrete-time system*

$$x(i + 1) = Ax(i) + Bu(i) \tag{6-95}$$

is uniformly completely controllable if and only if it is completely controllable.

For time-invariant systems it is useful to define the concept of controllable subspace.

Definition 6.4. *The **controllable subspace** of the linear time-invariant discrete-time system*

$$x(i + 1) = Ax(i) + Bu(i) \tag{6-96}$$

is the linear subspace consisting of the states that can be reached from the zero state within a finite number of steps.

The following characterization of the controllable subspace is quite convenient.

Theorem 6.9. *The controllable subspace of the n-dimensional time-invariant linear discrete-time system*

$$x(i + 1) = Ax(i) + Bu(i) \tag{6-97}$$

is the linear subspace spanned by the column vectors of the controllability matrix P.

Discrete-time systems, too, can be decomposed into a controllable and an uncontrollable part.

Theorem 6.10. *Consider the n-dimensional linear time-invariant discrete-time system*

$$x(i + 1) = Ax(i) + Bu(i). \tag{6-98}$$

Form a nonsingular transformation matrix $T = (T_1, T_2)$, where the columns of T_1 form a basis for the controllable subspace of the system, and the column vectors of T_2 together with those of T_1 span the whole n-dimensional space. Define the transformed state variable

$$x'(i) = T^{-1}x(i). \tag{6-99}$$

Then the transformed state variable satisfies the state difference equation

$$x'(i+1) = \begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix} x'(i) + \begin{pmatrix} B'_1 \\ 0 \end{pmatrix} u(i), \qquad \textbf{6-100}$$

where the pair $\{A'_{11}, B'_1\}$ *is completely controllable.*

Here the terminology "the pair $\{A, B\}$ is completely controllable" is shorthand for "the system $x(i+1) = Ax(i) + Bu(i)$ is completely controllable."

Also stabilizability can be defined for discrete-time systems.

Definition 6.5. *The linear time-invariant discrete-time system*

$$x(i+1) = Ax(i) + Bu(i) \qquad \textbf{6-101}$$

is ***stabilizable*** *if its unstable subspace is contained in its controllable subspace.*

Stabilizability may be tested as follows.

Theorem 6.11. *Suppose that the linear time-invariant discrete-time system*

$$x(i+1) = Ax(i) + Bu(i) \qquad \textbf{6-102}$$

is transformed according to Theorem 6.10 into the form **6-100**. *Then the system is stabilizable if and only if all the characteristic values of the matrix* A'_{22} *have moduli strictly less than 1.*

Analogously to the continuous-time case, we define the characteristic values of the matrix A'_{11} as the *controllable poles* of the sytem, and the remaining poles as the *uncontrollable poles*. Thus a system is stabilizable if and only if all its uncontrollable poles are stable (where a *stable pole* is defined as a characteristic value of the system with modulus strictly less than 1).

6.2.8 Reconstructibility

The definition of reconstructibility given in Section 1.7 can be applied to discrete-time systems if the continuous time variable t is replaced by the discrete variable i. The reconstructibility of a time-invariant linear discrete-time system can be tested as follows.

Theorem 6.12. *The* n*-dimensional time-invariant linear discrete-time system*

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i), \\ y(i) &= Cx(i), \end{aligned} \qquad \textbf{6-103}$$

is completely reconstructible if and only if the row vectors of the reconstructibility matrix

$$Q = \begin{pmatrix} C \\ CA \\ CA^2 \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{pmatrix} \tag{6-104}$$

span the whole n-dimensional space.

A proof of this theorem can be found in Meditch (1969). For general, time-varying systems the following test applies.

Theorem 6.13. *The linear discrete-time system*

$$\begin{aligned} x(i+1) &= A(i)x(i) + B(i)u(i), \\ y(i) &= C(i)x(i) \end{aligned} \tag{6-105}$$

is completely reconstructible if and only if for every i_1 there exists an $i_0 \leq i_1 - 1$ such that the symmetric nonnegative-definite matrix

$$M(i_0, i_1) = \sum_{i=i_0+1}^{i_1} \Phi^T(i, i_0 + 1)C^T(i)C(i)\Phi(i, i_0 + 1) \tag{6-106}$$

is nonsingular. Here $\Phi(i, i_0)$ is the transition matrix of the system.

A proof of this theorem is given by Meditch (1969).

Uniform complete reconstructibility can be defined as follows.

Definition 6.6. *The time-varying system* **6–105** *is* ***uniformly completely reconstructible*** *if there exist an integer $k \geq 1$ and positive constants α_0, α_1, β_0, and β_1 such that*

(a) $M(i_1 - k, i_1) > 0$ *for all* i_1; **6-107**

(b) $\alpha_0 I \leq M^{-1}(i_1 - k, i_1) \leq \alpha_1 I$ *for all* i_1; **6-108**

(c) $\beta_0 I \leq \Phi(i_1, i_1 - k)M^{-1}(i_1 - k, i_1)\Phi^T(i_1, i_1 - k) \leq \beta_1 I$ *for all* i_1. **6-109**

Here $M(i_0, i_1)$ is the matrix **6-106** *and $\Phi(i, i_0)$ is the transition matrix of the system.*

We are forced to introduce the inverse of $M(i_0, i_1)$ in order to avoid defining $\Phi(i, i_0)$ for i less than i_0.

For time-invariant systems we have:

Theorem 6.14. *The time-invariant linear discrete-time system*

$$x(i+1) = Ax(i), \qquad y(i) = Cx(i) \tag{6-110}$$

is uniformly completely reconstructible if and only if it is completely reconstructible.

For time-invariant systems we introduce the concept of unreconstructible subspace.

Definition 6.7. *The unreconstructible subspace of the n-dimensional linear time-invariant discrete-time system*

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i), \\ y(i) &= Cx(i) \end{aligned} \tag{6-111}$$

is the linear subspace consisting of the states x_0 for which

$$y(i; x_0, i_0, 0) = 0, \qquad i \geq i_0. \tag{6-112}$$

Here **6-112** denotes the response of the output variable y of the system to the initial state $x(i_0) = x_0$, with $u(i) = 0$, $i \geq i_0$. The following theorem gives more information about the unreconstructible subspace.

Theorem 6.15. *The unreconstructible subspace of the linear time-invariant discrete-time system*

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i), \\ y(i) &= Cx(i) \end{aligned} \tag{6-113}$$

is the null space of the reconstructibility matrix Q.

Using the concept of an unreconstructible subspace, discrete-time linear systems can also be decomposed into a reconstructible and an unreconstructible part.

Theorem 6.16. *Consider the linear time-invariant discrete-time system*

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i), \\ y(i) &= Cx(i). \end{aligned} \tag{6-114}$$

Form the nonsingular transformation matrix

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \tag{6-115}$$

where the rows of U_1 form a basis for the subspace which is spanned by the rows of the reconstructibility matrix Q of the system. U_2 is so chosen that its rows together with those of U_1 span the whole n-dimensional space. Define the transformed state variable

$$x'(t) = Ux(t). \tag{6-116}$$

Then in terms of the transformed state variable the system can be represented by the state difference equation

$$x'(i+1) = \begin{pmatrix} A'_{11} & 0 \\ A'_{21} & A'_{22} \end{pmatrix} x'(i) + \begin{pmatrix} B'_1 \\ B'_2 \end{pmatrix} u(i), \qquad \text{6-117}$$

$$y(i) = (C'_1, \quad 0)x'(i),$$

where the pair $\{A'_{11}, C'_1\}$ *is completely reconstructible.*

Here the terminology "the pair $\{A, C\}$ is completely reconstructible" means that the system $x(i+1) = Ax(i)$, $y(i) = Cx(i)$ is completely reconstructible.

A *detectable* discrete-time system is defined as follows.

Definition 6.8. *The linear time-invariant discrete-time system*

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i), \\ y(i) &= Cx(i), \end{aligned} \qquad \text{6-118}$$

is ***detectable*** *if its unreconstructible subspace is contained within its stable subspace.*

One way of testing for detectability is through the following result.

Theorem 6.17. *Consider the linear time-invariant discrete-time system*

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i), \\ y(i) &= Cx(i). \end{aligned} \qquad \text{6-119}$$

Suppose that it is transformed according to Theorem 6.16 into the form **6-117**. *Then the system is detectable if and only if all the characteristic values of the matrix* A'_{22} *have moduli strictly less than one.*

Analogously to the continuous-time case, we define the characteristic values of the matrix A'_{11} as the *reconstructible poles*, and the characteristic values of A'_{22} as the *unreconstructible poles* of the system. Then a system is detectable if and only if all its unreconstructible poles are stable.

6.2.9 Duality

As in the continuous-time case, discrete-time regulator and filtering theory turn out to be related through duality. It is convenient to introduce the following definition.

Definition 6.9. *Consider the linear discrete-time system*

$$\begin{aligned} x(i+1) &= A(i)x(i) + B(i)u(i), \\ y(i) &= C(i)x(i). \end{aligned} \qquad \text{6-120}$$

In addition, consider the system

$$\begin{aligned} x^*(i+1) &= A^T(i^*-i)x^*(i) + C^T(i^*-i)u^*(i), \\ y^*(i) &= B^T(i^*-i)x^*(i), \end{aligned} \qquad \textbf{6-121}$$

where i^* is an arbitrary fixed integer. Then the system **6-121** *is termed the* ***dual of the system*** **6-120** *with respect to i^*.*

Obviously, we have the following.

Theorem 6.18. *The dual of the system* **6-121** *with respect to i^* is the original system* **6-120**.

Controllability and reconstructibility of systems and their duals are related as follows.

Theorem 6.19. *Consider the system* **6-120** *and its dual* **6-121**:

(a) *The system* **6-120** *is completely controllable if and only if its dual is completely reconstructible.*
(b) *The system* **6-120** *is completely reconstructible if and only if its dual is completely controllable.*
(c) *Assume that* **6-120** *is time-invariant. Then* **6-120** *is stabilizable if and only if* **6-121** *is detectable.*
(d) *Assume that* **6-120** *is time-invariant. Then* **6-120** *is detectable if and only if* **6-121** *is stabilizable.*

The proof of this theorem is analogous to that of Theorem 1.41 (Section 1.8).

6.2.10 Phase-Variable Canonical Forms

Just as for continuous-time systems, phase-variable canonical forms can be defined for discrete-time systems. For single-input systems we have the following definition.

Definition 6.10. *A single-input time-invariant linear discrete-time system is in* ***phase-variable canonical form*** *if it is represented in the form*

$$x(i+1) = \begin{pmatrix} 0 & 1 & 0 \cdots\cdots & 0 \\ 0 & 0 & 1 \quad 0 \quad \cdots & 0 \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ 0 & \cdots & \cdots\cdots 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots\cdots & -\alpha_{n-1} \end{pmatrix} x(i) + \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix} \mu(i),$$

$$y(i) = Cx(i). \qquad \textbf{6-122}$$

Here the α_i, $i = 0, 1, \cdots, n-1$ are the coefficients of the characteristic polynomial

$$\sum_{i=0}^{n} \alpha_i z^i \tag{6-123}$$

of the system, where $\alpha_n = 1$. Any completely controllable time-invariant linear discrete-time system can be transformed into this form by the prescription of Theorem 1.43 (Section 1.9).

Similarly we introduce for single-output systems the following definition.

Definition 6.11. *A single-output time-invariant linear discrete-time system is in **dual phase-variable canonical form** if it is represented as follows*

$$x(i+1) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & 0 & \cdots & 0 & -\alpha_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 1 & -\alpha_{n-1} \end{pmatrix} x(i) + Bu(i),$$

$$\eta(i) = (0, \quad 0, \cdots, 0, \quad 1)x(i). \tag{6-124}$$

6.2.11 Discrete-Time Vector Stochastic Processes

In this section we give a very brief discussion of discrete-time vector stochastic processes, which is a different name for infinite sequences of stochastic vector variables of the form $v(i)$, $i = \cdots, -1, 0, 1, 2, \cdots$. Discrete-time vector stochastic processes can be characterized by specifying all joint probability distributions

$$P\{v(i_1) \le v_1, v(i_2) \le v_2, \cdots, v(i_m) \le v_m\} \tag{6-125}$$

for all real $v_1, v_2, \cdots, v_m$, for all integers $i_1, i_2, \cdots, i_m$, and all integers m. If

$$P\{v(i_1) \le v_1, v(i_2) \le v_2, \cdots, v(i_m) \le v_m\} \\ = P\{v(i_1+k) \le v_1, v(i_2+k) \le v_2, \cdots, v(i_m+k) \le v_m\} \tag{6-126}$$

for all real $v_1, v_2, \cdots, v_m$, for all integers $i_1, i_2, \cdots, i_m$, and for any integers m and k the process is called *stationary*. If the joint distributions **6-126** are all multidimensional Gaussian distributions, the process is termed *Gaussian*. We furthermore define:

Definition 6.12. *Consider the discrete-time vector stochastic process* $v(i)$. *Then we call*

$$m(i) = E\{v(i)\} \tag{6-127}$$

the mean of the process,

$$C_v(i, j) = E\{v(i)v^T(j)\} \tag{6-128}$$

the second-order joint moment matrix, and

$$R_v(i, j) = E\{[v(i) - m(i)][v(j) - m(j)]^T\} \tag{6-129}$$

the covariance matrix of the process. Finally,

$$Q(i) = E\{[v(i) - m(i)][v(i) - m(i)]^T\} = R_v(i, i) \tag{6-130}$$

is the variance matrix and $C_v(i, i)$ the second-order moment matrix of the process.

If the process v is stationary, its mean and variance matrix are independent of i, and its joint moment matrix $C_v(i, j)$ and its covariance matrix $R_v(i, j)$ depend upon $i - j$ only. A process that is not stationary, but that has the property that its mean is constant, its second-order moment matrix is finite for all i and its second-order joint moment matrix and covariance matrix depend on $i - j$ only, is called *wide-sense stationary.*

For wide-sense stationary discrete-time processes, we define the following.

Definition 6.13. *The power spectral density matrix $\Sigma_v(\theta)$, $-\pi \le \theta < \pi$, of a wide-sense stationary discrete-time process v is defined as*

$$\Sigma_v(\theta) = \sum_{i=-\infty}^{\infty} z^{-i} R_v(i), \qquad z = e^{j\theta}, \qquad -\pi \le \theta < \pi, \tag{6-131}$$

if it exists, where $R_v(i - k)$ is the covariance matrix of the process and where $j = \sqrt{-1}$.

The name power spectral density matrix stems from its close connection with the identically named quantity for continuous-time stochastic processes. The following fact sheds some light on this.

Theorem 6.20. *Let v be a wide-sense stationary zero mean discrete-time stochastic process with power spectral density matrix $\Sigma_v(\theta)$. Then*

$$E\{v(i)v^T(i)\} = R_v(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Sigma_v(\theta)\, d\theta. \tag{6-132}$$

A nonrigorous proof is as follows. We write

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Sigma_v(\theta)\, d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{i=-\infty}^{\infty} R_v(i) e^{-j\theta i}\right) d\theta \\ &= \sum_{i=-\infty}^{\infty} R_v(i) \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\theta i}\, d\theta\right) \\ &= R_v(0), \end{aligned} \tag{6-133}$$

since

$$\int_{-\pi}^{\pi} e^{-j\theta i}\, d\theta = \begin{cases} 2\pi & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{6-134}$$

Power spectral density matrices are especially useful when analyzing the response of time-invariant linear discrete-time systems when a realization of a discrete-time stochastic process serves as the input. We have the following result.

Theorem 6.21. *Consider an asymptotically stable time-invariant linear discrete-time system with z-transfer matrix $H(z)$. Let the input to the system be a realization of a wide-sense stationary discrete-time stochastic process u with power spectral density matrix $\Sigma_u(\theta)$, which is applied from time $-\infty$ on. Then the output y is a realization of a wide-sense stationary discrete-time stochastic process with power spectral density matrix*

$$\Sigma_y(\theta) = H(e^{j\theta})\Sigma_u(\theta)H^T(e^{-j\theta}), \qquad -\pi \le \theta < \pi. \qquad \text{6-135}$$

Example 6.7. *Sequence of mutually uncorrelated variables*

Suppose that the stochastic process $v(i)$, $i = \cdots, -1, 0, 1, 2, \cdots$, consists of a sequence of mutually uncorrelated, zero-mean, vector-valued stochastic variables with constant variance matices Q. Then the covariance matrix of the process is given by

$$R_v(i - j) = \begin{cases} Q & \text{for } i = j, \\ 0 & \text{for } i \ne j. \end{cases} \qquad \text{6-136}$$

This is a wide-sense stationary process. Its power spectral density matrix is

$$\Sigma_v(\theta) = Q. \qquad \text{6-137}$$

This process is the discrete-time equivalent of white noise.

Example 6.8. *Exponentially correlated noise*

Consider the scalar wide-sense stationary, zero-mean discrete-time stochastic process v with covariance function

$$R_v(i - k) = \sigma^2 \exp\left(-\left|\frac{(i-k)\Delta}{T}\right|\right). \qquad \text{6-138}$$

We refer to Δ as the sampling period and to T as the time constant of the process. The power spectral density function of the process is easily found to be

$$\Sigma_v(\theta) = \frac{\sigma^2(1 - e^{-2\Delta/T})}{(e^{j\theta} - e^{-\Delta/T})(e^{-j\theta} - e^{-\Delta/T})}, \qquad -\pi \le \theta < \pi. \qquad \text{6-139}$$

6.2.12 Linear Discrete-Time Systems Driven by White Noise

In the context of linear discrete-time systems, we often describe disturbances and other stochastically varying phenomena as the outputs of linear discrete-time systems of the form

$$\begin{aligned} x(i+1) &= A(i)x(i) + B(i)w(i), \\ y(i) &= C(i)x(i). \end{aligned} \qquad \textbf{6-140}$$

Here $x(i)$ is the state variable, $y(i)$ the output variable, and $w(i)$, $i = \cdots, -1, 0, 1, 2, \cdots$, a sequence of mutually uncorrelated, zero-mean, vector-valued stochastic vectors with variance matrix

$$E\{w(i)w^T(i)\} = V(i). \qquad \textbf{6-141}$$

As we saw in Example 6.7, the process w shows resemblance to the white noise process we considered in the continuous-time case, and we therefore refer to the process w as *discrete-time white noise*. We call $V(i)$ the variance matrix of the process. When $V(i)$ does not depend upon i, the discrete-time white noise process is wide-sense stationary. When $w(i)$ has a Gaussian probability distribution for each i, we refer to w as a *Gaussian discrete-time white noise process.*

Processes described by **6-140** may arise when continuous-time processes described as the outputs of continuous-time systems driven by white noise are sampled. Let the continuous-time variable $x(t)$ be described by

$$\dot{x}(t) = A(t)x(t) + B(t)w(t), \qquad \textbf{6-142}$$

where w is white noise with intensity $V(t)$. Then if t_i, $i = 0, 1, 2, \cdots$, is a sequence of sampling instants, we can write from **1-61**:

$$x(t_{i+1}) = \Phi(t_{i+1}, t_i)x(t_i) + \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, \tau)B(\tau)w(\tau)\, d\tau, \qquad \textbf{6-143}$$

where $\Phi(t, t_0)$ is the transition matrix of the differential system **6-142**. Now using the integration rules of Theorem 1.51 (Section 1.11.1) it can be seen that the quantities

$$\int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, \tau)B(\tau)w(\tau)\, d\tau, \qquad \textbf{6-144}$$

$i = 0, 1, 2, \cdots$, form a sequence of zero mean, mutually uncorrelated stochastic variables with variance matrices

$$\int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, \tau)B(\tau)V(\tau)B^T(\tau)\Phi^T(t_{i+1}, \tau)\, d\tau. \qquad \textbf{6-145}$$

It is observed that **6-143** is in the form **6-140**.

It is sometimes of interest to compute the variance matrix of the stochastic process x described by **6-140**. The following result is easily verified.

Theorem 6.22. *Let the stochastic discrete-time process x be the solution of the linear stachastic difference equation*

$$x(i+1) = A(i)x(i) + B(i)w(i), \qquad \text{6-146}$$

where $w(i)$, $i = -1, 0, 1, 2, \cdots$, is a sequence of mutually uncorrelated zero-mean, vector-valued stochastic variables with variance matrices $V(i)$. Suppose that $x(i_0) = x_0$ has mean m_0 and variance matrix Q_0. Then the mean af $x(i)$

$$m(i) = E\{x(i)\}, \qquad \text{6-147}$$

and the variance matrix af $x(i)$,

$$Q(i) = E\{[x(i) - m(i)][x(i) - m(i)]^T\}, \qquad \text{6-148}$$

can be given as follows. The mean is

$$m(i) = \Phi(i, i_0)m_0, \qquad i \geq i_0, \qquad \text{6-149}$$

where $\Phi(i, i_0)$ is the transition matrix of the difference equation **6-146,** *while $Q(i)$ is the solution of the matrix difference equation*

$$\begin{aligned} Q(i+1) &= A(i)Q(i)A^T(i) + B(i)V(i)B^T(i), \qquad i = i_0, i_0+1, \cdots, \\ Q(i_0) &= Q_0. \end{aligned} \qquad \text{6-150}$$

When the matrices A, B, and V are constant, the following can be stated about the steady-state behavior of the stochastic process x.

Theorem 6.23. *Let the discrete-time stochastic process x be the solution of the stochastic difference equation*

$$\begin{aligned} x(i+1) &= Ax(i) + Bw(i), \\ x(i_0) &= x_0. \end{aligned} \qquad \text{6-151}$$

where A and B are constant and where the uncorrelated sequence of zero-mean stochastic variables w has a constant variance matrix V. Then if all the characteristic values of A have moduli strictly less than 1, and $i_0 \to -\infty$, the cavariance motrix of the process tends to an asymptotic value $\bar{R}_x(i, j)$ which depends on $i - j$ only. The corresponding asymptotic variance matrix $\bar{Q}$ is the unique solution of the matrix equotion

$$\bar{Q} = A\bar{Q}A^T + BVB^T. \qquad \text{6-152}$$

In later sections we will be interested in quadratic expressions. The following results are useful.

Theorem 6.24. *Let the process x be the solution of*

$$\begin{aligned} x(i+1) &= A(i)x(i) + B(i)w(i), \\ x(i_0) &= x_0, \end{aligned} \qquad \text{6-153}$$

where the $w(i)$ are a sequence of mutually uncorrelated zero mean stochastic variables with variance matrices $V(i)$. Let $R(i)$ be a given sequence of nonnegative-definite symmetric matrices. Then

$$E\left\{\sum_{i=i_0}^{i_1} x^T(i)R(i)x(i)\right\} = \operatorname{tr}\left[E\{x_0x_0^T\}P(i_0) + \sum_{i=i_0}^{i_1-1} B(i)V(i)B^T(i)P(i+1)\right], \quad \text{6-154}$$

where the nonnegative-definite symmetric matrices $P(i)$ are the solution of the matrix difference equation

$$\begin{aligned} P(i) &= A^T(i)P(i+1)A(i) + R(i), \qquad i = i_1 - 1, i_1 - 2, \cdots, i_0, \\ P(i_1) &= R(i_1). \end{aligned} \quad \text{6-155}$$

If A and R are constant, and all the characteristic values of A have moduli strictly less than 1, $P(i)$ approaches a constant value $\bar{P}$ as $i_1 \to \infty$, where $\bar{P}$ is the unique solution of the matrix equation

$$\bar{P} = A^T\bar{P}A + R. \quad \text{6-156}$$

One method for obtaining the solutions to the linear matrix equations **6-152** and **6-156** is repeated application of **6-150** or **6-155**. Berger (1971) gives another method. Power (1969) gives a transformation that brings equations of the type **6-152** or **6-156** into the form

$$M_1X + XM_2^T = N_3, \quad \text{6-157}$$

or vice versa, so that methods of solution available for one of these equations can also be used for the other (see Section 1.11.3 for equations of the type **6-157**).

A special case occurs when all stochastic variables involved are Gaussian.

Theorem 6.25. *Consider the stochastic discrete-time process x described by*

$$\begin{aligned} x(i+1) &= A(i)x(i) + B(i)w(i), \\ x(i_0) &= x_0. \end{aligned} \quad \text{6-158}$$

Then if the mutually uncorrelated stochastic variables $w(i)$ are Gaussian and the initial state x_0 is Gaussian, x is a Gaussian process.

Example 6.9. *Exponentially correlated noise*

Consider the stochastic process described by the scalar difference equation

$$\xi(i+1) = \alpha\xi(i) + \omega(i), \qquad \xi(i_0) = \xi_0, \qquad i_0 \to -\infty, \quad \text{6-159}$$

where the $\omega(i)$ form a sequence of scalar uncorrelated stochastic variables with variance σ_ω^2 and where $|\alpha| < 1$. We consider ξ the output of a time-invariant discrete-time system with z-transfer function

$$\frac{1}{z - \alpha}. \quad \text{6-160}$$

and with the sequence ω as input. Since the power spectral density function of ω is

$$\Sigma_\omega(\theta) = \sigma_\omega^2, \tag{6-161}$$

we find for the spectral density matrix of ξ, according to **6-135**,

$$\Sigma_\xi(\theta) = \frac{\sigma_\omega^2}{(e^{j\theta} - \alpha)(e^{-j\theta} - \alpha)}. \tag{6-162}$$

We observe that **6-162** and **6-139** have identical appearances; therefore, **6-159** generates exponentially correlated noise. The steady-state variance σ_ξ^2 of the process ξ follows from **6-152**; in this case we have

$$\sigma_\xi^2 = \alpha^2\sigma_\xi^2 + \sigma_\omega^2 \tag{6-163}$$

or

$$\sigma_\xi^2 = \frac{\sigma_\omega^2}{1 - \alpha^2}. \tag{6-164}$$

Example 6.10. *Stirred tank with disturbances*

In Example 1.37 (Section 1.11.4), we considered a continuous-time model of the stirred tank with disturbances included. The stochastic state differential equation is given by

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{2\theta} & 0 & 0 & 0 \\ 0 & -\frac{1}{\theta} & \frac{F_{10}}{V_0} & \frac{F_{20}}{V_0} \\ 0 & 0 & -\frac{1}{\theta_1} & 0 \\ 0 & 0 & 0 & -\frac{1}{\theta_2} \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ \frac{c_{10} - c_0}{V_0} & \frac{c_{20} - c_0}{V_0} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} w(t), \tag{6-165}$$

where w is white noise with intensity

$$V = \begin{pmatrix} \dfrac{2\sigma_1^2}{\theta_1} & 0 \\ 0 & \dfrac{2\sigma_2^2}{\theta_2} \end{pmatrix}. \qquad \textbf{6-166}$$

Here the components of the state are, respectively, the incremental volume of fluid, the incremental concentration in the tank, the incremental concentration of the feed F_1, and the incremental concentration of the feed F_2. The variations in the concentrations of the feeds are represented as exponentially correlated noise processes with rms values σ_1 and σ_2 and time constants θ_1 and θ_2, respectively.

