5 OPTIMAL LINEAR OUTPUT FEEDBACK CONTROL SYSTEMS

5.1 INTRODUCTION

In Chapter 3 we considered the control of linear systems described by a state differential equation of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t). \tag{5-1}$$

An essential part of the theory of Chapter 3 is that it is assumed that the complete state vector $x(t)$ is available for measurement and feedback.

In this chapter we relax this assumption and study the much more realistic case where there is an observed variahle of the form

$$y(t) = C(t)x(t), \tag{5-2}$$

which is availahle for measurement and feedback. Control systems where the observed variable y serves as input to the controller, and not the state x, will be called *output feedback control systems.*

In view of the results of Chapter 4, it is not surprising that the optimal output feedback controller turns out to he a combination of an observer, through which the state of the system is reconstructed, and a control law which is an instantaneous, linear function of the reconstructed state. This control law is the same control law that would have been obtained if the state had been directly available for observation.

In Section 5.2 we consider a deterministic approach to the output feedback problem and we obtain regulators through a combination of asymptotically stable observers and linear, stabilizing control laws. In Section 5.3 a stochastic approach is taken, and optimal linear feedback regulators are derived as interconnections of optimal observers and optimal linear state feedback laws. In Section 5.4 tracking problems are studied. In Section 5.5 we consider regulators and tracking systems with nonzero set points and constant disturbances. Section 5.6 concerns the sensitivity of linear optimal feedback systems to disturbances and system variations, while the chapter concludes with Section 5.7, dealing with reduced-order feedback controllers.

5.2 THE REGULATION OF LINEAR SYSTEMS WITH INCOMPLETE MEASUREMENTS

5.2.1 The Structure of Output Feedback Control Systems

In this section we take a deterministic approach to the problem of regulating a linear system with incomplete measurements. Consider the system described by the state differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad \textbf{5-3}$$

while the observed variable is given by

$$y(t) = C(t)x(t). \qquad \textbf{5-4}$$

In Chapter 3 we considered control laws of the form

$$u(t) = -F(t)x(t), \qquad \textbf{5-5}$$

where it was assumed that the whole state $x(t)$ can be accurately measured. If the state is not directly available for measurement, a natural approach is first to construct an observer of the form

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + K(t)[y(t) - C(t)\hat{x}(t)], \qquad \textbf{5-6}$$

and then interconnect the control law with the *reconstructed* state $\hat{x}(t)$:

$$u(t) = -F(t)\hat{x}(t), \qquad \textbf{5-7}$$

where $F(t)$ is the same as in **5-5**. Figure 5.1 depicts the interconnection of the plant, the observer, and the control law. By substitution of the control law **5-7** into the observer equation **5-6**, the controller equations take the form

$$\begin{aligned} \dot{\hat{x}}(t) &= [A(t) - B(t)F(t) - K(t)C(t)]\hat{x}(t) + K(t)y(t), \\ u(t) &= -F(t)\hat{x}(t). \end{aligned} \qquad \textbf{5-8}$$

This leads to the simplified structure of Fig. 5.2.

The closed-loop system that results from interconnecting the plant with the controller is a linear system of dimension $2n$ (where n is the dimension of the state x), which can be described as

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{pmatrix} = \begin{pmatrix} A(t) & -B(t)F(t) \\ K(t)C(t) & A(t) - K(t)C(t) - B(t)F(t) \end{pmatrix} \begin{pmatrix} x(t) \\ \hat{x}(t) \end{pmatrix}. \qquad \textbf{5-9}$$

We now analyze the stability properties of the closed-loop system. To this end we consider the state $x(t)$ and the reconstruction error

$$e(t) = x(t) - \hat{x}(t). \qquad \textbf{5-10}$$

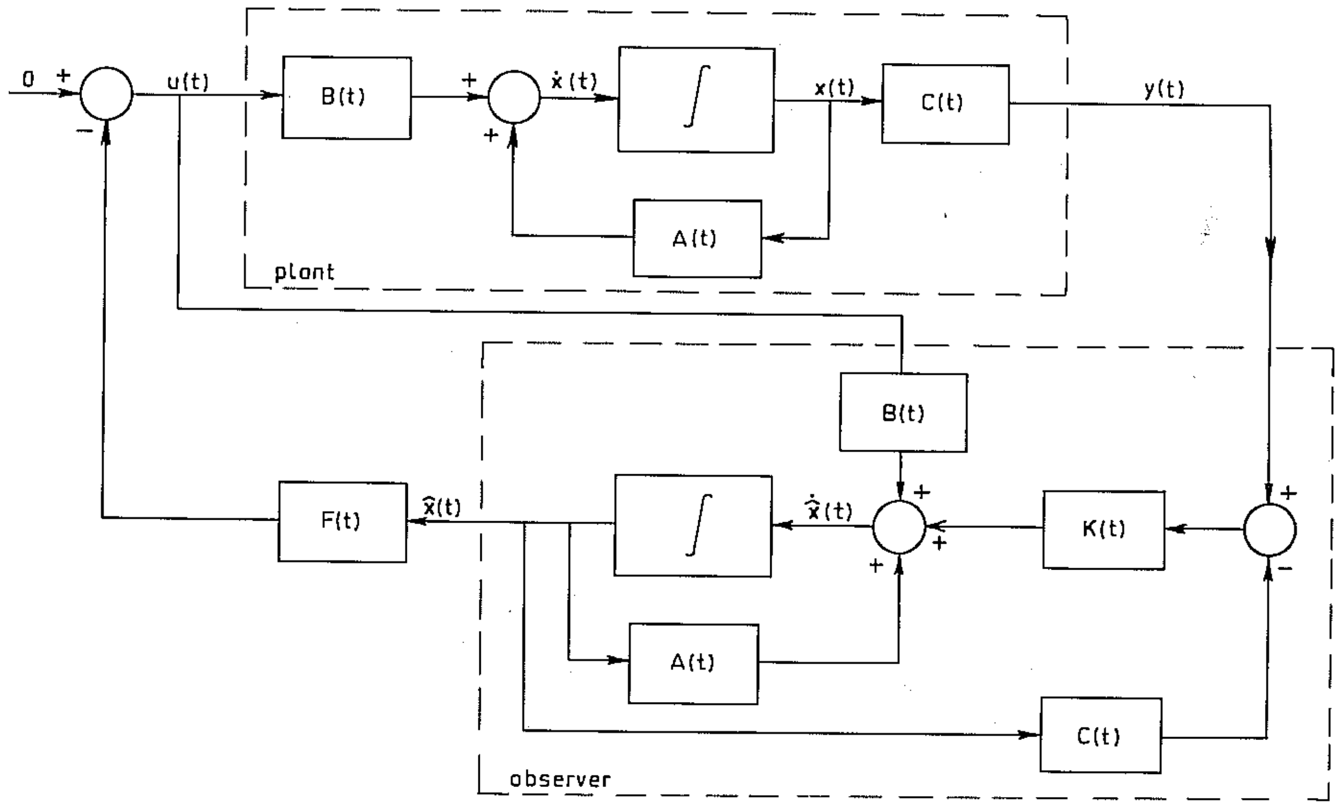

Fig. 5.1. The structure of an output feedback control system.

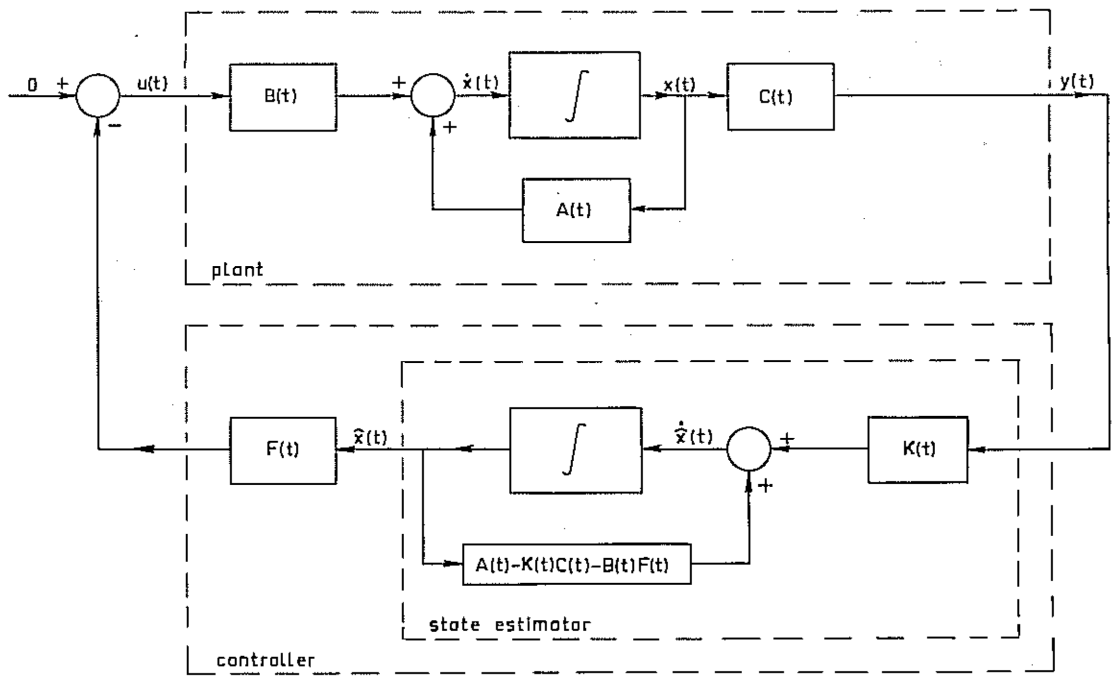

Fig. 5.2. The output feedback control system of Fig. 5.1 in a more compact form.

By subtracting **5-3** and **5-6**, it easily follows with the use of **5-4** that $e(t)$ satisfies

$$\dot{e}(t) = [A(t) - K(t)C(t)]e(t). \tag{5-11}$$

Substitution of $\hat{x}(t) = x(t) - e(t)$ into **5-3** and **5-7** yields

$$\dot{x}(t) = [A(t) - B(t)F(t)]x(t) + B(t)F(t)e(t). \tag{5-12}$$

When considering **5-11**, it is seen that $e(t)$ converges to zero, independent of the initial state, if a gain matrix $K(t)$ can be found that makes **5-11** asymptotically stable. However, finding a gain matrix $K(t)$ that makes **5-11** stable is equivalent to determining $K(t)$ such that the observer is asymptotically stable. As we know from Chapter 4, such a gain often can be found.

Next we consider **5-12**. If $B(t)$ and $F(t)$ are bounded and $e(t) \to 0$ as $t \to \infty$, $x(t)$ will always converge to zero if the system

$$\dot{x}(t) = [A(t) - B(t)F(t)]x(t) \tag{5-13}$$

is asymptotically stable. From Chapter 3 we know that often $F(t)$ can be determined so that **5-13** is asymptotically stable. Thus we have seen that it is usually possible to find gain matrices $F(t)$ and $K(t)$ such that Eqs. **5-11** and **5-12** constitnte an asymptotically stable system. Since the system **5-9** is obtained from the system described by **5-11** and **5-12** by a nonsingular linear transformation, it follows that it is usually possible to find gain matrices $F(t)$ and $K(t)$ such that the closed-loop control systems **5-9** is stable. In the following subsection the precise conditions under which this can be done are stated.

Finally, we remark the following. Combining **5-11** and **5-12** we obtain

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \end{pmatrix} = \begin{pmatrix} A(t) - B(t)F(t) & B(t)F(t) \\ 0 & A(t) - K(t)C(t) \end{pmatrix} \begin{pmatrix} x(t) \\ e(t) \end{pmatrix}. \tag{5-14}$$

Let us consider the time-invariant case, where all the matrices occurring in **5-14** are constant. Then the characteristic values of the system **5-14**, which are also the characteristic values of the system **5-9**, are the zeroes of

$$\det \begin{pmatrix} sI - A + BF & -BF \\ 0 & sI - A + KC \end{pmatrix}$$
$$= \det (sI - A + BF) \det (sI - A + KC). \tag{5-15}$$

The reason that the systems **5-9** and **5-14** have the same characteristic values is that their respective state vectors are related by a nonsingular linear transformation (see Problem 1.3). Consequently, the set of closed-loop characteristic values comprises the characteristic values of $A - BF$ (the *regulator poles*) and the characteristic values of $A - KC$ (the *observer poles*):

Theorem 5.1. *Consider the interconnection of the time-invariant system*

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t),\end{aligned} \tag{5-16}$$

the time-invariant observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)], \tag{5-17}$$

and the time-invariant control law

$$u(t) = -F\hat{x}(t). \tag{5-18}$$

Then the characteristic values of the interconnected system consist of the regulator poles (the characteristic values of $A - BF$) together with the observer poles (the characteristic values of $A - KC$).

These results show that we can consider the problem of determining an asymptotically stable observer and an asymptotically stable state feedback control law separately, since their interconnection results in an asymptotically stable control system.

Apart from stability considerations, are we otherwise justified in separately designing the observer and the control law? In Section 5.3 we formulate a stochastic optimal regulation problem. The solution of this stochastic version of the problem leads to an affirmative answer to the question just posed.

In this section we have considered full-order observers only. It can be shown that reduced-order observers interconnected with state feedback laws also lead to closed-loop poles that consist of the observer poles together with the controller poles.

Example 5.1. *Position control system.*

Consider the positioning system described by the state differential equation (see Example 2.1, Section 2.2.2, and Example 2.4, Section 2.3)

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t), \tag{5-19}$$

with

$$\begin{aligned}\kappa &= 0.787 \text{ rad/(V s}^2),\\ \alpha &= 4.6 \text{ s}^{-1}.\end{aligned} \tag{5-20}$$

The control law

$$\mu(t) = -(f_1, f_2)\hat{x}(t) \tag{5-21}$$

produces the regulator characteristic polynomial

$$\det(sI - A + BF) = s^2 + (\alpha + \kappa f_2)s + \kappa f_1. \tag{5-22}$$

By choosing

$$\begin{aligned} f_1 &= 254.1 \text{ V/rad}, \\ f_2 &= 19.57 \text{ V s/rad}, \end{aligned} \tag{5-23}$$

the regulator poles are placed at $-10 \pm j10 \text{ s}^{-1}$. Let us consider the observer

$$\dot{\hat{x}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}\hat{x}(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix}\mu(t) + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}[\eta(t) - (1, \quad 0)\hat{x}(t)], \tag{5-24}$$

where it is assumed that

$$\eta(t) = (1, \quad 0)x(t) \tag{5-25}$$

is the observed variable. The observer characteristic polynomial is

$$\det(sI - A + KC) = s^2 + (\alpha + k_1)s + \alpha k_1 + k_2. \tag{5-26}$$

To make the observer fast as compared to the regulator, we place the observer poles at $-50 \pm j50 \text{ s}^{-1}$. This yields for the gains:

$$\begin{aligned} k_1 &= 95.40 \text{ s}^{-1}, \\ k_2 &= 4561 \text{ s}^{-2}. \end{aligned} \tag{5-27}$$

In Fig. 5.3 we sketch the response of the output feedback system to the initial state $x(0) = \text{col}(0.1, 0)$, $\hat{x}(0) = 0$. For comparison we give in Fig. 5.4 the response of the corresponding state feedback system, where the control law **5-21** is directly connected to the state. We note that in the system with an observer, the observer very quickly catches up with the actual behavior of the state. Because of the slight time lag introduced by the observer, however, a greater input is required and the response is somewhat different from that of the system without an observer.

Example 5.2. *The pendulum positioning system.*

In this example we discuss the pendulum positioning system of Example 1.1 (Section 1.2.3). The state differential equation of this system is given by

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\dfrac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{g}{L'} & 0 & \dfrac{g}{L'} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ 0 \end{pmatrix}\mu(t). \tag{5-28}$$

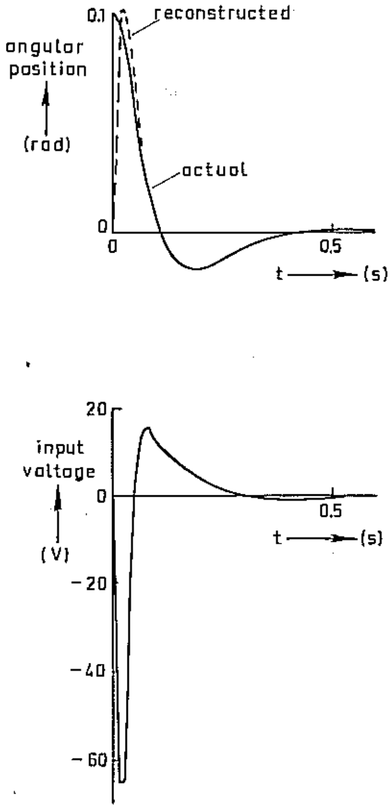

Fig. 5.3. The response and the input of the position control system with observer for $x(0) = \text{col}(0.1, 0)$; $\hat{x}(0) = \text{col}(0, 0)$.

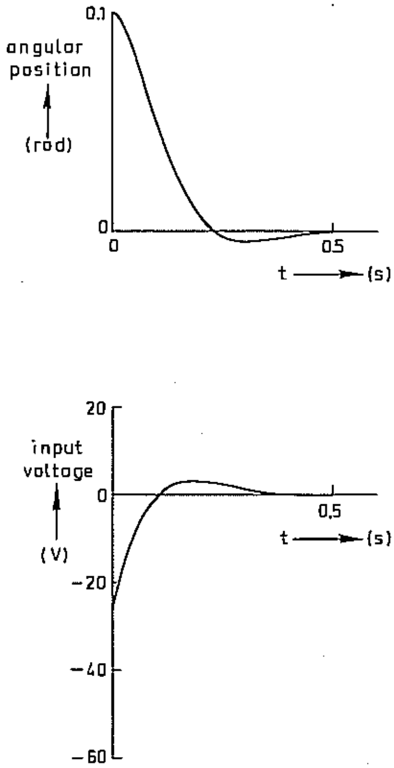

Fig. 5.4. The response and the input of the position control system with state feedback (without observer) for $x(0) = \text{col}(0.1, 0)$.