When we assume that the system is controlled by a process computer so that the valve settings change at instants separated by intervals Δ, the discrete-time version of the system description can he found according to the method described in the beginning of this section. Since this leads to somewhat involved expressions, we give only the outcome for the numerical values of Example 1.37 supplemented with the following values:

$$\begin{aligned} \sigma_1 &= 0.1 \text{ kmol/m}^3, \\ \sigma_2 &= 0.2 \text{ kmol/m}^3, \\ \theta_1 &= 40 \text{ s}, \\ \theta_2 &= 50 \text{ s}, \\ \Delta &= 5 \text{ s}. \end{aligned} \qquad \textbf{6-167}$$

With this the stochastic state difference equation is

$$x(i+1) = \begin{pmatrix} 0.9512 & 0 & 0 & 0 \\ 0 & 0.9048 & 0.0669 & 0.02262 \\ 0 & 0 & 0.8825 & 0 \\ 0 & 0 & 0 & 0.9048 \end{pmatrix} x(i) + \begin{pmatrix} 4.877 & 4.877 \\ -1.1895 & 3.569 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} u(i) + w(i), \qquad \textbf{6-168}$$

where $w(i)$, $i \geq i_0$, is a sequence of uncorrelated zero-mean stochastic vectors with variance matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.00004886 & 0.00009375 & 0.0001 \\ 0 & 0.00009375 & 0.002212 & 0 \\ 0 & 0.0001 & 0 & 0.007252 \end{pmatrix}. \qquad \textbf{6-169}$$

By repeated application of **6-150**, it is possible to find the steady-state value $\bar{Q}$ of the variance matrix of the state. Numerically, we obtain

$$\bar{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.00390 & 0.00339 & 0.00504 \\ 0 & 0.00339 & 0.0100 & 0 \\ 0 & 0.00504 & 0 & 0.0400 \end{pmatrix}. \qquad \textbf{6-170}$$

This means that the rms value of the variations in the tank volume is zero (this is obvious, since the concentration variations do not affect the flows), the rms value of the concentration in the tank is $\sqrt{0.00390} \simeq 0.0625$ kmol/m^3, and the rms values of the concentrations of the incoming feeds are 0.1 kmol/m^3 and 0.2 kmol/m^3, respectively. The latter two values are of course precisely σ_1 and σ_2.

6.3 ANALYSIS OF LINEAR DISCRETE-TIME CONTROL SYSTEMS

6.3.1 Introduction

In this section a brief review is given of the analysis of linear discrete-time control systems. The section closely parallels Chapter 2.

6.3.2 Discrete-Time Linear Control Systems

In this section we briefly describe discrete-time control problems, introduce the equations that will be used to characterize plant and controller, define the notions of the mean square tracking error and mean square input, and state the basic design objective. First, we introduce the *plant*, which is the system to be controlled and which is represented as a linear discrete-time system characterized by the equations

$$\begin{aligned} x(i+1) &= A(i)x(i) + B(i)u(i) + v_p(i), \\ x(i_0) &= x_0, \\ y(i) &= C(i)x(i) + E_1(i)u(i) + v_m(i), \\ z(i) &= D(i)x(i) + E_2(i)u(i), \\ &\text{for } i = i_0, i_0+1, \cdots. \end{aligned} \qquad \textbf{6-171}$$

Here x is the *state* of the plant, x_0 the *initial state*, u the *input variable*, y the *observed variable*, and z the *controlled variable*. Furthermore v_p represents the *disturbance variable* and v_m the *observation noise*. Finally, we associate with the plant a *reference variable* $r(i)$, $i = i_0, i_0 + 1, \cdots$. It is noted that in contrast to the continuous-time case we allow both the observed variable and the controlled variable to have a direct link from the plant input. The reason is that direct links easily arise in discrete-time systems obtained by sampling continuous-time systems where the sampling instants of the output variables do not coincide with the instants at which the input variable changes value (see Section 6.2.3). As in the continuous-time case, we consider separately *tracking problems*, where the controlled variable $z(i)$ is to follow a time-varying reference variable $r(i)$, and *regulator problems*, where the reference variable is constant or slowly varying.

Analogously to the continuous-time case, we consider *closed-loop* and *open-loop controllers*. The general closed-loop controller is taken as a linear discrete-time system described by the state difference equation and the output equation

$$\begin{aligned} q(i+1) &= L(i)q(i) + K_r(i)r(i) - K_f(i)y(i), \\ u(i) &= F(i)q(i) + H_r(i)r(i) - H_f(i)y(i). \end{aligned} \qquad \textbf{6-172}$$

We note that these equations imply that the controller is able to process the input data $r(i)$ and $y(i)$ instantaneously while generating the plant input $u(i)$. If there actually are appreciable processing delays, such as may be the case in computer control when high sampling rates are used, we assume that these delays have been accounted for when setting up the plant equations (see Section 6.2.3).

The general open-loop controller follows from **6-172** with K_f and H_f identical to zero.

Closely following the continuous-time theory, we judge the performance of a control system, open- or closed-loop, in terms of its *mean square tracking error* and its *mean square input*. The mean square tracking error is defined as

$$C_e(i) = E\{e^T(i)W_e(i)e(i)\}, \qquad \textbf{6-173}$$

where

$$e(i) = z(i) - r(i). \qquad \textbf{6-174}$$

$W_e(i)$ is a nonnegative-definite symmetric weighting matrix. Similarly, the mean square input is defined as

$$C_u(i) = E\{u^T(i)W_u(i)u(i)\}, \qquad \textbf{6-175}$$

where $W_u(i)$ is another nonnegative-definite weighting matrix. Our *basic objective* in designing a control system is to *reduce the mean square tracking error as much as possible, while at the same time keeping the mean square input down to a reasonable value.*

As in the continuous-time case, a requirement of primary importance is contained in the following design rule.

Design Objective 6.1. *A control system should be asymptotically stable.*

Discrete-time control systems, just as continuous-time control systems, have the property that an unstable plant can be stabilized by closed-loop control but never by open-loop control.

Example 6.11. *Digital position control system with proportional feedback*

As an example, we consider the digital positioning system of Example 6.2 (Section 6.2.3). This system is described by the state difference equation

$$x(i+1) = \begin{pmatrix} 1 & 0.08015 \\ 0 & 0.6313 \end{pmatrix} x(i) + \begin{pmatrix} 0.003396 \\ 0.06308 \end{pmatrix} \mu(i). \qquad \textbf{6-176}$$

Here the first component $\xi_1(i)$ of $x(i)$ is the angular position, and the second component $\xi_2(i)$ the angular velocity. Furthermore, $\mu(i)$ is the input voltage. Suppose that this system is made into a position servo by using proportional feedback as indicated in Fig. 6.7. Here the controlled variable $\zeta(i)$ is the position, and the input voltage is determined by the relation

$$\mu(i) = \lambda[r(i) - \zeta(i)]. \qquad \textbf{6-177}$$

In this expression $r(i)$ is the reference variable and λ a gain constant. We assume that there are no processing delays, so that the sampling instant of the

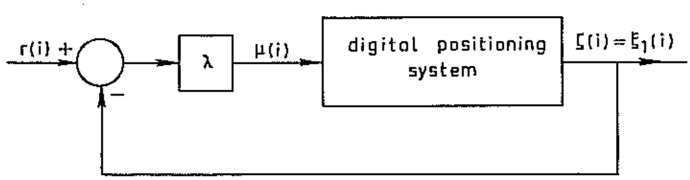

Fig. 6.7. A digital positioning system with proportional feedback.

output variable coincides with the instant at which a new control interval is initiated. Thus we have

$$\zeta(i) = (1, \quad 0)x(i). \tag{6-178}$$

In Example 6.6 (Section 6.2.6), it was found that the open-loop z-transfer function of the plant is given by

$$H(z) = \frac{0.003396(z + 0.8575)}{(z - 1)(z - 0.6313)}. \tag{6-179}$$

By using this it is easily found that the characteristic polynomial of the closed-loop system is given by

$$(z - 1)(z - 0.6313) + 0.003396\lambda(z + 0.8575). \tag{6-180}$$

In Fig. 6.8 the loci of the closed-loop roots are sketched. It is seen that when λ changes from 100 to 150 V/rad the closed-loop poles leave the unit circle,

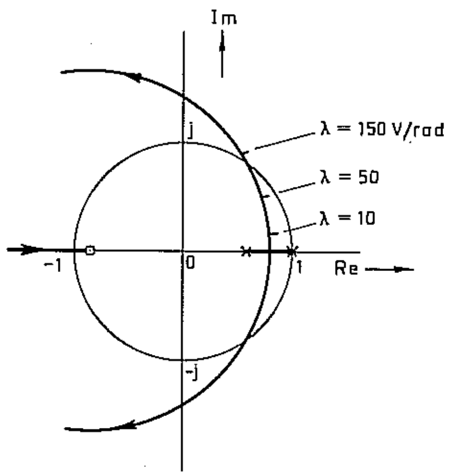

Fig. 6.8. The root loci of the digital position control system. ×, Open-loop poles; ○, open-loop zero.

hence the closed-loop system becomes unstable. Furthermore, it is to be expected that, in the stable region, as λ increases the system becomes more and more oscillatory since the closed-loop poles approach the unit circle more and more closely. To avoid resonance effects, while maximizing λ, the value of λ should be chosen somewhere between 10 and 50 V/rad.

6.3.3 The Steady-State and the Transient Analysis of the Tracking Properties

In this section the response of a linear discrete-time control system to the reference variable is studied. Both the steady-state response and the transient response are considered. The following assumptions are made.

1. *Design Objective* 6.1 *is satisfied, that is, the control system is asymptotically stable.*

2. *The control system is time-invariant and the weighting matrices* W_e *and* W_u *are constant.*

3. *The disturbance varioble* v_p *and the observation noise* v_m *are identical to zero.*

4. *The reference variable can be represented as*

$$r(i) = r_0 + r_v(i), \qquad i = i_0, i_0 + 1, \cdots, \tag{6-181}$$

where the ***constant part*** r_0 *is a stochastic vector with second-order moment matrix*

$$E\{r_0 r_0^T\} = R_0, \tag{6-182}$$

and the ***variable part*** r_v *is a wide-sense stationary zero-mean vector stochastic process with power spectral density matrix* $\Sigma_r(\theta)$.

Assuming zero initial conditions, we write for the z-transform $\mathbf{Z}(z)$ of the controlled variable and the z-transform $\mathbf{U}(z)$ of the input

$$\begin{aligned} \mathbf{Z}(z) &= T(z)\mathbf{R}(z), \\ \mathbf{U}(z) &= N(z)\mathbf{R}(z). \end{aligned} \tag{6-183}$$

Here $T(z)$ is the *transmission* of the system and $N(z)$ the transfer matrix from reference variable to input of the control system, while $\mathbf{R}(z)$ is the z-transform of the reference variable. The control system can beeither closed- or open-loop. Thus if $\mathbf{E}(z)$ is the z-transform of the tracking error $e(i) = z(i) - r(i)$, we have

$$\mathbf{E}(z) = [T(z) - I]\mathbf{R}(z). \tag{6-184}$$

To derive expressions for the steady-state mean square tracking error and input, we study the contributions of the constant part and the variable part of the reference variable separately. The constant part of the reference variable yields a steady-state response of the tracking error and the input as follows:

$$\begin{aligned} \lim_{i\to\infty} e(i) &= [T(1) - I]r_0, \\ \lim_{i\to\infty} u(i) &= N(1)r_0. \end{aligned} \tag{6-185}$$

From Section 6.2.11 it follows that in steady-state conditions the response of the tracking error to the variable part of the reference variable has the power spectral density matrix

$$[T(e^{j\theta}) - I]\Sigma_r(\theta)[T(e^{-j\theta}) - I]^T. \tag{6-186}$$

Consequently, the *steady-state mean square tracking error* can be expressed as

$$\begin{aligned} C_{e\infty} &= \lim_{i\to\infty} C_e(i) \\ &= E\{r_0^T[T(1) - I]^T W_e[T(1) - I]r_0\} \\ &\quad + \operatorname{tr}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[T(e^{j\theta}) - I]\Sigma_r(\theta)[T(e^{-j\theta}) - I]^T W_e \, d\theta\right\}. \end{aligned} \qquad \textbf{6-187}$$

This expression can be rewritten as

$$\begin{aligned} C_{e\infty} = \operatorname{tr}\Big\{&[T(1) - I]^T W_e[T(1) - I]R_0 \\ &+ \frac{1}{2\pi}\int_{-\pi}^{\pi}[T(e^{-j\theta}) - I]^T W_e[T(e^{j\theta}) - I]\Sigma_r(\theta)\, d\theta\Big\}. \end{aligned} \qquad \textbf{6-188}$$

Similarly, the *steady-state mean square input* can be expressed in the form

$$\begin{aligned} C_{u\infty} &= \lim_{i\to\infty} C_u(i) \\ &= \operatorname{tr}\left\{N^T(1)W_u N(1)R_0 + \frac{1}{2\pi}\int_{-\pi}^{\pi} N^T(e^{-j\theta})W_u N(e^{j\theta})\Sigma_r(\theta)\, d\theta\right\}. \end{aligned} \qquad \textbf{6-189}$$

Before further analyzing these expressions, we introduce the following additional assumption.

5. *The constant part ond the variable part of the reference variable have uncorrelated components, that is, both R_0 and $\Sigma_r(\theta)$ are diagonal ond can be written in the form*

$$\begin{aligned} R_0 &= \operatorname{diag}(R_{0,1}, R_{0,2}, \cdots, R_{0,p}), \\ \Sigma_r(\theta) &= \operatorname{diag}[\Sigma_{r,1}(\theta), \Sigma_{r,2}(\theta), \cdots, \Sigma_{r,p}(\theta)], \end{aligned} \qquad \textbf{6-190}$$

where p is the dimension of the reference variable and the controlled variable.

With this assumption we write for **6-188**:

$$\begin{aligned} C_{e\infty} &= \sum_{i=1}^{p} R_{0,i}\{[T(1) - I]^T W_e[T(1) - I]\}_{ii} \\ &\quad + \frac{1}{2\pi}\sum_{i=1}^{p}\int_{-\pi}^{\pi}\Sigma_{r,i}(\theta)\{[T(e^{-j\theta}) - I]^T W_e[T(e^{j\theta}) - I]\}_{ii}\, d\theta, \end{aligned} \qquad \textbf{6-191}$$

where $\{M\}_{ii}$ denotes the i-th diagonal entry of the matrix M. Following Chapter 2, we now introduce the following notions.

Definition 6.14. *Let $\rho(i)$, $i = \cdots, -1, 0, 1, 2, \cdots$, be a scalar wide-sense stationary discrete-time stochastic process with power spectral density function $\Sigma_\rho(\theta)$. Then the* ***normalized frequency band*** *Θ of this process is defined as the* *set of normalized frequencies θ, $0 \leq \theta \leq \pi$, for which*

$$\Sigma_p(\theta) \geq \alpha. \tag{6-192}$$

Here α is so chosen that the frequency band contains a given fraction $1 - \varepsilon$, where ε is small with respect to 1, of half the power of the process, that is,

$$\int_{\theta \in \Theta} \Sigma_p(\theta)\, d\theta = (1 - \varepsilon) \int_0^{\pi} \Sigma_p(\theta)\, d\theta. \tag{6-193}$$

As in Chapter 2, when the frequency band is an interval $[\theta_1, \theta_2]$, we define $\theta_2 - \theta_1$ as the *normalized bandwidth* of the process. When the frequency band is an interval $[0, \theta_c]$, we define θ_c as the *normalized cutoff frequency* of the process.

In the special case where the discrete-time process is derived from a continuous-time process by sampling, the (not normalized) bandwidth and cutoff frequency follow from the corresponding normalized quantities by the relation

$$\omega = \theta/\Delta, \tag{6-194}$$

where Δ is the sampling period and ω the (not normalized) angular frequency.

Before returning to our discussion of the steady-state mean square tracking error we introduce another concept.

Definition 6.15. *Let $T(z)$ be the transmission of an asymptotically stable time-invariant linear discrete-time control system. Then we define the* ***normalized frequency band of the i-th link*** *of the control system as the set of normalized frequencies θ, $0 \leq \theta \leq \pi$, for which*

$$\{[T(e^{-j\theta}) - I]^T W_e [T(e^{j\theta}) - I]\}_{ii} \leq \varepsilon^2 W_{e,ii}. \tag{6-195}$$

Here ε is a given number which is small with respect to 1, W_e is the weighting matrix for the mean square tracking error, and $W_{e,ii}$ the i-th diagonal entry of W_e.

Here as well we speak of the *bandwidth* and the *cutoff frequency* of the i-th link, if they exist. If the discrete-time system is derived from a continuous-time system by sampling, the (not normalized) bandwidth and cutoff frequency can be obtained by the relation **6-194.**

We can now phrase the following advice, which follows from a consideration of **6-191.**

Design Objective 6.2. *Let $T(z)$ be the $p \times p$ transmission of an asymptotically stable time-invariant linear discrete-time control system, for which both the constant and the variable part of the reference variable have uncorrelated components. Then in order to obtain a small steady-state mean square tracking error, the frequency band of each of the p links should contain the frequency* *band of the corresponding component of the reference variable. If the i-th component of the reference variable, $i = 1, 2, \cdots, p$, is likely to have a nonzero constant part, $\{[T(1) - I]^T W_e [T(1) - I]\}_{ii}$ should be small, preferably zero.*

Let us now consider the steady-state mean square input as given by **6-189**. Under assumption 5 this expression can be rewritten as

$$C_{u\infty} = \sum_{i=1}^{p} R_{0,i}\{N^T(1)W_u N(1)\}_{ii} + \frac{1}{2\pi}\sum_{i=1}^{p}\int_{\theta=-\pi}^{\pi} \Sigma_{r,i}(\theta)\{N^T(e^{-j\theta})W_u N(e^{j\theta})\}_{ii}\, d\theta. \qquad \textbf{6-196}$$

Since $C_{u\infty}$ should not be made too large, we extract the following advice.

Design Objective 6.3. *In order to obtain a small steady-state mean square input in an asymptotically stable time-invariant linear discrete-time control system with a p-dimensional reference variable with uncorrelated components,*

$$\{N^T(e^{-j\theta})W_u N(e^{j\theta})\}_{ii} \qquad \textbf{6-197}$$

should be made small over the normalized frequency band of the i-th component of the reference variable, for $i = 1, 2, \cdots, p$.

As in Chapter 2, we do not impose restrictions on the first term of **6-196** because only the fluctuations of the input variable about its set point need be considered.

We conclude this section with a discussion of the *transient* behavior of the response of the control system to the reference variable. As in the continuous-time case, we define the *settling time* of the mean square tracking error, the mean square input, or any other quantity, as *the time it takes this quantity to reach its steady-state value within a specified accuracy.* This settling time can be expressed as a number of intervals, or in seconds when the sampling interval is known. Obviously, it is desirable that the mean square tracking error of a control system settle down to its steady-state value as soon as possible after start-up or after upsets. We thus have the following design rule.

Design Objective 6.4. *The settling time of the mean square tracking error of a discrete-time control system should be as short as possible.*

The transient behavior of the mean square tracking error, the mean square input, and other quantities of interest can be computed in a manner similar to the continuous-time approach. For the various stochastic processes that influence the evolution of the control system, mathematical models are assumed in the form of discrete-time systems driven by discrete-time white noise. The variance matrix of the state of the system that results by augmenting the control system difference equation with these models can be computed according to Theorem 6.22 (Section 6.2.12). This variance matrix yields all the data required. The example at the end of this section illustrates the procedure. Often, however, a satisfactory estimate of tbe settling time of a given quantity can be obtained by evaluating the transient behavior of the response of the control system to the constant part of the reference variable alone; this tben becomes a simple matter of computing step responses.

For time-invariant control systems, information about the settling time can often be derived from the location of the closed-loop characteristic values of the system. From Section 6.2.4 we know that all responses are linear combinations of functions of the form λ^i, $i = i_0, i_0 + 1, \cdots$, where λ is a characteristic value. Since the time it takes $|\lambda|^i$ to reach 1 % of its initial value of 1 is (assuming that $|\lambda| < 1$)

$$\frac{2}{\log_{10}\left(\dfrac{1}{|\lambda|}\right)} \tag{6-198}$$

time intervals, an estimate of the 1 % settling time of an asymptotically stable linear time-invariant discrete-time control system is

$$\max_l \left\{\frac{2}{\log_{10}\left(\dfrac{1}{|\lambda_l|}\right)}\right\} \tag{6-199}$$

time intervals, where λ_l, $l = 1, 2, \cdots, n$, are the characteristic values of the control system. As with continuous-time systems, this formula may give misleading results inasmuch as some of the characteristic values may not appear in the response of certain variables.

We conclude this section by pointing out that when a discrete-time control system is used to describe a sampled continuous-time system the settling time as obtained from the discrete-time description may give a completely erroneous impression of the settling time for the continuous-time system. This is because it occasionally happens that a sampled system exhibits quite satisfactory behavior at the sampling instants, while *between* the sampling instants large overshoots appear that do not settle down for a long time. We shall meet examples of such situations in later sections.

Example 6.12. *Digital position control system with proportional feedback*

We illustrate the results of this section for a single-input single-output system only, for which we take the digital position control system of Example 6.11. Here the steady-state tracking properties can be analyzed by considering the scalar transmission $T(z)$, which is easily computed and turns out to be given by

$$T(z) = \frac{0.003396\lambda(z + 0.8575)}{(z - 1)(z - 0.6313) + 0.003396\lambda(z + 0.8575)}. \qquad \textbf{6-200}$$

In Fig. 6.9 plots are given of $|T(e^{j\omega\Delta})|$ for $\Delta = 0.1$ s, and for values of λ between 5 and 100 V/rad. It is seen from these plots that the most favorable value of λ is about 15 V/rad; for this value the system bandwidth is maximal without the occurrence of undesirable resonance effects.

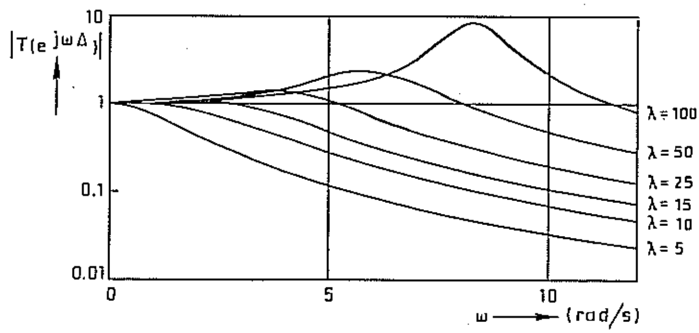

Fig. 6.9. The transmissions of the digital position control system for various values of the gain factor λ.

To compute the mean square tracking error and the mean square input voltage, we assume that the reference variable can be described by the model

$$r(i + 1) = 0.9802r(i) + w(i). \qquad \textbf{6-201}$$

Here w forms a sequence of scalar uncorrelated stochastic variables with variance 0.0392 rad^2. With a sampling interval of 0.1 s, this represents a sampled exponentially correlated noise process with a time constant of 5 s. The steady-state rms value of r can be found to be 1 rad (see Example 6.9).

With the simple feedback scheme of Example 6.11, the input to the plant is given by

$$\mu(i) = \lambda r(i) - \lambda \xi_1(i), \qquad \textbf{6-202}$$

which results in the closed-loop difference equation

$$x(i + 1) = \begin{pmatrix} 0.94906 & 0.08015 \\ -0.9462 & 0.6313 \end{pmatrix} x(i) + \begin{pmatrix} 0.05094 \\ 0.9462 \end{pmatrix} r(i). \qquad \textbf{6-203}$$

Here the value $\lambda = 15$ V/rad has been substituted. Augmenting this equation with **6-201**, we obtain

$$\begin{pmatrix} \xi_1(i+1) \\ \xi_2(i+1) \\ r(i+1) \end{pmatrix} = \begin{pmatrix} 0.94906 & 0.08015 & 0.05094 \\ -0.9462 & 0.6313 & 0.9462 \\ 0 & 0 & 0.9802 \end{pmatrix} \begin{pmatrix} \xi_1(i) \\ \xi_2(i) \\ r(i) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} w(i). \qquad \textbf{6-204}$$

We now define the variance matrix

$$Q(i) = E\left\{ \begin{pmatrix} \xi_1(i) \\ \xi_2(i) \\ r(i) \end{pmatrix} (\xi_1(i), \xi_2(i), r(i)) \right\}. \qquad \textbf{6-205}$$

Here it is assumed that $E\{x(i_0)\} = 0$ and $E\{r(i_0)\} = 0$, so that $x(i)$ and $r(i)$ have zero means for all i. Denoting the entries of $Q(i)$ as $Q_{jk}(i)$, $j, k = 1, 2, 3$, the mean square tracking error can be expressed as

$$\begin{aligned} C_e(i) &= E\{[\xi_1(i) - r(i)]^2\} \\ &= E\{\xi_1^2(i)\} - 2E\{\xi_1(i)r(i)\} + E\{r^2(i)\} \\ &= Q_{11}(i) - 2Q_{13}(i) + Q_{33}(i) \\ &= \operatorname{tr}\left\{ Q(i) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \right\}. \end{aligned} \qquad \textbf{6-206}$$

For the mean square input, we have

$$C_u(i) = E\{\mu^2(i)\} = E\{\lambda^2[r(i) - \xi_1(i)]^2\} = \lambda^2 C_e(i). \qquad \textbf{6-207}$$

For the variance matrix $Q(i)$, we obtain from Theorem 6.22 the matrix difference equation

$$Q(i+1) = MQ(i)M^T + NVN^T, \qquad \textbf{6-208}$$

where M is the 3×3 matrix and N the 3×1 matrix in **6-204**. V is the variance of $w(i)$. For the initial condition of this matrix difference equation, we choose

$$Q(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad \textbf{6-209}$$

This choice of $Q(0)$ implies that at $i = 0$ the plant is at rest, while the initial variance of the reference variable equals the steady-state variance 1 rad^2. Figure 6.10 pictures the evolution of the rms tracking error and the rms input voltage. It is seen that the settling time is somewhere between 10 and 20 sampling intervals.

It is also seen that the steady-state rms tracking error is nearly 0.4 rad, which is quite a large value. This means that the reference variable is not very well tracked. To explain this we note that continuous-time exponentially correlated noise with a time constant of 5 s (from which the reference variable is

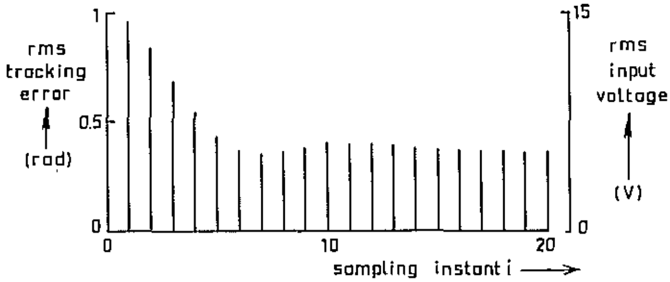

Fig. 6.10. Rms tracking error and rms input voltage for the digital position control system.

derived) has a 1 % cutoff frequency of 63.66/5 = 12.7 rad/s (see Section 2.5.2). The digital position servo is too slow to track this reference variable properly since its 1 % cutoff frequency is perhaps 1 rad/s. We also see, however, that the steady-state rms input voltage is about 4 V. By assuming that the maximally allowable rms input voltage is 25 V, it is clear that there is considerable room for improvement.

Finally, in Fig. 6.11 we show the response of the position digital system to a step of 1 rad in the reference variable. This plot confirms that the settling time of the tracking error is somewhere between 10 and 20 time intervals,

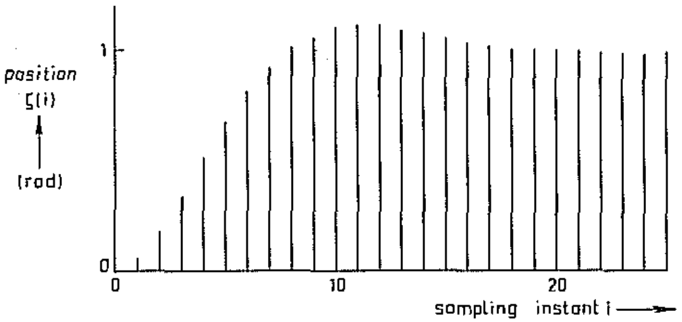

Fig. 6.11. The response of the digital position control system to a step in the reference variable of 1 rad.

depending upon the accuracy required. From the root locus of Fig. 6.8, we see that the distance of the closed-loop poles from the origin is about 0.8. The corresponding estimated 1% settling time according to **6-199** is 20.6 time intervals.

6.3.4 Further Aspects of Linear Discrete-Time Control System Performance

In this section we briefly discuss other aspects of the performance of linear discrete-time control systems. They are: *the effect of disturbances*; *the effect of observation noise*; *and the effect of plant parameter uncertainty*. We can carry out an analysis very similar to that for the continuous-time case. We very briefly summarize the results of this analysis. To describe the effect of the disturbances on the mean square tracking error in the single-input single-output case, it turns out to be useful to introduce the *sensitivity function*

$$S(z) = \frac{1}{1 + H(z)G(z)}, \qquad \textbf{6-210}$$

where

$$H(z) = D(zI - A)^{-1}B + E \qquad \textbf{6-211}$$

is the open-loop transfer function of the plant, and

$$G(z) = F(zI - L)^{-1}K_f + H_f \qquad \textbf{6-212}$$

is the transfer function of the feedback link of the controller. Here it is assumed that the controlled variable of the plant is also the observed variable, that is, in **6-171** $C = D$ and $E_1 = E_2 = E$. To reduce the effect of the disturbances, it turns out that $|S(e^{j\theta})|$ must be made small over the frequency band of the equivalent disturbance at the controlled variable. If

$$|S(e^{j\theta})| \leq 1 \qquad \text{for all } 0 \leq \theta < \pi, \qquad \textbf{6-213}$$

the closed-loop system always reduces the effect of disturbances, no matter what their statistical properties are. If constant disturbances are to be suppressed, $S(1)$ should be made small (this statement is not true without qualification if the matrix A has a characteristic value at 1). In the case of a multiinput multioutput system, the sensitivity function **6-210** is replaced with the *sensitivity matrix*

$$S(z) = [I + H(z)G(z)]^{-1}, \qquad \textbf{6-214}$$

and the condition **6-213** is replaced with the condition

$$S^T(e^{-j\theta})W_eS(e^{j\theta}) \leq W_e \qquad \text{for all } 0 \leq \theta < \pi, \qquad \textbf{6-215}$$

where W_e is the weighting matrix of the mean square tracking error.

In the scalar case, making $S(e^{j\theta})$ small over a prescribed frequency band can be achieved by making the controller transfer function $G(e^{j\theta})$ large over that frequency band. This conflicts, however, with the requirement that the mean square input be restricted, that the effect of the observation noise be restrained, and, possibly, with the requirement of stability. A compromise must be found.

The condition that $S(e^{j\theta})$ be small over as large a frequency band as possible also ensures that the closed-loop system receives protection against parameter variations. Here the condition **6-213**, or **6-215** in the multivariable case, guarantees that the effect of small parameter variations in the closed-loop system is always less than in an equivalent open-loop system.

6.4 OPTIMAL LINEAR DISCRETE-TIME STATE FEEDBACK CONTROL SYSTEMS

6.4.1 Introduction

In this section a review is given of linear optimal control theory for discrete-time systems, where it is assumed that the state of the system can be completely and accurately observed at all times. As in the continuous-time case, much of the attention is focused upon the regulator problem, although the tracking problem is discussed as well. The section is organized along the lines of Chapter 3.

6.4.2 Stability Improvement by State Feedback

In Section 3.2 we proved that a continuous-time linear system can be stabilized by an appropriate feedback law if the system is completely controllable or stabilizable. The same is true for discrete-time systems.

Theorem 6.26. *Let*

$$x(i+1) = Ax(i) + Bu(i) \qquad \textbf{6-216}$$

represent a time-invariant linear discrete-time system. Consider the time-invariant control law

$$u(i) = -Fx(i). \qquad \textbf{6-217}$$

Then the closed-loop characteristic values, that is, the characteristic values of $A - BF$, can be arbitrarily located in the complex plane (within the restriction that complex characteristic values occur in complex conjugate pairs) by choosing F suitably if and only if **6-216** *is completely controllable. It is possible to choose F such that the closed-loop system is stable if and only if* **6-216** *is stabilizable.*

Since the proof of the theorem depends entirely on the properties of the matrix $A - BF$, it is essentially identical to that for continuous-time systems. Moreover, the computational methods of assigning closed-loop poles are the same as those for continuous-time systems.

A case of special interest occurs when all closed-loop characteristic values are assigned to the origin. The characteristic polynomial of $A - BF$ then is of the form

$$\det(\lambda I - A + BF) = \lambda^n, \qquad \textbf{6-218}$$

where n is the dimension of the system. Since according to the Cayley–Hamilton theorem every matrix satisfies its own characteristic equation, we must have

$$(A - BF)^n = 0. \qquad \textbf{6-219}$$

In matrix theory it is said that this matrix is *nilpotent* with index n. Let us consider what implications this has. The state at the instant i can be expressed as

$$x(i) = (A - BF)^i x(0). \qquad \textbf{6-220}$$

This shows that, if **6-219** is satisfied, any initial state $x(0)$ is reduced to the zero state at or before the instant n, that is, in n steps or less (Cadzow, 1968; Farison and Fu, 1970). We say that a system with this property exhibits a *state deadbeat response*. In Section 6.4.7 we encounter systems with *output* deadbeat responses.

The preceding shows that the state of any completely controllable time-invariant discrete-time system can be forced to the zero state in at most n steps, where n is the dimension of the system. It may very well be, however, that the control law that assigns all closed-loop poles to the origin leads to excessively large input amplitudes or to an undesirable transient behavior.

We summarize the present results as follows.

Theorem 6.27. *Let the state difference equation*

$$x(i + 1) = Ax(i) + Bu(i) \qquad \textbf{6-221}$$

represent a completely controllable, time-invariant, n-dimensional, linear discrete-time system. Then any initial state can be reduced to the zero state in at most n steps, that is, for every $x(0)$ *there exists an input that makes* $x(n) = 0$. *This can be achieved through the time-invariant feedback law*

$$u(i) = -Fx(i), \qquad \textbf{6-222}$$

where F is so chosen that the matrix $A - BF$ *has all its characteristic values at the origin.*

Example 6.13. *Digital position control system*

The digital positioning system of Example 6.2 (Section 6.2.3) is described by the state difference equation

$$x(i+1) = \begin{pmatrix} 1 & 0.08015 \\ 0 & 0.6313 \end{pmatrix} x(i) + \begin{pmatrix} 0.003396 \\ 0.06308 \end{pmatrix} \mu(i). \qquad \textbf{6-223}$$

The system has the characteristic polynomial

$$(z-1)(z-0.6313) = z^2 - 1.6313z + 0.6313. \qquad \textbf{6-224}$$

In phase-variable canonical form the system can therefore be represented as

$$x'(i+1) = \begin{pmatrix} 0 & 1 \\ -0.6313 & 1.6313 \end{pmatrix} x'(i) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mu(i). \qquad \textbf{6-225}$$

The transformed state $x'(i)$ is related to the original state $x(i)$ by $x(i) = Tx'(i)$, where by Theorem 1.43 (Section 1.9) the matrix T can be found to be

$$T = \begin{pmatrix} 0.002912 & 0.003396 \\ -0.06308 & 0.06308 \end{pmatrix}. \qquad \textbf{6-226}$$

It is immediately seen that in terms of the transformed state the state deadbeat control law is given by

$$\mu(i) = -(-0.6313,\ 1.6313)x'(i). \qquad \textbf{6-227}$$

In terms of the original state, we have

$$\mu(i) = -(-0.6313,\ 1.6313)T^{-1}x(i), \qquad \textbf{6-228}$$

or

$$\mu(i) = -(158.5,\ 17.33)x(i). \qquad \textbf{6-229}$$

In Fig. 6.12 the complete response of the deadbeat digital position control system to an initial condition $x(0) = \text{col}\,(0.1, 0)$ is sketched, not only at the sampling instants, but also at the intermediate times. This response has been obtained by simulating the continuous-time positioning system while it is controlled with piecewise constant inputs obtained from the discrete-time control law **6-229**. It is seen that the system is completely at rest after two sampling periods.

6.4.3 The Linear Discrete-Time Optimal Regulator Problem

Analogously to the continuous-time problem, we define the discrete-time regulator problem as follows.

Definition 6.16. *Consider the discrete-time linear system*

$$x(i+1) = A(i)x(i) + B(i)u(i), \qquad \textbf{6-230}$$

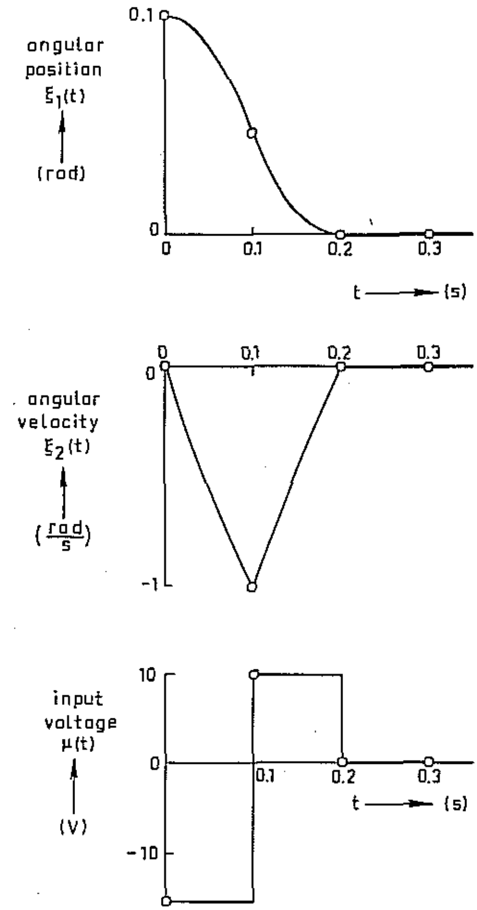

Fig. 6.12. State deadbeat response of the digital position control system.

where

$$x(i_0) = x_0, \qquad \text{6-231}$$

with the controlled variable

$$z(i) = D(i)x(i). \qquad \text{6-232}$$

Consider as well the criterion

$$\sum_{i=i_0}^{i_1-1} [z^T(i+1)R_3(i+1)z(i+1) + u^T(i)R_2(i)u(i)] + x^T(i_1)P_1x(i_1), \qquad \text{6-233}$$

where $R_3(i+1) > 0$ *and* $R_2(i) > 0$ *for* $i = i_0, i_0 + 1, \cdots, i_1 - 1$, *and* $P_1 \geq 0$. *Then the problem of determining the input* $u(i)$ *for* $i = i_0, i_0 + 1, \cdots, i_1 - 1$, *is called the* ***discrete-time deterministic linear optimal regulator problem.***

If all matrices occurring in the problem formulation are constant, we refer to it as the ***time-invariant discrete-time linear optimal regulator problem.***

It is noted that the two terms following the summation sign in the criterion do not have the same index. This is motivated as follows. The initial value of the controlled variable $z(i_0)$ depends entirely upon the initial state $x(i_0)$ and cannot be changed. Therefore there is no point in including a term with $z(i_0)$ in the criterion. Similarly, the final value of the input $u(i_1)$ affects only the system behavior beyond the terminal instant i_1; therefore the term involving $u(i_1)$ can be excluded as well. For an extended criterion, where the criterion contains a cross-term, see Problem 6.1.

It is also noted that the controlled variable does not contain a direct link in the problem formulation of Definition 6.16, although as we saw in Section 6.2.3 such a direct link easily arises when a continuous-time system is discretized. The omission of a direct link can be motivated by the fact that usually some freedom exists in selecting the controlled variable, so that often it is justifiable to make the instants at which the controlled variable is to be controlled coincide with the sampling instants. In this case no direct link enters into the controlled variable (see Section 6.2.3). Regulator problems where the controlled variable does have a direct link, however, are easily converted to the formulation of Problem 6.1.

In deriving the optimal control law, our approach is different from the continuous-time case where we used elementary calculus of variations; here we invoke dynamic programming (Bellman, 1957; Kalman and Koepcke, 1958). Let us define the scalar function $\sigma[x(i), i]$ as follows:

$$\sigma[x(i), i] = \begin{cases} \min\limits_{u(i),\cdots,u(i_1-1)} \left\{ \sum\limits_{j=i}^{i_1-1} [z^T(j+1)R_3(j+1)z(j+1) \right. \\ \qquad\qquad \left. + u^T(j)R_2(j)u(j)] + x^T(i_1)P_1x(i_1) \right\} \\ \qquad\qquad\qquad\qquad \text{for } i = i_0, i_0+1, \cdots, i_1-1, \\ x^T(i_1)P_1x(i_1) \qquad\qquad \text{for } i = i_1. \end{cases} \qquad \textbf{6-234}$$

We see that $\sigma[x(i), i]$ represents the minimal value of the criterion, computed over the period $i, i+1, \cdots, i_1$, when at the instant i the system is in the state $x(i)$. We derive an iterative equation for this function. Consider the instant $i-1$. Then if the input $u(i-1)$ is arbitrarily selected, but $u(i), u(i+1), \cdots, u(i_1-1)$ are chosen optimally with respect to the state at time i, we can write for the criterion over the period $i-1, i, \cdots, i_1$:

$$\sum_{j=i-1}^{i_1-1} [z^T(j+1)R_3(j+1)z(j+1) + u^T(j)R_2(j)u(j)] + x^T(i_1)P_1x(i_1)$$
$$= [z^T(i)R_3(i)z(i) + u^T(i-1)R_2(i-1)u(i-1)] + \sigma[x(i), i]. \qquad \textbf{6-235}$$

Obviously, to determine $u^0(i-1)$, the optimal input at time $i-1$, we must choose $u(i-1)$ so that the expression

$$z^T(i)R_3z(i) + u^T(i-1)R_2u(i-1) + \sigma[x(i), i] \qquad \textbf{6-236}$$

is minimized. The minimal value of **6-236** must of course be the minimal value of the criterion evaluated over the control periods $i-1, i, \cdots, i_1-1$. Consequently, we have the equality

$$\sigma[x(i-1), i-1)] = \min_{u(i-1)} \{z^T(i)R_3(i)z(i) + u^T(i-1)R_2(i-1)u(i-1) + \sigma[x(i), i]\}. \qquad \textbf{6-237}$$

By using **6-230** and **6-232** and rationalizing the notation, this expression takes the form

$$\sigma(x, i-1) = \min_u \{[A(i-1)x + B(i-1)u]^T R_1(i)[A(i-1)x + B(i-1)u] + u^T R_2(i-1)u + \sigma([A(i-1)x + B(i-1)u], i)\}, \qquad \textbf{6-238}$$

where

$$R_1(i) = D^T(i)R_3(i)D(i). \qquad \textbf{6-239}$$

This is an iterative equation in the function $\sigma(x, i)$. It can be solved in the order $\sigma(x, i_1)$, $\sigma(x, i_1-1)$, $\sigma(x, i_1-2), \cdots$, since $\sigma(x, i_1)$ is given by **6-234**. Let us attempt to find a solution of the form

$$\sigma(x, i) = x^T P(i) x, \qquad \textbf{6-240}$$

where $P(i)$, $i = i_0, i_0+1, \cdots, i_1$, is a sequence of matrices to be determined. From **6-234** we immediately see that

$$P(i_1) = P_1. \qquad \textbf{6-241}$$

Substitution of **6-240** into **6-238** and minimization shows that the optimal input is given by

$$u(i-1) = -F(i-1)x(i-1), \qquad i = i_0+1, \cdots, i_1, \qquad \textbf{6-242}$$

where the gain matrix $F(i-1)$ follows from

$$F(i-1) = \{R_2(i-1) + B^T(i-1)[R_1(i) + P(i)]B(i-1)\}^{-1} \cdot B^T(i-1)[R_1(i) + P(i)]A(i-1). \qquad \textbf{6-243}$$

The inverse matrix in this expression always exists since $R_2(i-1) > 0$ and a nonnegative-definite matrix is added. Substitution of **6-242** into **6-238** yields with **6-243** the following difference equation in $P(i)$:

$$P(i-1) = A^T(i-1)[R_1(i) + P(i)][A(i-1) - B(i-1)F(i-1)], \qquad i = i_0+1, \cdots, i_1. \qquad \textbf{6-244}$$

It is easily verified that the right-hand side is a symmetric matrix.

We sum up these results as follows.

Theorem 6.28. *Consider the discrete-time deterministic linear optimal regulator problem. The optimal input is given by*

$$u(i) = -F(i)x(i), \qquad i = i_0, i_0 + 1, \cdots, i_1 - 1, \qquad \textbf{6-245}$$

where

$$F(i) = \{R_2(i) + B^T(i)[R_1(i + 1) + P(i + 1)]B(i)\}^{-1} \cdot B^T(i)[R_1(i + 1) + P(i + 1)]A(i). \qquad \textbf{6-246}$$

Here the inverse always exists and

$$R_1(i) = D^T(i)R_3(i)D(i), \qquad i = i_0 + 1, i_0 + 2, \cdots, i_1. \qquad \textbf{6-247}$$

The sequence of matrices $P(i)$, $i = i_0, i_0 + 1, \cdots, i_1 - 1$, *satisfies the matrix difference equation*

$$P(i) = A^T(i)[R_1(i + 1) + P(i + 1)][A(i) - B(i)F(i)], \qquad i = i_0, i_0 + 1, \cdots, i_1 - 1, \qquad \textbf{6-248}$$

with the terminal condition

$$P(i_1) = P_1. \qquad \textbf{6-249}$$

The value of the criterion **6-233** *achieved with this control law is given by*

$$x^T(i_0)P(i_0)x(i_0). \qquad \textbf{6-250}$$

We note that the difference equation **6-248** is conveniently solved backward, where first $F(i)$ is computed from $P(i + 1)$ through **6-246**, and then $P(i)$ from $P(i + 1)$ and $F(i)$ through **6-248**. This presents no difficulties when the aid of a digital computer is invoked. Equation **6-248** is the equivalent of the continuous-time Riccati equation.

It is not difficult to show that under the conditions of Definition 6.16 the solution of the discrete-time deterministic linear optimal regulator problem as given in Theorem 6.28 always exists and is unique.

Example 6.14. *Digital position control system*

Let us consider the digital positioning system of Example 6.2 (Section 6.2.3). We take as the controlled variable the position, that is, we let

$$\zeta(i) = (1, 0)x(i). \qquad \textbf{6-251}$$

The following criterion is selected. Minimize

$$\sum_{i=0}^{i_1-1} [\zeta^2(i+1) + \rho\mu^2(i)]. \qquad \textbf{6-252}$$

Table 6.1 shows the behavior of the gain vector $F(i)$ for $i_1 = 10$ and $\rho = 0.00002$. We see that as i decreases, $F(i)$ approaches a steady-state value

$$\bar{F} = (110.4,\ 12.66). \qquad \textbf{6-253}$$

The response of the corresponding steady-state closed-loop system to the initial state $x(0) = \text{col}\,(0.1, 0)$ is given in Fig. 6.13.

Table 6.1 Behavior of the Feedback Gain Vector $F(i)$ for the Digital Position Control System

i	$F(i)$
9	(107.7, 8.63)
8	(114.0, 12.66)
7	(109.4, 12.58)
6	(110.3, 12.64)
5	(110.4, 12.66)
4	(110.4, 12.66)
3	(110.4, 12.66)
2	(110.4, 12.66)
1	(110.4, 12.66)
0	(110.4, 12.66)

6.4.4 Steady-State Solution of the Discrete-Time Regulator Problem

In this section we study the case where the control period extends from i_0 to infinity. The following results are in essence identical to those for the continuous-time case.

Theorem 6.29. *Consider the discrete-time deterministic linear optimal regulator problem and its solution as given in Theorem 6.28. Assume that $A(i)$, $B(i)$, $R_1(i+1)$, and $R_2(i)$ are bounded for $i \geq i_0$, and suppose that*

$$R_3(i+1) \geq \alpha I, \qquad R_2(i) \geq \beta I, \qquad i \geq i_0, \qquad \textbf{6-254}$$

where α and β are positive constants.

(i) *Then if the system* **6-230** *is either*

  (a) *completely controllable, or*

  (b) *exponentially stable,*

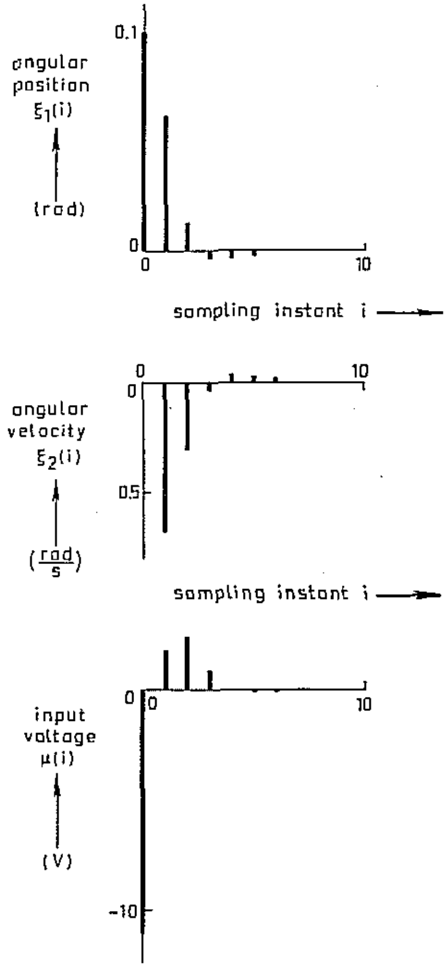

Fig. 6.13. Response of the optimal digital position control system to the initial condition $x(0) = \text{col}(0.1, 0)$.

the solution $P(i)$ *of the difference equations* **6-246** *and* **6-248** *with the terminal condition* $P(i_1) = 0$ *converges to a nonnegative-definite sequence of matrices* $\bar{P}(i)$ *as* $i_1 \to \infty$, *which is a solution of the difference equations* **6-246** *and* **6-248.**
(ii) *Moreover, if the system* **6-230, 6-232** *is either*

(c) *both uniformly completely controllable and uniformly completely reconstructible, or*

(d) *exponentially stable,*

the solution $P(i)$ *of the difference equations* **6-246** *and* **6-248** *with the terminal condition* $P(i_1) = P_1$ *converges to* $\bar{P}(i)$ *as* $i_1 \to \infty$ *for any* $P_1 \geq 0$.

The stability of the steady-state control law that corresponds to the steady-state solution $\bar{P}$ is ascertained from the following result.

Theorem 6.30. *Consider the discrete-time deterministic linear optimal regulator problem and suppose that the assumptions of Theorem 6.29 concerning* A, B, R_1, R_3, *and* R_2 *are satisfied. Then if the system* **6-230**, **6-232** *is either*

(a) *uniformly completely controllable and uniformly completely reconstructible, or*

(b) *exponentially stable,*

the following facts hold.

(i) *The steady-state optimal control law*

$$u(i) = -\bar{F}(i)x(i), \qquad \textbf{6-255}$$

where $\bar{F}(i)$ *is obtained by substituting* $\bar{P}(i)$ *for* $P(i)$ *in* **6-246**, *is exponentially stable.*

(ii) *The steady-state optimal control law* **6-255** *minimizes*

$$\lim_{i_1 \to \infty} \left\{ \sum_{i=i_0}^{i_1} [z^T(i+1)R_3(i+1)z(i+1) + u^T(i)R_2(i)u(i)] + x^T(i_1)P_1x(i_1) \right\} \qquad \textbf{6-256}$$

for all $P_1 \geq 0$. *The minimal value of* **6-256**, *which is achieved by the steady-state optimal control law, is given by*

$$x^T(i_0)\bar{P}(i_0)x(i_0). \qquad \textbf{6-257}$$

The proofs of these theorems can be given along the lines of Kalman's proofs (Kalman, 1960) for continuous-time systems. The duals of these theorems (for reconstruction) are considered by Deyst and Price (1968). In the time-invariant case, the following facts hold (Caines and Mayne, 1970, 1971).

Theorem 6.31. *Consider the time-invariant discrete-time linear optimal regulator problem. Then if the system is both stabilizable and detectable the following facts hold.*

(i) *The solution* $P(i)$ *of the difference equations* **6-246** *and* **6-248** *with the terminal condition* $P(i_1) = P_1$ *converges to a constant steady-state solution* $\bar{P}$ *as* $i_1 \to \infty$ *for any* $P_1 \geq 0$.