The components of the state are

$$\begin{aligned} \xi_1(t) &= s(t), \\ \xi_2(t) &= \dot{s}(t), \\ \xi_3(t) &= s(t) + L'\phi(t), \\ \xi_4(t) &= \dot{s}(t) + L'\dot{\phi}(t). \end{aligned} \qquad \text{5-29}$$

Here $s(t)$ is the displacement of the carriage and $\phi(t)$ the angle the pendulum makes with the vertical. We assume that both these quantities can be measured. This yields for the observed variable

$$y(t) = \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\dfrac{1}{L'} & 0 & \dfrac{1}{L'} & 0 \end{pmatrix} x(t). \qquad \text{5-30}$$

The main function of the control system is to stabilize the system. We therefore choose as the controlled variable the position of the pendulum

$$\zeta(t) = \xi_3(t) = s(t) + L'\phi(t). \tag{5-31}$$

We first select the regulator poles by solving the regulator problem with the criterion

$$\int_{t_0}^{\infty} [\zeta^2(t) + \rho\mu^2(t)]\, dt. \tag{5-32}$$

To determine an appropriate value of ρ, we select it such that the estimated radius ω_0 of the faraway poles such as given in Theorem 3.11 (Section 3.8.1) is $10\ \mathrm{s}^{-1}$. This yields a settling time of roughly $10/\omega_0 = 1$ s. It follows from the numerical values of Example 1.1 that the oscillation period of the pendulum is $2\pi\sqrt{L'/g} \simeq 1.84$ s, so that we have chosen the settling time somewhat less than the oscillation period.

To compute ρ from ω_0, we must know the transfer function $H(s)$ of the system from the input force μ to the controlled variable ζ. This transfer function is given by

$$H(s) = \frac{-\dfrac{g}{L'M}}{s\left(s + \dfrac{F}{M}\right)\left(s^2 - \dfrac{g}{L'}\right)}. \tag{5-33}$$

It follows with **3-486** that

$$\omega_0 = \left[\frac{(g/L'M)^2}{\rho}\right]^{1/8}. \tag{5-34}$$

With the numerical values of Example 1.1, it can be found that we must choose

$$\rho = 10^{-6}\ \mathrm{m^2/N^2} \tag{5-35}$$

to make ω_0 approximately $10\ \mathrm{s}^{-1}$. It can be computed that the resulting steady-state gain matrix is given by

$$\bar{F} = (389.0,\quad 26.91,\quad -1389,\quad -282.4), \tag{5-36}$$

while the closed-loop poles are $-9.870 \pm j3.861$ and $-4.085 \pm j9.329\ \mathrm{s}^{-1}$. Figure 5.5 gives the response of the state feedback control system to the initial state $s(0) = 0$, $\dot{s}(0) = 0$, $\phi(0) = 0.1$ rad ($\simeq 6°$), $\dot{\phi}(0) = 0$. It is seen that the input force assumes values up to about 100 N, the carriage displacement undergoes an excursion of about 0.3 m, and the maximal pendulum displacement is about 0.08 m.

Assuming that this performance is acceptable, we now proceed to determine an observer for the system. Since we have two observed variables, there is considerable freedom in choosing the observer gain matrix in order to

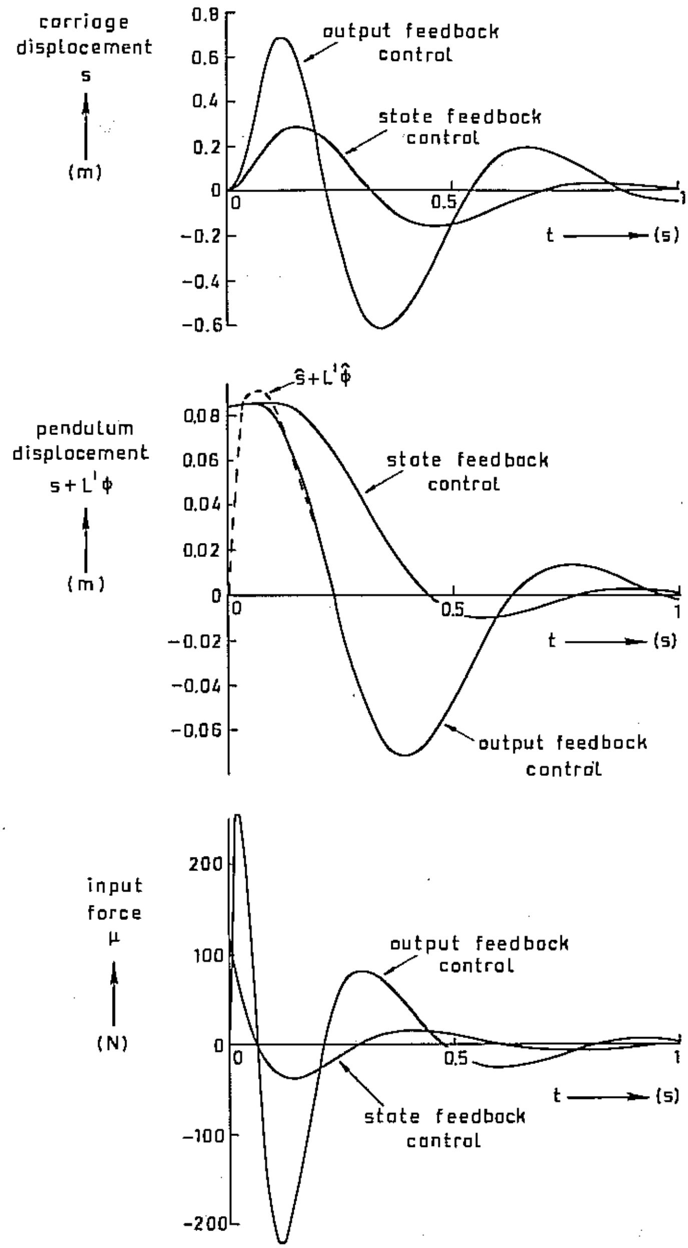

Fig. 5.5. Responses of the state feedback and output feedback pendulum-balancing systems to the initial state $x(0) = \text{col}(0, 0, 0.0842, 0)$ [the observer initial state is $\hat{x}(0) = 0$].

attain a given set of observer poles. To simplify the problem we impose the restriction that the first component of the observed variable (the displacement) is used only to reconstruct the state of the carriage (i.e., ξ_1 and ξ_2), and the second component of the observed variable is used only to reconstruct the motion of the pendulum (i.e., ξ_3 and ξ_4). Thus we assume the following structure of the observer:

$$\dot{\hat{x}}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{g}{L'} & 0 & \frac{g}{L'} & 0 \end{pmatrix} \hat{x}(t) + \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ 0 \end{pmatrix} \mu(t)$$
$$+ \begin{pmatrix} k_1 & 0 \\ k_2 & 0 \\ 0 & k_3 \\ 0 & k_4 \end{pmatrix} \left[y(t) - \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{L'} & 0 & \frac{1}{L'} & 0 \end{pmatrix} \hat{x}(t) \right]. \qquad \text{5-37}$$

Here the gains k_1, k_2, k_3, and k_4 are to be determined. It is easily found that with the structure of 5-37 the observer characteristic polynomial is given by

$$\left[s^2 + s\left(k_1 + \frac{F}{M}\right) + \frac{F}{M} k_1 + k_2 \right] \left[s^2 + s \frac{k_3}{L'} + \frac{k_4 - g}{L'} \right]. \qquad \text{5-38}$$

It is clearly seen that one pair of poles governs the speed of reconstruction of the motion of the carriage, and the other that of the pendulum. We now choose the gains k_1 to k_4 such that both pairs of poles are somewhat further away from the origin than the regulator poles obtained above. There is no point in choosing the observer poles very far away, since the resulting high observer gains will give difficulties in the implementation without improving the control system response very much. We thus select both pairs of observer poles as

$$21.2(-1 \pm j)\ \mathrm{s}^{-1}.$$

The distance of these poles to the origin is $30\ \mathrm{s}^{-1}$. It can be found with the numerical values of Example 1.1 that to achieve these observer poles the gains must be chosen as

$$\begin{aligned} k_1 &= 41.4, \qquad & k_3 &= 35.6, \\ k_2 &= 859, \qquad & k_4 &= 767. \end{aligned} \qquad \text{5-39}$$

Figure 5.5 also gives the response of the interconnection of the resulting observer with the control law and the pendulum positioning system to the same initial conditions as before, with $\hat{x}(0) = 0$. The estimate $\hat{s}(t)$ of the carriage displacement is not shown in the figure since it coincides with the actual carriage displacement right from the beginning owing to the special set of initial conditions. It is seen that the estimate $\hat{s} + L'\hat{\phi}$ of the pendulum displacement $s + L'\phi$ very quickly catches up with the correct value. Nevertheless, because of the slight time lag in the reconstruction process, the motion of the output feedback pendulum balancing system is more violent than in the state feedback case. From a practical point of view, this control system is probably not acceptable because the motion is too violent and the system moves well out of the range where the linearization is valid; very likely the pendulum will topple. A solution can be sought in decreasing ρ so as to damp the motion of the system. An alternative solution is to make the observer faster, but this may cause difficulties with noise in the system.

5.2.2* Conditions for Pole Assignment and Stabilization of Output Feedback Control Systems

In this section we state the precise conditions on the system described by **5-3** and **5-4** such that there exist an observer **5-6** and a control law **5-7** that make the closed-loop control system **5-9** asymptotically stable (G. W. Johnson, 1969; Potter and VanderVelde, 1969):

Theorem 5.2. *Consider the interconnection of the system*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t), \end{aligned} \qquad \textbf{5-40}$$

the observer

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + K(t)[y(t) - C(t)\hat{x}(t)], \qquad \textbf{5-41}$$

and the control law

$$u(t) = -F(t)\hat{x}(t). \qquad \textbf{5-42}$$

Then sufficient conditions for the existence of gain matrices $K(t)$ and $F(t)$, $t \geq t_0$, such that the interconnected system is exponentially stable, are that the system **5-40** *be uniformly completely controllable and uniformly completely reconstructible or that it be exponentially stable. In the time-invariant situation (i.e., all matrices occurring in* **5-40, 5-41,** *and* **5-42** *are constant), necessary and sufficient conditions for the existence of stabilizing gain matrices K and F are that the system* **5-40** *be both stabilizable and detectable. In the time-invariant case, necessary and sufficient conditions for arbitrary assignment of both the regulator and the observer poles (within the restriction that complex* *poles occur in complex conjugate pairs) are that the system be completely controllable and completely reconstructible.*

The proof of this theorem is based upon Theorems 3.1 (Section 3.2.2), 3.2 (Section 3.2.2), 3.6 (Section 3.4.2), 4.3 (Section 4.2.2), 4.4 (Section 4.2.2), and 4.10 (Section 4.4.3).

5.3 OPTIMAL LINEAR REGULATORS WITH INCOMPLETE AND NOISY MEASUREMENTS

5.3.1 Problem Formulation and Solution

In this section we formulate the optimal linear regulator problem when the observations of the system are *incomplete* and *inaccurate*, that is, the complete state vector cannot be measured, and the measurements that are available are noisy. In addition, we assume that the system is subject to stochastically varying disturbances. The precise formulation of this problem is as follows.

Definition 5.1. *Consider the system*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + w_1(t), \qquad t \geq t_0, \\ x(t_0) &= x_0, \end{aligned} \tag{5-43}$$

where x_0 is a stochastic vector with mean $\bar{x}_0$ and variance matrix Q_0. The observed variable is given by

$$y(t) = C(t)x(t) + w_2(t), \qquad t \geq t_0. \tag{5-44}$$

The joint stochastic process col (w_1, w_2) *is a white noise process with intensity*

$$\begin{pmatrix} V_1(t) & V_{12}(t) \\ V_{12}^T(t) & V_2(t) \end{pmatrix}, \qquad t \geq t_0. \tag{5-45}$$

The controlled variable can be expressed as

$$z(t) = D(t)x(t), \qquad t \geq t_0. \tag{5-46}$$

Then the ***stochastic linear optimal output feedback regulator problem*** *is the problem of finding the functional*

$$u(t) = f[y(\tau), t_0 \leq \tau \leq t], \qquad t_0 \leq t \leq t_1, \tag{5-47}$$

such that the criterion

$$\sigma = E\left\{\int_{t_0}^{t_1} [z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\, dt + x^T(t_1)P_1x(t_1)\right\} \tag{5-48}$$

is minimized. Here $R_3(t)$, $R_2(t)$, and P_1 are symmetric weighting matrices such that $R_2(t) > 0$, $R_3(t) > 0$, $t_0 \leq t \leq t_1$, and $P_1 \geq 0$.

The solution of this problem is, as expected, the combination of the solutions of the stochastic optimal regulator problem of Chapter 3 (Theorem 3.9, Section 3.6.3) and the optimal reconstruction problem of Chapter 4. This rather deep result is known as the *separation principle* and is stated in the following theorem.

Theorem 5.3. *The optimal linear solution of the stochastic linear optimal output feedback regulator problem is the same as the solution of the corresponding stochastic optimal state feedback regulator problem (Theorem 3.9, Section 3.6.3) except that in the control law the state $x(t)$ is replaced with its minimum mean square linear estimator $\hat{x}(t)$, that is, the input is chosen as*

$$u(t) = -F^0(t)\hat{x}(t), \qquad \textbf{5-49}$$

where $F^0(t)$ is the gain matrix given by **3-344** *and $\hat{x}(t)$ is the output of the optimal observer derived in Sections 4.3.2, 4.3.3, and 4.3.4 for the nonsingular uncorrelated, nonsingular correlated, and the singulor cases, respectively.*

An outline of the proof of this theorem for the nonsingular uncorrelated case is given in Section 5.3.3. We remark that the solution as indicated is the best *linear* solution. It can be proved (Wonham, 1968b, 1970b; Fleming, 1969; Kushner, 1967, 1971) that, if the processes w_1 and w_2 are Gaussian white noise processes and the initial state x_0 is Gaussian, the optimal linear solution is *the* optimal solution (without qualification).

Restricting ourselves to the case where the problem of estimating the state is nonsingular and the state excitation and observation noises are uncorrelated, we now write out in detail the solution to the stochastic linear output feedback regulator problem. For the input we have

$$u(t) = -F^0(t)\hat{x}(t), \qquad \textbf{5-50}$$

with

$$F^0(t) = R_2^{-1}(t)B^T(t)P(t). \qquad \textbf{5-51}$$

Here $P(t)$ is the solution of the Riccati equation

$$\begin{aligned} -\dot{P}(t) &= D^T(t)R_3(t)D(t) - P(t)B(t)R_2^{-1}(t)B^T(t)P(t) \\ &\qquad + A^T(t)P(t) + P(t)A(t), \qquad \textbf{5-52} \\ P(t_1) &= P_1. \end{aligned}$$

The estimate $\hat{x}(t)$ is obtained as the solution of

$$\begin{aligned} \dot{\hat{x}}(t) &= A(t)\hat{x}(t) + B(t)u(t) + K^0(t)[y(t) - C(t)\hat{x}(t)], \\ \hat{x}(t_0) &= \bar{x}_0, \end{aligned} \qquad \textbf{5-53}$$

where

$$K^0(t) = Q(t)C^T(t)V_2^{-1}(t). \tag{5-54}$$

The variance matrix $Q(t)$ is the solution of the Riccati equation

$$\begin{aligned}\dot{Q}(t) &= V_1(t) - Q(t)C^T(t)V_2^{-1}(t)C(t)Q(t) + A(t)Q(t) + Q(t)A^T(t),\\ Q(t_0) &= Q_0.\end{aligned} \tag{5-55}$$

Figure 5.6 gives a block diagram of this stochastic optimal output feedback control system.