(ii) *The steady-state optimal control law is time-invariant and asymptotically stable.*

(iii) *The steady-state optimal control law minimizes* **6-256** *for all* $P_1 \geq 0$. *The minimal value of this expression is given by*

$$x^T(i_0)\bar{P}x(i_0). \qquad \textbf{6-258}$$

In conclusion, we derive a result that is useful when studying the closed-loop pole locations of the steady-state time-invariant optimal regulator. Define the quantity

$$p(i) = [R_1(i+1) + P(i+1)]x(i+1), \qquad i = i_0, i_0+1, \cdots, i_1 - 1, \quad \textbf{6-259}$$

where R_1 and P are as given in Theorem 6.28. We derive a difference equation for $p(i)$. From the terminal condition **6-249**, it immediately follows that

$$p(i_1 - 1) = [R_1(i_1) + P_1]x(i_1). \quad \textbf{6-260}$$

Furthermore, we have with the aid of **6-248**

$$\begin{aligned} p(i-1) &= R_1(i)x(i) + P(i)x(i) \\ &= R_1(i)x(i) + A^T(i)[R_1(i+1) + P(i+1)][A(i) - B(i)F(i)]x(i) \\ &= R_1(i)x(i) + A^T(i)[R_1(i+1) + P(i+1)]x(i+1) \\ &= R_1(i)x(i) + A^T(i)p(i). \end{aligned} \quad \textbf{6-261}$$

Finally, we express $u^0(i)$ in terms of $p(i)$. Consider the following string of equalities

$$\begin{aligned} -R_2^{-1}(i)B^T(i)p(i) &= -R_2^{-1}(i)B^T(i)[R_1(i+1) + P(i+1)]x(i+1) \\ &= -R_2^{-1}(i)B^T(i)[R_1(i+1) + P(i+1)][A(i)x(i) + B(i)u^0(i)] \\ &= -R_2^{-1}(i)B^T(i)[R_1(i+1) + P(i+1)]A(i)x(i) \\ &\quad -R_2^{-1}(i)B^T(i)[R_1(i+1) + P(i+1)]B(i)u^0(i). \end{aligned} \quad \textbf{6-262}$$

Now from **6-246** it follows that

$$\begin{aligned} B^T(i)&[R_1(i+1) + P(i+1)]A(i)x(i) \\ &= \{R_2(i) + B^T(i)[R_1(i+1) + P(i+1)]B(i)\}F(i)x(i) \\ &= -\{R_2(i) + B^T(i)[R_1(i+1) + P(i+1)]B(i)\}u^0(i). \end{aligned} \quad \textbf{6-263}$$

Substitution of this into **6-262** yields

$$-R_2^{-1}(i)B^T(i)p(i) = u^0(i). \quad \textbf{6-264}$$

Inserting $u^0(i)$ as given here into the state difference equation, we obtain the following two-point boundary-value problem

$$\begin{aligned} x(i+1) &= A(i)x(i) - B(i)R_2^{-1}(i)B^T(i)p(i), \quad && i = i_0, i_0+1, \cdots, i_1 - 1, \\ p(i-1) &= R_1(i)x(i) + A^T(i)p(i), && i = i_0+1, i_0+2, \cdots, i_1 - 1 \\ x(i_0) &= x_0, \\ p(i_1 - 1) &= [R_1(i_1) + P_1]x(i_1). \end{aligned} \quad \textbf{6-265}$$

We could have derived these equations directly by a variational approach to the discrete-time regulator problem, analogously to the continuous-time version.

Let us now consider the time-invariant steady-state case. Then $p(i)$ is defined by

$$p(i) = (R_1 + \bar{P})x(i+1), \qquad i = i_0, i_0 + 1, \cdots. \qquad \textbf{6-266}$$

In the time-invariant case the difference equations **6-265** take the form

$$\begin{aligned} x(i+1) &= Ax(i) - BR_2^{-1}B^Tp(i), \qquad && i = i_0, i_0+1, \cdots, \\ p(i-1) &= R_1x(i) + A^Tp(i), && i = i_0+1, i_0+2, \cdots. \end{aligned} \qquad \textbf{6-267}$$

Without loss of generality we take $i_0 = 0$; thus we rewrite **6-267** as

$$\begin{aligned} x(i+1) &= Ax(i) - BR_2^{-1}B^Tp(i), \qquad && i = 0, 1, 2, \cdots, \\ p(i) &= R_1x(i+1) + A^Tp(i+1), && i = 0, 1, 2, \cdots. \end{aligned} \qquad \textbf{6-268}$$

We study these difference equations by z-transformation. Application of the z-transformation to both equations yields

$$\begin{aligned} z\mathbf{X}(z) - zx_0 &= A\mathbf{X}(z) - BR_2^{-1}B^T\mathbf{P}(z), \\ \mathbf{P}(z) &= zR_1\mathbf{X}(z) - zR_1x_0 + zA^T\mathbf{P}(z) - zA^Tp_0, \end{aligned} \qquad \textbf{6-269}$$

where $x_0 = x(0)$, $p_0 = p(0)$, and $\mathbf{X}(z)$ and $P(z)$ are the z-transforms of x and p, respectively. Solving for $\mathbf{X}(z)$ and $P(z)$, we write

$$\begin{pmatrix} \mathbf{X}(z) \\ \mathbf{P}(z) \end{pmatrix} = \begin{pmatrix} zI - A & BR_2^{-1}B^T \\ -R_1 & z^{-1}I - A^T \end{pmatrix}^{-1} \begin{pmatrix} zx_0 \\ -R_1x_0 - A^Tp_0 \end{pmatrix}. \qquad \textbf{6-270}$$

When considering this expression, we note that each component of $\mathbf{X}(z)$ and $P(z)$ is a rational function in z with singularities at those values of z where

$$\det \begin{pmatrix} zI - A & BR_2^{-1}B^T \\ -R_1 & z^{-1}I - A^T \end{pmatrix} = 0. \qquad \textbf{6-271}$$

Let z_j, $j = 1, 2, \cdots$, denote the roots of this expression, the left-hand side of which is a polynomial in z and $1/z$. If z_j is a root, $1/z_j$ also is a root. Moreover, zero can never be a root of **6-271** and there are at most $2n$ roots (n is the dimension of the state x). It follows that both $x(i)$ and $p(i)$ can be described as linear combinations of expressions of the form $z_j^{\,i}$, $iz_j^{\,i}$, $i^2z_j^{\,i}$, $\cdots$, for all values of j. Terms of the form $i^kz_j^{\,i}$, $k = 0, 1, \cdots, l-1$, occur when z_j has multiplicity l. Now we know that under suitable conditions stated in Theorem 6.31 the steady-state response of the closed-loop regulator is asymptotically stable. This means that the initial conditions of the difference equations **6-268** are such that the coefficients of the terms in $x(i)$ with powers of z_j with $|z_j| \geq 1$ are zero. Consequently, $x(i)$ is a linear combination of powers of those roots z_j for which $|z_j| < 1$. This means that these roots are characteristic values of the closed-loop regulator. Now, since **6-271** may have less than $2n$ roots, there may be less than n roots with moduli strictly less than 1 (it is seen in Section 6.4.7 that this is the case only when A has one or more characteristic values zero). This leads to the conclusion that the remaining characteristic values of the closed-loop regulator are zero, since z appears in the denominators of the expression on the right-hand side of **6-270** after inversion of the matrix.

We will need these results later (Section 6.4.7) to analyze the behavior of the closed-loop characteristic values. We summarize as follows.

Theorem 6.32. *Consider the time-invariant discrete-time deterministic linear optimal regulator problem. Suppose that the n-dimensional system*

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i), \\ z(i) &= Dx(i), \end{aligned} \qquad \textbf{6-272}$$

is stabilizable and detectable. Let z_j, $j = 1, 2, \cdots, r$, *with* $r \le n$, *denote those roots of*

$$\det \begin{pmatrix} zI - A & BR_2^{-1}B^T \\ -D^TR_3D & z^{-1}I - A^T \end{pmatrix} = 0 \qquad \textbf{6-273}$$

that have moduli strictly less than 1. Then z_j, $j = 1, 2, \cdots, r$, *constitute* r *of the characteristic values of the closed-loop steady-state optimal regulator. The remaining* $n - r$ *characteristic values are zero.*

Using an approach related to that of this section, Vaughan (1970) gives a method for finding the steady-state solution of the regulator problem by diagonalization.

Example 6.15. *Stirred tank*

Consider the problem of regulating the stirred tank of Example 6.3 (Section 6.2.3) which is described by the state difference equation

$$x(i+1) = \begin{pmatrix} 0.9512 & 0 \\ 0 & 0.9048 \end{pmatrix} x(i) + \begin{pmatrix} 4.877 & 4.877 \\ -1.1895 & 3.569 \end{pmatrix} u(i). \qquad \textbf{6-274}$$

We choose as controlled variables the outgoing flow and the concentration, that is,

$$z(i) = \begin{pmatrix} 0.01 & 0 \\ 0 & 1 \end{pmatrix} x(i). \qquad \textbf{6-275}$$

The criterion is given by

$$\sum_{i=0}^{\infty} [z^T(i+1)R_3z(i+1) + u^T(i)R_2u(i)]. \qquad \textbf{6-276}$$

Exactly as in the continuous-time case of Example 3.9 (Section 3.4.1), we choose for the weighting matrices

$$R_3 = \begin{pmatrix} 50 & 0 \\ 0 & 0.02 \end{pmatrix} \quad \text{and} \quad R_2 = \rho \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{pmatrix}, \qquad \textbf{6-277}$$

where ρ is a scalar constant to be determined.

The steady-state feedback gain matrix can be found by repeated application of **6-246** and **6-248**. For $\rho = 1$ numerical computation yields

$$\bar{F} = \begin{pmatrix} 0.07125 & -0.7029 \\ 0.01357 & 0.04548 \end{pmatrix}. \qquad \textbf{6-278}$$

The closed-loop characteristic values are ~~$0.5982 \pm j0.08988$~~ 0.5083 and 0.6881. Figure 6.14 shows the response of the closed-loop system to the initial conditions $x(0) = \text{col}\,(0.1, 0)$ and $x(0) = \text{col}\,(0, 0.1)$. The response is quite similar to that of the corresponding continuous-time regulator as given in Fig. 3.11 (Section 3.4.1).

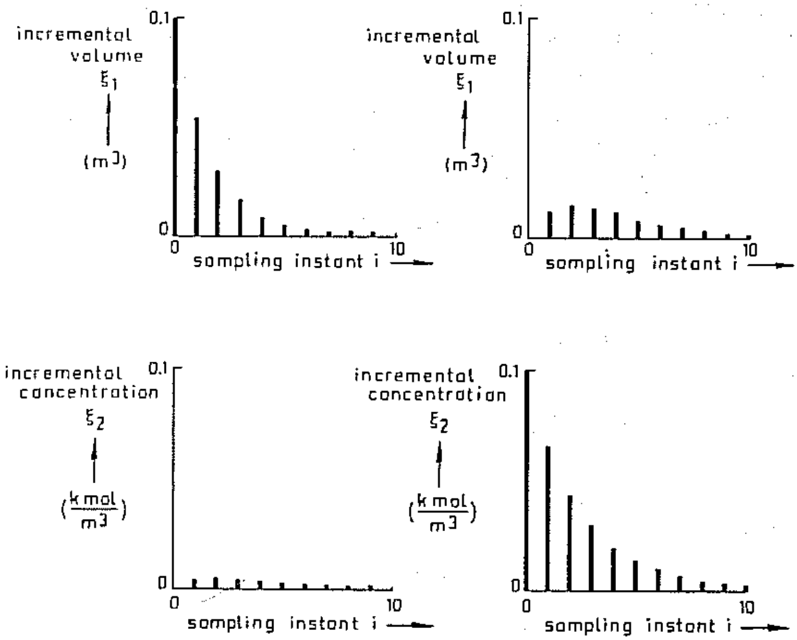

Fig. 6.14. Closed-loop responses of the regulated stirred tank, discrete-time version. Left column: Responses of volume and concentration to the initial conditions $\xi_1(0) = 0.1\ m^3$ and $\xi_2(0) = 0\ kmol/m^3$. Right column: Responses of volume and concentration to the initial conditions $\xi_1(0) = 0\ m^3$ and $\xi_2(0) = 0.1\ kmol/m^3$.

6.4.5 The Stochastic Discrete-Time Linear Optimal Regulator

The stochastic discrete-time linear optimal regulator problem is formulated as follows.

Definition 6.17. *Consider the discrete-time linear system*

$$x(i+1) = A(i)x(i) + B(i)u(i) + w(i),$$
$$x(i_0) = x_0, \qquad \textbf{6-279}$$

where $w(i)$, $i = i_0, i_0 + 1, \cdots, i_1 - 1$, *constitutes a sequence of uncorrelated, zero-mean stochastic variables with variance matrices* $V(i)$, $i = i_0, \cdots, i_1 - 1$. *Let*

$$z(i) = D(i)x(i) \qquad \textbf{6-280}$$

be the controlled variable. Then the problem of minimizing the criterion

$$E\left\{\sum_{i=i_0}^{i_1-1}[z^T(i+1)R_3(i+1)z(i+1) + u^T(i)R_2(i)u(i)] + x^T(i_1)P_1x(i_1)\right\}, \qquad \textbf{6-281}$$

where $R_3(i+1) > 0$, $R_2(i) > 0$ *for* $i = i_0, \cdots, i_1 - 1$ *and* $P_1 \geq 0$, *is termed* ***the stochastic discrete-time linear optimal regulator problem.*** *If all the matrices in the problem formulatian are constant, we refer to it as the* ***time-invariant stochastic discrete-time linear optimal regulator problem.***

As in the continuous-time case, the solution of the stochastic regulator problem is identical to that of the deterministic equivalent (Åström, Koepcke, and Tung, 1962; Tou, 1964; Kushner, 1971).

Theorem 6.33. *The criterion* **6-281** *of the stochastic discrete-time linear optimal regulator problem is minimized by choosing the input according to the control law*

$$u(i) = -F(i)x(i), \qquad i = i_0, i_0 + 1, \cdots, i_1 - 1, \qquad \textbf{6-282}$$

where

$$F(i) = \{R_2(i) + B^T(i)[R_1(i+1) + P(i+1)]B(i)\}^{-1} \cdot B^T(i)[R_1(i+1) + P(i+1)]A(i). \qquad \textbf{6-283}$$

The sequence of matrices $P(i)$, $i = i_0, \cdots, i_1 - 1$, *is the solution of the matrix difference equation*

$$P(i) = A^T(i)[R_1(i+1) + P(i+1)][A(i) - B(i)F(i)],$$
$$i = i_0, i_0 + 1, \cdots, i_1 - 1, \qquad \textbf{6-284}$$

with the terminal condition

$$P(i_1) = P_1. \qquad \textbf{6-285}$$

Here

$$R_1(i) = D^T(i)R_3(i)D(i). \qquad \textbf{6-286}$$

The value of the criterion **6-281** *achieved with this control law is given by*

$$x_0^T P(i_0) x_0 + \sum_{j=i_0+1}^{i_1} \operatorname{tr}\{V(j-1)[P(j) + R_1(j)]\}. \qquad \textbf{6-287}$$

This theorem can be proved by a relatively straightforward extension of the dynamic programming argument of Section 6.4.3. We note that Theorem 6.33 gives the linear control law **6-282** as *the* optimal solution, without further qualification. This is in contrast to the continuous-time case (Theorem 3.9, Section 3.6.3), where we restricted ourself to linear control laws.

As in the continuous-time case, the stochastic regulator problem encompasses regulator problems with disturbances, tracking problems, and tracking problems with disturbances. Here as well, the structure of the solutions of each of these special versions of the problem is such that the feedback gain from the state of the plant is not affected by the properties of the disturbances of the reference variable (see Problems 6.2 and 6.3).

Here too we can investigate in what sense the steady-state control law is optimal. As in the continuous-time case, it can be surmised that, if it exists, the steady-state control law minimizes

$$\lim_{N\to\infty} \frac{1}{N} E\left\{\sum_{i=i_0}^{i_0+N-1} [z^T(i+1)R_3(i+1)z(i+1) + u^T(i)R_2(i)u(i)]\right\} \qquad \textbf{6-288}$$

(assuming that this expression exists for the steady-state optimal control law) with respect to all linear control laws for which this expressions exists. The minimal value of **6-288** is given by

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=i_0+1}^{i_0+N} \operatorname{tr}\{[R_1(j) + \bar{P}(j)]V(j-1)\}, \qquad \textbf{6-289}$$

where $\bar{P}(j), j \geq i_0$, is the steady-state solution of **6-284**. In the time-invariant case, the steady-state control law moreover minimizes

$$\lim_{i_0\to-\infty} E\{z^T(i+1)R_3 z(i+1) + u^T(i)R_2 u(i)\} \qquad \textbf{6-290}$$

with respect to all time-invariant control laws. The minimal value of **6-290** is given by

$$\operatorname{tr}[(R_1 + \bar{P})V]. \qquad \textbf{6-291}$$

Kushner (1971) discusses these facts.

Example 6.16. *Stirred tank with disturbances*

In Example 6.10 (Section 6.2.12), we modeled the stirred tank with disturbances in the incoming concentrations through the stochastic difference equation **6-168**. If we choose for the components of the controlled variable the outgoing flow and the concentration in the tank, we have

$$z(i) = \begin{pmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x(i). \qquad \textbf{6-292}$$

We consider the criterion

$$E\left\{\sum_{i=0}^{N-1} [z^T(i+1)R_3 z(i+1) + u^T(i)R_2 u(i)\right\}, \qquad \textbf{6-293}$$

where the weighting matrices R_3 and R_2 are selected as in Example 6.15. For $\rho = 1$ numerical computation yields the steady-state feedback gain matrix

$$\bar{F} = \begin{pmatrix} 0.07125 & -0.07029 & -0.009772 & -0.003381 \\ 0.01357 & 0.04548 & 0.008671 & 0.003052 \end{pmatrix}. \qquad \textbf{6-294}$$

Comparison with the solution of Example 6.15 shows that, as in the continuous-time case, the feedback link of the control law (represented by the first two columns of $\bar{F}$) is not affected by introducing the disturbances into the model (see Problem 6.2).

The steady-state *rms* values of the outgoing flow, the concentration, and the incoming flows can be computed by setting up the closed-loop system state difference equation and solving for $\bar{Q}$, the steady-state variance matrix of the state of the augmented system.

6.4.6 Linear Discrete-Time Regulators with Nonzero Set Points and Constant Disturbances

In this section we study linear discrete-time regulators with nonzero set points and constant disturbances. We limit ourselves to time-invariant systems and first consider nonzero set point regulators. Suppose that the system

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i), \\ z(i) &= Dx(i), \end{aligned} \qquad \textbf{6-295}$$

must be operated about the set point

$$z(i) = z_0, \qquad \textbf{6-296}$$

where z_0 is a given constant vector. As in the continuous-time case of Section 3.7.1, we introduce the shifted state, input, and controlled variables. Then the steady-state control law that returns the system from any initial condition to the set point optimally, in the sense that a criterion of the form

$$\sum_{i=i_0}^{\infty} [z'^T(i+1)R_3 z'(i+1) + u'^T(i)R_2 u'(i)] \qquad \textbf{6-297}$$

is minimized, is of the form

$$u'(i) = -\bar{F}x'(i), \tag{6-298}$$

where u', x', and z' are the shifted input, state, and controlled variables, respectively, and where $\bar{F}$ is the steady-state feedback gain matrix. In terms of the original system variables, this control law must take tbe form

$$u(i) = -\bar{F}x(i) + u_0', \tag{6-299}$$

where u_0' is a constant vector. With this control law the closed-loop system is described by

$$\begin{aligned} x(i+1) &= \bar{A}x(i) + Bu_0', \\ z(i) &= Dx(i), \end{aligned} \tag{6-300}$$

where

$$\bar{A} = A - B\bar{F}. \tag{6-301}$$

Assuming that the closed-loop system is asymptotically stable, the controlled variable will approach a constant steady-state value

$$\lim_{i\to\infty} z(i) = H_c(1)u_0', \tag{6-302}$$

where $H_c(z)$ is the *closed-loop transfer matrix*

$$H_c(z) = D(zI - \bar{A})^{-1}B. \tag{6-303}$$

The expression **6-302** shows that a zero steady-state error is obtained when u_0' is chosen as

$$u_0' = H_c^{-1}(1)z_0, \tag{6-304}$$

provided the inverse exists, where it is assumed that dim (u) = dim (z). We call the control law

$$u(i) = -\bar{F}x(i) + H_c^{-1}(1)z_r(i) \tag{6-305}$$

the *nonzero set point optimal control law*.

We see that the existence of this control law is determined by the existence of the inverse of $H_c(1)$. Completely analogously to the continuous-time case, it can be shown that

$$\det\,[H_c(z)] = \frac{\psi(z)}{\phi_c(z)}, \tag{6-306}$$

where $\phi_c(z)$ is the closed-loop characteristic polynomial

$$\phi_c(z) = \det\,(zI - A + B\bar{F}), \tag{6-307}$$

and where $\psi(z)$ is the open-loop numerator polynomial; that is, $\psi(z)$ follows from

$$\det\,[H(z)] = \frac{\psi(z)}{\phi(z)}. \tag{6-308}$$

Here

$$H(z) = D(zI - A)^{-1}B \tag{6-309}$$

is the open-loop transfer matrix and

$$\phi(z) = \det(zI - A) \tag{6-310}$$

is the open-loop characteristic polynomial. The relation **6-306** shows that $H_c^{-1}(1)$ exists provided $\psi(1) \neq 0$. Since $H(e^{j\theta})$ describes the frequency response of the open-loop system, this condition is equivalent to requiring that the open-loop frequency response matrix have a numerator polynomial that does not vanish at $\theta = 0$.

We summarize as follows.

Theorem 6.34. *Consider the time-invariant discrete-time linear system*

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i), \\ z(i) &= Dx(i), \end{aligned} \tag{6-311}$$

where $\dim(z) = \dim(u)$. *Consider any asymptotically stable time-invariant control law*

$$u(i) = -Fx(i) + u_0'. \tag{6-312}$$

Let $H(z)$ be the open-loop transfer matrix

$$H(z) = D(zI - A)^{-1}B \tag{6-313}$$

and $H_c(z)$ the closed-loop transfer matrix

$$H_c(z) = D(zI - A + BF)^{-1}B. \tag{6-314}$$

Then $H_c(1)$ is nonsingular and the controlled variable $z(i)$ can under steady-state conditions be maintained at any constant set point z_0 by choosing

$$u_0' = H_c^{-1}(1)z_0 \tag{6-315}$$

if and only if $H(z)$ has a nonzero numerator polynomial that has no zeroes at $z = 1$.

It is noted that this theorem holds not only for the optimal control law, but for any stable control law.

Next we very briefly consider regulators with constant disturbances. We suppose that the plant is described by the state difference and output equations

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i) + v_0, \\ z(i) &= Dx(i), \end{aligned} \tag{6-316}$$

where v_0 is a constant vector. Shifting the state and input variables, we reach the conclusion that the control law that returns the shifted state optimally to zero must be of the form

$$u(i) = -\bar{F}x(i) + u_0', \qquad \textbf{6-317}$$

where u_0' is a suitable constant vector. The steady-state response of the controlled variable with this control law is given by

$$\lim_{i\to\infty} z(i) = H_c(1)u_0' + D(I - \bar{A})^{-1}v_0, \qquad \textbf{6-318}$$

where $H_c(z) = D(zI - A + B\bar{F})^{-1}B$. It is possible to make the steady-state response **6-318** equal to zero by choosing

$$u_0' = -H_c^{-1}(1)D(I - \bar{A})^{-1}v_0. \qquad \textbf{6-319}$$

provided dim (z) = dim (u) and $H_c(1)$ is nonsingular. Thus the *zero-steady-state-error optimal control law* is given by

$$u(i) = -\bar{F}x(i) - H_c^{-1}(1)D(I - \bar{A})^{-1}v_0. \qquad \textbf{6-320}$$

The conditions for the existence of $H_c^{-1}(1)$ are given in Theorem 6.34.

The disadvantage of the control law **6-320** is that its application requires accurate measurement of the constant disturbance v_0. This difficulty can be circumvented by appending to the system an "integral state" q (compare Section 3.7.2), defined by the difference relation

$$q(i + 1) = q(i) + z(i), \qquad i \geq i_0, \qquad \textbf{6-321}$$

with $q(i_0)$ given. Then it can easily be seen that any asymptotically stable control law of the form

$$u(i) = -F_1x(i) - F_2q(i) \qquad \textbf{6-322}$$

suppresses the effect of constant disturbances on the controlled variable, that is, $z(i)$ assumes the value zero in steady-state conditions no matter what the value of v_0 is in **6-316**. Necessary and sufficient conditions for the existence of such an asymptotically stable control law are that the system **6-316** be stabilizable, and [assuming that dim (u) = dim (z)] that the open-loop transfer matrix possess no zeroes at the origin.

Example 6.17. *Digital position control system*

In Example 6.6 (Section 6.2.6), we saw that the digital positioning system of Example 6.2 (Section 6.2.3) has the transfer function

$$H(z) = \frac{0.003396(z + 0.8575)}{(z - 1)(z - 0.6313)}. \qquad \textbf{6-323}$$

Because the numerator polynomial of this transfer function does not have a zero at $z = 1$, a nonzero set point optimal controller can be obtained. In

Example 6.14 (Section 6.4.3), we obtained the steady-state feedback gain vector $\bar{F} = (110.4,\ 12.66)$. It is easily verified that the corresponding nonzero set point optimal control law is given by

$$\mu(i) = -\bar{F}x(i) + 110.4\zeta_0, \qquad \textbf{6-324}$$

where ζ_0 is the (scalar) set point. Figure 6.15 shows the response of the closed-loop system to a step in the set point, not only at the sampling instants but also at intermediate times, obtained by simulation of the

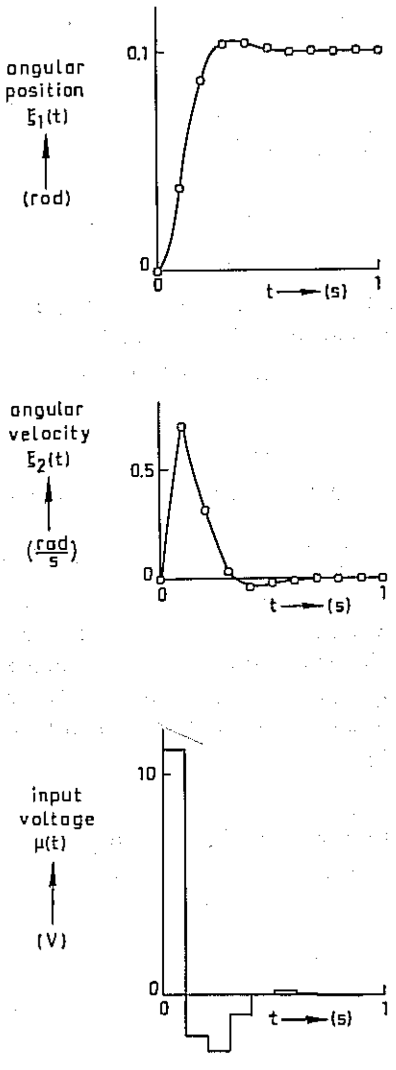

Fig. 6.15. Responses of the digital position control system to a step of 0.1 rad in the set point.

continuous-time system. The system exhibits an excellent response, not quite as fast as the deadbeat response of Fig. 6.12, but with smaller input amplitudes.

6.4.7 Asymptotic Properties of Time-Invariant Optimal Control Laws

In this section we study the asymptotic properties of time-invariant steady-state optimal control laws when in the criterion the weighting matrix R_2 is replaced with

$$R_2 = \rho N, \tag{6-325}$$

where $\rho \downarrow 0$. Let us first consider the behavior of the closed-loop poles. In Theorem 6.32 (Section 6.4.4) we saw that the nonzero closed-loop characteristic values are those roots of the equation

$$\det \begin{pmatrix} zI - A & BR_2^{-1}B^T \\ -R_1 & z^{-1}I - A^T \end{pmatrix} = 0 \tag{6-326}$$

that have moduli less than 1, where $R_1 = D^T R_3 D$. Using Lemmas 1.2 (Section 1.5.4) and 1.1 (Section 1.5.3), we write

$$\begin{aligned}
\det &\begin{pmatrix} zI - A & BR_2^{-1}B^T \\ -R_1 & z^{-1}I - A^T \end{pmatrix} \\
&= \det(zI - A) \det[z^{-1}I - A^T + R_1(zI - A)^{-1}BR_2^{-1}B^T] \\
&= \det(zI - A)\det(z^{-1}I - A^T) \\
&\quad \cdot \det[I + R_1(zI - A)^{-1}BR_2^{-1}B^T(z^{-1}I - A^T)^{-1}] \\
&= \det(zI - A)\det(z^{-1}I - A^T) \\
&\quad \cdot \det[I + R_2^{-1}B^T(z^{-1}I - A^T)^{-1}R_1(zI - A)^{-1}B] \\
&= \det(zI - A)\det(z^{-1}I - A^T) \\
&\quad \cdot \det\left[I + \frac{1}{\rho}N^{-1}B^T(z^{-1}I - A^T)^{-1}D^TR_3D(zI - A)^{-1}B\right] \\
&= \phi(z)\phi(z^{-1}) \det\left[I + \frac{1}{\rho}N^{-1}H^T(z^{-1})R_3H(z)\right],
\end{aligned} \tag{6-327}$$

where

$$\phi(z) = \det(zI - A) \tag{6-328}$$

is the open-loop characteristic polynomial, and

$$H(z) = D(zI - A)^{-1}B \tag{6-329}$$

is the open-loop transfer matrix.

To study the behavior of the closed-loop characteristic values, let us first consider the single-input single-output case. We assume that the scalar transfer function $H(z)$ can be written as

$$H(z) = \frac{\psi(z)}{\phi(z)}, \tag{6-330}$$

where

$$\phi(z) = z^{n-q} \prod_{i=1}^{q} (z - \pi_i), \qquad \pi_i \neq 0, \qquad i = 1, 2, \cdots, q, \tag{6-331}$$

with $q \leq n$, is the characteristic polynomial of the system, and where

$$\psi(z) = \alpha z^{s-p} \prod_{i=1}^{p} (z - \nu_i), \qquad \nu_i \neq 0, \qquad i = 1, 2, \cdots, p, \tag{6-332}$$

with $p \leq s \leq n - 1$, is the numerator polynomial of the system. Then **6-327** takes the form (assuming $R_3 = 1$ and $N = 1$):

$$\prod_{i=1}^{q} (z - \pi_i)\left(\frac{1}{z} - \pi_i\right) + \frac{\alpha^2}{\rho} \prod_{i=0}^{p} (z - \nu_i)\left(\frac{1}{z} - \nu_i\right). \tag{6-333}$$

To apply standard root locus techniques, we bring this expression into the form

$$\prod_{i=1}^{q} (z - \pi_i)\left(z - \frac{1}{\pi_i}\right) + \frac{\alpha^2}{\rho} \frac{\prod_{i=1}^{p} (-\nu_i)}{\prod_{i=1}^{q} (-\pi_i)} z^{q-p} \prod_{i=1}^{p} (z - \nu_i)\left(z - \frac{1}{\nu_i}\right). \tag{6-334}$$

We conclude the following concerning the loci of the $2q$ roots of this expression, where we assume that $q \geq p$ (see Problem 6.4 for the case $q < p$).

1. The $2q$ loci originate for $\rho = \infty$ at π_i and $1/\pi_i$, $i = 1, 2, \cdots, q$.
2. As $\rho \downarrow 0$, the loci behave as follows.
   (a) p roots approach the zeroes ν_i, $i = 1, 2, \cdots, p$;
   (b) p roots approach the inverse zeroes $1/\nu_i$, $i = 1, 2, \cdots, p$;
   (c) $q - p$ roots approach 0;
   (d) the remaining $q - p$ roots approach infinity.
3. Those roots that go to infinity as $\rho \downarrow 0$ asymptotically are at a distance

$$\left| \frac{\alpha^2}{\rho} \frac{\prod_{i=1}^{p} \nu_i}{\prod_{i=1}^{q} \pi_i} \right|^{1/(q-p)} \tag{6-335}$$

from the origin. Consequently, those roots that go to zero are asymptotically at a distance

$$\left| \frac{\rho}{\alpha^2} \frac{\prod_{i=1}^{q} \pi_i}{\prod_{i=1}^{p} \nu_i} \right|^{1/(q-p)} \tag{6-336}$$

from the origin.

Information about the optimal closed-loop poles is obtained by selecting those roots that have moduli less than 1. We conclude the following.

Theorem 6.35. *Consider the steady-state solution of the time-invariant single-input single-output discrete-time linear regulator problem. Let the open-loop transfer function be given by*

$$H(z) = \frac{\alpha z^{s-p} \prod_{i=1}^{p} (z - \nu_i)}{z^{n-q} \prod_{i=1}^{q} (z - \pi_i)}, \qquad \alpha \neq 0, \tag{6-337}$$

where the $\pi_i \neq 0$, $i = 1, 2, \cdots, q$, are the nonzero open-loop characteristic values, and $\nu_i \neq 0$, $i = 1, 2, \cdots, p$, the nonzero zeroes. Suppose that $n \geq q \geq p$, $n - 1 \geq s \geq p$ and that in the criterion **6-233** *we have $R_3 = 1$ and $R_2 = \rho$. Then the following holds.*

(a) *Of the n closed-loop characteristic values $n - q$ are always at the origin.*

(b) *As $\rho \downarrow 0$, of the q remaining closed-loop characteristic values p approach the numbers $\hat{\nu}_i$, $i = 1, 2, \cdots, p$, where*

$$\hat{\nu}_i = \begin{cases} \nu_i & \text{if} \quad |\nu_i| \leq 1, \\ \dfrac{1}{\nu_i} & \text{if} \quad |\nu_i| > 1. \end{cases} \tag{6-338}$$

(c) *As $\rho \downarrow 0$, the $q - p$ other closed-loop characteristic values go to zero. These closed-loop poles asymptotically are at a distance*

$$\left| \frac{\rho}{\alpha^2} \frac{\prod_{i=1}^{q} \pi_i}{\prod_{i=1}^{p} \nu_i} \right|^{1/(q-p)} \tag{6-339}$$

from the origin.

(d) *As $\rho \to \infty$, the q nonzero closed-loop characteristic values approach the numbers $\hat{\pi}_i$, $i = 1, 2, \cdots, q$, where*

$$\hat{\pi}_i = \begin{cases} \pi_i & \textit{if} \quad |\pi_i| \leq 1, \\ \dfrac{1}{\pi_i} & \textit{if} \quad |\pi_i| > 1. \end{cases} \tag{6-340}$$

Let us now consider the behavior of the nonzero set point optimal control law derived in Section 6.4.6. For a single-input single-output system, it is easily seen that the system transfer function from the (scalar) set point $\zeta_0(i)$ (now assumed to be variable) to the controlled variable $\zeta(i)$ is given by

$$T(z) = \frac{H_c(z)}{H_c(1)}, \qquad \textbf{6-341}$$

where $H_c(z)$ is the closed-loop transfer function. As in the continuous-time case (Section 3.8.2), it is easily verified that we can write

$$H_c(z) = \frac{\psi(z)}{\phi_c(z)}, \qquad \textbf{6-342}$$

where $\psi(z)$ is the open-loop transfer function numerator polynomial and $\phi_c(z)$ the closed-loop characteristic polynomial. For $\psi(z)$ we have

$$\psi(z) = \alpha z^{s-p} \prod_{i=1}^{p} (z - \nu_i), \qquad \textbf{6-343}$$

while in the limit $\rho \downarrow 0$ we write for the closed-loop characteristic polynomial

$$\phi_c(z) = z^{n-p} \prod_{i=1}^{p} (z - \hat{\nu}_i). \qquad \textbf{6-344}$$

Substitution into **6-342** and **6-341** shows that in the limit $\rho \downarrow 0$ the control system transfer function can be written as

$$T_0(z) = \frac{1}{z^{n-s}} \prod_{i=1}^{p} \left(\frac{z - \nu_i}{z - \hat{\nu}_i} \right) \prod_{k=1}^{p} \left(\frac{1 - \hat{\nu}_k}{1 - \nu_k} \right). \qquad \textbf{6-345}$$

Now if the open-loop transfer function has no zeroes outside the unit circle. the limiting control system transfer function reduces to

$$T_0(z) = \frac{1}{z^{n-s}}. \qquad \textbf{6-346}$$

This represents a pure delay, that is, the controlled variable and the variable set point are related as follows:

$$\zeta(i) = \zeta_0[i - (n - s)]. \qquad \textbf{6-347}$$

We summarize as follows.

Theorem 6.36. *Consider the nonzero set point optimal control law, as described in Section 6.4.6., for a single-input single-output system. Let $R_3 = 1$ and $R_2 = \rho$. Then as $\rho \downarrow 0$, the control system transmission (that is, the transfer function of the closed-loop system from the set point to the controlled* *variable) approaches*

$$T_0(z) = \frac{1}{z^{n-s}} \prod_{i=1}^{p} \left(\frac{z - \nu_i}{z - \hat{\nu}_i}\right) \prod_{k=1}^{p} \left(\frac{1 - \hat{\nu}_k}{1 - \nu_k}\right), \qquad \textbf{6-348}$$

where the $\hat{\nu}_i$, $i = 1, 2, \cdots, p$ *are derived from the nonzero open-loop zeroes* ν_i, $i = 1, 2, \cdots, p$, *as indicated in* **6-338**, *and where* n *is the dimension of the system and* s *the degree of the numerator polynomial of the system. If the open-loop transfer function has no zeroes outside the unit circle, the limiting system transfer function is*

$$T_0(z) = \frac{1}{z^{n-s}}, \qquad \textbf{6-349}$$

which represents a pure delay.

We see that, if the open-loop system has no zeroes outside the unit circle, the limiting closed-loop system has the property that the response of the controlled variable to a step in the set point achieves a zero tracking error after $n - s$ time intervals. We refer to this as *output deodbeat response.*

We now discuss the asymptotic behavior of the closed-loop characteristic values for multiinput systems. Referring back to **6-327**, we consider the roots of

$$\phi(z)\phi(z^{-1}) \det \left[I + \frac{1}{\rho} N^{-1}H^T(z^{-1})R_3H(z)\right]. \qquad \textbf{6-350}$$

Apparently, for $\rho = \infty$ those roots of this expression that are finite are the roots of

$$\phi(z)\phi(z^{-1}). \qquad \textbf{6-351}$$

Let us write

$$\phi(z) = z^{n-q} \prod_{i=1}^{q} (z - \pi_i), \qquad \textbf{6-352}$$

and assume that $\pi_i \neq 0$, $i = 1, 2, \cdots, q$. Then we have

$$\phi(z)\phi(z^{-1}) = \prod_{i=1}^{q} (z - \pi_i)(z^{-1} - \pi_i), \qquad \textbf{6-353}$$

which shows that $2q$ root loci of **6-350** originate for $\rho = \infty$ at the nonzero characteristic values of the open-loop system and their inverses.

Let us now consider the roots of **6-350** as $\rho \downarrow 0$. Clearly, those roots that stay finite approach the zeroes of

$$\phi(z)\phi(z^{-1}) \det [H^T(z^{-1})R_3H(z)]. \qquad \textbf{6-354}$$

Let us now assume that the input and the controlled variable have the same dimensions, so that $H(z)$ is a square transfer matrix, with

$$\det [H(z)] = \frac{\psi(z)}{\phi(z)}. \tag{6-355}$$

Then the zeroes of **6-354** are the zeroes of

$$\psi(z^{-1})\psi(z). \tag{6-356}$$

Let us write the numerator polynomial $\psi(z)$ in the form

$$\psi(z) = \alpha z^{s-p} \prod_{i=1}^{p} (z - \nu_i), \tag{6-357}$$

where $\nu_i \neq 0$, $i = 1, 2, \cdots, p$. Then **6-356** can be written as

$$\alpha^2 \prod_{i=1}^{p} (z - \nu_i)(z^{-1} - \nu_i). \tag{6-358}$$

This shows that $2p$ root loci of **6-350** terminate for $\rho = 0$ at the nonzero zeroes ν_i, $i = 1, 2, \cdots, p$, and the inverse zeroes $1/\nu_i$, $1 = 1, 2, \cdots, p$.

Let us suppose that $q \geq p$ (for the case $q < p$, see Problem 6.4). Then there are $2q$ root loci of **6-350**, which originate for $\rho = \infty$ at the nonzero open-loop poles and their inverses. As we have seen, $2p$ loci terminate for $\rho = 0$ at the nonzero open-loop zeroes and their inverses. Of the remaining $2q - 2p$ loci, $q - p$ must go to infinity as $\rho \downarrow 0$, while the other $q - p$ loci approach the origin.

The nonzero closed-loop poles are those roots of **6-350** that lie inside the unit circle. We conclude the following.

Theorem 6.37. *Consider the steady-state solution of the time-invariant regulator problem. Suppose that* dim (u) = dim (z) *and let* $H(z)$ *be the open-loop transfer matrix*

$$H(z) = D(zI - A)^{-1}B. \tag{6-359}$$

Furthermore, let

$$\det [H(z)] = \frac{\psi(z)}{\phi(z)}, \tag{6-360}$$

where

$$\phi(z) = z^{n-q} \prod_{i=1}^{q} (z - \pi_i), \tag{6-361}$$

with $\pi_i \neq 0$, $i = 1, 2, \cdots, q$, *is the open-loop characteristic polynamial. In additian, suppose that*

$$\psi(z) = \alpha z^{s-p} \prod_{i=1}^{p} (z - \nu_i), \tag{6-362}$$

with $p \leq q$, *and where* $\nu_i \neq 0$, $i = 1, 2, \cdots, p$. *Finally, set* $R_2 = \rho N$ *where* $N > 0$ *and* ρ *is a positive scalar. Then we have the following.*

(a) *Of the* n *closed-loop poles,* $n - q$ *always are at the origin.*

(b) *As* $\rho \downarrow 0$, *of the remaining* q *closed-loop poles,* p *approach the numbers* $\hat{\nu}_i$, $i = 1, 2, \cdots, p$, *where*

$$\hat{\nu}_i = \begin{cases} \nu_i & \text{if } |\nu_i| \leq 1, \\ \dfrac{1}{\nu_i} & \text{if } |\nu_i| > 1. \end{cases} \tag{6-363}$$

(c) *As* $\rho \downarrow 0$, *the* $q - p$ *other closed-loop poles go to zero.*

(d) *As* $\rho \to \infty$, *the* q *nonzero closed-loop poles approach the numbers* $\hat{\pi}_i$, $i = 1, 2, \cdots, q$, *where*

$$\hat{\pi}_i = \begin{cases} \pi_i & \text{if } |\pi_i| \leq 1, \\ \dfrac{1}{\pi_i} & \text{if } |\pi_i| > 1. \end{cases} \tag{6-364}$$

We note that contrary to the continuous-time case the closed-loop poles remain finite as the weighting matrix R_2 approaches the zero matrix. Similarly, the feedback gain matrix F also remains finite. Often, but not always, the limiting feedback gain matrix can be found by setting $R_2 = 0$ in the difference equations **6-246** and **6-248** and iterating until the steady-state value is found (see the examples, and also Pearson, 1965; Rappaport and Silverman, 1971).

For the response of the closed-loop system with this limiting feedback law, the following is to be expected. As we have seen, the limiting closed-loop system asymptotically has $n - p$ characteristic values at the origin. If the open-loop zeroes are all inside the unit circle, they cancel the corresponding limiting closed-loop poles. This means that the response is determined by the $n - p$ poles at the origin, resulting in a deadbeat response of the controlled variable after $n - p$ steps. We call this an output deadbeat response, in contrast to the state deadbeat response discussed in Section 6.4.2. If a system exhibits an output deadheat response, the output reaches the desired value exactly after a finite number of steps, but the system as a whole may remain in motion for quite a long time, as one of the examples at the end of this section illustrates. If the open-loop system has zeroes outside the unit circle, the cancellation effect does not occur and as a result the limiting regulator does not exhibit a deadbeat response.

It is noted that these remarks are conjectures, based on analogy with the continuous-time case. A complete theory is missing as yet. The examples at the end of the section confirm the conjectures. An essential difference between the discrete-time theory and the continuous-time theory is that in the discrete-time case the steady-state solution $\bar{P}$ of the matrix equation **6-248** generally does not approach the zero matrix as R_2 goes to zero, even if the open-loop transfer matrix possesses no zeroes outside the unit circle.

Example 6.18. *Digital position control system*

Let us consider the digital positioning system of Example 6.2 (Section 6.2.3). From Example 6.6 (Section 6.2.6), we know that the open-loop transfer function is

$$H(z) = \frac{0.003396(z + 0.8575)}{(z - 1)(z - 0.6313)}. \qquad \textbf{6-365}$$

It follows from Theorem 6.37 that the optimal closed-loop poles approach 0 and -0.8575 as $\rho \downarrow 0$. It is not difficult to find the loci of the closed-loop characteristic values. Expression **6-334** takes for this system the form

$$(z - 1)(z - 0.6313)(z - 1)(z - 1.584) + \frac{0.00001566}{\rho} z(z + 0.8575)(z + 1.166). \qquad \textbf{6-366}$$

The loci of the roots of this expression are sketched in Fig. 6.16. Those loci that lie inside the unit circle are the loci of the closed-loop poles. It can be

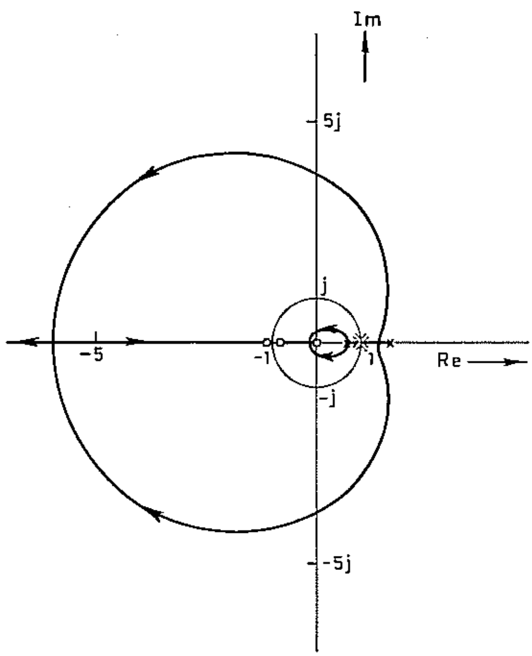

Fig. 6.16. Loci of the closed-loop poles and the inverse closed-loop poles for the digital position control system.

found that the limiting feedback gain matrix $\bar{F}_0$ for $\rho = 0$ is given by

$$\bar{F}_0 = (294.5, \quad 23.60). \tag{6-367}$$

Let us determine the corresponding nonzero set point optimal control law. We have for the limiting closed-loop transfer function

$$H_c(z) = \frac{\psi(z)}{\phi_c(z)} = \frac{0.003396(z + 0.8575)}{z(z + 0.8575)} = \frac{0.003396}{z}. \tag{6-368}$$

Consequently, $H_c(1) = 0.003396$ and the nonzero set point optimal control law is

$$\mu(i) = -\bar{F}_0 x(i) + 294.5\zeta_0(i). \tag{6-369}$$

Figure 6.17 gives the response of the system to a step in the set point, not only at the sampling instants but also at intermediate times. Comparing with the state deadbeat response of the same system as derived in Example 6.13, we observe the following.

(a) When considering only the response of the angular position at the sampling instants, the system shows an output deadbeat response after one sampling interval. In between the response exhibits a bad overshoot, however, and the actual settling time is in the order of 2 s, rather than 0.1 s.

(b) The input amplitude and the angular velocity assume large values.

These disadvantages are characteristic for output deadbeat control systems. Better results are achieved by not letting ρ go to zero. For $\rho = 0.00002$ the closed-loop poles are at 0.2288 ± 0.3184. The step response of the corresponding closed-loop system is given in Example 6.17 (Fig. 6.15) and is obviously much better than that of Fig. 6.17.

The disadvantages of the output deadbeat response are less pronounced when a larger sampling interval Δ is chosen. This causes the open-loop zero at -0.8575 to move closer to the origin; as a result the output deadbeat control system as a whole comes to rest much faster. For an alternative solution, which explicitly takes into account the behavior of the system between the sampling instants, see Problem 6.5.

Example 6.19. *Stirred tank with time delay*

Consider the stirred tank with time delay of Example 6.4 (Section 6.2.3). As the components of the controlled variable we choose the outgoing flow and concentration; hence

$$z(i) = \begin{pmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x(i). \tag{6-370}$$

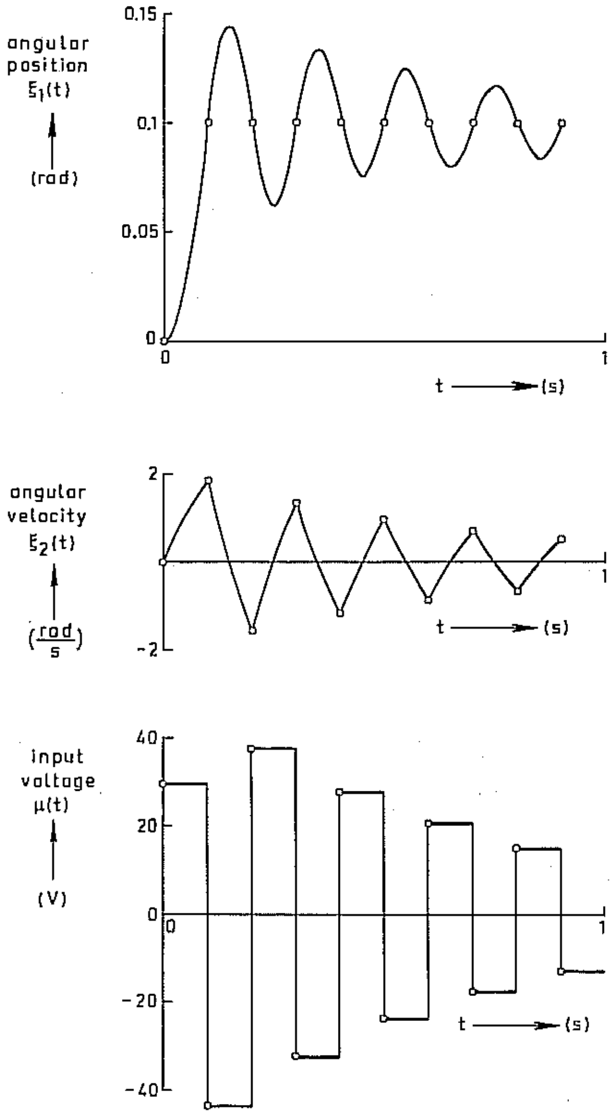

Fig. 6.17. Response of the output deadbeat digital position control system to a step in the set point of 0.1 rad.

It can be found that the open-loop transfer matrix of the system is

$$H(z) = \begin{pmatrix} \dfrac{4.877}{z - 0.9512} & \dfrac{4.877}{z - 0.9512} \\ \dfrac{-1.1895}{z(z - 0.9048)} & \dfrac{3.569}{z(z - 0.9048)} \end{pmatrix}. \qquad \text{6-371}$$

The determinant of the transfer matrix is

$$\det [H(z)] = \frac{26.62}{z(z - 0.9512)(z - 0.9048)}. \qquad \text{6-372}$$

Because the open-loop characteristic polynomial is given by

$$\phi(z) = z^2(z - 0.9512)(z - 0.9048), \qquad \text{6-373}$$

the numerator polynomial of the transfer matrix is

$$\psi(z) = 26.62z. \qquad \text{6-374}$$

As a result, two closed-loop poles are always at the origin. The loci of the two other poles originate for $\rho = \infty$ at 0.9512 and 0.9048, respectively, and both approach the origin as $\rho \downarrow 0$. This means that in this case the output deadbeat control law is also a state deadbeat control law.

Let us consider the criterion

$$\sum_{i=0}^{\infty} [z^T(i + 1)R_3 z(i + 1) + u^T(i)R_2 u(i)], \qquad \text{6-375}$$

where, as in previous examples,

$$R_3 = \begin{pmatrix} 50 & 0 \\ 0 & 0.02 \end{pmatrix} \quad \text{and} \quad R_2 = \rho \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{pmatrix}. \qquad \text{6-376}$$

When one attempts to compute the limiting feedback law for $\rho = 0$ by setting $R_2 = 0$ in the difference equation for $P(i)$ and $F(i)$, difficulties occur because for certain choices of P_1 the matrix

$$R_2 + B^T[R_1 + P(i + 1)]B \qquad \text{6-377}$$

becomes singular at the first iteration. This can be avoided by choosing a very small value for ρ (e.g., $\rho = 10^{-6}$). By using this technique numerical computation yields the limiting feedback gain matrix

$$\bar{F}_0 = \begin{pmatrix} 0.1463 & -0.1720 & 0.2262 & -0.6786 \\ 0.04875 & 0.1720 & -0.2262 & 0.6786 \end{pmatrix}. \qquad \text{6-378}$$

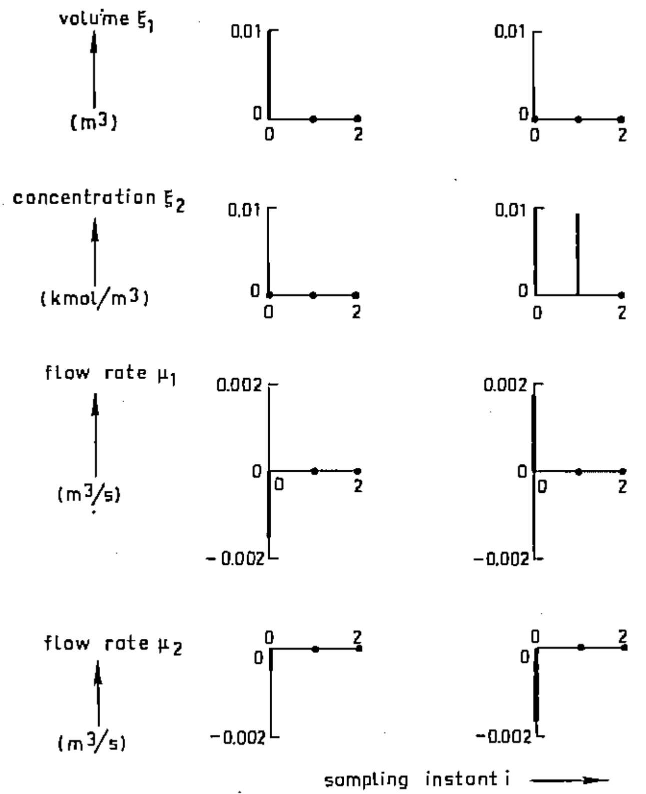

Fig. 6.18. Deadbeat response of the stirred tank with time delay. Left column: Responses of volume, concentration, feed no. 1, and feed no. 2 to the initial condition $\xi_1(0) = 0.01$ m^3, while all other components of the initial state are zero. Right column: Responses of volume, concentration, feed no. 1, and feed no. 2 to the initial condition $\xi_2(0) = 0.01$ kmol/m^3, while all other components of the initial state are zero.