5.3.2 Evaluation of the Performance of Optimal Output Feedback Regulators

We proceed by analyzing the performance of optimal output feedback control systems, still limiting ourselves to the nonsingular case with uncorrelated state excitation and observation noises. The interconnection of the system **5-43**, the optimal observer **5-53**, and the control law **5-50** forms a system of dimension $2n$, where n is the dimension of the state x. Let us define, as before, the reconstruction error

$$e(t) = x(t) - \hat{x}(t). \tag{5-56}$$

It is easily obtained from Eqs. **5-43**, **5-53**, and **5-50** that the augmented vector col $[e(t), \hat{x}(t)]$ satisfies the differential equation

$$\begin{pmatrix}\dot{e}(t)\\ \dot{\hat{x}}(t)\end{pmatrix} = \begin{pmatrix}A(t) - K^0(t)C(t) & 0\\ K^0(t)C(t) & A(t) - B(t)F^0(t)\end{pmatrix}\begin{pmatrix}e(t)\\ \hat{x}(t)\end{pmatrix} + \begin{pmatrix}I & -K^0(t)\\ 0 & K^0(t)\end{pmatrix}\begin{pmatrix}w_1(t)\\ w_2(t)\end{pmatrix}, \tag{5-57}$$

with the initial condition

$$\begin{pmatrix}e(t_0)\\ \hat{x}(t_0)\end{pmatrix} = \begin{pmatrix}x(t_0) - \bar{x}_0\\ \bar{x}_0\end{pmatrix}. \tag{5-58}$$

The reason that we consider col $(e, \hat{x})$ is that the variance matrix of this augmented vector is relatively easily found, as we shall see. All mean square quantities of interest can then be obtained from this variance matrix. Let us denote the variance matrix of col $[e(t), \hat{x}(t)]$ as

$$E\left\{\begin{pmatrix}e(t) - E\{e(t)\}\\ \hat{x}(t) - E\{\hat{x}(t)\}\end{pmatrix}([e(t) - E\{e(t)\}]^T, [\hat{x}(t) - E\{\hat{x}(t)\}]^T)\right\} = \begin{pmatrix}Q_{11}(t) & Q_{12}(t)\\ Q_{12}^T(t) & Q_{22}(t)\end{pmatrix}. \tag{5-59}$$

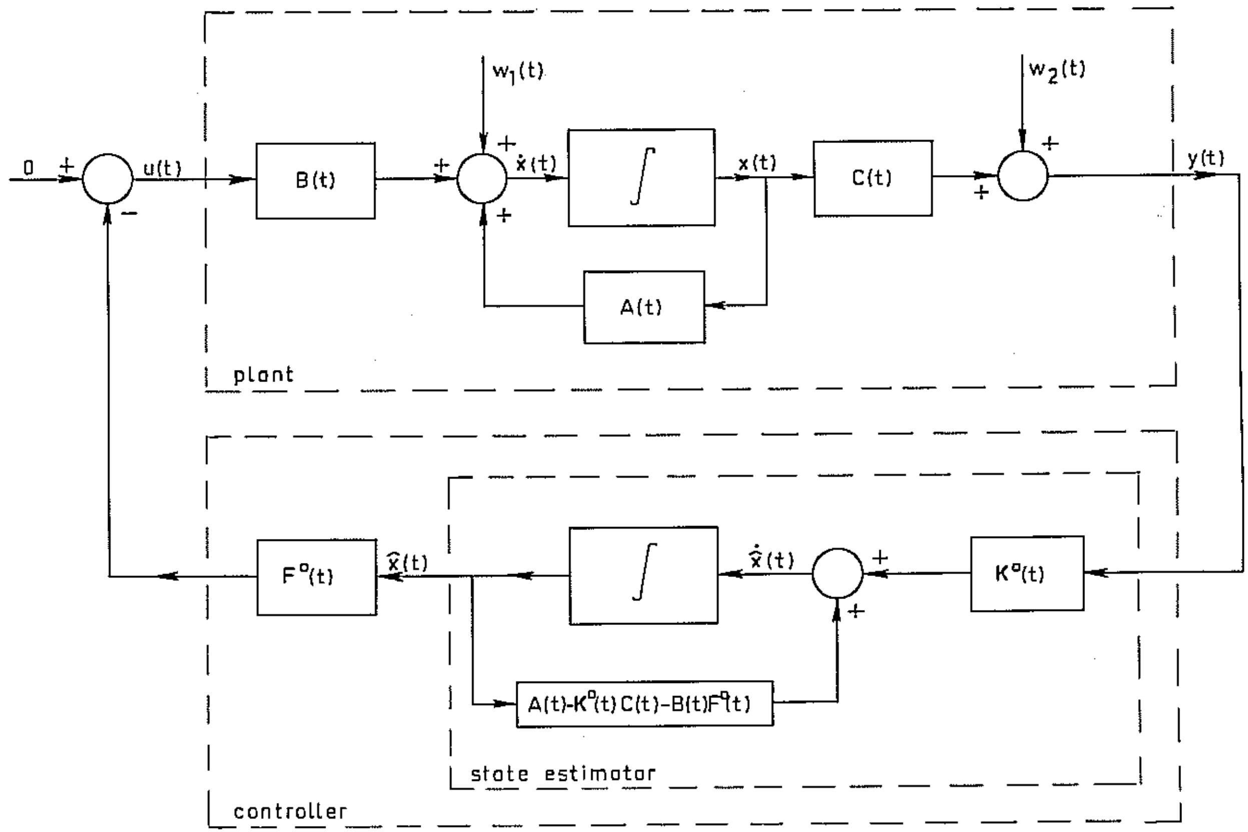

Fig. 5.6. The optimal linear regulator with incomplete and noisy measurements.

The differential equations for the matrices Q_{11}, Q_{12}, and Q_{22} can be obtained by application of Theorem 1.52 (Section 1.11.2). It easily follows that these matrices satisfy the equations:

$$\begin{aligned}
\dot{Q}_{11}(t) &= [A(t) - K^0(t)C(t)]Q_{11}(t) + Q_{11}(t)[A(t) - K^0(t)C(t)]^T \\
&\qquad + V_1(t) + K^0(t)V_2(t)K^{0T}(t), \\
\dot{Q}_{12}(t) &= Q_{11}(t)C^T(t)K^{0T}(t) + Q_{12}(t)[A(t) - B(t)F^0(t)]^T \\
&\qquad + [A(t) - K^0(t)C(t)]Q_{12}(t) - K^0(t)V_2(t)K^{0T}(t), \\
\dot{Q}_{22}(t) &= Q_{12}^T(t)C^T(t)K^{0T}(t) + Q_{22}(t)[A(t) - B(t)F^0(t)]^T + K^0(t)C(t)Q_{12}(t) \\
&\qquad + [A(t) - B(t)F^0(t)]Q_{22}(t) + K^0(t)V_2(t)K^{0T}(t),
\end{aligned} \tag{5-60}$$

with the initial conditions

$$Q_{11}(t_0) = Q_0, \qquad Q_{12}(t_0) = 0, \qquad Q_{22}(t_0) = 0. \tag{5-61}$$

When considering these equations, we immediately note that of course

$$Q_{11}(t) = Q(t), \qquad t \geq t_0. \tag{5-62}$$

As a result, in the differential equation for $Q_{12}(t)$ the terms $Q_{11}(t)C^T(t)K^{0T}(t)$ and $-K^0(t)V_2(t)K^{0T}(t)$ cancel because $K^0(t) = Q(t)C^T(t)V_2^{-1}(t)$. What is left of the equation for $Q_{12}(t)$ is a homogeneous differential equation in $Q_{12}(t)$ with the initial condition $Q_{12}(t_0) = 0$, which of course has the solution

$$Q_{12}(t) = 0, \qquad t \geq t_0. \tag{5-63}$$

Apparently, $e(t)$ and $\hat{x}(t)$ are uncorrelated stochastic processes. This is why we have chosen to work with the joint process col $(e, \hat{x})$. Note that $e(t)$ and $\hat{x}(t)$ are uncorrelated no matter how the input to the plant is chosen. The reason for this is that the behavior of the reconstruction error e is independent of that of the input u, and the contribution of the input $u(\tau)$, $t_0 \leq \tau \leq t$, to the reconstructed state $\hat{x}(t)$ is a known quantity which is subtracted to compute the covariance of $e(t)$ and $\hat{x}(t)$. We use this fact in the proof of the separation principle in Section 5.3.3.

The differential equation for $Q_{22}(t)$ now simplifies to

$$\begin{aligned}
\dot{Q}_{22}(t) &= [A(t) - B(t)F^0(t)]Q_{22}(t) + Q_{22}(t)[A(t) - B(t)F^0(t)]^T \\
&\qquad + K^0(t)V_2(t)K^{0T}(t),
\end{aligned} \tag{5-64}$$

with the initial condition

$$Q_{22}(t_0) = 0. \tag{5-65}$$

Once we have computed $Q_{22}(t)$, the variance matrix of the joint process col $(e, \hat{x})$ is known, and all mean square quantities or integrated mean square quantities of interest can be obtained, since

$$x(t) = e(t) + \hat{x}(t). \tag{5-66}$$

Thus we can compute the mean square regulation error as

$$\begin{aligned} E\{z^T(t)W_e(t)z(t)\} &= E\{x^T(t)D^T(t)W_e(t)D(t)x(t) \\ &= \operatorname{tr}\,[D^T(t)W_e(t)D(t)E\{x(t)x^T(t)\}] \qquad \text{5-67} \\ &= \operatorname{tr}\,\{D^T(t)W_e(t)D(t)[\bar{x}(t)\bar{x}^T(t) + Q_{11}(t) + Q_{22}(t)]\}, \end{aligned}$$

where $W_e(t)$ is the weighting matrix and $\bar{x}(t)$ is the mean of $x(t)$. Similarly, we can compute the mean square input as

$$\begin{aligned} E\{u^T(t)W_u(t)u(t)\} &= E\{\hat{x}^T(t)F^{0T}(t)W_u(t)F^0(t)\hat{x}(t)\} \\ &= \operatorname{tr}\,[F^{0T}(t)W_u(t)F^0(t)E\{\hat{x}(t)\hat{x}^T(t)\}] \\ &= \operatorname{tr}\,\{F^{0T}(t)W_u(t)F^0(t)[\bar{x}(t)\bar{x}^T(t) + Q_{22}(t)]\}, \qquad \text{5-68} \end{aligned}$$

where $W_u(t)$ is the weighting matrix of the mean square input.

It follows that in order to compute the optimal regulator gain matrix $F^0(t)$, the optimal filter gain matrix $K^0(t)$, the mean square regulation error, and the mean square input one must solve *three* $n \times n$ matrix differential equations: the Riccati equation **5-52** to obtain $P(t)$ and from this $F^0(t)$, the Riccati equation **5-55** to determine $Q(t)$ and from this $K^0(t)$, and finally the linear matrix differential equation **5-64** to obtain the variance matrix $Q_{22}(t)$ of $\hat{x}(t)$. In the next theorem, however, we state that if the mean square regulation error and the mean square input are not required separately, but only the value of the criterion σ as given by **5-48** is required, then merely the basic Riccati equations for $P(t)$ and $Q(t)$ need be solved.

Theorem 5.4. *Consider the stochastic regulator problem of Definition 5.1. Suppose that*

$$V_2(t) > 0, \qquad V_{12}(t) = 0 \quad \textit{for all } t. \qquad \text{5-69}$$

Then the following facts hold:

(a) *All mean square quantities of interest can be obtained from the variance matrix* diag $[Q(t), Q_{22}(t)]$ *of* col $[e(t), \hat{x}(t)]$, *where* $e(t) = x(t) - \hat{x}(t)$, $Q(t)$ *is the variance matrix of* $e(t)$, *and* $Q_{22}(t)$ *can be obtained as the solution of the matrix differential equation*

$$\begin{aligned} \dot{Q}_{22}(t) = [A(t) - B(t)F^0(t)]Q_{22}(t) + Q_{22}(t)[A(t) - B(t)F^0(t)]^T \\ + K^0(t)V_2(t)K^{0T}(t), \qquad t \geq t_0, \qquad \text{5-70} \end{aligned}$$

$$Q_{22}(t_0) = 0.$$

(b) *The minimal value of the criterion* **5-48** *can be expressed in the following two alternative forms*

$$\sigma^0 = \bar{x}_0^T P(t_0)\bar{x}_0 + \operatorname{tr}\left\{\int_{t_0}^{t_1}[P(t)K^0(t)V_2(t)K^{0T}(t) + Q(t)R_1(t)]\,dt + P_1Q(t_1)\right\}$$

5-71

and

$$\sigma^0 = \bar{x}_0^T P(t_0)\bar{x}_0 + \operatorname{tr}\left\{P(t_0)Q_0 + \int_{t_0}^{t_1}[P(t)V_1(t) + Q(t)F^{0T}(t)R_2(t)F^0(t)]\,dt\right\}. \tag{5-72}$$

Here we have abbreviated

$$R_1(t) = D^T(t)R_3(t)D(t), \tag{5-73}$$

and $P(t)$ and $Q(t)$ are the solutions of the Riccati equations **5-52** *and* **5-55**, *respectively.*

(c) *Furthermore, if the aptimal observer and regulator Riccati equations have the steady-state solutions $\bar{Q}(t)$ and $\bar{P}(t)$ as $t_0 \to -\infty$ and $t_1 \to \infty$, respectively, then the time-averaged criterion*

$$\bar{\sigma} = \lim_{\substack{t_0 \to -\infty \\ t_1 \to \infty}} \frac{1}{t_1 - t_0} E\left\{\int_{t_0}^{t_1}[z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\,dt\right\}, \tag{5-74}$$

if it exists, can be expressed in the alternative forms

$$\bar{\sigma} = \lim_{\substack{t_0 \to -\infty \\ t_1 \to \infty}} \frac{1}{t_1 - t_0} \operatorname{tr}\left\{\int_{t_0}^{t_1}[\bar{P}(t)\bar{K}(t)V_2(t)\bar{K}^T(t) + \bar{Q}(t)R_1(t)]\,dt\right\} \tag{5-75}$$

and

$$\bar{\sigma} = \lim_{\substack{t_0 \to -\infty \\ t \to \infty}} \frac{1}{t_1 - t_0} \operatorname{tr}\left\{\int_{t_0}^{t_1}[\bar{P}(t)V_1(t) + \bar{Q}(t)\bar{F}^T(t)R_2(t)\bar{F}(t)]\,dt\right\}. \tag{5-76}$$

Here $\bar{K}(t)$ and $\bar{F}(t)$ are the gains corresponding to the steady-state solutians $\bar{Q}(t)$ and $\bar{P}(t)$, respectively.

(d) *Finally, in the time-invariant case, where $\bar{Q}(t)$ and $\bar{P}(t)$ and thus also $\bar{F}(t)$ and $\bar{K}(t)$ are constant matrices, the following expressions hold:*

$$\begin{aligned}\bar{\sigma} &= E\{z^T(t)R_3z(t) + u^T(t)R_2u(t)\}\\ &= \operatorname{tr}[\bar{P}\bar{K}V_2\bar{K}^T + \bar{Q}R_1] &&\text{5-77a}\\ &= \operatorname{tr}[\bar{P}V_1 + \bar{Q}\bar{F}^TR_2\bar{F}]. &&\text{5-77b}\end{aligned}$$

This theorem can be proved as follows. Setting $W_e(t) = R_3(t)$ and $W_u(t) = R_2(t)$ in **5-67** and **5-68**, we write for the criterion

$$\begin{aligned}&E\left\{\int_{t_0}^{t_1}[z^T(t)R_3(t)z(t) + u^T(t)R_2(t)u(t)]\,dt + x^T(t_1)P_1x(t_1)\right\}\\ &\quad= \int_{t_0}^{t_1}[\bar{z}^T(t)R_3(t)\bar{z}(t) + \bar{u}^T(t)R_2(t)\bar{u}(t)]\,dt + \bar{x}^T(t_1)P_1\bar{x}(t_1)\\ &\qquad+ \operatorname{tr}\left\{\int_{t_0}^{t_1}[R_1(t)[Q(t) + Q_{22}(t)] + F^{0T}(t)R_2(t)F^0(t)Q_{22}(t)]\,dt\right.\\ &\qquad\qquad\left.+ P_1[Q(t_1) + Q_{22}(t_1)]\right\}. \end{aligned} \tag{5-78}$$

Let us separately consider the expression

$$\operatorname{tr}\left\{\int_{t_0}^{t_1}[R_1(t) + F^{0T}(t)R_2(t)F^0(t)]Q_{22}(t)\,dt + P_1Q_{22}(t_1)\right\}, \qquad \textbf{5-79}$$

where, as we know, $Q_{22}(t)$ is the solution of the matrix differential equation

$$\begin{aligned}\dot{Q}_{22}(t) &= [A(t) - B(t)F^0(t)]Q_{22}(t) + Q_{22}(t)[A(t) - B(t)F^0(t)]^T \\ &\qquad + K^0(t)V_2(t)K^{0T}(t), \qquad \textbf{5-80} \\ Q_{22}(t_0) &= 0.\end{aligned}$$

It is not difficult to show (Problem 5.5) that **5-79** can be written in the form

$$\operatorname{tr}\left\{\int_{t_0}^{t_1} S(t)K^0(t)V_2(t)K^{0T}(t)\,dt\right\}, \qquad \textbf{5-81}$$

where $S(t)$ is the solution of the matrix differential equation

$$\begin{aligned}-\dot{S}(t) &= [A(t) - B(t)F^0(t)]^TS(t) + S(t)[A(t) - B(t)F^0(t)] + R_1(t) \\ &\qquad + F^{0T}(t)R_2(t)F^0(t), \qquad \textbf{5-82} \\ S(t_1) &= P_1\end{aligned}$$

Obviously, the solution of this differential equation is

$$S(t) = P(t), \qquad t \le t_1. \qquad \textbf{5-83}$$

Combining these results, and using the fact that the first two terms of the right-hand side of **5-78** can be replaced with $\bar{x}_0^TP(t_0)\bar{x}_0$, we obtain the desired expression **5-71** from **5-78**.

The alternative expression **5-72** for the criterion can be obtained by substituting

$$R_1(t) = P(t)B(t)R_2^{-1}(t)B^T(t)P(t) - A^T(t)P(t) - P(t)A(t) - \dot{P}(t) \qquad \textbf{5-84}$$

into **5-71** and integrating by parts. The proofs of parts (c) and (d) of Theorem 5.4 follow from **5-71** and **5-72** by letting $t_0 \to -\infty$ and $t_1 \to \infty$.

Of course in any practical situation in which $t_1 - t_0$ is large, we use the steady-state gain matrices $\bar{K}(t)$ and $\bar{F}(t)$ even when $t_1 - t_0$ is not infinite. Particularly, we do so in the time-invariant case, where $\bar{K}$ and $\bar{F}$ are constant. From optimal regulator and observer theory and in view of Section 5.2, we know that the resulting *steady-state output feedback control system* is asymptotically stable whenever the corresponding state feedback regulator and observer are asymptotically stable.

Before concluding this section with an example, two remarks are made. First, we note that in the time-invariant steady-state case the following lower bounds follow from **5-77a** and **5-77b**:

$$\lim_{R_2 \to 0} \bar{\sigma} \geq \operatorname{tr}(\bar{Q}R_1), \qquad \textbf{5-85a}$$

$$\lim_{V_2 \to 0} \bar{\sigma} \geq \operatorname{tr}(\bar{P}V_1). \qquad \textbf{5-85b}$$

These inequalities can be interpreted as follows. Even if we do not at all weight the input u, and thus do not constrain the input amplitude, the criterion $\bar{\sigma}$ still cannot be less than tr $(\bar{Q}R_1)$ according to **5-85a**. This minimum contribution to the criterion is caused by the unavoidable inaccuracy in reconstructing the state. Similarly, even when no measurement noise is present, that is, V_2 approaches zero, the criterion $\bar{\sigma}$ cannot be less then tr $(\bar{P}V_1)$. This value is not surprising since it is exactly the value of the criterion for the state feedback stochastic regulator (see Theorem 3.9, Section 3.6.3).

The second remark concerns the locations of the control system poles in the time-invariant steady-state case. In Section 5.2 we saw that the control system poles consist of the regulator poles and the observer poles. It seems a good rule of thumb that the weighting matrices R_2 and V_2 be chosen so that the regulator poles and the observer poles have distances to the origin of roughly the same order of magnitude. It seems to be wasteful to have very fast regulation when the reconstruction process is slow, and vice versa. In particular, when there is a great deal of observation noise as compared to the state excitation noise, the observer poles are relatively close to the origin and the reconstruction process is slow. When we now make the regulator just a little faster than the observer, it is to be expected that the regulator can keep up with the observer. A further increase in the speed of the regulator will merely increase the mean square input without decreasing the mean square regulation error appreciably. On the other hand, when there is very little observation noise, the limiting factor in the design will be the permissible mean square input. This will constrain the speed of the regulator, and there will be very little point in choosing an observer that is very much faster, even though the noise conditions would permit it.

Example 5.3. *Position control system.*

Let us consider the position control system discussed in many previous examples. Its state differential equation is

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t). \qquad \textbf{5-86}$$

Here $x(t) = \operatorname{col}[\xi_1(t), \xi_2(t)]$, with $\xi_1(t)$ the angular position and $\xi_2(t)$ the angular velocity of the system. The input variable $\mu(t)$ is the input voltage.

The controlled variable is the position, hence is given by

$$\zeta(t) = (1, \quad 0)x(t). \tag{5-87}$$

In Example 3.8 (Section 3.4.1), we solved the deterministic regulator problem with the criterion

$$\int_{t_0}^{t_1} [\zeta^2(t) + \rho\mu^2(t)]\, dt. \tag{5-88}$$

With the numerical values

$$\begin{aligned} \kappa &= 0.787 \text{ rad/(V s}^2), \\ \alpha &= 4.6 \text{ s}^{-1}, \\ \rho &= 0.00002 \text{ rad}^2/\text{V}^2, \end{aligned} \tag{5-89}$$

we found the steady-state feedback gain matrix

$$\bar{F} = (223.6, \quad 18.69). \tag{5-90}$$

The steady-state solution of the regulator Riccati equation is given by

$$\bar{P} = \begin{pmatrix} 0.1098 & 0.005682 \\ 0.005682 & 0.0004753 \end{pmatrix}. \tag{5-91}$$

The closed-loop regulator poles are $-9.66 \pm j9.09\ \text{s}^{-1}$. From Fig. 3.9 (Section 3.4.1), we know that the settling time of the system is of the order of 0.3 s, while an initial deviation in the position of 0.1 rad causes an input voltage with an initial peak value of 25 V.

In Example 4.4 (Section 4.3.2), we assumed that the system is disturbed by an external torque on the shaft $\tau_d(t)$. This results in the following modification of the state differential equation:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t) + \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \tau_d(t), \tag{5-92}$$

where $1/\gamma$ is the rotational moment of inertia of the rotating parts. It was furthermore assumed that the observed variable is given by

$$\eta(t) = (1, \quad 0)x(t) + \nu_m(t), \tag{5-93}$$

where $\nu_m(t)$ represents the observation noise. This expression implies that the angular displacement is measured. Under the assumption that $\tau_d(t)$ and $\nu_m(t)$ are adequately represented as uncorrelated white noise processes with intensities

$$V_d = 10 \text{ N}^2 \text{ m}^2 \text{ s} \tag{5-94}$$

and

$$V_m = 10^{-7} \text{ rad}^2 \text{ s}, \tag{5-95}$$

respectively, we found in Example 4.4 that with $\gamma = 0.1\ \text{kg}^{-1}\ \text{m}^{-2}$ the steady-state optimal observer is given by

$$\dot{\hat{x}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}\hat{x}(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix}\mu(t) + \bar{K}[\eta(t) - (1, \quad 0)\hat{x}(t)], \tag{5-96}$$

where the steady-state gain matrix is

$$\bar{K} = \begin{pmatrix} 40.36 \\ 814.3 \end{pmatrix}. \tag{5-97}$$

The observer poles are $-22.48 \pm j22.24\ \text{s}^{-1}$, while the steady-state variance matrix is given by

$$\bar{Q} = \begin{pmatrix} 0.04036 \times 10^{-4} & 0.8143 \times 10^{-4} \\ 0.8143 \times 10^{-4} & 36.61 \times 10^{-4} \end{pmatrix}. \tag{5-98}$$

With

$$\mu(t) = -\bar{F}\hat{x}(t), \tag{5-99}$$

the steady-state optimal output feedback controller is described by

$$\begin{aligned} \dot{\hat{x}}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}\hat{x}(t) - \begin{pmatrix} 0 \\ \kappa \end{pmatrix}\bar{F}\hat{x}(t) + \bar{K}[\eta(t) - (1, \quad 0)\hat{x}(t)], \\ \mu(t) &= -\bar{F}\hat{x}(t). \end{aligned} \tag{5-100}$$

It follows that

$$\lim_{t_0 \to -\infty} E\{\zeta^2(t) + \rho\mu^2(t)\} = \operatorname{tr}(\bar{P}\bar{K}V_2\bar{K}^T + \bar{Q}R_1) = 0.00009080\ \text{rad}^2. \tag{5-101}$$

From this result we find the following bounds on the steady-state rms tracking error and rms input voltage:

$$\lim_{t_0 \to -\infty} \sqrt{E\{\zeta^2(t)\}} < \sqrt{0.00009080} \simeq 0.0095\ \text{rad}, \tag{5-102a}$$

$$\lim_{t_0 \to -\infty} E\{\rho\mu^2(t)\} < 0.00009080\ \text{rad}^2, \tag{5-102b}$$

so that

$$\lim_{t_0 \to -\infty} \sqrt{E\{\mu^2(t)\}} < \sqrt{\frac{0.00009080}{\rho}} \simeq 2.13\ \text{V}. \tag{5-103}$$

The exact values of the steady-state rms tracking error and rms input voltage must be obtained by solving for the steady-state variance matrix of the augmented state col $[x(t), \hat{x}(t)]$. As outlined in the text (Section 5.3.2), this is most efficiently done by first computing the steady-state variance matrix diag $(\bar{Q}_{11}, \bar{Q}_{22})$ of col $[e(t), \hat{x}(t)]$, which requires only the solution of an additional 2×2 linear matrix equation. It can be found that the steady-state variance matrix $\bar{\Pi}$ of col $[x(t), \hat{x}(t)]$ is given by

$$\bar{\Pi} = \begin{pmatrix} \bar{Q}_{11} + \bar{Q}_{22} & \bar{Q}_{22} \\ \bar{Q}_{22} & \bar{Q}_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 0.00004562 & 0 & 0.00004158 & -0.00008145 \\ 0 & 0.006119 & -0.00008145 & 0.002458 \\ 0.00004158 & -0.00008145 & 0.00004158 & -0.00008145 \\ -0.00008145 & 0.002458 & -0.00008145 & 0.002458 \end{pmatrix}. \tag{5-104}$$

This yields for the steady-state mean square tracking error

$$\lim_{t_0 \to -\infty} E\{\zeta^2(t)\} = \mathrm{tr}\left[\bar{\Pi} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} (1, \quad 0, \quad 0, \quad 0)\right] = 0.00004562 \text{ rad}^2, \tag{5-105}$$

so that the rms tracking error is $\sqrt{0.00004562} \simeq 0.00674$ rad. We see that this is somewhat less than the bound **5-102**. Similarly, we obtain for the mean square input voltage

$$\lim_{t_0 \to -\infty} E\{\mu^2(t)\} = \mathrm{tr}\left[\bar{\Pi} \begin{pmatrix} 0 \\ \bar{F}^T \end{pmatrix} (0, \quad \bar{F})\right] = 2.258 \text{ V}^2, \tag{5-106}$$

so that the rms input voltage is about 1.5 V. It depends, of course, on the specifications of the system whether or not this performance is satisfactory.

It is noted that the regulator poles $(-9.66 \pm j9.09)$ and the observer poles $(-22.48 \pm j22.24)$ are of the same order of magnitude, which is a desirable situation. Had we found that, for example, the observer poles are very far away as compared to the regulator poles, we could have moved the observer poles closer to the origin without appreciable loss in performance.

5.3.3* Proof of the Separation Principle

In this section we prove the separation principle as stated in Theorem 5.3 for the nonsingular uncorrelated case, that is, we assume that the intensity $V_2(t)$ of the observation noise is positive-definite and that $V_{12}(t) = 0$ on $[t_0, t_1]$. It is relatively straightforward to prove that the solution as given is the best *linear* solution of the stochastic linear output feedback regulator problem. Denoting

$$R_1(t) = D^T(t)R_3(t)D(t), \tag{5-107}$$

we write

$$\begin{aligned}
E[z^T(t)R_3(t)z(t)]& \\
&= E[x^T(t)R_1(t)x(t)] \\
&= E\{[x(t) - \hat{x}(t) + \hat{x}(t)]^T R_1(t)[x(t) - \hat{x}(t) + \hat{x}(t)]\} \\
&= E\{[x(t) - \hat{x}(t)]^T R_1(t)[x(t) - \hat{x}(t)]\} \\
&\quad + 2E\{[x(t) - \hat{x}(t)]^T R_1(t)\hat{x}(t)\} + E\{\hat{x}^T(t)R_1(t)\hat{x}(t)\}.
\end{aligned} \tag{5-108}$$

Here $\hat{x}(t)$ is the minimum mean square linear estimator of $x(t)$ operating on $y(\tau)$ and $u(\tau)$, $t_0 \le \tau \le t$. From optimal observer theory we know that

$$E\{[x(t) - \hat{x}(t)]^T R_1(t)[x(t) - \hat{x}(t)]\} = \text{tr}\,[R_1(t)Q(t)], \tag{5-109}$$

where $Q(t)$ is the variance matrix of the reconstruction error $x(t) - \hat{x}(t)$. Furthermore,

$$E\{[x(t) - \hat{x}(t)]^T R_1(t)\hat{x}(t)\} = \text{tr}\,[E\{[x(t) - \hat{x}(t)]\hat{x}^T(t)\}R_1(t)] = 0, \tag{5-110}$$

since as we have seen in Section 5.3.2 the quantities $e(t) = x(t) - \hat{x}(t)$ and $\hat{x}(t)$ are uncorrelated. Thus we find that we can write

$$\begin{aligned}
E\{x^T(t)R_1(t)x(t)\} &= \text{tr}\,[R_1(t)Q(t)] + E\{\hat{x}^T(t)R_1(t)\hat{x}(t)\}, \\
E\{x^T(t_1)P_1x(t_1)\} &= \text{tr}\,[P_1Q(t_1)] + E\{\hat{x}^T(t_1)P_1\hat{x}(t_1)\}.
\end{aligned} \tag{5-111}$$

Using **5-111**, we write for the criterion **5-48**:

$$E\left\{\int_{t_0}^{t_1}[\hat{x}^T(t)R_1(t)\hat{x}(t) + u^T(t)R_2(t)u(t)]\,dt + \hat{x}^T(t_1)P_1\hat{x}(t_1)\right\} \\
+ \text{tr}\left[\int_{t_0}^{t_1}R_1(t)Q(t)\,dt + P_1Q(t_1)\right]. \tag{5-112}$$

We observe that the last two terms in this expression are independent of the control applied to the system. Also from optimal observer theory, we know that we can write (since by assumption the reconstruction problem is nonsingular)

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + K^0(t)[y(t) - C(t)\hat{x}(t)], \tag{5-113}$$

where $K^0(t)$ is the optimal gain matrix. However, in Section 4.3.6 we found that the innovation process $y(t) - C(t)\hat{x}(t)$ is a white noise process with intensity $V_2(t)$. ~~Then the problem of minimizing the criterion **5-112**, with the behavior of $\hat{x}(t)$ described by **5-113**, is a stochastic linear regulator problem where the complete state can be observed, such as described in Section 3.6.1.~~ It follows from Theorem 3.9 that the optimal *linear* solution of this state feedback stochastic regulator problem is the linear control law

$$u(t) = -F^0(t)\hat{x}(t), \tag{5-114}$$

where $F^0(t)$ is given by 5-51.

This terminates the proof of Theorem 5.3 for the case where the reconstruction problem is nonsingular and the state excitation and observation noises are uncorrelated. The proof can be extended to the singular correlated case.

5.4 LINEAR OPTIMAL TRACKING SYSTEMS WITH INCOMPLETE AND NOISY MEASUREMENTS

In Section 3.6.2 we considered tracking problems as special cases of stochastic state feedback regulator problems. Necessarily, we found control laws that require that both the state of the plant and the state of the reference variable are available. In this section we consider a similar problem, but it is assumed that only certain linear combinations of the components of the state can be measured, which moreover are contaminated with additive noise. We furthermore assume that only the reference variable itself can be measured, also contaminated with white noise.

We thus adopt the following model for the reference variable $z_r(t)$:

$$z_r(t) = D_r(t)x_r(t), \tag{5-115}$$

where

$$\dot{x}_r(t) = A_r(t)x_r(t) + w_{r1}(t). \tag{5-116}$$

In this expression w_{r1} is white noise with intensity $V_{r1}(t)$. It is furthermore assumed that we observe

$$y_r(t) = z_r(t) + w_{r2}(t). \tag{5-117}$$

Here w_{r2} is white noise with intensity $V_{r2}(t)$.

The system to be controlled is described by the state differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w_1(t), \tag{5-118}$$

where w_1 is white noise with intensity $V_1(t)$. The system has the controlled variable

$$z(t) = D(t)x(t) \tag{5-119}$$

and the observed variable

$$y(t) = C(t)x(t) + w_2(t). \tag{5-120}$$

Here w_2 is white noise with intensity $V_2(t)$. We assume that $V_{r2}(t) > 0$, $V_2(t) > 0$, $t_0 \le t \le t_1$.

To obtain an optimization problem, we consider the criterion

$$E\left\{\int_{t_0}^{t_1}[[z(t)-z_r(t)]^TR_3(t)[z(t)-z_r(t)]+u^T(t)R_2(t)u(t)]\,dt\right\}. \qquad \textbf{5-121}$$

Here $R_3(t) > 0$, $R_2(t) > 0$, $t_0 \leq t \leq t_1$. The first term of the integrand serves to force the controlled variable $z(t)$ to follow the reference variable $z_r(t)$, while the second term constrains the amplitudes of the input.

We now phrase the stochastic optimal tracking problem with incomplete and noisy observations as follows.

Definition 5.2. *Consider the system*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w_1(t), \qquad t \geq t_0, \qquad \textbf{5-122}$$

where $x(t_0)$ is a stochastic variable with mean $\bar{x}_0$ and variance matrix Q_0, and w_1 is white noise with intensity $V_1(t)$. The controlled variable is

$$z(t) = D(t)x(t), \qquad \textbf{5-123}$$

and the observed variable is

$$y(t) = C(t)x(t) + w_2(t), \qquad \textbf{5-124}$$

where w_2 is white noise with intensity $V_2(t)$, with $V_2(t) > 0$, $t_0 \leq t \leq t_1$. Consider furthermore the reference variable

$$z_r(t) = D_r(t)x_r(t), \qquad \textbf{5-125}$$

where

$$\dot{x}_r(t) = A_r(t)x_r(t) + w_{r1}(t), \qquad t \geq t_0. \qquad \textbf{5-126}$$

Here $x_r(t_0)$ is a stochastic variable with mean $\bar{x}_{r0}$ and variance matrix Q_{r0}, and w_{r1} is white noise with intensity $V_{r1}(t)$. The observed variable for the x_r process is

$$y_r(t) = C_r(t)x_r(t) + w_{r2}(t), \qquad \textbf{5-127}$$

where w_{r2} is white noise with intensity $V_{r2}(t) > 0$, $t_0 \leq t \leq t_1$. Then the ***optimal linear tracking problem with incomplete and noisy observations*** *is the problem of choosing the input to the system* **5-122** *as a function of $y(\tau)$ and $y_r(\tau)$, $t_0 \leq \tau \leq t$, such that the criterion*

$$E\left\{\int_{t_0}^{t_1}[[z(t)-z_r(t)]^TR_3(t)[z(t)-z_r(t)]+u^T(t)R_2(t)u(t)]\,dt\right\} \qquad \textbf{5-128}$$

is minimized, where $R_3(t) > 0$ and $R_2(t) > 0$ for $t_0 \leq t \leq t_1$.

To solve the problem we combine the reference model and the plant in an augmented system. In terms of the augmented state $\tilde{x}(t) = \text{col}\,[x(t), x_r(t)]$, we write

$$\dot{\tilde{x}}(t) = \begin{pmatrix} A(t) & 0 \\ 0 & A_r(t) \end{pmatrix}\tilde{x}(t) + \begin{pmatrix} B(t) \\ 0 \end{pmatrix}u(t) + \begin{pmatrix} w_1(t) \\ w_{r1}(t) \end{pmatrix}. \qquad \textbf{5-129}$$

The observed variable for the augmented system is

$$\begin{pmatrix} y(t) \\ y_r(t) \end{pmatrix} = \begin{pmatrix} C(t) & 0 \\ 0 & C_r(t) \end{pmatrix} \tilde{x}(t) + \begin{pmatrix} w_2(t) \\ w_{r2}(t) \end{pmatrix}. \qquad \textbf{5-130}$$

For the criterion we write

$$E\left\{\int_{t_0}^{t_1} [\tilde{x}^T(t)\tilde{D}^T(t)R_3(t)\tilde{D}(t)\tilde{x}(t) + u^T(t)R_2(t)u(t)]\, dt\right\}, \qquad \textbf{5-131}$$

where

$$\tilde{D}(t) = [D(t), \quad -D_r(t)]. \qquad \textbf{5-132}$$

The tracking problem is now in the form of a standard stochastic regulator problem and can be solved by application of Theorem 5.3. It follows that we can write

$$u(t) = -F^0(t)\begin{pmatrix} \hat{x}(t) \\ \hat{x}_r(t) \end{pmatrix}. \qquad \textbf{5-133}$$

If we assume that all the white noise processes and initial values associated with the plant and the reference process are uncorrelated, two separate observers can be constructed, one for the state of the plant and one for the state of the reference process. Furthermore, we know from Section 3.6.3 that because of the special structure of the tracking problem we can write

$$F^0(t) = [F_1(t), \quad -F_2(t)], \qquad \textbf{5-134}$$

where the partitioning is consistent with the other partitionings, and where the feedback gain matrix $F_1(t)$ is completely independent of the properties of the reference process.