In Fig. 6.18 the deadbeat response to two initial conditions is sketched. It is observed that initial errors in the volume ξ_1 are reduced to zero in one sampling period. For the concentration ξ_2 two sampling periods are required; this is because of the inherent delay in the system.

6.4.8 Sensitivity

In Section 3.9 we saw that the continuous-time time-invariant closed-loop regulator possesses the property that it always decreases the effect of disturbances and parameter variations as compared to the open-loop system. It is shown in this section by a counter example that this is not generally the case for discrete-time systems. The same example shows, however, that protection over a wide range of frequencies can still be obtained.

Example 6.20. *Digital angular velocity control*

Consider the angular velocity control system of Example 3.3 (Section 3.3.1), which is described by the scalar state differential equation

$$\dot{\xi}(t) = -\alpha\xi(t) + \kappa\mu(t). \qquad \textbf{6-379}$$

Let us assume that the input is piecewise constant over intervals of duration Δ. Then the resulting discrete-time system is described by

$$\xi(i+1) = e^{-\alpha\Delta}\xi(i) + \frac{\kappa}{\alpha}(1 - e^{-\alpha\Delta})\mu(i), \qquad \textbf{6-380}$$

where we have replaced $\xi(i\Delta)$ with $\xi(i)$ and $\mu(i\Delta)$ with $\mu(i)$. With the numerical values $\alpha = 0.5\ \mathrm{s}^{-1}$, $\kappa = 150\ \mathrm{rad}/(\mathrm{V\ s}^2)$, and $\Delta = 0.1$ s, we obtain

$$\xi(i+1) = 0.9512\xi(i) + 14.64\mu(i). \qquad \textbf{6-381}$$

The controlled variable $\zeta(i)$ is the angular velocity $\xi(i)$, that is,

$$\zeta(i) = \xi(i). \qquad \textbf{6-382}$$

Let us consider the problem of minimizing

$$\sum_{i=0}^{\infty} [\zeta^2(i+1) + \rho\mu^2(i)]. \qquad \textbf{6-383}$$

It is easily found that with $\rho = 1000$ the steady-state solution is given by

$$\bar{P} = 1.456,$$
$$\bar{F} = 0.02240. \qquad \textbf{6-384}$$

The return difference of the closed-loop system is

$$J(z) = I + (zI - A)^{-1}B\bar{F}, \qquad \textbf{6-385}$$

which can be found to be

$$J(z) = \frac{z - 0.6232}{z - 0.9512}. \qquad \textbf{6-386}$$

To determine the behavior of $J(z)$ for z on the unit circle, set

$$z = e^{j\omega\Delta},$$

where $\Delta = 0.1$ s is the sampling interval. With this we find

$$|J(e^{j\omega\Delta})|^2 = \frac{1.388 - 1.246\cos(\omega\Delta)}{1.905 - 1.902\cos(\omega\Delta)}. \qquad \textbf{6-387}$$

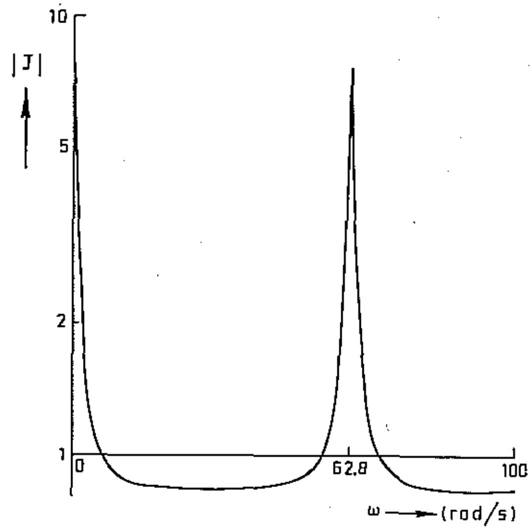

Fig. 6.19. Behavior of the return difference for a first-order discrete-time regulator.

Figure 6.19 gives a plot of the behavior of $|J(e^{j\omega\Delta})|$. We see that sensitivity reduction is achieved for low frequencies up to about 7 rad/s, but by no means for all frequencies. If the significant disturbances occur within the frequency band up to 7 rad/s, however, the sensitivity reduction may very well be adequate.

6.5 OPTIMAL LINEAR RECONSTRUCTION OF THE STATE OF LINEAR DISCRETE-TIME SYSTEMS

6.5.1 Introduction

This section is devoted to a review of the optimal reconstruction of the state of linear discrete-time systems. The section parallels Chapter 4.

6.5.2 The Formulation of Linear Discrete-Time Reconstruction Problems

In this section we discuss the formulation of linear discrete-time reconstruction problems. We pay special attention to this question since there are certain differences from the continuous-time case. As before, we take the point of view that the linear discrete-time system under consideration is obtained by operating a linear continuous-time system with a piecewise constant input, as indicated in Fig. 6.20. The instants at which the input changes value are given by t_i, $i = 0, 1, 2, \cdots$, which we call the *control*

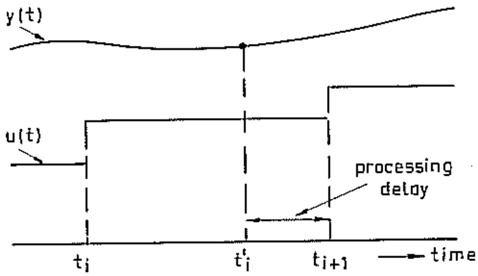

Fig. 6.20. Relationship of control actuation instant t_i and observation instant t'_i.

instants. These instants form the basic time grid. We furthermore introduce the *observation instants* t'_i, $i = 0, 1, 2, \cdots$, which are the instants at which the observed variable $y(t)$ of the continuous-time system is sampled. It is assumed that the observation instant t'_i always *precedes* the control instant t_{i+1}. The difference $t_{i+1} - t'_i$ will be called the *processing delay*; in the case of a control system, it is the time that is available to process the observation $y(t'_i)$ in order to determine the input $u(t_{i+1})$.

Suppose that the continuous-time system is described by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w_1(t), \qquad t \geq t_0, \qquad \textbf{6-388}$$

where w_1 is white noise with time-varying intensity $V_1(t)$. We furthermore assume that the observed variable is given by

$$y(t'_i) = C(t'_i)x(t'_i) + w_2(t'_i), \qquad i = 0, 1, 2, \cdots, \qquad \textbf{6-389}$$

where the $w_2(t'_i)$, $i = 0, 1, 2, \cdots$, form a sequence of uncorrelated stochastic vectors. To obtain the discrete-time description of the system, we write

$$x(t_{i+1}) = \Phi(t_{i+1}, t_i)x(t_i) + \left[\int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, \tau)B(\tau)\, d\tau\right]u(t_i) + \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, \tau)w_1(\tau)\, d\tau, \qquad \textbf{6-390}$$

and

$$y(t'_i) = C(t'_i)\Phi(t'_i, t_i)x(t_i) + \left[C(t'_i)\int_{t_i}^{t_i'} \Phi(t'_i, \tau)B(\tau)\, d\tau\right]u(t_i) + C(t'_i)\int_{t_i}^{t_i'} \Phi(t'_i, \tau)w_1(\tau)\, d\tau + w_2(t'_i), \qquad \textbf{6-391}$$

where in both cases $i = 0, 1, 2, \cdots$, and where $\Phi(t, t_0)$ is the transition matrix of the system **6-388**. We see that the two equations **6-390** and **6-391** are of the form

$$x^+(i+1) = A_d(i)x^+(i) + B_d(i)u^+(i) + w_1^+(i),$$
$$y^+(i) = C_d(i)x^+(i) + E_d(i)u^+(i) + w_2^+(i). \qquad \textbf{6-392}$$

This method of setting up the discrete-time version of the problem has the following characteristics.

1. In the discrete-time version of the reconstruction problem, we assume that $y^+(i)$ is the latest observation that can be processed to obtain a reconstructed value for $x^+(i+1)$.

2. The output equation generally contains a *direct link*. As can be seen from **6-391**, the direct link is absent [i.e., $E_d(i) = 0$] when the processing delay takes up the whole interval (t_i, t_{i+1}).

3. Even if in the continuous-time problem the state excitation noise w_1 and the observation noise w_2 are uncorrelated, the state excitation noise w_1^+ and the observation noise w_2^+ of the discrete-time version of the problem will be *correlated*, because, as can be seen from **6-390**, **6-391**, and **6-392**, both $w_1^+(i)$ and $w_2^+(i)$ depend upon $w_1(t)$ for $t_i \le t \le t_i'$. Clearly, $w_1^+(i)$ and $w_2^+(i)$ are uncorrelated only if $t_i' = t_i$, that is, if the processing delay takes up the whole interval (t_i, t_{i+1}).

Example 6.21. *The digital positioning system*

Let us consider the digital positioning system of Example 6.2 (Section 6.2.3). It has been assumed that the sampling period is Δ. We now assume that the observed variable is the angular displacement ξ_1, so that in the continuous-time version

$$C = (1, 0). \qquad \textbf{6-393}$$

We moreover assume that there is a processing delay Δ_d, so that the observations are taken at an interval Δ_d before the instants at which control actuation takes place. Disregarding the noises that are possibly present, it is easily found with the use of **6-391** that the observation equation takes the form

$$\eta^+(i) = \left[1, \frac{1}{\alpha}(1 - e^{-\alpha\Delta'})\right]x^+(i) + \frac{\kappa}{\alpha}\left(\Delta' - \frac{1}{\alpha} + \frac{1}{\alpha}e^{-\alpha\Delta'}\right)\mu^+(i), \qquad \textbf{6-394}$$

where

$$\Delta' = \Delta - \Delta_d. \qquad \textbf{6-395}$$

With the numerical value

$$\Delta_d = 0.02 \text{ s}, \qquad \textbf{6-396}$$

we obtain for the observation equation

$$\eta^+(i) = (1, 0.06608)x^+(i) + 0.002381\mu^+(i). \qquad \textbf{6-397}$$

6.5.3 Discrete-Time Observers

In this section we consider dynamical systems that are able to reconstruct the state of another system that is being observed.

Definition 6.18. *The system*

$$\hat{x}(i+1) = \hat{A}(i)\hat{x}(i) + \hat{B}(i)u(i) + \hat{C}(i)y(i) \qquad \text{6-398}$$

is a ***full-order observer*** *for the system*

$$\begin{aligned} x(i+1) &= A(i)x(i) + B(i)u(i), \\ y(i) &= C(i)x(i) + E(i)u(i), \end{aligned} \qquad \text{6-399}$$

if

$$\hat{x}(i_0) = x(i_0) \qquad \text{6-400}$$

implies

$$\hat{x}(i) = x(i), \qquad i \geq i_0, \qquad \text{6-401}$$

for all $u(i)$, $i \geq i_0$.

It is noted that consistent with the reasoning of Section 6.5.2 the latest observation that the observer processes for obtaining $x(i+1)$ is $y(i)$. The following theorem gives more information about the structure of an observer.

Theorem 6.38. *The system* **6-398** *is a full order observer for the system* **6-399** *if and only if*

$$\begin{aligned} \hat{A}(i) &= A(i) - K(i)C(i), \\ \hat{B}(i) &= B(i) - K(i)E(i), \\ \hat{C}(i) &= K(i), \end{aligned} \qquad \text{6-402}$$

all for $i \geq i_0$, *where* $K(i)$ *is an arbitrary time-varying matrix.*

This theorem is easily proved by subtracting the state difference equations **6-399** and **6-398**. With **6-402** the observer can be represented as follows:

$$\hat{x}(i+1) = A(i)\hat{x}(i) + B(i)u(i) + K(i)[y(i) - C(i)\hat{x}(i) - E(i)u(i)]. \qquad \text{6-403}$$

The observer consists of a model of the system, with as extra driving variable an input which is proportional to the difference $y(i) - \hat{y}(i)$ of the observed variable $y(i)$ and its predicted value

$$\hat{y}(i) = C(i)\hat{x}(i) + E(i)u(i). \qquad \text{6-404}$$

We now discuss the stability of the observer and the behavior of the reconstruction error $e(i) = x(i) - \hat{x}(i)$.

Theorem 6.39. *Consider the observer* **6-398** *for the system* **6-399**. *Then the reconstruction error*

$$e(i) = x(i) - \hat{x}(i) \qquad \textbf{6-405}$$

satisfies the difference equation

$$e(i + 1) = [A(i) - K(i)C(i)]e(i), \qquad i \geq i_0. \qquad \textbf{6-406}$$

The reconstruction error has the property that

$$e(i) \to 0, \qquad as \qquad i \to \infty, \qquad \textbf{6-407}$$

for all $e(i_0)$, *if and only if the observer is asymptotically stable.*

The difference equation **6-406** is easily found by subtracting the state difference equations in **6-399** and **6-398**. The behavior of $A(i) - K(i)C(i)$ determines both the stability of the observer and the behavior of the reconstruction error; hence the second part of the theorem.

As in the continuous-time case, we now consider the question: When does there exist a gain matrix K that stabilizes the observer and thus ensures that the reconstruction error will always eventually approach zero? Limiting ourselves to time-invariant systems, we have the following result.

Theorem 6.40. *Consider the time-invariant observer*

$$\hat{x}(i + 1) = A\hat{x}(i) + Bu(i) + K[y(i) - C\hat{x}(i) - Eu(i)] \qquad \textbf{6-408}$$

for the time-invariant system

$$\begin{aligned} x(i + 1) &= Ax(i) + Bu(i), \\ y(i) &= Cx(i) + Eu(i). \end{aligned} \qquad \textbf{6-409}$$

Then the observer poles (that is, the characteristic values of $A - KC$*) can be arbitrarily located in the complex plane (within the restriction that complex poles occur in complex conjugate pairs) by suitably choosing the gain matrix* K *if and only if the system* **6-409** *is completely reconstructible.*

The proof of this theorem immediately follows from the continuous-time equivalent (Theorem 4.3, Section 4.2.2). For systems that are only detectable, we have the following result.

Theorem 6.41. *Consider the time-invariant observer* **6-408** *for the time-invariant system* **6-409**. *Then a gain matrix* K *can be found such that the observer is asymptotically stable if and only if the system* **6-409** *is detectable.*

A case of special interest occurs when the observer poles are all located at the origin, that is, all the characteristic values of $A - KC$ are zero. Then the characteristic polynomial of $A - KC$ is given by

$$\det [\lambda I - (A - KC)] = \lambda^n, \tag{6-410}$$

so that by the Cayley–Hamilton theorem

$$(A - KC)^n = 0. \tag{6-411}$$

It follows by repeated application of the difference equation **6-406** for the reconstruction error that now

$$e(n) = (A - KC)^n e(0) = 0 \tag{6-412}$$

for every $e(0)$, which means that every initial value of the reconstruction error is reduced to zero in at most n steps. In analogy with deadbeat control laws, we refer to observers with this property as *deadbeat observers*. Such observers produce a completely accurate reconstruction of the state after at most n steps.

Finally, we point out that if the system **6-409** has a scalar observed variable y, a unique solution of the gain matrix K is obtained for a given set of observer poles. In the case of multioutput systems, however, in general many different gain matrices exist that result in the same set of observer poles.

The observers considered so far in this section are systems of the same dimension as the system to be observed. Because of the output equation $y(i) = C(i)x(i) + E(i)u(i)$, we have available m equations in the unknown state $x(i)$ (assuming that y has dimension m); clearly, it must be possible to construct a reduced-order observer of dimension $n - m$ to reconstruct $x(i)$ completely. This observer can be constructed more or less analogously to the continuous-time case (Section 4.2.3).

Example 6.22. *Digital positioning system*

Consider the digital positioning system of Example 6.2 (Section 6.2.3), which is described by the state difference equation

$$x(i + 1) = \begin{pmatrix} 1 & 0.08015 \\ 0 & 0.6313 \end{pmatrix} x(i) + \begin{pmatrix} 0.003396 \\ 0.06308 \end{pmatrix} \mu(i). \tag{6-413}$$

As in Example 6.21, we assume that the observed variable is the angular position but that there is a processing delay of 0.02 s. This yields for the observed variable:

$$\eta(i) = (1, 0.06608)x(i) + 0.002381\mu(i). \tag{6-414}$$

It is easily verified that the system is completely reconstructible so that

Theorem 6.40 applies. Let us write $K = \text{col}(k_1, k_2)$. Then we find

$$A - KC = \begin{pmatrix} 1 - k_1 & 0.08015 - 0.06608k_1 \\ -k_2 & 0.6313 - 0.06608k_2 \end{pmatrix}. \qquad \textbf{6-415}$$

This matrix has the characteristic polynomial

$$z^2 + (-1.6313 + k_1 + 0.06608k_2)z + (0.6313 - 0.6313k_1 + 0.01407k_2). \qquad \textbf{6-416}$$

We obtain a deadbeat observer by setting

$$\begin{aligned} -1.6313 + k_1 + 0.06608k_2 &= 0, \\ 0.6313 - 0.6313k_1 + 0.01407k_2 &= 0. \end{aligned} \qquad \textbf{6-417}$$

This results in the gain matrix

$$K = \begin{pmatrix} 1.159 \\ 7.143 \end{pmatrix}. \qquad \textbf{6-418}$$

An observer with this gain reduces any initial reconstruction error to zero in at most two steps.

6.5.4 Optimal Discrete-Time Linear Observers

In this section we study discrete-time observers that are *optimal* in a well-defined sense. To this end we assume that the system under consideration is affected by disturbances and that the observations are contaminated by observation noise. We then find observers such that the reconstructed state is optimal in the sense that the mean square reconstruction error is minimized. We formulate our problem as follows.

Definition 6.19. *Consider the system*

$$\begin{aligned} x(i + 1) &= A(i)x(i) + B(i)u(i) + w_1(i), \\ y(i) &= C(i)x(i) + E(i)u(i) + w_2(i), \qquad i \geq i_0. \end{aligned} \qquad \textbf{6-419}$$

Here col $[w_1(i), w_2(i)]$, $i \geq i_0$, *forms a sequence of zero-mean, uncorrelated vector stochastic variables with variance matrices*

$$\begin{pmatrix} V_1(i) & V_{12}(i) \\ V_{12}^T(i) & V_2(i) \end{pmatrix}, \qquad i \geq i_0. \qquad \textbf{6-420}$$

Furthermore, $x(i_0)$ *is a vector stochastic variable, uncorrelated with* w_1 *and* w_2, *with*

$$E\{x(i_0)\} = \bar{x}_0, \qquad E\{[x(i_0) - \bar{x}_0][x(i_0) - \bar{x}_0]^T\} = Q_0. \qquad \textbf{6-421}$$

Consider the observer

$$\hat{x}(i+1) = A(i)\hat{x}(i) + B(i)u(i) + K(i)[y(i) - C(i)\hat{x}(i) - E(i)u(i)] \tag{6-422}$$

for this system. Then the problem of finding the sequence of matrices $K^0(i_0)$, $K^0(i_0+1), \cdots, K^0(i-1)$, *and the initial condition* $\hat{x}(i_0)$, *so as to minimize*

$$E\{e^T(i)W(i)e(i)\}, \tag{6-423}$$

where $e(i) = x(i) - \hat{x}(i)$, *and where* $W(i)$ *is a positive-definite symmetric weighting matrix, is termed the* ***discrete-time optimal observer problem.*** *If*

$$V_2(i) > 0, \qquad i \geq i_0,$$

the optimal observer problem is called ***nonsingular.***

To solve the discrete-time optimal observer problem, we first establish the difference equation that is satisfied by the reconstruction error $e(i)$. Subtraction of the system state difference equation **6-419** and the observer equation **6-422** yields

$$e(i+1) = [A(i) - K(i)C(i)]e(i) + w_1(i) - K(i)w_2(i), \qquad i \geq i_0. \tag{6-424}$$

Let us now denote by $\tilde{Q}(i)$ the variance matrix of $e(i)$, and by $\bar{e}(i)$ the mean of $e(i)$. Then we write

$$E\{e(i)e^T(i)\} = \tilde{Q}(i) + \bar{e}(i)\bar{e}^T(i), \tag{6-425}$$

so that

$$E\{e^T(i)W(i)e(i)\} = \bar{e}^T(i)W(i)\bar{e}(i) + \text{tr}\,[\tilde{Q}(i)W(i)]. \tag{6-426}$$

The first term of this expression is obviously minimized by making $\bar{e}(i) = 0$. This can be achieved by letting $\bar{e}(i_0) = 0$, which in turn is done by choosing

$$\hat{x}(i_0) = \bar{x}_0. \tag{6-427}$$

The second term in **6-425** can be minimized independently of the first term. With the aid of Theorem 6.22 (Section 6.2.12), it follows from **6-424** that $\tilde{Q}$ satisfies the recurrence relation

$$\begin{aligned}\tilde{Q}(i+1) = {} & [A(i) - K(i)C(i)]\tilde{Q}(i)[A(i) - K(i)C(i)]^T \\ & + V_1(i) - V_{12}(i)K^T(i) - K(i)V_{12}^T(i) + K(i)V_2(i)K^T(i), \\ & \qquad i \geq i_0,\end{aligned} \tag{6-428}$$

with

$$\tilde{Q}(i_0) = Q_0. \tag{6-429}$$

Repeated application of this recurrence relation will give us $\tilde{Q}(i+1)$ as a function of $K(i), K(i-1), \cdots, K(i_0)$. Let us now consider the problem of minimizing tr $[\tilde{Q}(i+1)W(i+1)]$ with respect to $K(i_0), K(i_0+1), \cdots, K(i)$. This is equivalent to minimizing $\tilde{Q}(i+1)$, that is, finding a sequence of matrices $K^0(i_0), K^0(i_0+1), \cdots, K^0(i)$ such that for the corresponding value $Q(i+1)$ of $\tilde{Q}(i+1)$ we have $Q(i+1) \leq \tilde{Q}(i+1)$. Now **6-428** gives us $\tilde{Q}(i+1)$ as a function of $K(i)$ and $\tilde{Q}(i)$, where $\tilde{Q}(i)$ is a function of $K(i_0), \cdots, K(i-1)$. Clearly, for given $K(i)$, $\tilde{Q}(i+1)$ is a monotone function of $\tilde{Q}(i)$, that is, if $Q(i) \leq \tilde{Q}(i)$ then $Q(i+1) \leq \tilde{Q}(i+1)$, where $Q(i+1)$ is obtained from $Q(i)$ by **6-428**. Therefore, $\tilde{Q}(i+1)$ can be minimized by first minimizing $\tilde{Q}(i)$ with respect to $K(i_0), K(i_0+1), \cdots, K(i-1)$, substituting the minimal value $Q(i)$ of $\tilde{Q}(i)$ into **6-428**, and then minimizing $\tilde{Q}(i+1)$ with respect to $K(i)$.

Let us suppose that the minimal value $Q(i)$ of $\tilde{Q}(i)$ has been found. Substituting $Q(i)$ for $\tilde{Q}(i)$ into **6-428** and completing the square, we obtain

$$\begin{aligned}\tilde{Q}(i+1) = {} & [K - (AQC^T + V_{12})(V_2 + CQC^T)^{-1}](V_2 + CQC^T) \\ & \cdot [K - (AQC^T + V_{12})(V_2 + CQC^T)^{-1}]^T \\ & - (AQC^T + V_{12})(V_2 + CQC^T)^{-1}(CQA^T + V_{12}^T) \\ & + AQA^T + V_1,\end{aligned} \qquad \textbf{6-430}$$

where for brevity we have omitted the arguments i on the right-hand side and where it has been assumed that

$$V_2(i) + C(i)Q(i)C^T(i) \qquad \textbf{6-431}$$

is nonsingular. This assumption is always justified in the nonsingular observer problem, where $V_2(i) > 0$. When considering **6-430**, we note that $\tilde{Q}(i+1)$ is minimized with respect to $K(i)$ if we choose $K(i)$ as $K^0(i)$, where

$$K^0(i) = [A(i)Q(i)C^T(i) + V_{12}(i)][V_2(i) + C(i)Q(i)C^T(i)]^{-1}. \qquad \textbf{6-432}$$

The corresponding value of $\tilde{Q}(i+1)$ is given by

$$Q(i+1) = [A(i) - K^0(i)C(i)]Q(i)A^T(i) + V_1(i) - K^0(i)V_{12}^T(i), \qquad \textbf{6-433}$$

with

$$Q(i_0) = Q_0. \qquad \textbf{6-434}$$

The relations **6-432** and **6-433** together with the initial condition **6-434** enable us to compute the sequence of gain matrices recurrently, starting with $K(i_0)$.

We summarize our conclusions as follows.

Theorem 6.42. *The optimal gain matrices $K^0(i)$, $i \geq i_0$, for the nonsingular optimal observer problem can be obtained from the recurrence relations*

$$\begin{aligned}K^0(i) &= [A(i)Q(i)C^T(i) + V_{12}(i)][V_2(i) + C(i)Q(i)C^T(i)]^{-1}, \\ Q(i+1) &= [A(i) - K^0(i)C(i)]Q(i)A^T(i) + V_1(i) - K^0(i)V_{12}^T(i),\end{aligned} \qquad \textbf{6-435}$$

both for $i \geq i_0$, with the initial condition

$$Q(i_0) = Q_0. \qquad \textbf{6-436}$$

The initial condition of the observer should be chosen as

$$\hat{x}(i_0) = \bar{x}_0. \qquad \textbf{6-437}$$

The matrix $Q(i)$ is the variance matrix of the reconstruction error $e(i) = x(i) - \hat{x}(i)$. For the optimal observer the mean square reconstruction error is given by

$$E\{e^T(i)W(i)e(i)\} = \mathrm{tr}\,[Q(i)W(i)]. \tag{6-438}$$

Singular optimal observation problems can be handled in a manner that is more or less analogous to the continuous-time case (Brammer, 1968; Tse and Athans, 1970). Discrete-time observation problems where the state excitation noise and the observation noise are colored rather than white noise processes (Jazwinski, 1970) can be reduced to singular or nonsingular optimal observer problems.