Figure 5.7 gives the block diagram of the optimal tracking system, still under the assumption that two separate observers can be used. It is seen that

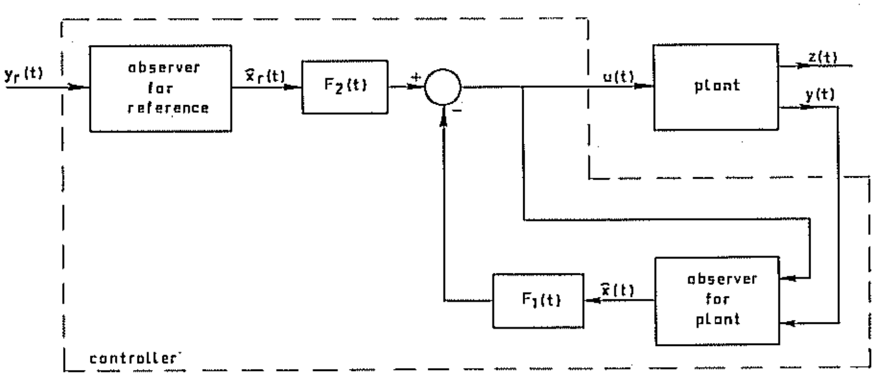

Fig. 5.7. The structure of the optimal tracking system.

the feedback link of the controller is completely independent of the properties of the reference variable.

We conclude this section with an examination of the transmission $T(s)$ of the system in the steady-state time-invariant case. A simple way to find this transfer matrix is as follows. Set $x(0) = \hat{x}(0) = 0$, and assume that the system is free of noise. It follows that $x(t) = \hat{x}(t)$ for $t \geq 0$. We can thus completely omit the plant observer in the computation of $T(s)$ and substitute $x(t)$ wherever we find $\hat{x}(t)$. We thus have the following relations:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t),\\
z(t) &= Dx(t),\\
u(t) &= -\bar{F}_1 x(t) + \bar{F}_2 \hat{x}_r(t),\\
\dot{\hat{x}}(t) &= A_r \hat{x}_r(t) + \bar{K}_r[y_r(t) - C_r \hat{x}_r(t)].
\end{aligned}
\qquad \text{5-135}
$$

It easily follows that

$$\mathbf{Z}(s) = T(s)\mathbf{Y}_r(s), \qquad \text{5-136}$$

where $\mathbf{Z}(s)$ and $\mathbf{Y}_r(s)$ are the Laplace transforms of $z(t)$ and $y_r(t)$, and where

$$T(s) = D(sI - A + B\bar{F}_1)^{-1}B\bar{F}_2(sI - A_r + \bar{K}_r C_r)^{-1}\bar{K}_r. \qquad \text{5-137}$$

In general $T(0)$ does not equal the unit matrix, so that step changes in the reference cause a steady-state error. The reason for this is that the present control system has not been designed for steps in the reference variable. If it is important that the control system have a zero steady-state error to constant references, the design method suggested in the next section should be adopted. We finally note that in the transmission only the regulator poles and the reference observer poles occur, while the plant observer poles have been canceled.

Example 5.4. *Position servo*

We return to the by now familiar positioning system. Consider the problem of designing a control system such that the angular position tracks a reference variable. For the system itself, the disturbances, and the observation noise we use the equations and numerical data of Example 5.3 (Section 5.3.2). We model the reference variable as exponentially correlated noise:

$$\zeta_r(t) = \xi_r(t), \qquad \text{5-138}$$

with

$$\dot{\xi}_r(t) = -\frac{1}{\theta}\xi_r(t) + w_{r1}(t), \qquad t \geq t_0, \qquad \text{5-139}$$

Here w_{r1} is scalar white noise with constant intensity V_{r1}. It is assumed that the reference variable is observed with additive white noise, so that we measure

$$\eta_r(t) = \xi_r(t) + w_{r2}(t), \tag{5-140}$$

where w_{r2} has constant intensity V_{r2} and is uncorrelated with w_{r1}. The steady-state optimal observer for the reference process is easily computed. It is described by

$$\dot{\hat{\xi}}_r(t) = -\frac{1}{\theta}\,\hat{\xi}_r(t) + \bar{K}_r[\eta_r(t) - \hat{\xi}_r(t)], \tag{5-141}$$

where

$$\bar{K}_r = -\frac{1}{\theta} + \sqrt{\frac{1}{\theta^2} + \frac{V_{r1}}{V_{r2}}}. \tag{5-142}$$

The optimization criterion is expressed as

$$E\left\{\int_{t_0}^{t_1} [[\zeta(t) - \zeta_r(t)]^2 + \rho\mu^2(t)]\, dt\right\}. \tag{5-143}$$

The resulting steady-state control law is given by

$$\mu(t) = -\bar{F}_1\hat{x}(t) + \bar{F}_2\hat{\xi}_r(t). \tag{5-144}$$

$\bar{P}_{11}$ and $\bar{F}_1$ have been computed in Example 3.8 (Section 3.4.1), in which we obtained the following results:

$$\bar{P}_{11} = \begin{pmatrix} \dfrac{\sqrt{\rho}}{\kappa}\sqrt{\alpha^2 + \dfrac{2\kappa}{\sqrt{\rho}}} & \dfrac{\sqrt{\rho}}{\kappa} \\ \dfrac{\sqrt{\rho}}{\kappa} & \dfrac{\rho}{\kappa^2}\left(-\alpha + \sqrt{\alpha^2 + \dfrac{2\kappa}{\sqrt{\rho}}}\right) \end{pmatrix}, \tag{5-145}$$

$$\bar{F}_1 = \left(\frac{1}{\sqrt{\rho}}, \frac{1}{\kappa}\left(-\alpha + \sqrt{\alpha^2 + \frac{2\kappa}{\sqrt{\rho}}}\right)\right).$$

Using the results of Section 3.6.3, it can be found that

$$\bar{F}_2 = \frac{\dfrac{\kappa}{\rho}}{\dfrac{\kappa}{\sqrt{\rho}} + \dfrac{1}{\theta^2} + \dfrac{1}{\theta}\left(\alpha^2 + \dfrac{2\kappa}{\sqrt{\rho}}\right)^{1/2}}. \tag{5-146}$$

Since we now have the reference observer and the regulator gains available, we can use **5-137** to calculate the transmission $T(s)$ of the closed-loop tracking system. We obtain

$$T(s) = \frac{\kappa}{s^2 + s\left(\alpha^2 + \dfrac{2\kappa}{\sqrt{\rho}}\right)^{1/2} + \dfrac{\kappa}{\sqrt{\rho}}} \; \frac{\dfrac{\kappa}{\rho}}{\dfrac{\kappa}{\sqrt{\rho}} + \dfrac{1}{\theta^2} + \dfrac{1}{\theta}\left(\alpha^2 + \dfrac{2\kappa}{\sqrt{\rho}}\right)^{1/2}} \cdot \frac{-\dfrac{1}{\theta} + \left(\dfrac{1}{\theta^2} + \dfrac{V_{r1}}{V_{r2}}\right)^{1/2}}{s + \left(\dfrac{1}{\theta^2} + \dfrac{V_{r1}}{V_{r2}}\right)^{1/2}}. \qquad \text{5-147}$$

We note that the break frequency of the transmission is the least of the break frequency of the closed-loop plant and the break frequency of the reference observer. The break frequency of the closed-loop plant is ω_0, where $\omega_0^2 = \kappa/\sqrt{\rho}$, while the break frequency of the reference observer is

$$\left(\frac{1}{\theta^2} + \frac{V_{r1}}{V_{r2}}\right)^{1/2}. \qquad \text{5-148}$$

Which break frequency is the lowest depends upon the "signal-to-noise" ratio V_{r1}/V_{r2} of the reference variable and the value of ρ, which in turn is determined by the allowable input amplitudes to the plant. Let us first consider the effect of V_{r1}/V_{r2}. If the reference variable is accurately measured, (i.e., V_{r2} is small) the reference observer break frequency is high and the closed-loop feedback system break frequency will prevail. On the other hand, if the reference variable is inaccurately measured, the reference observer limits the total bandwidth of the system.

When we next consider the effect of the weighting factor ρ, we see that if ρ is small, that is, large input amplitudes are allowed, the closed-loop system break frequency is high and the reference observer determines the break frequency. Conversely, if ρ is large, the break frequency is limited by the closed-loop plant.

Let us assume the following numerical values for the reference process:

$$\begin{aligned} \theta &= 5 \text{ s}, \\ V_{r1} &= 0.4 \text{ rad}^2/\text{s}. \end{aligned} \qquad \text{5-149}$$

This makes the reference variable break frequency 0.2 rad/s, while the reference variable rms value is 1 rad. Let us furthermore assume that the reference variable measurement noise w_{r2} is exponentially correlated noise with rms value 0.181 rad and time constant 0.025 s. This makes the break frequency of the reference variable measurement noise 40 rad/s. Since this break frequency is quite high as compared to 0.2 rad/s, we approximate the measurement noise as white noise with density

$$V_{r2} = 2(0.1)^2 0.0816 = 0.001636 \text{ rad}^2/\text{s}. \tag{5-150}$$

With the numerical values **5-149** and **5-150**, we find for the reference observer break frequency the value

$$\left(\frac{1}{\theta^2} + \frac{V_{r1}}{V_{r2}}\right)^{1/2} \simeq 15.6 \text{ rad/s}. \tag{5-151}$$

Since the break frequency of the reference observer is less than the break frequency of 40 rad/s of the reference measurement noise, we conclude that it is justified to approximate this measurement noise as white noise.

We finally must determine the most suitable value of the weighting factor ρ. In order to do this, we evaluate the control law for various values of ρ and compute the corresponding rms tracking errors and rms input voltages. Omitting the disturbing torque τ_d and the system measurement noise ν_m we write for the system equations

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + b\mu(t),\\
\mu(t) &= -\bar{F}_1 x(t) + \bar{F}_2 \hat{\xi}_r(t),\\
\dot{\hat{\xi}}_r(t) &= -\frac{1}{\theta}\hat{\xi}_r(t) + \bar{K}_r[\eta_r(t) - \hat{\xi}_r(t)],\\
\dot{\xi}_r(t) &= -\frac{1}{\theta}\xi_r(t) + w_{r1}(t),\\
\eta_r(t) &= \xi_r(t) + w_{r2}(t).
\end{aligned} \tag{5-152}$$

Combining all these relations we obtain the augmented differential equation

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\hat{\xi}}_r(t) \\ \dot{\xi}_r(t) \end{pmatrix} = \begin{pmatrix} A - b\bar{F}_1 & b\bar{F}_2 & 0 \\ 0 & -\frac{1}{\theta} - \bar{K}_r & \bar{K}_r \\ 0 & 0 & -\frac{1}{\theta} \end{pmatrix} \begin{pmatrix} x(t) \\ \hat{\xi}_r(t) \\ \xi_r(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{K}_r w_{r2}(t) \\ w_{r1}(t) \end{pmatrix}. \tag{5-153}$$

From this equation we can set up and solve the steady-state variance matrix of the augmented state col $[x(t), \hat{\xi}_r(t), \xi_r(t)]$, and from this the steady-state rms tracking error and rms input voltage can be computed. Of course we can also use the technique of Section 5.3.2. Table 5.1 lists the results for decreasing values of the weighting coefficient ρ. Note that the contributions of the reference excitation noise w_{r1} and the reference measurement noise w_{r2} are given separately, together with their total contribution.

If the maximally allowable input voltage is about 100 V, the weighting coefficient ρ should certainly not be chosen less than 0.00001; for this value the rms input voltage is nearly 50 V. The corresponding rms tracking error is about 0.27 rad, which is still quite a large value as compared to the rms value of the reference variable of 1 rad. If this rms value is too large, the requirements on the reference variable bandwidth must be lowered. It should be remarked, however, that the values obtained for the rms tracking error and the rms input are probably larger than the actual values encountered, since modelling stochastic processes by exponentially correlated noise usually leads to power spectral density functions that decrease much slower with increasing frequency than actual density functions.

For $\rho = 0.00001$ it can be computed from **5-152** that the zero-frequency transmission is given by $T(0) = 0.8338$. This means that the proposed control system shows a considerable steady-state error when subjected to a constant reference variable. This phenomenon occurs, first, because exponentially correlated noise has relatively much of its power at high frequencies and, second, because the term that weights the input in the optimization criterion tends to keep the input small, at the expense of the tracking accuracy. In the following section we discuss how tracking systems with a zero steady-state error can be obtained.

The rms values given in Table 5.1 do not include the contributions of the system disturbances and observation errors. Our findings in Example 5.3 suggest, however, that these contributions are negligible as compared to those of the reference variable.

5.5 REGULATORS AND TRACKING SYSTEMS WITH NONZERO SET POINTS AND CONSTANT DISTURBANCES

5.5.1 Nonzero Set Points

As we saw in Chapter 2, sometimes it is important to design tracking systems that show a zero steady-state error response to constant values of the reference variable. The design method of the preceding section can never produce such tracking systems, since the term in the optimization criterion that weights the input always forces the input to a smaller value, at the expense of a nonzero tracking error. For small weights on the input, the steady-state tracking error decreases, but it never disappears completely. In this section we approach the problem of obtaining a zero steady-state tracking error,

Table 5.1 The Effect of the Weighting Factor ρ on the Performance of the Position Servo System

ρ	Contribution of reference variable to rms tracking error (rad)	Contribution of reference measurement noise to rms tracking error (rad)	Total rms tracking error (rad)	Contribution of reference variable to rms input voltage (V)	Contribution of reference measurement noise to rms input voltage (V)	Total rms input voltage (V)
0.1	0.8720	0.0038	0.8720	1.438	0.222	1.455
0.01	0.6884	0.0125	0.6885	4.365	0.825	4.442
0.001	0.4942	0.0280	0.4950	10.32	2.69	10.67
0.0001	0.3524	0.0472	0.3556	21.84	8.15	23.31
0.00001	0.2596	0.0664	0.2680	43.03	23.08	48.82

as in Section 3.7.1, from the point of view of a variable set point. Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \textbf{5-154}$$

with the controlled variable

$$z(t) = Dx(t). \qquad \textbf{5-155}$$

In Section 3.7.1 we derived the nonzero set point optimal control law

$$u(t) = -\bar{F}x(t) + H_c^{-1}(0)z_0. \qquad \textbf{5-156}$$

$\bar{F}$ is the steady-state gain matrix for the criterion

$$\int_{t_0}^{\infty} [z^T(t)R_3z(t) + u^T(t)R_2u(t)]\,dt, \qquad \textbf{5-157}$$

while $H_c(s)$ is the closed-loop transfer matrix

$$H_c(s) = D(sI - A + B\bar{F})^{-1}B. \qquad \textbf{5-158}$$

It is assumed that the dimension of u equals that of z, and that the open-loop transfer matrix $H(s) = D(sI - A)^{-1}B$ has no zeroes at the origin. These assumptions guarantee the existence of $H_c^{-1}(0)$. Finally, z_0 is the set point for the controlled variable. The control law **5-156** causes the control system to reach the set point optimally from any initial state, and to make an optimal transition to the new set point whenever z_0 changes.

Let us now consider a stochastic version of the nonzero set point regulator problem. We assume that the plant is described by

$$\dot{x}(t) = Ax(t) + Bu(t) + w_1(t), \qquad \textbf{5-159}$$

where w_1 is white noise. The controlled variable again is

$$z(t) = Dx(t), \qquad \textbf{5-160}$$

but we introduce an observed variable

$$y(t) = Cx(t) + w_2(t), \qquad \textbf{5-161}$$

where w_2 is also white noise. Suppose that the set point z_0 for the controlled variable of this system is accurately known. Then the nonzero set point steady-state optimal controller for this system obviously is

$$\begin{aligned} u(t) &= -\bar{F}\hat{x}(t) + H_c^{-1}(0)z_0, \\ \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + \bar{K}[y(t) - C\hat{x}(t)], \end{aligned} \qquad \textbf{5-162}$$

where $\bar{K}$ is the steady-state optimal observer gain and where $\bar{F}$ and $H_c(s)$ are as given before. If no state excitation noise and observation noise are present, the controlled variable will eventually approach z_0 as t increases.

The control law is optimal in the sense that the steady-state value of

$$E\{z^T(t)R_3z(t) + u^T(t)R_2u(t)\} \tag{5-163}$$

is minimized, where z and u are taken relative to their set points. When the set point changes, an optimal transition to the new set point is made.

The controller described by **5-162** may give quite good results when the set point z_0 is a slowly varying quantity. Unsatisfactory results may be obtained when the set point occasionally undergoes step changes. This may result in the input having too large a transient, necessitating reduction in the loop gain of the system. This in turn deteriorates the disturbance suppression properties of the system. This difficulty can be remedied by interpreting quick changes in the set point as "noise." Thus we write the control law **5-162** in the form

$$u(t) = -\bar{F}\hat{x}(t) + H_c^{-1}(0)\hat{z}_0(t), \tag{5-164}$$

where $\hat{z}_0(t)$ is the estimated set point. The observed set point, $r(t)$, is represented as

$$r(t) = z_0(t) + w_u(t), \tag{5-165}$$

where w_u is white noise and z_0 is the actual set point. In order to determine $\hat{z}_0(t)$ (compare Example 4.3, Section 4.3.2, on the estimation of a constant), we model z_0 as

$$\dot{z}_0(t) = w_0(t), \tag{5-166}$$

where w_0 is another white noise process. The steady-state optimal observer for the set point will be of the form

$$\dot{\hat{z}}_0(t) = \bar{K}_0[r(t) - \hat{z}_0(t)], \tag{5-167}$$

where $\bar{K}_0$ is the appropriate steady-state observer gain matrix.

The controller defined by **5-164** and **5-167** has the property that, if no noise is present and the observed set point $r(t)$ is constant, the controlled variable will in the steady state precisely equal $r(t)$. This follows from **5-167**, since in the steady state $\hat{z}_0(t) \equiv r(t)$ so that in **5-164** $\hat{z}_0(t)$ is replaced with $r(t)$, which in turn causes $z(t)$ to assume the value $r(t)$. It is seen that in the case where r, z_0, u, and z are scalar the *prefilter* (see Fig. 5.8) defined by **5-164** and **5-167** is nothing but a first-order filter. In the multidimensional case a generalization of this first-order filter is obtained. When the components of the uncorrelated white noise processes w_0 and w_u are assumed to be uncorrelated as well, it is easily seen that $\bar{K}_0$ is diagonal, so that the prefilter consists simply of a parallel bank of scalar first-order filters. It is suggested that the time constants of these filters be determined on the basis of the desired response to steps in the components of the reference variable and in relation to likely step sizes and permissible input amplitudes.