We remark finally that in the literature a version of the discrete-time linear optimal observer problem is usually given that is different from the one considered here in that it is assumed that $y(i + 1)$ rather than $y(i)$ is the latest observation available for reconstructing $x(i + 1)$. In Problem 6.6 it is shown how the solution of this alternative version of the problem can be derived from the present version.

In this section we have considered optimal observers. As in the continuous-time case, it can be proved (see, e.g., Meditch, 1969) that the optimal observer is actually the *minimum mean square linear estimator* of $x(i + 1)$ given the data $u(j)$ and $y(j)$, $j = i_0, i_0 + 1, \cdots, i$; that is, we cannot find any other linear operator on these data that yields an estimate with a smaller mean square reconstruction error. Moreover, if the initial state x_0 is Gaussian, and the white noise sequences w_1 and w_2 are jointly Gaussian, the optimal observer is *the* minimum mean square estimator of $x(i + 1)$ given $u(j)$, $y(j)$, $j = i_0, i_0 + 1, \cdots, i$; that is, it is impossible to determine any other estimator operating on these data that has a smaller mean square reconstruction error (see, e.g., Jazwinski, 1970).

Example 6.23. *Stirred tank with disturbances*

In Example 6.10 (Section 6.2.12), we considered a discrete-time version of the stirred tank. The plant is described by the state difference equation

$$x(i+1) = \begin{pmatrix} 0.9512 & 0 & 0 & 0 \\ 0 & 0.9048 & 0.0669 & 0.02262 \\ 0 & 0 & 0.8825 & 0 \\ 0 & 0 & 0 & 0.9048 \end{pmatrix} x(i) + \begin{pmatrix} 4.877 & 4.877 \\ -1.1895 & 3.569 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} u(i) + w_1(i), \tag{6-439}$$

where $w_1(i)$, $i \geq i_0$, is a sequence of uncorrelated zero-mean stochastic variables with the variance matrix **6-169**. The components of the state are the incremental volume of the fluid in the tank, the incremental concentration in the tank, and the incremental concentrations of the two incoming feeds. We assume that we can observe at each instant of time i the incremental volume, as well as the incremental concentration in the tank. Both observations are contaminated with uncorrelated, zero-mean observation errors with standard deviations of 0.001 m^3 and 0.001 kmol/m^3, respectively. Furthermore, we assume that the whole sampling interval is used to process the data, so that the observation equation takes the form

$$y(i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x(i) + w_2(i), \qquad \textbf{6-440}$$

where $w_2(i)$, $i \geq i_0$, have the variance matrix

$$\begin{pmatrix} 10^{-6} & 0 \\ 0 & 10^{-6} \end{pmatrix}. \qquad \textbf{6-441}$$

The processes w_1 and w_2 are uncorrelated. In Example 6.10 we found that the steady-state variance matrix of the state of the system is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.00369 & 0.00339 & 0.00504 \\ 0 & 0.00339 & 0.0100 & 0 \\ 0 & 0.00504 & 0 & 0.0400 \end{pmatrix}. \qquad \textbf{6-442}$$

Using this variance matrix as the initial variance matrix $Q(0) = Q_0$, the recurrence relations **6-435** can be solved. Figure 6.21 gives the evolution of the rms reconstruction errors of the last three components of the state as obtained from the evolution of $Q(i)$, $i \geq 0$. The rms reconstruction error of the first component of the state, the volume, of course remains zero all the time, since the volume does not fluctuate and thus we know its value exactly at all times.

It is seen from the plots that the concentrations of the feeds cannot be reconstructed very accurately because the rms reconstruction errors approach steady-state values that are hardly less than the rms values of the fluctuations in the concentrations of the feeds themselves. The rms reconstruction error of the concentration of the tank approaches a steady-state value of about 0.0083 kmol/m^3. The reason that this error is larger than the standard deviation of 0.001 kmol/m^3 of the observation error is the presence of the

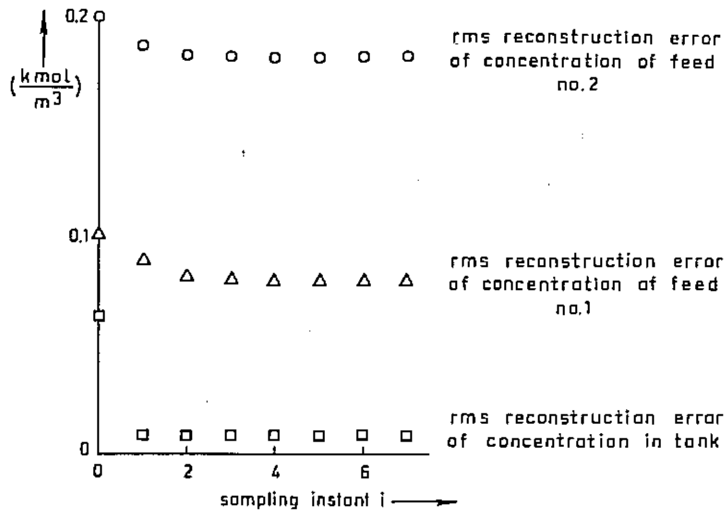

Fig. 6.21. Behavior of the rms reconstruction errors for the stirred tank with disturbances.

processing delay—the observer must predict the concentration a full sampling interval ahead.

6.5.5 Innovations

In this section we state the following fact, which is more or less analogous to the corresponding continuous-time result.

Theorem 6.43. *Consider the optimal observer of Theorem 6.42. Then the* ***innovation process***

$$y(i) - E(i)u(i) - C(i)\hat{x}(i), \qquad i \geq i_0, \tag{6-443}$$

is a sequence of zero-mean uncorrelated stochastic vectors with variance matrices

$$C(i)Q(i)C^T(i) + V_2(i), \qquad i \geq i_0. \tag{6-444}$$

That the innovation sequence is discrete-time white noise can be proved analogonsly to the continuous-time case. That the variance matrix of **6-443** is given by **6-444** follows by inspection.

6.5.6 Duality of the Optimal Observer and Regulator Problems; Steady-State Properties of the Optimal Observer

In this subsection we expose the duality of the linear discrete-time optimal regulator and observer problems. Here the following results are available.

Theorem 6.44. *Consider the linear discrete-time optimal regulator problem (DORP) of Definition 6.16 (Section 6.4.3) and the linear discrete-time optimal observer problem (DOOP) of Definition 6.19 (Section 6.5.4). Let in the observer problem $V_1(i)$ be given by*

$$V_1(i) = G(i)V_3(i)G^T(i), \qquad i \geq i_0, \tag{6-445}$$

where

$$V_3(i) > 0, \qquad i \geq i_0. \tag{6-446}$$

Suppose also that the state excitation noise and the observation noise are uncorrelated in the DOOP, that is,

$$V_{12}(i) = 0, \qquad i \geq i_0. \tag{6-447}$$

Let the various matrices occurring in the DORP and the DOOP be related as follows:

$A(i)$	*of the DORP equals*	$A^T(i^* - i)$	*of the DOOP,*
$B(i)$	*of the DORP equals*	$C^T(i^* - i)$	*of the DOOP,*
$D(i+1)$	*of the DORP equals*	$G^T(i^* - i)$	*of the DOOP,*
$R_3(i+1)$	*of the DORP equals*	$V_3(i^* - i)$	*of the DOOP,*
$R_2(i)$	*of the DORP equals*	$V_2(i^* - i)$	*of the DOOP,*
P_1	*of the DORP equals*	Q_0	*of the DOOP,*

all for $i \leq i_1 - 1$. Here

$$i^* = i_0 + i_1 - 1. \tag{6-448}$$

Under these conditions the solutions of the DORP (Theorem 6.28, Section 6.4.3) and the DOOP (Theorem 6.42, Section 6.5.4) are related as follows.

(a) *$P(i+1)$ of the DORP equals $Q(i^* - 1) - V_1(i^* - i)$ of the DOOP for $i \leq i_1 - 1$;*
(b) *$F(i)$ of the DORP equals $K^{0T}(i^* - i)$ of the DOOP for $i \leq i_1 - 1$;*
(c) *The closed-loop regulator of the DORP,*

$$x(i+1) = [A(i) - B(i)F(i)]x(i), \tag{6-449}$$

and the unforced reconstruction error equation of the DOOP,

$$e(i+1) = [A(i) - K^0(i)C(i)]e(i), \tag{6-450}$$

are dual with respect to i^* in the sense of Definition 6.9.

The proof of this theorem follows by a comparison of the recursive matrix equations that determine the solutions of the regulator and observer problems. Because of duality, computer programs for regulator problems can be used for observer problems, and vice versa. Moreover, by using duality it is very simple to derive the following results concerning the steady-state properties of the nonsingular optimal observer with uncorrelated state excitation and observation noises from the corresponding properties of the optimal regulator.

Theorem 6.45. *Consider the nonsingular optimal observer problem with uncorrelated state excitation and observation noises of Definition 6.19 (Section 6.5.4). Assume that $A(i)$, $C(i)$, $V_1(i) = G(i)V_3(i)G^T(i)$ and $V_2(i)$ are bounded for all i, and that*

$$V_3(i) \geq \alpha I, \qquad V_2(i) \geq \beta I, \qquad \text{for all } i, \tag{6-451}$$

where α and β are positive constants.

(i) *Then if the system* **6-419** *is either*

(a) *completely reconstructible, or*

(b) *exponentially stable,*

and the initial variance $Q_0 = 0$, the variance $Q(i)$ of the reconstruction error converges to a steady-state solution $\bar{Q}(i)$ as $i_0 \to -\infty$, which satisfies the matrix difference equations **6-435.**

(ii) *Moreover, if the system*

$$x(i+1) = A(i)x(i) + G(i)w_3(i), \qquad y(i) = C(i)x(i), \tag{6-452}$$

is either

(c) *both uniformly completely reconstructible and uniformly completely controllable (from w_3), or*

(d) *exponentially stable,*

the variance $Q(i)$ of the reconstruction error converges to $\bar{Q}(i)$ for $i_0 \to -\infty$ for any initial variance $Q_0 \geq 0$.

(iii) *If either condition* (c) *or* (d) *holds, the steady-state optimal observer, which is obtained by using the gain matrix $\bar{K}$ corresponding to the steady-state variance $\bar{Q}$, is exponentially stable.*

(iv) *Finally, if either condition* (c) *or* (d) *holds, the steady-state observer minimizes*

$$\lim_{i_0 \to -\infty} E\{e^T(i)W(i)e(i)\} \tag{6-453}$$

for every initial variance Q_0. The minimal value of **6-453**, *which is achieved by the steady-state optimal observer, is given by*

$$\text{tr}\,[\bar{Q}(i)W(i)]. \tag{6-454}$$

Similarly, it follows by "dualizing" Theorem 6.31 (Section 6.4.4) that, in the time-invariant nonsingular optimal observer problem with uncorrelated state excitation and observation noises, the properties mentioned under (ii), (iii), and (iv) hold provided the system **6-452** is both detectable and stabilizable.

We leave it as an exercise for the reader to state the dual of Theorem 6.37 (Section 6.4.7) concerning the asymptotic behavior of the regulator poles.

6.6 OPTIMAL LINEAR DISCRETE-TIME OUTPUT FEEDBACK SYSTEMS

6.6.1 Introduction

In this section we consider the design of optimal linear discrete-time control systems where the state of the plant cannot be completely and accurately observed, so that an observer must be connected. This section parallels Chapter 5.

6.6.2 The Regulation of Systems with Incomplete Measurements

Consider a linear discrete-time system described by the state difference equation

$$x(i+1) = A(i)x(i) + B(i)u(i), \qquad \textbf{6-455}$$

with the controlled variable

$$z(i) = D(i)x(i). \qquad \textbf{6-456}$$

In Section 6.4 we considered controlling this system with state feedback control laws of the form

$$u(i) = -F(i)x(i). \qquad \textbf{6-457}$$

Very often it is not possible to measure the complete state accurately, however, but only an observed variable of the form

$$y(i) = C(i)x(i) + E(i)u(i) \qquad \textbf{6-458}$$

is available. Assuming, as before, that $y(i)$ is the latest observation available for reconstructing $x(i+1)$, we can connect an observer to this system of the form

$$\hat{x}(i+1) = A(i)\hat{x}(i) + B(i)u(i) + K(i)[y(i) - E(i)u(i) - C(i)\hat{x}(i)]. \qquad \textbf{6-459}$$

Then a most natural thing to do is to replace the state x in **6-457** with its reconstructed value $\hat{x}$:

$$u(i) = -F(i)\hat{x}(i). \qquad \textbf{6-460}$$

We first consider the stability of the interconnection of the plant given by **6-455** and **6-458**, the observer **6-459**, and the control law **6-460**. We have the following result, completely analogous to the continuous-time result of Theorem 5.2 (Section 5.2.2).

Theorem 6.46. *Consider the interconnection of the system described by* **6-455** *and* **6-458**, *the observer* **6-459**, *and the control law* **6-460**. *Then sufficient conditions for the existence of gain matrices $F(i)$ and $K(i)$, $i \geq i_0$, such that the interconnected system is exponentially stable are that the system described by* **6-455** *and* **6-458** *be uniformly completely controllable and uniformly completely reconstructible, or that it be exponentially stable. In the time-invariant case (i.e., all matrices occurring in* **6-455**, **6-458**, **6-459**, *and* **6-460** *are constant) necessary and sufficient conditions for the existence of stabilizing gain matrices K and F are that the system given by* **6-455** *and* **6-458** *be both stabilizable and detectable. Moreover, in the time-invariant case, necessary and sufficient conditions for arbitrarily assigning all the closed-loop poles in the complex plane (within the restriction that complex poles occur in complex conjugate pairs) by suitably choosing the gain matrices K and F are that the system be both completely reconstructible and completely controllable.*

The proof of this theorem follows by recognizing that the reconstruction error

$$e(i) = x(i) - \hat{x}(i) \qquad \textbf{6-461}$$

satisfies the difference equation

$$e(i + 1) = [A(i) - K(i)C(i)]e(i). \qquad \textbf{6-462}$$

Substitution of $\hat{x}(i) = x(i) + e(i)$ into **6-460** yields for **6-455**

$$x(i + 1) = [A(i) - B(i)F(i)]x(i) + B(i)F(i)e(i). \qquad \textbf{6-463}$$

Theorem 6.46 then follows by application of Theorem 6.29 (Section 6.4.4), Theorem 6.45 (Section 6.5.4), Theorem 6.26 (Section 6.4.2), and Theorem 6.41 (Section 6.5.3). We moreover see from **6-462** and **6-463** that in the time-invariant case the characteristic values of the interconnected system comprise the characteristic values of $A - BF$ (the *regulator poles*) and the characteristic values of $A - KC$ (the *observer poles*).

A case of special interest occurs when in the time-invariant case all the regulator poles as well as the observer poles are assigned to the origin. Then we know from Section 6.5.3 that the observer will reconstruct the state completely accurately in at most n steps (assuming that n is the dimension of the state x), and it follows from Section 6.4.2 that after this the regulator will drive the system to the zero state in at most another n steps. Thus we have obtained an output feedback control system that reduces any initial state to the origin in at most $2n$ steps. We call such systems *output feedback state deadbeat control systems*.

Example 6.24. *Digital position output feedback state deadbeat control system*

Let us consider the digital positioning system of Example 6.2 (Section 6.2.3). In Example 6.13 (Section 6.3.3) we derived the state deadbeat control

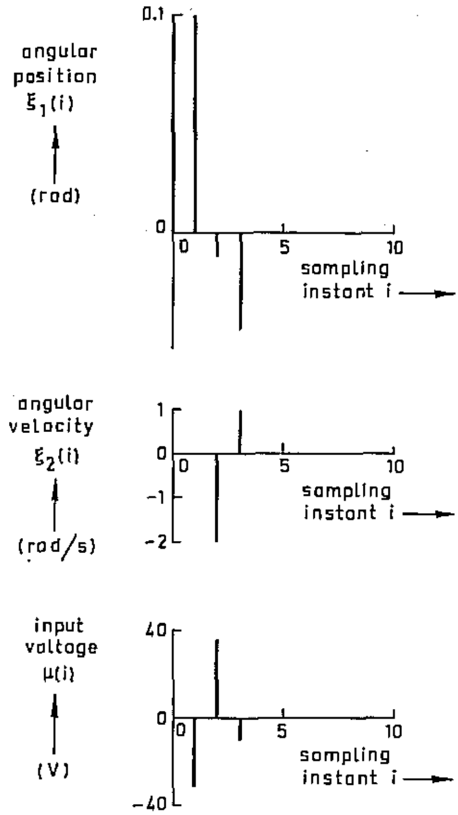

Fig. 6.22. Response of the output feedback state deadbeat position control system from the initial state col$[x(0), \hat{x}(0)]$ = col(0.1, 0, 0, 0). The responses are shown at the sampling instants only and not at intermediate times.

law for this system, while in Example 6.22 (Section 6.5.3) we found the deadbeat observer. In Fig. 6.22 we give the response of the interconnection of deadbeat control law, the deadbeat observer, and the system to the initial state

$$x(0) = \text{col}\,(0.1,\quad 0), \qquad \hat{x}(0) = 0. \tag{6-464}$$

It is seen that the initial state is reduced to the zero state in four steps. Comparison with the state feedback deadbeat response of the same system, as depicted in Fig. 6.12 (Section 6.3.3), shows that the output feedback control system exhibits relatively large excursions of the state before it returns to the zero state, and requires larger input amplitudes.

6.6.3 Optimal Linear Discrete-Time Regulators with Incomplete and Noisy Measurements

We begin this section by defining the central problem.

Definition 6.20. *Consider the linear discrete-time system*

$$x(i+1) = A(i)x(i) + B(i)u(i) + w_1(i), \qquad x(i_0) = x_0, \quad i \geq i_0, \tag{6-465}$$

where x_0 *is a stochastic vector with mean* $\bar{x}_0$ *and variance matrix* Q_0. *The observed variable of the system is*

$$y(i) = C(i)x(i) + E(i)u(i) + w_2(i). \tag{6-466}$$

The variables col $[w_1(i), w_2(i)]$ *form a sequence of uncorrelated stochastic vectors, uncorrelated with* x_0, *with zero means and variance matrices*

$$E\left\{\begin{pmatrix} w_1(i) \\ w_2(i) \end{pmatrix}(w_1^T(i), \quad w_2^T(i))\right\} = \begin{pmatrix} V_1(i) & V_{12}(i) \\ V_{12}^T(i) & V_2(i) \end{pmatrix}, \qquad i \geq i_0. \tag{6-467}$$

The controlled variable can be expressed as

$$z(i) = D(i)x(i). \tag{6-468}$$

Then the ***stochastic linear discrete-time optimal output feedback regulator problem*** *is the problem of finding the functional*

$$u(i) = f[y(i_0), y(i_0+1), \cdots, y(i-1), i], \qquad i_0 \leq i \leq i_1 - 1, \tag{6-469}$$

such that the criterion

$$\sigma = E\left\{\sum_{i=i_0}^{i_1-1}[z^T(i+1)R_3(i+1)z(i+1) + u^T(i)R_2(i)u(i)] + x^T(i_1)P_1x(i_1)\right\} \tag{6-470}$$

is minimized. Here $R_3(i+1) > 0$ *and* $R_2(i) > 0$ *for* $i_0 \leq i \leq i_1 - 1$, *and* $P_1 \geq 0$.

As in the continuous-time case, the solution of this problem satisfies the separation principle (Gunckel and Franklin, 1963; Åström, 1970; Kushner, 1971).

Theorem 6.47. *The solution of the stochastic linear discrete-time optimal output feedback problem is as follows. The optimal input is given by*

$$u(i) = -F(i)\hat{x}(i), \qquad i_0 \leq i \leq i_1 - 1, \tag{6-471}$$

where $F(i), i_0 \leq i \leq i_1 - 1$, *is the sequence of gain matrices for the deterministic optimal regulator as given in Theorem 6.28 (Section 6.4.3). Furthermore,* $\hat{x}(i)$

is the minimum mean-square linear estimator of $x(i)$ given $y(j)$, $i_0 \le j \le i-1$; $\hat{x}(i)$ for the nonsingular case [i.e., $V_2(i) > 0$, $i_0 \le i \le i_1 - 1$] can be obtained as the output of the optimal observer as described in Theorem 6.42 (Section 6.5.4).

We note that this theorem states *the* optimal solution to the stochastic linear discrete-time optimal output feedback problem and not just the optimal *linear* solution, as in the continuous-time equivalent of the present theorem (Theorem 5.3, Section 5.3.1). Theorem 6.47 can be proved analogously to the continuous-time equivalent.

We now consider the computation of the criterion **6-470**, where we restrict ourselves to the nonsingular case. The closed-loop control system is described by the relations

$$\begin{aligned} x(i+1) &= A(i)x(i) + B(i)u(i) + w_1(i),\\ y(i) &= C(i)x(i) + E(i)u(i) + w_2(i),\\ u(i) &= -F(i)\hat{x}(i),\\ \hat{x}(i+1) &= A(i)\hat{x}(i) + B(i)u(i) + K(i)[y(i) - E(i)u(i) - C(i)\hat{x}(i)]. \end{aligned} \qquad \textbf{6-472}$$

In terms of the reconstruction error,

$$e(i) = x(i) - \hat{x}(i), \qquad \textbf{6-473}$$

and the observer state $\hat{x}(i)$, **6-472** can be rewritten in the form

$$\begin{pmatrix} e(i+1) \\ \hat{x}(i+1) \end{pmatrix} = \begin{pmatrix} A(i) - K(i)C(i) & 0 \\ K(i)C(i) & A(i) - B(i)F(i) \end{pmatrix} \begin{pmatrix} e(i) \\ \hat{x}(i) \end{pmatrix} + \begin{pmatrix} I & -K(i) \\ 0 & K(i) \end{pmatrix} \begin{pmatrix} w_1(i) \\ w_2(i) \end{pmatrix}, \qquad \textbf{6-474}$$

with the initial condition

$$\begin{pmatrix} e(i_0) \\ \hat{x}(i_0) \end{pmatrix} = \begin{pmatrix} x(i_0) - \bar{x}_0 \\ \bar{x}_0 \end{pmatrix}. \qquad \textbf{6-475}$$

Defining the variance matrix of col $[e(i), \hat{x}(i)]$ as

$$E\left\{\begin{pmatrix} e(i) - E\{e(i)\} \\ \hat{x}(i) - E\{\hat{x}(i)\} \end{pmatrix}(e^T(i) - E\{e^T(i)\},\ \ \hat{x}^T(i) - E\{\hat{x}^T(i)\})\right\} = \begin{pmatrix} Q_{11}(i) & Q_{12}(i) \\ Q_{12}^T(i) & Q_{22}(i) \end{pmatrix}, \qquad i \ge i_0, \qquad \textbf{6-476}$$

it can be found by application of Theorem 6.22 (Section 6.2.12) that the matrices $Q_{jk}(i), j, k = 1, 2$, satisfy difference equations, of which we give only that for Q_{22}:

$$
\begin{aligned}
Q_{22}(i+1) = {} & K(i)C(i)Q_{11}(i)C^T(i)K^T(i) \\
& + [A(i) - B(i)F(i)]Q_{12}^T(i)C^T(i)K^T(i) \\
& + K(i)C(i)Q_{12}(i)[A(i) - B(i)F(i)]^T \\
& + [A(i) - B(i)F(i)]Q_{22}(i)[A(i) - B(i)F(i)]^T \\
& + K(i)V_2(i)K^T(i), \qquad i \geq i_0,
\end{aligned} \qquad \textbf{6-477}
$$

with the initial condition

$$
Q_{22}(i_0) = 0. \qquad \textbf{6-478}
$$

Now obviously $Q_{11}(i) = Q(i)$, where $Q(i)$ is the variance matrix of the reconstruction error. Moreover, by setting up the difference equation for Q_{12}, it can be proved that $Q_{12}(i) = 0$, $i_0 \leq i \leq i_1 - 1$, which means that analogously with the continuous-time case the quantities $e(i)$ and $\hat{x}(i)$ are *uncorrelated* for $i_0 \leq i \leq i_1 - 1$. As a result, Q_{22} can be found from the difference equation

$$
\begin{aligned}
Q_{22}(i+1) &= K(i)[C(i)Q(i)C^T(i) + V_2(i)]K^T(i) \\
&\qquad + [A(i) - B(i)F(i)]Q_{22}(i)[A(i) - B(i)F(i)]^T, \\
Q_{22}(i_0) &= 0.
\end{aligned} \qquad \textbf{6-479}
$$

When the variance matrix of col $[e(i), \hat{x}(i)]$ is known, all mean square and rms quantities of interest can be computed. In particular, we consider the criterion **6-470**. In terms of the variance matrix of col $(e, \hat{x})$ we write for the criterion:

$$
\begin{aligned}
\sigma = \bar{x}_0^T P(i_0)\bar{x}_0 + \operatorname{tr} \Bigg\{ & \sum_{i=i_0}^{i_1-1} \{R_1(i+1)[Q(i+1) + Q_{22}(i+1)] \\
& + F^T(i)R_2(i)F(i)Q_{22}(i)\} + P_1[Q(i_1) + Q_{22}(i_1)] \Bigg\},
\end{aligned} \qquad \textbf{6-480}
$$

where

$$
R_1(i) = D^T(i)R_3(i)D(i), \qquad \textbf{6-481}
$$

and $P(i)$ is defined in **6-248**. Let us separately consider the terms

$$
\begin{aligned}
& \operatorname{tr} \left\{ \sum_{i=i_0}^{i_1-1} [R_1(i+1)Q_{22}(i+1) + F^T(i)R_2(i)F(i)Q_{22}(i)] + P_1 Q_{22}(i_1) \right\} \\
& = \operatorname{tr} \left\{ \sum_{i=i_0+1}^{i_1-1} [R_1(i) + F^T(i)R_2(i)F(i)]Q_{22}(i) + [P_1 + R_1(i_1)]Q_{22}(i_1) \right\},
\end{aligned} \qquad \textbf{6-482}
$$

where we have used the fact that $Q_{22}(i_0) = 0$. Now using the results of Problem 6.7, **6-482** can be rewritten as

$$\text{tr}\left\{Q_{22}(i_0)\tilde{P}(i_0) + \sum_{j=i_0+1}^{i_1} \tilde{P}(j)K(j-1) \cdot [C(j-1)Q(j-1)C^T(j-1) + V_2(j-1)]K^T(j-1)\right\}, \qquad \textbf{6-483}$$

where $\tilde{P}$ satisfies the matrix difference equation

$$\tilde{P}(i-1) = [A(i-1) - B(i-1)F(i-1)]^T\tilde{P}(i) \cdot [A(i-1) - B(i-1)F(i-1)] + R_1(i) + F^T(i)R_2(i)F(i), \qquad \textbf{6-484}$$

$$\tilde{P}(i_1) = P_1 + R_1(i_1).$$

It is not difficult to recognize that $\tilde{P}(i) = P(i) + R_1(i)$, $i_0 + 1 \leq i \leq i_1$. By using this, substitution of **6-483** into **6-480** yields for the criterion

$$\sigma = \bar{x}_0{}^T P(i_0)\bar{x}_0 + \sum_{i=i_0}^{i_1-1} \text{tr}\left\{R_1(i+1)Q(i+1) + [P(i+1) + R_1(i+1)]K(i) \cdot [C(i)Q(i)C^T(i) + V_2(i)]K^T(i)\right\} + \text{tr}\,[P_1 Q(i_1)]. \qquad \textbf{6-485}$$

By suitable manipulations it can be found that the criterion can be expressed in the alternative form:

$$\sigma = \bar{x}_0{}^T P(i_0)\bar{x}_0 + \text{tr}\,[P(i_0)Q_0] + \sum_{i=i_0}^{i_1-1} \text{tr}\left\{[R_1(i+1) + P(i+1)]V_1(i) + Q(i)F^T(i) \cdot \{R_2(i) + B^T(i)[R_1(i+1) + P(i+1)]B(i)\}F(i)\right\}. \qquad \textbf{6-486}$$

We can now state the following theorem.