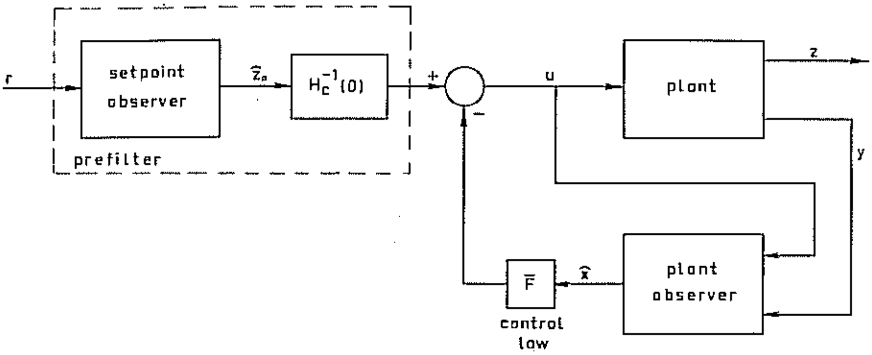

Fig. 5.8. Nonzero set point optimal controller with set point observer.

Example 5.5. *The positioning system*

In Example 5.3 (Section 5.3.2), we found a zero set point optimal controller for the positioning system. Let us determine the corresponding nonzero set point control system. We first determine the nonzero set point optimal control law. It follows from Example 3.8 (Section 3.4.1) that the closed-loop transfer function $H_c(s)$ is given by

$$H_c(s) = \frac{\kappa}{s^2 + s\left(\alpha^2 + \dfrac{2\kappa}{\sqrt{\rho}}\right)^{1/2} + \dfrac{\kappa}{\sqrt{\rho}}}. \qquad \textbf{5-168}$$

Consequently, the nonzero set point control law **5-164** is

$$\mu(t) = -F\hat{x}(t) + \frac{1}{\sqrt{\rho}}\hat{\zeta}_0(t), \qquad \textbf{5-169}$$

where $\hat{\zeta}_0(t)$ is the estimated set point. Let us design for step changes in the observed set point. The observer **5-167** for the set point is of the form

$$\dot{\hat{\zeta}}_0(t) = k_0[r(t) - \hat{\zeta}_0(t)], \qquad \textbf{5-170}$$

where $r(t)$ is the reference variable and k_0 a scalar gain factor. Using the numerical values of Example 5.3, we give in Fig. 5.9 the responses of the nonzero set point control system defined by **5-169** and **5-170** to a step of 1 rad in the reference variable $r(t)$ for various values of the gain k_0. Assuming that an input voltage of up to 100 V is tolerable, we see that a suitable value of k_0 is about $20\ \mathrm{s}^{-1}$. The corresponding time constant of the prefilter is $1/k_0 = 0.05$ s.

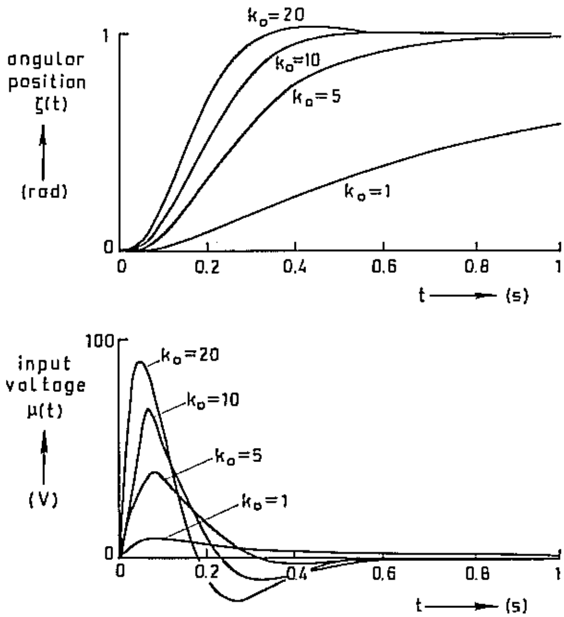

Fig. 5.9. Responses of the position control system as nonzero set point control system to a step in the set point of 1 rad for various values of the prefilter gain k_0.

5.5.2* Constant Disturbances

In the preceding section we discussed nonzero set point regulators. In the present section the question of constant disturbances is investigated, which is somewhat similar to the nonzero set point problem. The approach presented in this section is somewhat different from that in Section 3.7.2. As in Section 3.7.2, however, controllers with integrating action will be obtained.

Constant disturbances frequently occur in control problems. Often they are caused by inaccuracies in determining consistent nominal values of the input, the state, and the controlled variable. These disturbances can usually be represented through an additional constant forcing term v_0 in the state differential equation as follows:

$$\dot{x}(t) = Ax(t) + Bu(t) + v_0. \qquad \textbf{5-171}$$

As in the preceding section, we limit our discussion to the time-invariant case. For the controlled variable we write

$$z(t) = Dx(t). \qquad \textbf{5-172}$$

Let us assume, for the time being, that the complete state $x(t)$ can be observed at all times. Then we can consider the control law

$$u(t) = -\bar{F}x(t) + u_0, \tag{5-173}$$

where $\bar{F}$ is a gain matrix selected according to some quadratic optimization criterion of the usual form and where the constant vector u_0 is to be chosen such that in steady-state conditions the contribution of the constant disturbance v_0 to the controlled variable z is canceled.

With the control law **5-173**, the closed-loop system equations are

$$\begin{aligned} \dot{x}(t) &= (A - B\bar{F})x(t) + Bu_0 + v_0, \\ z(t) &= Dx(t). \end{aligned} \tag{5-174}$$

Since the closed-loop system will be assumed to be asymptotically stable, the controlled variable eventually approaches a constant value, which is easily seen to be given by

$$\lim_{t\to\infty} z(t) = D(-\bar{A})^{-1}Bu_0 + D(-\bar{A})^{-1}v_0. \tag{5-175}$$

Here we have abbreviated

$$\bar{A} = A - B\bar{F}. \tag{5-176}$$

Does there exist a u_0 such that the steady-state value of $z(t)$ as given by **5-175** is zero? As in the nonzero set point problem, three cases must be distinguished:

(a) *The dimension of z is greater than that of u:* In this case the vector equation

$$D(-\bar{A})^{-1}Bu_0 + D(-\bar{A})^{-1}v_0 = 0 \tag{5-177}$$

represents more equations than there are variables, which means that in general no solution exists. This is the case where it is attempted to control the variable $z(t)$ with an input $u(t)$ of smaller dimension and too few degrees of freedom are available.

(b) *The dimensions of u and z are the same:* In this case **5-177** can be solved for u_0 as follows:

$$u_0 = -H_c^{-1}(0)D(-\bar{A})^{-1}v_0. \tag{5-178}$$

Here $H_c(s)$ is the closed-loop transfer matrix

$$H_c(s) = D(sI - A - B\bar{F})^{-1}B. \tag{5-179}$$

As we know from Theorem 3.10 (Section 3.7), the inverse of $H_c(0)$ exists if the open-loop system transfer matrix $D(sI - A)^{-1}B$ has no zeroes at the origin.

(c) *The dimension of z is less than that of u:* In this case there are too many degrees of freedom and the dimension of z can be increased by adding components to the controlled variable.

In case (b), where dim (z) = dim (u), the control law

$$u(t) = -\bar{F}x(t) - H_c^{-1}(0)D(-\bar{A})^{-1}v_0 \qquad \textbf{5-180}$$

has the property that constant disturbances are compensated in an optimal manner. This control law, which has been given by Eklund (1969), will be referred to as the *zero-steady-state-error optimal control law*. As we have seen, it exists when dim (z) = dim (u) and the open-loop system has no zeroes at the origin.

Let us now suppose that in addition to v_0 fluctuating disturbances act upon the system as well, and that the system state can only be incompletely and inaccurately observed. We thus replace the state differential equation with

$$\dot{x}(t) = Ax(t) + Bu(t) + v_0 + w_1(t), \qquad \textbf{5-181}$$

where v_0 is the constant disturbance and w_1 white noise with intensity V_1. Furthermore, we assume that we have for the observed variable

$$y(t) = Cx(t) + w_2(t), \qquad \textbf{5-182}$$

where w_2 is white noise with intensity V_2.

In this situation the control law **5-180** must be replaced by

$$u(t) = -\bar{F}\hat{x}(t) - H_c^{-1}(0)D(-\bar{A})^{-1}\hat{v}_0, \qquad \textbf{5-183}$$

where $\hat{x}(t)$ and $\hat{v}_0$ are the minimum mean square estimates of $x(t)$ and v_0. An optimal observer can be obtained by modeling the constant disturbance through

$$\dot{v}_0(t) = 0. \qquad \textbf{5-184}$$

The resulting *steady-state* optimal observer, however, will have a zero gain matrix for updating the estimate of v_0, since according to the model **5-184** the value of v_0 never changes (compare Example 4.3, Section 4.3.2, concerning the estimation of a constant). Since in practice v_0 varies slowly, or occasionally changes value, it is better to model v_0 through

$$\dot{v}_0(t) = w_0(t), \qquad \textbf{5-185}$$

where the intensity V_0 of the white noise w_0 is so chosen that the increase in the fluctuations of v_0 reflects the likely variations in the slowly varying disturbance. When this model is used, the resulting steady-state optimal observer continues to track $v_0(t)$ and is of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + \hat{v}_0(t) + \bar{K}_1[y(t) - C\hat{x}(t)], \\ \dot{\hat{v}}_0(t) &= \bar{K}_2[y(t) - C\hat{x}(t)]. \end{aligned} \qquad \textbf{5-186}$$

The control system that results from combining this observer with the control law **5-183** has the property that in the absence of other disturbances and observation noise the constant disturbance is always compensated so that a zero steady-state regulation or tracking error results (Eklund, 1969). As expected, this is achieved by "integrating action" of the controller (see Problem 2.3). The procedure of this section enables us to introduce this integrating action and at the same time improve the transient response of the control system and the suppression of fluctuating disturbances. The procedure is equally easily applied to multivariable as to single-input single-output systems.

It is not difficult to see that the procedure of this section can be combined with that of Section 5.5.1 when encountering tracking or regulating systems subject to nonzero set points as well as constant disturbances, by choosing the input as

$$u(t) = -\bar{F}\hat{x}(t) - H_c^{-1}(0)D(-\bar{A})^{-1}\hat{v}_0 + H_c^{-1}(0)\hat{z}_0. \qquad \textbf{5-187}$$

Here $\hat{z}_0$ is either the estimated set point and can be obtained as described in Section 5.5.1, or is the actual set point.

We remark that often is it is possible to trace back the constant disturbances to one or two sources. In such a case we can replace v_0 with

$$v_0 = Gv_1, \qquad \textbf{5-188}$$

where G is a given matrix and v_1 a constant disturbance of a smaller dimension than v_0. By modeling v_1 as integrated white noise, the dimension of the observer can be considerably decreased in this manner.

Example 5.6. *Integral control of the positioning system*

In this example we devise an integral control system for the positioning system. We assume that a constant disturbance can enter into the system in the form of a constant torque τ_0 on the shaft in addition to a disturbing torque τ_d which varies quickly. Thus we modify the state differential equation **5-92** of Example 5.3 (Section 5.3.2) to

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t) + \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \tau_d(t) + \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \tau_0. \qquad \textbf{5-189}$$

As in Example 5.3, we represent the variable part of the disturbing torque as white noise with intensity V_d.

It is easily seen from **5-189** that the zero-steady-state-error optimal control law is given by

$$\mu(t) = -\bar{F}\hat{x}(t) - \frac{\gamma}{\kappa}\hat{\tau}_0, \qquad \textbf{5-190}$$

where $\bar{F}$ is an appropriate steady-state optimal feedback gain matrix, and $\hat{\tau}_0$ is an estimate of τ_0.

To obtain an observer we model the constant part of the disturbance as

$$\dot{\tau}_0(t) = w_0(t), \tag{5-191}$$

where the white noise w_0 has intensity V_0. As in Example 5.3, the observed variable is given by

$$\eta(t) = (1, 0)x(t) + v_m(t), \tag{5-192}$$

where v_m is white noise with intensity V_m. The steady-state optimal observer thus has the form

$$\begin{aligned}\dot{\hat{x}}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}\hat{x}(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix}\mu(t) + \begin{pmatrix} 0 \\ \gamma \end{pmatrix}\hat{\tau}_0(t) + \begin{pmatrix} \bar{k}_1 \\ \bar{k}_2 \end{pmatrix}[\eta(t) - (1, 0)\hat{x}(t)], \\ \dot{\hat{\tau}}_0(t) &= \bar{k}_3[\eta(t) - (1, 0)\hat{x}(t)],\end{aligned} \tag{5-193}$$

where the scalar gains $\bar{k}_1$, $\bar{k}_2$, and $\bar{k}_3$ follow from the steady-state solution of the appropriate observer Riccati equation. With the numerical values of Example 5.3, and with the additional numerical value

$$V_0 = 60 \text{ N}^2\text{ m}^2\text{ s}^{-1}, \tag{5-194}$$

it follows that these gains are given by

$$\bar{k}_1 = 42.74, \qquad \bar{k}_2 = 913.2, \qquad \bar{k}_3 = 24495. \tag{5-195}$$

The assumption **5-194** implies that the rms value of the increment of τ_0 during a period of 1 s is $\sqrt{60} \simeq 7.75$ Nm. This torque is equivalent to an

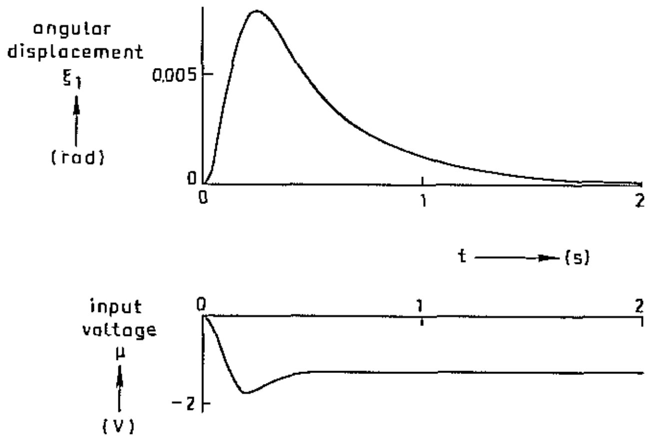

Fig. 5.10. Response of the zero steady-state error position control system to a constant torque of 10 N m on the shaft.

input voltage of nearly 1 V. The observer poles corresponding to the gains **5-195** are $-22.44 \pm j22.27$ and $-2.450\ \mathrm{s}^{-1}$.

By substituting the control law **5-190** into the observer equations **5-193**, it is easily found that the controller has a pole at the origin, hence exhibit integrating action, as expected. For $\bar{F}$ we choose the steady-state optimal gain matrix **5-90** derived in Example 5.3. The corresponding regulator poles are $-9.66 \pm j9.09\ \mathrm{s}^{-1}$. In Fig. 5.10 we give the response of the control system from zero initial conditions to a constant disturbance $\tau_0 = 10$ Nm. It is seen that the maximum deviation of the angular displacement caused by this constant torque is not more than about 0.008 rad.

5.6* SENSITIVITY OF TIME-INVARIANT OPTIMAL LINEAR OUTPUT FEEDBACK CONTROL SYSTEMS

In Chapter 3, Section 3.9, we saw that time-invariant linear optimal state feedback systems are insensitive to disturbances and parameter variations in the sense that the return difference matrix $J(s)$, obtained by opening the feedback loop at the state, satisfies an inequality of the form

$$J^T(-j\omega)WJ(j\omega) \geq W, \qquad \text{for all real } \omega, \tag{5-196}$$

where W is the weighting matrix $\bar{F}^T R_2 \bar{F}$.

In this section we see that optimal output feedback systems generally do not possess such a property, although it can be closely approximated. Consider the time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) + w_1(t), \tag{5-197}$$

where w_1 is white noise with constant intensity V_1. The observed variable is given by

$$y(t) = Cx(t) + w_2(t), \tag{5-198}$$

where w_2 is white noise uncorrelated with w_1 with constant intensity V_2. The controlled variable is

$$z(t) = Dx(t), \tag{5-199}$$

while the optimization criterion is specified as

$$E\left\{\int_{t_0}^{t_1} [z^T(t)R_3z(t) + u^T(t)R_2u(t)]\, dt\right\}, \tag{5-200}$$

with R_3 and R_2 symmetric, constant, positive-definite weighting matrices.

To simplify the analysis, we assume that the controlled variable is also the observed variable (apart from the observation noise), that is, $C = D$. Then

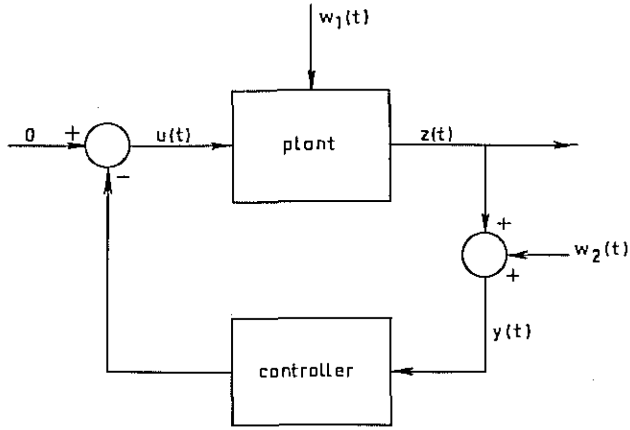

Fig. 5.11. Simplified output feedback control configuration.

we can schematically represent the control configuration as in Fig. 5.11, where observer and control law have been combined into the controller. Let us now consider the steady-state controller that results by letting $t_0 \to -\infty$ and $t_1 \to \infty$. Then the steady-state observer is described by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + \bar{K}[y(t) - D\hat{x}(t)], \qquad \textbf{5-201}$$

where $\bar{K}$ is the steady-state observer gain matrix. Laplace transformation of **5-201** and solution for the transform $\hat{\mathbf{X}}(s)$ of $\hat{x}(t)$ yields

$$\hat{\mathbf{X}}(s) = (sI - A + \bar{K}D)^{-1}[B\mathbf{U}(s) + \bar{K}\mathbf{Y}(s)], \qquad \textbf{5-202}$$

where $\mathbf{U}(s)$ and $\mathbf{Y}(s)$ are the Laplace transforms of $u(t)$ and $y(t)$, respectively. All initial conditions are assumed to be zero. For the input we have in terms of Laplace transforms

$$\mathbf{U}(s) = -\bar{F}\hat{\mathbf{X}}(s), \qquad \textbf{5-203}$$

where $\bar{F}$ is the steady-state feedback gain matrix. Substitution of **5-203** into **5-202** and solution for $\mathbf{U}(s)$ yields

$$\mathbf{U}(s) = -G(s)\mathbf{Y}(s), \qquad \textbf{5-204}$$

where

$$G(s) = [I + \bar{F}(sI - A + \bar{K}D)^{-1}B]^{-1}\bar{F}(sI - A + \bar{K}D)^{-1}\bar{K}. \qquad \textbf{5-205}$$

We now consider the return difference matrix

$$J(s) = I + H(s)G(s) \qquad \textbf{5-206}$$

for the control system, where

$$H(s) = D(sI - A)^{-1}B \qquad \textbf{5-207}$$

is the plant transfer matrix. Generally, there does not exist a nonnegative-definite weighting matrix W such that an inequality of the form

$$J^T(-j\omega)WJ(j\omega) \geq W \qquad \textbf{5-208}$$

is satisfied for all real frequencies ω. Indeed, it can easily be proved (see Problem 5.6) that in the single-input single-output case **5-208** is *never* satisfied for all ω when $W > 0$. Of course the inequality **5-208** must hold in some useful frequency range, adapted to the frequency band of the disturbances acting upon the plant, since it follows from the optimality of the controller that the specific disturbances for which the control system has been designed are attenuated.

We now prove, however, that under certain conditions satisfaction of **5-208** for all frequencies can be obtained asymptotically. Consider the algebraic Riccati equation

$$0 = D^T R_3 D - \bar{P} B R_2^{-1} B^T \bar{P} + A^T \bar{P} + \bar{P} A, \qquad \textbf{5-209}$$

which must be solved to obtain the regulation gain $\bar{F} = R_2^{-1} B^T \bar{P}$. Suppose that

$$R_2 = \rho N, \qquad \textbf{5-210}$$

where ρ is a positive scalar and N a positive-definite matrix. Then it follows from Theorem 3.14 (Section 3.8.3) that if $\dim(z) = \dim(u)$, and the open-loop transfer matrix $H(s) = D(sI - A)^{-1}B$ has zeroes with nonpositive real parts only, as $\rho \downarrow 0$ the desired solution $\bar{P}$ of **5-209** approaches the zero matrix. This implies that

$$\lim_{\rho \downarrow 0} \bar{P} B \frac{1}{\rho} N^{-1} B^T \bar{P} = D^T R_3 D, \qquad \textbf{5-211}$$

or

$$\lim_{\rho \downarrow 0} \rho \bar{F}^T N \bar{F} = D^T R_3 D. \qquad \textbf{5-212}$$

Now the general solution of the matrix equation $X^T X = M^T M$, where X and M have equal dimensions, can be written in the form $X = UM$, where U is an arbitrary unitary matrix, that is, U satisfies $U^T U = I$. We therefore conclude from **5-212** that as $\rho \downarrow 0$ the gain matrix $\bar{F}$ asymptotically behaves as

$$\bar{F} \to \frac{1}{\sqrt{\rho}} N^{-1/2} U R_3^{1/2} D. \qquad \textbf{5-213}$$

As a result,

$$G(s) \to [D(sI - A + \bar{K}D)^{-1}B]^{-1} D(sI - A + \bar{K}D)^{-1}\bar{K}, \qquad \textbf{5-214}$$

as $\rho \downarrow 0$. It is not difficult to prove that

$$[D(sI - A + \bar{K}D)^{-1}B]^{-1}D(sI - A + \bar{K}D)^{-1}\bar{K} = [D(sI - A)^{-1}B]^{-1}D(sI - A)^{-1}\bar{K}. \quad \textbf{5-215}$$

With this it follows for the return difference matrix $J(s)$ of the configuration of Fig. 5.11 that as $R_2 \to 0$

$$J(s) \to J_0(s), \quad \textbf{5-216}$$

where

$$J_0(s) = I + D(sI - A)^{-1}\bar{K}. \quad \textbf{5-217}$$

We now derive an inequality for the asymptotic return difference matrix $J_0(s)$. The steady-state variance matrix $\bar{Q}$ satisfies the algebraic Riccati equation

$$0 = V_1 - \bar{Q}D^TV_2^{-1}D\bar{Q} + A\bar{Q} + \bar{Q}A^T, \quad \textbf{5-218}$$

assuming that the state excitation noise and observation noise are uncorrelated, that $V_2 > 0$, and that the Riccati differential equation possesses a steady-state solution. We can now go through manipulations very similar to those in Section 3.9, where we dealt with the sensitivity of the state feedback regulator. Addition and subtraction of $s\bar{Q}$ and rearrangement yield

$$0 = V_1 - \bar{Q}D^TV_2^{-1}D\bar{Q} - (sI - A)\bar{Q} - \bar{Q}(-sI - A^T). \quad \textbf{5-219}$$

Premultiplication by $D(sI - A)^{-1}$ and postmultiplication by $(-sI - A^T)^{-1}D^T$ give

$$0 = D(sI - A)^{-1}(V_1 - \bar{Q}D^TV_2^{-1}D\bar{Q})(-sI - A^T)^{-1}D^T - D\bar{Q}(-sI - A^T)^{-1}D^T - D(sI - A)^{-1}\bar{Q}D^T. \quad \textbf{5-220}$$

By adding and subtracting an extra term V_2, this expression can be rearranged into the form

$$[I + D(sI - A)^{-1}\bar{Q}D^TV_2^{-1}]V_2[I + V_2^{-1}D\bar{Q}(-sI - A^T)^{-1}D^T] = V_2 + D(sI - A)^{-1}V_1(-sI - A^T)^{-1}D^T. \quad \textbf{5-221}$$

Since $\bar{Q}D^TV_2^{-1} = \bar{K}$ we immediately recognize that this expression implies the equality

$$J_0(s)V_2J_0^T(-s) = V_2 + D(sI - A)^{-1}V_1(-sI - A^T)^{-1}D^T. \quad \textbf{5-222}$$

Substituting $s = j\omega$ we see that the second term on the right-hand side is a nonnegative-definite Hermitian matrix; thus we have

$$J_0(j\omega)V_2J_0^T(-j\omega) \geq V_2 \qquad \text{for all real } \omega. \quad \textbf{5-223}$$

It follows from Theorem 2.2 (Section 2.10) that

$$S_0^T(-j\omega)V_2^{-1}S_0(j\omega) \leq V_2^{-1} \qquad \text{for a real } \omega, \quad \textbf{5-224}$$

where $S_0(s)$ is the asymptotic sensitivity matrix:

$$S_0(s) = J_0^{-1}(s). \tag{5-225}$$

We also have

$$J_0^T(-j\omega)V_2^{-1}J_0(j\omega) \geq V_2^{-1}.$$

We thus have the following result (Kwakernaak, 1969).

Theorem 5.5. *Consider the steady-state time-invariant stochastic optimal output feedback regulator. Suppose that the observed variable is also the controlled variable, that is,*

$$\begin{aligned} y(t) &= Dx(t) + w_2(t), \\ z(t) &= Dx(t). \end{aligned} \tag{5-226}$$

Also assume that the state excitation noise $w_1(t)$ and the observation noise $w_2(t)$ are uncorrelated, that the observation problem is nonsingular, that is, $V_2 > 0$, and that the steady-state output feedback regulator is asymptotically stable. Then if dim (u) = dim (z), and the open-loop transfer matrix $H(s) = D(sI - A)^{-1}B$ possesses no right-half plane zeroes, the return difference matrix of the closed-loop system asymptotically approaches $J_0(s)$ as $R_2 \to 0$, where

$$J_0(s) = I + D(sI - A)^{-1}\bar{K}. \tag{5-227}$$

$\bar{K}$ is the steady-state observer gain matrix. The asymptotic return difference matrix satisfies the relation

$$J_0(s)V_2J_0^T(-s) = V_2 + D(sI - A)^{-1}V_1(-sI - A^T)^{-1}D^T. \tag{5-228}$$

The asymptotic return difference matrix $J_0(s)$ and its inverse, the asymptotic sensitivity matrix $S_0(s) = J_0^{-1}(s)$, satisfy the inequalities

$$\begin{aligned} J_0(j\omega)V_2J_0^T(-j\omega) &\geq V_2 && \text{for all real } \omega, \\ S_0^T(-j\omega)V_2^{-1}S_0(j\omega) &\leq V_2^{-1} && \text{for all real } \omega, \\ J_0^T(-j\omega)V_2^{-1}J_0(j\omega) &\geq V_2^{-1} && \text{for all real } \omega. \end{aligned} \tag{5-229}$$

This theorem shows that asymptotically the sensitivity matrix of the output feedback regulator system satisfies an inequality of the form **5-196**, which means that in the asymptotic control system disturbances are always reduced as compared to the open-loop steady-state equivalent control system no matter what the power spectral density matrix of the disturbances. It also means that the asymptotic control system reduces the effect of all (sufficiently small) plant variations as compared to the open-loop steady-state equivalent The following points are worth noting:

(i) The weighting matrix in the sensitivity criterion is V_2^{-1}. This is not surprising. Let us assume for simplicity that V_2 is diagonal. Then if one of the diagonal elements of V_2 is small, the corresponding component of the observed variable can be accurately measured, which means that the gain in the corresponding feedback loop can be allowed to be large. This will have a favorable effect on the suppression of disturbances and plant variations at this output, which in turn is reflected by a large weighting coefficient in the sensitivity criterion.

(ii) The theorem is not valid for systems that possess open-loop zeroes in the right-half plane.

(iii) In practical cases it is never possible to choose R_2 very small. This means that the sensitivity criterion is violated over a certain frequency range. Examples show that this is usually the case in the high-frequency region. It is to be expected that the sensitivity reduction is not spoiled too badly when R_2 is chosen so small that the faraway regulator poles are much further away from the origin than the observer poles.

(iv) The right-hand side of **5-228** can be evaluated directly without solving Riccati equations. It can be used to determine the behavior of the return difference matrix, in particular in the single-input single-output case.

(v) It can be shown (Kwakernaak, 1969), that a result similar to Theorem 5.5 holds when

$$y(t) = \begin{pmatrix} D \\ M \end{pmatrix} x(t) + w_2(t), \tag{5-230}$$

that is, $y(t)$ *includes* the controlled variable $z(t)$.

Example 5.7. *Position control system*

Again we consider the positioning system described by the state differential equation

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t) + \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \tau_d(t). \tag{5-231}$$

Here $\tau_d(t)$ is white noise with intensity V_d. The observed variable is

$$\eta(t) = (1, 0)x(t) + \nu_m(t), \tag{5-232}$$

where $\nu_m(t)$ is white noise with intensity V_m. The controlled variable is

$$\zeta(t) = (1, 0)x(t). \tag{5-233}$$

The system satisfies the assumptions of Theorem 5.5, since the controlled variable is the observed variable, the state excitation and observation noise are assumed to be uncorrelated, and the open-loop transfer function,

$$H(s) = \frac{\kappa}{s(s + \alpha)}, \tag{5-234}$$

possesses no right-half plane zeroes. To compute the asymptotic return difference $J_0(s)$, we evaluate **5-228**, which easily yields

$$\begin{aligned} J_0(s)J_0(-s) &= 1 + \frac{\gamma^2 V_d/V_m}{-s^2(-s^2+\alpha^2)} \\ &= \frac{s^4 - \alpha^2 s^2 + \gamma^2 V_d/V_m}{-s^2(-s^2+\alpha^2)}. \end{aligned} \tag{5-235}$$

Substitution of $s = j\omega$ provides us with the relation

$$|J_0(j\omega)|^2 = \frac{\omega^4 + \alpha^2\omega^2 + \gamma^2 V_d/V_m}{\omega^2(\omega^2+\alpha^2)} \tag{5-236}$$

or

$$|S_0(j\omega)|^2 = \frac{\omega^2(\omega^2+\alpha^2)}{\omega^4 + \alpha^2\omega^2 + \gamma^2 V_d/V_m}, \tag{5-237}$$

which shows that $|S_0(j\omega)| < 1$ for all real ω.

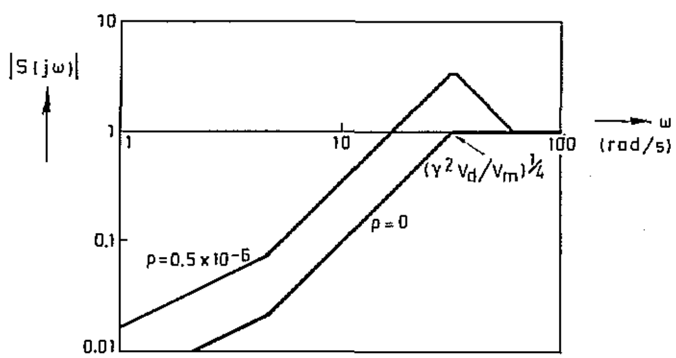

Fig. 5.12. Asymptotic Bode plots of the sensitivity function of the position control system for $\rho = 0$ and $\rho = 0.5 \times 10^{-6}$.

Figure 5.12 gives an asymptotic Bode plot of $|S(j\omega)|$ which shows that the limiting controller provides protection against all disturbances and parameter variations up to a frequency of about $(\gamma^2 V_d/V_m)^{1/4}$. With the numerical values

$$\begin{aligned} \gamma &= 0.1 \text{ kg}^{-1}\text{ m}^{-2}, \\ V_d &= 10 \text{ N}^2\text{ m}^2\text{ s}, \\ V_m &= 10^{-7}\text{ rad}^2\text{ s}, \end{aligned} \tag{5-238}$$

this break frequency is about 31.6 rad/s.

The frequency range over which disturbance protection is obtained is reduced when the weighting factor ρ in the criterion

$$E\left\{\int_{t_0}^{t_1}[\zeta^2(t) + \rho\mu^2(t)]\,dt\right\} \tag{5-239}$$

is chosen greater than zero. It is reasonable to assume that the disturbance reduction is not affected so long as the regulator break frequency is much larger than the observer break frequency. Since the regulator break frequency (Example 5.3, Section 5.3.2) is $(\kappa/\sqrt{\rho})^{1/2}$, we conclude that with $\kappa = 0.787$ rad/(Vs^2) the value of ρ should be 0.5×10^{-6} or less (for this value of ρ the regulator break frequency is 33.4 rad/s). It can be computed, using Theorem 5.4 (Section 5.3.2), that with this value of ρ we have

$$\lim_{t\to\infty} E\{\zeta^2(t) + \rho\mu^2(t)\} = 0.00001906 \text{ rad}^2. \tag{5-240}$$

It follows that the rms input voltage is bounded by

$$\sqrt{E\{\mu^2(t)\}} \le \sqrt{\frac{0.00001906}{\rho}} \simeq 6.17 \text{ V}, \tag{5-241}$$

which is quite an acceptable value when input amplitudes of up to 100 V are permissible. It can be calculated that the sensitivity function of the steady-state controller for this value of ρ is given by

$$S(s) = \frac{s(s + 4.6)(s^2 + 87.9s + 3859)}{(s^2 + 47.5s + 1125)(s^2 + 44.96s + 1000)}. \tag{5-242}$$

The asymptotic Bode plot of $|S(j\omega)|$ is given in Fig. 5.12 as well and is compared to the plot for $\rho = 0$. It is seen that the disturbance attenuation cutoff frequency is shifted from about 30 to about 20 rad/s, while disturbances in the frequency range near 30 rad/s are slightly amplified instead of attenuated. By making ρ smaller than 0.5×10^{-6}, the asymptotic sensitivity function can be more closely approximated.

Using the methods of Section 5.5.1, it is easy to determine the nonzero set point optimal controller for this system. Figure 5.13 gives the response of the resulting nonzero set point output feedback control system to a step of 0.1 rad in the set point of the angular position, from zero initial conditions, for the nominal parameter values, and for two sets of off-nominal values. As in Example 3.25 (Section 3.9), the off-nominal values of the plant constants α and κ are assumed to be caused by changes in the inertial load of the dc motor. It is seen that the effect of the parameter changes is moderate.

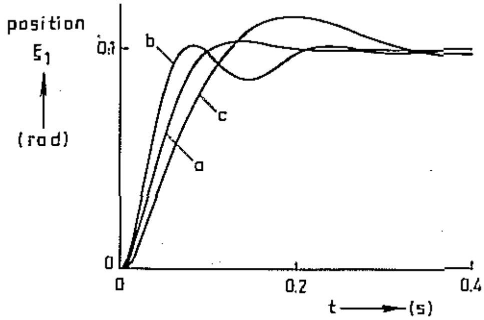

Fig. 5.13. The effect of parameter variations on the response of the output feedback position control system. (*a*) Nominal load; (*b*) inertial load $\frac{2}{3}$ of nominal; (*c*) inertial load $\frac{3}{2}$ of nominal.

5.7* LINEAR OPTIMAL OUTPUT FEEDBACK CONTROLLERS OF REDUCED DIMENSIONS

5.7.1* Introduction

In Section 5.3.1 we obtained the solution of the stochastic linear optimal output feedback regulator problem. It is immediately clear that the dimension of the controller by itself equals the dimension of the plant, since the optimal observer has the dimension of the plant. This may be a severe drawback of the design methods suggested, since in some cases a controller of much lower dimension would render quite satisfactory, although not optimal, performance. Moreover, the dimension of the mathematical model of a system is a number that very much depends on the accuracy of the model. The model may incorporate some marginal effects that drastically increase the dimension of the model without much improvement in the accuracy of the model. When this is the case, there seems to be no reason why the dimension of the controller should also be increased.

Motivated by the fact that the complexity and cost of the controller increase with its dimension, we intend to investigate in this section methods for obtaining controllers of lower dimensions than those prescribed by the methods of Section 5.3. One obvious way to approach the problem of designing controllers of low dimension is to describe the plant by a cruder mathematical model, of lower dimension. Methods are available (see e.g., Mitra, 1967; Chen and Shieh, 1968b; Davison, 1968a; Aoki, 1968; Kuppurajulu and Elangovan, 1970; Fossard, 1970; Chidambara and Schainker, 1971) for reducing the dimension of the model while retaining only the "significant modes" of the model. In this case the methods of Section 5.3 result in controllers of lower dimension. There are instances, however, in which it is not easy to achieve a reduction of the dimension of the plant. There are also situations where dimension reduction by neglecting the "parasitic" effects leads to the design of a controller that makes the actual control system unstable (Sannuti and Kokotović, 1969).