Theorem 6.48. *Consider the stochastic output feedback regulator problem of Definition 6.20. Suppose that* $V_2(i) > 0$ *for all* i. *Then the following facts hold.*

(a) *The minimal value of the criterion* **6-470** *can be expressed in the alternative forms* **6-485** *and* **6-486**.

(b) *In the time-invariant case, in which the optimal observer and regulator problems have steady-state solutions as* $i_0 \rightarrow -\infty$ *and* $i_1 \rightarrow \infty$, *characterized by* $\bar{Q}$ *and* $\bar{P}$, *with corresponding steady-state gain matrices* $\bar{K}$ *and* $\bar{F}$, *the* *following holds:*

$$\lim_{\substack{i_0 \to -\infty \\ i_1 \to \infty}} \frac{1}{i_1 - i_0} E\left\{\sum_{i=i_0}^{i_1-1} [z^T(i+1)R_3(i+1)z(i+1) + u^T(i)R_2(i)u(i)]\right\}$$
$$= \lim_{i_0 \to -\infty} E\{z^T(i+1)R_3z(i+1) + u^T(i)R_2u(i)\}$$
$$= \operatorname{tr}[R_1\bar{Q} + (\bar{P} + R_1)\bar{K}(C\bar{Q}C^T + V_2)\bar{K}^T]$$
$$= \operatorname{tr}\{(R_1 + \bar{P})V_1 + \bar{Q}\bar{F}^T[R_2 + B^T(R_1 + \bar{P})B]\bar{F}\}. \quad \textbf{6-487}$$

(c) *All mean square quantities of interest can be obtained from the variance matrix* diag $[Q(i), Q_{22}(i)]$ *of* col $[e(i), \hat{x}(i)]$. *Here* $e(i) = x(i) - \hat{x}(i)$, $Q(i)$ *is the variance matrix of* $e(i)$, *and* $Q_{22}(i)$ *can be obtained as the solution of the matrix difference equation*

$$Q_{22}(i+1) = [A(i) - B(i)F(i)]Q_{22}(i)[A(i) - B(i)F(i)]^T$$
$$+ K(i)[C(i)Q(i)C^T(i) + V_2(i)]K^T(i), \qquad i \geq i_0, \quad \textbf{6-488}$$
$$Q_{22}(i_0) = 0.$$

The proof of part (b) of this theorem follows by application of part (a).

The general stochastic regulator problem can be specialized to tracking problems, regulation problems for systems with disturbances, and tracking problems for systems with disturbances, completely analogous to what we have discussed for the continuous-time case.

6.6.4 Nonzero Set Points and Constant Disturbances

The techniques developed in Section 5.5 for dealing with time-invariant regulators and tracking systems with nonzero set points and constant disturbances can also be applied to the discrete-time case. We first consider the case where the system has a nonzero set point z_0 for the controlled variable. The system state difference equation is

$$x(i+1) = Ax(i) + Bu(i) + w_1(i), \qquad i \geq i_0, \qquad \textbf{6-489}$$

the controlled variable is

$$z(i) = Dx(i), \qquad i \geq i_0, \qquad \textbf{6-490}$$

and the observed variable is

$$y(i) = Cx(i) + Eu(i) + w_2(i), \qquad i \geq i_0. \qquad \textbf{6-491}$$

The joint process col (w_1, w_2) is given as in Definition 6.20 (Section 6.6.3). From Section 6.4.6 it follows that the nonzero set point controller is specified by

$$u(i) = -\bar{F}\hat{x}(i) + H_c^{-1}(1)\hat{z}_0, \qquad \textbf{6-492}$$

where $\bar{F}$ is a suitable feedback gain matrix, and

$$H_c(z) = D(zI - A + B\bar{F})^{-1}B \qquad \textbf{6-493}$$

is the (square) closed-loop transfer matrix (assuming that dim $(z) =$ dim (u)). Furthermore, $\hat{x}(i)$ is the minimum mean square estimator of $x(i)$ and $\hat{z}_0$ that of z_0.

How $\hat{z}_0$ is obtained depends on how we model the set point. If we assume that the set point varies according to

$$z_0(i + 1) = z_0(i) + w_0(i), \qquad \textbf{6-494}$$

and that we observe

$$r(i) = z_0(i) + w_s(i), \qquad \textbf{6-495}$$

where col (w_0, w_s) constitutes a white noise sequence, the steady-state optimal observer for the set point is of the form

$$\hat{z}_0(i + 1) = \hat{z}_0(i) + \bar{K}_r[r(i) - \hat{z}_0(i)]. \qquad \textbf{6-496}$$

This observer in conjunction with the control law **6-492** yields a zero-steady-state-error response when the reference variable $r(i)$ is constant.

Constant disturbances can be dealt with as follows. Let the state difference equation be given by

$$x(i + 1) = Ax(i) + Bu(i) + v_0 + w_1(i), \qquad \textbf{6-497}$$

where v_0 is a constant disturbance. The controlled variable and observed variable are as given before. Then from Section 6.4.6, we obtain the zero-steady-state-error control law

$$u(i) = -\bar{F}\hat{x}(i) - H_c^{-1}(1)D(I - \bar{A})^{-1}\hat{v}_0, \qquad \textbf{6-498}$$

with all quantities defined as before, $\bar{A} = A - B\bar{F}$, and $\hat{v}_0$ an estimate of v_0. In order to obtain $\hat{v}_0$, we model the constant disturbance as

$$v_0(i + 1) = v_0(i) + w_0(i), \qquad \textbf{6-499}$$

where w_0 constitutes a white noise sequence. The steady-state optimal observer for $x(i)$ and $z_0(i)$ will be of the form

$$\begin{aligned} \hat{x}(i + 1) &= A\hat{x}(i) + Bu(i) + \hat{v}_0(i) + \bar{K}_1[y(i) - C\hat{x}(i) - E(i)u(i)], \\ \hat{v}_0(i + 1) &= \hat{v}_0(i) + \bar{K}_2[y(i) - C\hat{x}(i) - E(i)u(i)]. \end{aligned} \qquad \textbf{6-500}$$

This observer together with the control law **6-498** produces a zero-steady-state-error response to a constant disturbance. This is a form of integral control.

Example 6.25. *Integral control of the digital positioning system*

Consider the digital positioning system of previous examples. In Example 6.14 (Section 6.4.3), we obtained the state feedback control law

$$u(i) = -\bar{F}x(i) = -(110.4, \quad 12.66)x(i). \qquad \textbf{6-501}$$

Assuming that the servo motor is subject to constant disturbances in the form of constant torques on the shaft, we must include a term of the form

$$v_0 = \begin{pmatrix} 0.003396 \\ 0.06308 \end{pmatrix} \alpha \qquad \textbf{6-502}$$

in the state difference equation **6-26**, where α is a constant. It is easily seen that with the state feedback law **6-501** this leads to the zero-steady-state-error control law

$$\mu(i) = -\bar{F}\hat{x}(i) - \hat{\alpha}(i). \qquad \textbf{6-503}$$

The observer **6-500** is in this case of the form

$$\begin{aligned} \hat{x}(i+1) &= \begin{pmatrix} 1 & 0.08015 \\ 0 & 0.6313 \end{pmatrix} \hat{x}(i) + \begin{pmatrix} 0.003396 \\ 0.06308 \end{pmatrix} [\mu(i) + \hat{\alpha}(i)] \\ &\quad + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} [\eta(i) - (1, \quad 0)\hat{x}(i)], \\ \hat{\alpha}(i+1) &= \hat{\alpha}(i) + k_3[\eta(i) - (1, \quad 0)\hat{x}(i)]. \end{aligned} \qquad \textbf{6-504}$$

Here it has been assumed that

$$\eta(i) = (1, \quad 0)x(i) \qquad \textbf{6-505}$$

is the observed variable (i.e., the whole sampling interval is used for processing the data), and k_1, k_2, and k_3 are scalar gains to be selected. We choose these gains such that the observer is a deadbeat observer; this results in the following values:

$$k_1 = 2.6313, \qquad k_2 = 18.60, \qquad k_3 = 158.4. \qquad \textbf{6-506}$$

Figure 6.23 shows the response of the resulting zero-steady-state-error control system from zero initial conditions to a relatively large constant disturbance of 10 V (i.e., the disturbing torque is equivalent to a constant additive input voltage of 10 V). It is seen that the magnitude of the disturbance is identified after three sampling intervals, and that it takes the system another three to four sampling intervals to compensate fully for the disturbance.

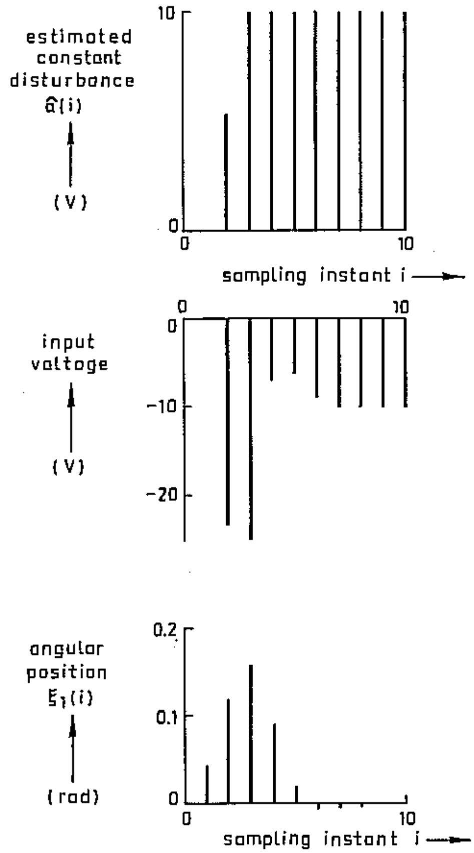

Fig. 6.23. Response of the digital positioning system with integral control from zero initial conditions to a constant disturbance.

6.7 CONCLUSIONS

In this chapter we have summarized the main results of linear optimal control theory for discrete-time systems. As we have seen, in many instances the continuous-time theory can be extended to the discrete-time case in a fairly straightforward manner. This chapter explicitly reviews most of the results needed in linear discrete-time control system design.

Although in many respects the discrete-time theory parallels the continuous-time theory, there are a few differences. One of the striking dissimilarities is that, in theory, continuous-time control systems can be made arbitrarily fast. This cannot be achieved with discrete-time systems, where the speed of action is restricted by the sampling interval. The fastest type of control that can be achieved with discrete-time systems is deadbeat control.

In this chapter we have usually considered linear discrete-time systems thought to be derived from continuous-time systems by sampling. We have not paid very much attention to what happens *between* the sampling interval instants, however, except by pointing out in one or two examples that the behavior at the sampling instants may be misleading for what happens in between. This is a reason for caution. As we have seen in the same examples, it is often possible to modify the discrete-time problem formulation to obtain a more acceptable design.

The most fruitful applications of linear discrete-time control theory lie in the area of computer process control, a rapidly advancing field.

6.8 PROBLEMS

6.1. *A modified discrete-time regulator problem*

Consider the linear discrete-time system

$$x(i+1) = A(i)x(i) + B(i)u(i), \qquad \textbf{6-507}$$

with the modified criterion

$$\sum_{i=i_0}^{i_1-1} [x^T(i)R_1(i)x(i) + 2x^T(i)R_{12}(i)u(i) + u^T(i)R_2(i)u(i)] + x^T(i_1)P_1x(i_1). \qquad \textbf{6-508}$$

Show that minimizing **6-508** for the system **6-507** is equivalent to a standard discrete-time regulator problem where the criterion

$$\sum_{i=i_0}^{i_1-1} [x^T(i+1)R_1'(i+1)x(i+1) + u'^T(i)R_2(i)u'(i)] + x^T(i_1)P_1x(i_1) \qquad \textbf{6-509}$$

is minimized for the system

$$x(i+1) = A'(i)x(i) + B(i)u'(i), \qquad \textbf{6-510}$$

with

$$R_1'(i) = \begin{cases} R_1(i) - R_{12}(i)R_2^{-1}(i)R_{12}^T(i), & i = i_0+1, i_0+2, \cdots, i_1-1, \\ 0, & i = i_1, \end{cases}$$

$$u'(i) = u(i) + R_2^{-1}(i)R_{12}^T(i)x(i), \qquad i = i_0, i_0+1, \cdots, i_1-1, \qquad \textbf{6-511}$$

$$A'(i) = A(i) - B(i)R_2^{-1}(i)R_{12}^T(i), \qquad i = i_0, i_0+1, \cdots, i_1-1.$$

6.2. *Stochastic state feedback regulator problems structured as regulator problems with disturbances*

Consider the linear discrete-time system

$$\begin{aligned} x(i+1) &= A(i)x(i) + B(i)u(i) + v(i), \\ z(i) &= D(i)x(i). \end{aligned} \qquad \textbf{6-512}$$

Here the disturbance variable v is modeled as

$$\begin{aligned} v(i) &= D_d(i)x_d(i), \\ x_d(i+1) &= A_d(i)x_d(i) + w_d(i), \end{aligned} \qquad \textbf{6-513}$$

where the $w_d(i)$, $i \geq i_0$, form a sequence of uncorrelated stochastic vectors with given variance matrices. Consider also the criterion

$$E\left\{\sum_{i=i_0}^{i_1-1}[z^T(i+1)R_3(i+1)z(i+1) + u^T(i)R_2(i)u(i)] + x^T(i_1)P_1x(i_1)\right\}. \qquad \textbf{6-514}$$

(a) Show how the problem of controlling the system such that the criterion **6-514** is minimized can be converted into a standard stochastic regulator problem.

(b) Show that the optimal control law can be expressed as

$$u(i) = -F(i)x(i) - F_d(i)x_d(i), \qquad i = i_0, i_0+1, \cdots, i_1-1, \qquad \textbf{6-515}$$

where the feedback gain matrices $F(i)$, $i = i_0, \cdots, i_1 - 1$, are completely independent of the properties of the disturbance variable.

6.3. *Stochastic state feedback regulator problems structured as tracking problems*

Consider the linear discrete-time system

$$\begin{aligned} x(i+1) &= A(i)x(i) + B(i)u(i), \\ z(i) &= D(i)x(i). \end{aligned} \qquad \textbf{6-516}$$

Consider also a reference variable z_r, which is modeled through the equations

$$\begin{aligned} z_r(i) &= D_r(i)x_r(i), \\ x_r(i+1) &= A_r(i)x_r(i) + w_r(i), \end{aligned} \qquad \textbf{6-517}$$

where $w_r(i)$, $i \geq i_0$, forms a sequence of uncorrelated stochastic vectors with variance matrices $V_r(i)$. Consider as well the criterion

$$E\left\{\sum_{i=i_0}^{i_1-1}[z(i+1) - z_r(i+1)]^T R_3(i+1)[z(i+1) - z_r(i+1)] + u^T(i)R_2(i)u(i)\right\}. \qquad \textbf{6-518}$$

(a) Show how the problem of controlling the system such that the criterion **6-518** is minimized can be converted into a standard stochastic discrete-time optimal regulator problem.

(b) Show that the optimal control law can be expressed in the form

$$u(i) = -F(i)x(i) + F_r(i)x_r(i), \qquad i = i_0, i_0 + 1, \cdots, i_1 - 1, \qquad \textbf{6-519}$$

where the feedback gain matrices $F(i)$, $i = i_0, \cdots, i_1 - 1$, are completely independent of the properties of the reference variable.

6.4. *The closed-loop regulator poles*

Prove the following generalization of Theorem 6.37 (Section 6.4.7). Consider the steady-state solution of the time-invariant linear discrete-time optimal regulator problem. Suppose that dim (z) = dim (u) and let

$$H(z) = D(zI - A)^{-1}B,$$

$$\det\,[H(z)] = \frac{\psi(z)}{\phi(z)},$$

$$\phi(z) = z^{n-q}\prod_{i=1}^{q}(z - \pi_i), \qquad \text{with } \pi_i \neq 0,\ i = 1, 2, \cdots, q, \qquad \textbf{6-520}$$

$$\psi(z) = z^{n-p}\prod_{i=1}^{p}(z - \nu_i), \qquad \text{with } \nu_i \neq 0,\ i = 1, 2, \cdots, p,$$

and

$$R_2 = \rho N,$$

with $N > 0$ and ρ a positive scalar. Finally, set $r = \max(p, q)$. Then:

(a) Of the n closed-loop regulator poles, $n - r$ always stay at the origin.

(b) As $\rho \downarrow 0$, of the remaining r closed-loop poles, p approach the numbers $\hat{\nu}_i$, $i = 1, 2, \cdots, p$, which are defined as in **6-363.**

(c) As $\rho \downarrow 0$, the $r - p$ other closed-loop poles approach the origin.

(d) As $\rho \downarrow \infty$, of the r nonzero closed-loop poles, q approach the numbers $\hat{\pi}_i$, $i = 1, \cdots, q$, which are defined as in **6-364.**

(e) As $\rho \downarrow \infty$, the $r - p$ other nonzero closed-loop poles approach the origin.

6.5. *Mixed continuous-time discrete-time regulator problem*

Consider the discrete-time system that results from applying a piecewise constant input to the continuous-time system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t). \qquad \textbf{6-521}$$

Use the procedure and notation of Section 6.2.3 in going from the continuous-time to the discrete-time version. Suppose now that one wishes to take into account the behavior of the system between the sampling instants and consider therefore the *integral* criterion (rather than a *sum* criterion)

$$\int_{t_{i_0}}^{t_{i_1}} [x^T(t)R_1(t)x(t) + u^T(t)R_2(t)u(t)]\,dt + x^T(t_{i_1})P_1x(t_{i_1}). \qquad \textbf{6-522}$$

Here t_{i_0} is the first sampling instant and t_{i_1} the last.

(a) Show that minimizing the criterion **6-522**, while the system **6-521** is commanded by stepwise constant inputs, is equivalent to minimizing an expression of the form

$$\sum_{i=i_0}^{i_1-1} [x^T(t_i)R_1'(i)x(t_i) + 2x^T(t_i)R_{12}'(i)u(t_i) + u^T(t_i)R_2'(i)u(t_i)] + x^T(t_{i_1})P_1x(t_{i_1}) \qquad \textbf{6-523}$$

for the discrete-time system

$$x(t_{i+1}) = \Phi(t_{i+1}, t_i)x(t_i) + \left[\int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, \tau)B(\tau)\,d\tau\right]u(t_i), \qquad \textbf{6-524}$$

where $\Phi(t, t_0)$ is the transition matrix of the system **6-521**. Derive expressions for $R_1'(i)$, $R_{12}'(i)$, and $R_2'(i)$.

(b) Suppose that A, B, R_1, and R_2 are constant matrices and also let the sampling interval $t_{i+1} - t_i = \Delta$ be constant. Show that if the sampling interval is small first approximations to R_1', R_{12}', and R_2' are given by

$$\begin{aligned} R_1' &\simeq R_1\Delta, \\ R_{12}' &\simeq \tfrac{1}{2}R_1B\Delta^2, \\ R_2' &\simeq (R_2 + \tfrac{1}{3}B^TR_1B\Delta^2)\Delta. \end{aligned} \qquad \textbf{6-525}$$

6.6. *Alternative version of the discrete-time optimal observer problem*
Consider the system

$$\begin{aligned} x(i+1) &= A(i)x(i) + B(i)u(i) + w_1(i), \\ y(i) &= C(i)x(i) + E(i)u(i) + w_2(i), \qquad i \geq i_0, \end{aligned} \qquad \textbf{6-526}$$

where col $[w_1(i), w_2(i)]$, $i \geq i_0$, forms a sequence of zero-mean uncorrelated vector stochastic variables with variance matrices

$$\begin{pmatrix} V_1(i) & V_{12}(i) \\ V_{12}^T(i) & V_2(i) \end{pmatrix}, \qquad i \geq i_0. \qquad \textbf{6-527}$$

Furthermore, $x(i_0)$ is a vector stochastic variable, uncorrelated with w_1 and w_2, with mean $\bar{x}_0$ and variance matrix Q_0. Show that the best linear estimator of $x(i)$ operating on $y(j)$, $i_0 \leq j \leq i$ (not $i - 1$, as in the version of Section 6.5), can be described as follows:

$$\begin{aligned} \hat{x}(i+1) = {} & [I - K(i+1)C(i+1)][A(i)\hat{x}(i) + B(i)u(i)] \\ & + K(i+1)[y(i+1) - E(i+1)u(i+1)], \qquad i \geq i_0. \end{aligned} \qquad \textbf{6-528}$$

Here the gain matrices K are obtained from the iterative relations

$$
\begin{aligned}
K(i+1) &= [S(i+1)C^T(i+1) + V_{12}(i)][C(i+1)S(i+1)C^T(i+1) \\
&\qquad + C(i+1)V_{12}(i) + V_{12}^T(i)C^T(i+1) + V_2(i)]^{-1}, \\
S(i+1) &= A(i)Q(i)A^T(i) + V_1(i), \\
Q(i+1) &= [I - K(i+1)C(i+1)]S(i+1) - K(i+1)V_{12}^T(i),
\end{aligned}
\qquad \textbf{6-529}
$$

all for $i \geq i_0$. Here $Q(i)$ is the variance matrix of the reconstruction error $x(i) - \hat{x}(i)$, and $S(i)$ is an auxiliary matrix. The initial condition for **6-528** is given by

$$\hat{x}(i_0) = [I - K(i_0)C(i_0)]\bar{x}_0 + K(i_0)[y(i_0) - E(i_0)u(i_0)], \qquad \textbf{6-530}$$

where

$$K(i_0) = Q_0 C^T(i_0)[C(i_0)Q_0 C^T(i_0) + V_2(i_0)]^{-1}. \qquad \textbf{6-531}$$

The initial variance matrix, which serves as initial condition for the iterative equations **6-529**, is given by

$$Q(i_0) = [I - K(i_0)C(i_0)]Q_0. \qquad \textbf{6-532}$$

Hint: To derive the observer equation, express $y(i+1)$ in terms of $x(i)$ and use the standard version of the observer problem given in the text.

6.7. *Property of a matrix difference equation*

Consider the matrix difference equation

$$Q(i+1) = A(i)Q(i)A^T(i) + R(i), \qquad i_0 \leq i \leq i_1 - 1, \qquad \textbf{6-533}$$

together with the linear expression

$$\text{tr}\left[\sum_{j=i_0}^{i_1-1} Q(j)S(j) + P_1 Q(i_1)\right]. \qquad \textbf{6-534}$$

Prove that this expression can also be written as

$$\text{tr}\left[Q_0 P(i_0) + \sum_{j=i_0+1}^{i_1} R(j-1)P(j)\right], \qquad \textbf{6-535}$$

where the sequence of matrices $P(j)$, $i_0 \leq j \leq i_1$, satisfies the matrix difference equation

$$
\begin{aligned}
P(i-1) &= A^T(i-1)P(i)A(i-1) + S(i-1), \qquad i_0 + 1 \leq i \leq i_1, \\
P(i_1) &= P_1.
\end{aligned}
\qquad \textbf{6-536}
$$

6.8. *Linear discrete-time optimal output feedback controllers of reduced dimensions*

Consider the linear time-invariant discrete-time system

$$\begin{aligned} x(i+1) &= Ax(i) + Bu(i) + w_1(i), \qquad x(i_0) = x_0, \\ z(i) &= D(i)x(i), \\ y(i) &= C(i)x(i) + E(i)u(i) + w_2(i), \end{aligned} \tag{6-537}$$

all for $i \geq i_0$, where col $[w_1(i), w_2(i)]$, $i \geq i_0$, forms a sequence of uncorrelated stochastic vectors uncorrelated with x_0. Consider for this system the time-invariant controller

$$\begin{aligned} q(i+1) &= Lq(i) + K_1 y(i), \\ u(i) &= -Fq(i) - K_2 y(i). \end{aligned} \tag{6-538}$$

Assume that the interconnection of controller and plant is asymptotically stable.

(a) Develop matrix relations that can be used to compute expressions of the form

$$\lim_{i_0 \to -\infty} E\{z^T(i) R_3 z(i)\} \tag{6-539}$$

and

$$\lim_{i_0 \to -\infty} E\{u^T(i) R_2 u(i)\}. \tag{6-540}$$

Presuming that computer programs can be developed that determine the controller matrices L, K_1, F, and K_2 such that **6-539** is minimized while **6-540** is constrained to a given value, outline a method for determining discrete-time optimal output feedback controllers of reduced dimensions. (Compare the continuous-time approach discussed in Section 5.7.)

(b) When gradient methods are used to solve numerically the optimization problem of (a), the following result is useful. Let M, N, and R be given matrices of compatible dimensions, each depending upon a parameter γ. Let $\bar{S}$ be the solution of the linear matrix equation

$$\bar{S} = M\bar{S}M^T + N, \tag{6-541}$$

and consider the scalar

$$\text{tr}\,(\bar{S}R) \tag{6-542}$$

as a function of γ. Then the gradient of **6-542** with respect to γ is given by

$$\frac{\partial}{\partial \gamma}[\text{tr}\,(\bar{S}R)] = \text{tr}\left(\frac{\partial R}{\partial \gamma}\bar{S} + \frac{\partial N}{\partial \gamma}\bar{U} + 2\bar{U}\frac{\partial M}{\partial \gamma}\bar{S}M^T\right), \tag{6-543}$$

where $\bar{U}$ is the solution of the adjoint matrix equation

$$\bar{U} = M^T\bar{U}M + R. \tag{6-544}$$

Prove this.