Our approach to the problem of designing low-dimensional controllers is therefore as follows. We use mathematical models for the systems which are as accurate as possible, without hesitating to include marginal effects that may or may not have significance. However we limit the dimension of the controller to some fixed number m, less than n, where n is the dimension of the plant model. In fact, we attempt to select the smallest m that still produces a satisfactory control system. We feel that this method is more dependable than that of reducing the dimension of the plant. This approach was originally suggested by Newton, Gould, and Kaiser (1957), and was further pursued by Sage and Eisenberg (1966), Sims and Melsa (1970), Johnson and Athans (1970), and others.

5.7.2* Controllers of Reduced Dimensions

Consider the system described by the equations

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + w_1(t), \qquad x(t_0) = x_0, \\ y(t) &= C(t)x(t) + w_2(t), \end{aligned} \tag{5-243}$$

where, as usual, $x(t)$ is an n-dimensional state vector, $u(t)$ is a k-dimensional input variable, $y(t)$ is an l-dimensional observed variable, and w_1 and w_2 are white noise processes. The joint process col (w_1, w_2) has the intensity $V(t)$. It is furthermore assumed that the initial state x_0 is a stochastic vector, uncorrelated with w_1 and w_2, with mean $\bar{x}_0$ and variance matrix Q_0.

We now consider a controller for the system given above described by

$$\begin{aligned} \dot{q}(t) &= L(t)q(t) + K(t)y(t), \qquad q(t_0) = q_0, \\ u(t) &= -F(t)q(t), \end{aligned} \tag{5-244}$$

where q is the m-dimensional state vector of the controller. The observed variable y serves as input to the controller, and the input to the plant u is the output of the controller. It is noted that we do not allow a direct link in the controller. The reason is that a direct link causes the white observation noise w_2 to penetrate directly into the input variable u, which results in infinite input amplitudes since white noise has infinite amplitudes.

We are now in a position to formulate the linear optimal output feedback control problem for controllers of reduced dimensions (Kwakernaak and Sivan, 1971a):

Definition 5.3. *Consider the system* **5-243** *with the statistical data given. Then the* ***optimal output feedback control problem for a controller of reduced dimension*** *is to find, for a given integer m, with $1 \leq m \leq n$, and a given final time t_1, matrix functions $L(t)$, $K(t)$, and $F(t)$, $t_0 \leq t \leq t_1$, and the probability distribution of q_0, so as to minimize σ_m, where*

$$\sigma_m = E\left\{\int_{t_0}^{t_1}[x^T(t)R_1(t)x(t) + u^T(t)R_2(t)u(t)]\,dt\right\}. \qquad \textbf{5-245}$$

Here $R_1(t)$ and $R_2(t)$, $t_0 \leq t \leq t_1$, are given matrices, nonnegative-definite and positive-definite, respectively, for all t.

In the special case in which $m = n$, the solution to this problem follows from Theorem 5.3 which states that $F(t)$ and $K(t)$ in **5-244** are the optimal regulator and observer gains, respectively, and

$$L(t) = A(t) - B(t)F(t) - K(t)C(t). \qquad \textbf{5-246}$$

It is easy to recognize that σ_m, $m = 1, 2, \cdots$, forms a monotonically nonincreasing sequence of numbers, that is,

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots, \qquad \textbf{5-247}$$

since an m-dimensional controller is a special case of an $(m+1)$-dimensional controller. Also, for $m \geq n$ the value of σ_m no longer decreases, since we know from Theorem 5.3 that the optimal controller (without restriction on its dimension) has the dimension n; thus we have

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_{n-1} \geq \sigma_n = \sigma_{n+1} = \sigma_{n+2} = \cdots. \qquad \textbf{5-248}$$

One way to approach the problem of Definition 5.3 is to convert it to a deterministic dynamic optimization problem. This can be done as follows. Let us combine the plant equation **5-243** with the controller equation **5-244**. The control system is then described by the augmented state differential equation

$$\begin{pmatrix}\dot{x}(t)\\ \dot{q}(t)\end{pmatrix} = \begin{pmatrix}A(t) & -B(t)F(t)\\ K(t)C(t) & L(t)\end{pmatrix}\begin{pmatrix}x(t)\\ q(t)\end{pmatrix} + \begin{pmatrix}I & 0\\ 0 & K(t)\end{pmatrix}\begin{pmatrix}w_1(t)\\ w_2(t)\end{pmatrix}. \qquad \textbf{5-249}$$

We now introduce the second-order joint moment matrix

$$S(t) = E\left\{\begin{pmatrix}x(t)\\ q(t)\end{pmatrix}(x^T(t),\quad q^T(t))\right\}. \qquad \textbf{5-250}$$

It follows from Theorem 1.52 (Section 1.11.2) that $S(t)$ is the solution of the matrix differential equation

$$\dot{S}(t) = M(t)S(t) + S(t)M^T(t) + N(t)V(t)N^T(t),\qquad S(t_0) = S_0, \tag{5-251}$$

where

$$M(t) = \begin{pmatrix} A(t) & -B(t)F(t) \\ K(t)C(t) & L(t) \end{pmatrix}, \qquad N(t) = \begin{pmatrix} I & 0 \\ 0 & K(t) \end{pmatrix},$$
$$S_0 = E\left\{\begin{pmatrix} x_0x_0^T & x_0q_0^T \\ q_0x_0^T & q_0q_0^T \end{pmatrix}\right\}. \tag{5-252}$$

Using the matrix function $S(t)$, the criterion **5-245** can be rewritten in the form

$$\sigma_m = \operatorname{tr}\left\{\int_{t_0}^{t_1} [S_{11}(t)R_1(t) + S_{22}(t)F^T(t)R_2(t)F(t)]\,dt\right\}, \tag{5-253}$$

where $S_{11}(t)$ and $S_{22}(t)$ are the $n \times n$ and $m \times m$ diagonal blocks of $S(t)$, respectively.

The problem of determining the optimal behaviors of the matrix functions $L(t)$, $F(t)$, and $K(t)$ and the probability distribution of q_0 has now been reduced to the problem of choosing these matrix functions and S_0 such that σ_m as given by **5-253** is minimized, where the matrix function $S(t)$ follows from **5-251**. Application of dynamic optimization techniques to this problem (Sims and Melsa, 1970) results in a two-point boundary value problem for nonlinear matrix differential equations; this problem can be quite formidable from a computational point of view.

In order to simplify the problem, we now confine ourselves to time-invariant systems and formulate a steady-state version of the problem that is numerically more tractable and, moreover, is more easily implemented. Let us thus assume that the matrices A, B, C, V, R_1, and R_2 are constant. Furthermore, we also restrict the choice of controller to time-invariant controllers with constant matrices L, K, and F. Assuming that the interconnection of plant and controller is asymptotically stable, the limit

$$\bar{\sigma}_m = \lim_{t_0 \to -\infty} E\{x^T(t)R_1x(t) + u^T(t)R_2u(t)\} \tag{5-254}$$

will exist. As before, the subscript m refers to the dimension of the controller. We now consider the problem of choosing the constant matrices L, K, and F (of prescribed dimensions) such that $\bar{\sigma}_m$ is minimized.

As before, we can argue that

$$\bar{\sigma}_1 \geq \bar{\sigma}_2 \geq \bar{\sigma}_3 \geq \cdots \geq \bar{\sigma}_{n-1} \geq \bar{\sigma}_n = \bar{\sigma}_{n+1} = \bar{\sigma}_{n+2} = \cdots. \tag{5-255}$$

The minimal value that can ever be obtained is achieved for $m = n$, since as we know from Theorem 5.4 (Section 5.3.2) the criterion **5-254** is minimized by the interconnection of the steady-state optimal observer with the steady-state optimal control law.

The problem of minimizing **5-254** with respect to L, K, and F can be converted into a mathematical programming problem as follows. Since by assumption the closed-loop control system is asymptotically stable, that is, the constant matrix M has all its characteristic values strictly within the left-half complex plane, as $t_0 \to -\infty$ the variance matrix $S(t)$ of the augmented state approaches a constant steady-state value $\bar{S}$ that is the unique solution of the linear matrix equation

$$M\bar{S} + \bar{S}M^T + NVN^T = 0. \qquad \textbf{5-256}$$

Also, $\bar{\sigma}_m$ can be expressed as

$$\bar{\sigma}_m = \mathrm{tr}\,(\bar{S}_{11}R_1 + \bar{S}_{22}F^TR_2F), \qquad \textbf{5-257}$$

where $\bar{S}_{11}$ and $\bar{S}_{22}$ are the $n \times n$ and $m \times m$ diagonal blocks of $\bar{S}$, respectively.

Thus the problem of solving the steady-state version of the linear time-invariant optimal feedback control problem for controllers of reduced dimension is reduced to determining constant matrices L, K, and F of prescribed dimensions that minimize

$$\bar{\sigma}_m = \mathrm{tr}\,(\bar{S}_{11}R_1 + \bar{S}_{22}F^TR_2F), \qquad \textbf{5-258}$$

and satisfy the constraints

$$\text{(i)} \qquad M\bar{S} + \bar{S}M^T + NVN^T = 0, \qquad \textbf{5-259a}$$

$$\text{(ii)} \qquad \mathrm{Re}\,[\lambda_i(M)] < 0, \qquad i = 1, 2, \cdots, n + m. \qquad \textbf{5-259b}$$

Here the $\lambda_i(M)$, $i = 1, 2, \cdots, n + m$, denote the characteristic values of the matrix M, and Re stands for "the real part of."

It is noted that the problem of finding time-varying matrices $L(t)$, $K(t)$, and $F(t)$, $t_0 \leq t \leq t_1$, that minimize the criterion σ_m always has a solution as long as the matrix $A(t)$ is continuous, and all other matrices occurring in the problem formulation are piecewise continuous. The steady-state version of the problem, however, that is, the problem of *minimizing* $\bar{\sigma}_m$ *with respect* to the constant matrices L, K, and F, has a solution only if for the given dimension m of the controller there exist matrices L, K, and F such that the compound matrix M is asymptotically stable. For $m = n$ necessary and sufficient conditions on the matrices A, B, and C so that there exist matrices L, K, and F that render M asymptotically stable are that $\{A, B\}$ be stabilizable and $\{A, C\}$ detectable (Section 5.2.2). For $m < n$ such conditions are not known, although it is known what is the least dimension of the controller such that all closed-loop poles can be arbitrarily assigned (see, e.g., Brash and Pearson, 1970).

In the following subsection some guidelines for the numerical determination of the matrices L, K, and F are given. We conclude this section with a note on the selection of the proper dimension of the controller. Assume that for given R_1 and R_2 the optimization problem has been solved for $m = 1, 2, \cdots, n$, and that $\bar{\sigma}_1, \bar{\sigma}_2, \cdots, \bar{\sigma}_n$ have been computed. Is it really meaningful to compare the values of $\bar{\sigma}_1, \bar{\sigma}_2, \cdots, \bar{\sigma}_n$, and thus decide upon the most desirable of m as the number that gives a sufficiently small value of $\bar{\sigma}_m$? The answer is that this is probably not meaningful since the designs all have different mean square inputs. The maximally allowable mean square input, however, is a prescribed number, which is not related to the complexity of the controller selected. Therefore, a more meaningful comparison results when for each m the weighting matrix R_2 is so adjusted that the maximally allowable mean square input is obtained. This can be achieved by letting

$$R_2 = \rho_m R_{20}, \tag{5-260}$$

where ρ_m is a positive scalar and R_{20} a positive-definite weighting matrix which determines the relative importance of the components of the input. Then we rephrase our problem as follows. For given m, R_1, and R_{20}, minimize the criterion

$$\bar{\sigma}_m = \operatorname{tr}(\bar{S}_{11}R_1 + \rho_m \bar{S}_{22}F^T R_{20} F), \tag{5-261}$$

with respect to the constant matrices L, K, and F, subject to the constraints (i) and (ii), where ρ_m is so chosen that

$$\operatorname{tr}(\bar{S}_{22}F^T R_{20} F) \tag{5-262}$$

equals the given maximally allowable mean square input.

5.7.3* Numerical Determination of Optimal Controllers of Reduced Dimensions

In this section some results are given that are useful in obtaining an efficient computer program for the solution of the steady-state version of the linear time-invariant optimal output feedback control problem for a controller of reduced dimension as outlined in the preceding subsection. In particular, we describe a method for computing the gradient of the objective function (in this case $\bar{\sigma}_m$) with respect to the unknown parameters (in this case the entries of the matrices L, K, and F). This gradient can be used in any standard function minimization algorithm employing gradients, such as the conjugate gradient method or the Powell-Fletcher technique [see, e.g., Pierre (1969) or Beveridge and Schechter (1970) for extensive reviews of unconstrained optimization methods].

Gradient methods are particularly useful for solving the present function minimization problem, since the gradient can easily be computed, as we shall see. Moreover, meeting constraint (ii), which expresses that the control system be asymptotically stable, is quite simple when care is taken to choose the starting values of L, K, and F such that (ii) is satisfied, and we move with sufficiently small steps along the search directions prescribed. This is because as the boundary of the region where the control system is stable is approached, the criterion becomes infinite, and this provides a natural barrier against moving out of the stability region.

A remark on the representation of the controller is in order at this point. Clearly, the value of the criterion $\bar{\sigma}_m$ is determined only by the *external* representation of the controller, that is, its transfer matrix $F(sI - L)^{-1}K$, or, equivalently, its impulse response matrix $F \exp [L(t - \tau)]K$. It is well-known that for a given external representation many *internal* representations (in the form of a state differential equation together with an output equation) are possible. Therefore, when the optimization problem is set up starting from an internal representation of the controller, as we prefer to do, and all the entries of the matrices L, K, and F are taken as free parameters, the minimizing values of L, K, and F are not at all unique. This may give numerical difficulties. Moreover, the dimension of the function minimization problem is unnecessarily increased. These difficulties can be overcome by choosing a canonical representation of the controller equations. For example, when the controller is a single-input system, the phase canonical form of the state equations (see Section 1.9) has the minimal number of free parameters. Similarly, when the controller is a single-output system, the dual phase canonical form (see also Section 1.9) has the minimal number of free parameters. For multiinput multioutput systems related canonical forms can be used (Bucy and Ackermann, 1970). It is noted, however, that considerable reduction in the number of free parameters can often be achieved by imposing structural constraints on the controller, for example, by blocking certain feedback paths that can be expected to be of minor significance.

We discuss finally the evaluation of the gradient of $\bar{\sigma}_m$ with respect to the entries of L, K, and F. Let γ be one of the free parameters. Then introducing the matrix

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & FR_2F^T \end{pmatrix}, \tag{5-263}$$

the gradient of $\bar{\sigma}_m$ with respect to γ can be written as

$$\frac{\partial \bar{\sigma}_m}{\partial \gamma} = \frac{\partial}{\partial \gamma} [\mathrm{tr}\,(\bar{S}R)] = \mathrm{tr}\left(\frac{\partial \bar{S}}{\partial \gamma} R + \bar{S}\frac{\partial R}{\partial \gamma}\right). \tag{5-264}$$

Furthermore, taking the partial derivative of **5-259a** with respect to the same parameter we find that

$$\frac{\partial M}{\partial \gamma}\bar{S} + M\frac{\partial \bar{S}}{\partial \gamma} + \frac{\partial \bar{S}}{\partial \gamma}M^T + \bar{S}\frac{\partial M^T}{\partial \gamma} + \frac{\partial}{\partial \gamma}(NVN^T) = 0. \tag{5-265}$$

At this point it is convenient to introduce a linear matrix equation which is adjoint to **5-259a** and is given by

$$M^T\bar{U} + \bar{U}M + R = 0. \qquad \textbf{5-266}$$

Using the fact that for any matrices A, B, and C of compatible dimensions $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and $\operatorname{tr}(C) = \operatorname{tr}(C^T)$, we write with the aid of **5-265** and **5-266** for **5-264**

$$\begin{aligned}
\frac{\partial \bar{\sigma}_m}{\partial \gamma} &= \operatorname{tr}\left[\frac{\partial \bar{S}}{\partial \gamma}(-M^T\bar{U} - \bar{U}M) + \bar{S}\frac{\partial R}{\partial \gamma}\right] \\
&= \operatorname{tr}\left[\bar{U}\left(-M\frac{\partial \bar{S}}{\partial \gamma} - \frac{\partial \bar{S}}{\partial \gamma}M^T\right) + \bar{S}\frac{\partial R}{\partial \gamma}\right] \\
&= \operatorname{tr}\left\{\bar{U}\left[\frac{\partial M}{\partial \gamma}\bar{S} + \bar{S}\frac{\partial M^T}{\partial \gamma} + \frac{\partial}{\partial \gamma}(NVN^T)\right] + \bar{S}\frac{\partial R}{\partial \gamma}\right\} \\
&= \operatorname{tr}\left[2\frac{\partial M}{\partial \gamma}\bar{S}\bar{U} + \bar{U}\frac{\partial}{\partial \gamma}(NVN^T) + \bar{S}\frac{\partial R}{\partial \gamma}\right].
\end{aligned} \qquad \textbf{5-267}$$

Thus in order to compute the gradient of the objective function $\bar{\sigma}_m$ with respect to γ, one of the free parameters, the two linear matrix equations **5-259a** and **2-266** must be solved for $\bar{S}$ and $\bar{U}$, respectively, and the resulting values must be inserted into **5-267**. When a different parameter is considered, the bulk of the computational effort, which consists of solving the two matrix equations, need not be repeated. In Section 1.11.3 we discussed numerical methods for solving linear matrix equations of the type at hand.

Example 5.8. *Position control system*

In this example we design a position control system with a constraint on the dimension of the controller. The system to be controlled is the dc motor of Example 5.3 (Section 5.3.2), which is described by the state differential and observed variable equations

$$\begin{aligned}
\dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix}\mu(t) + \begin{pmatrix} 0 \\ \gamma \end{pmatrix}\tau_d(t), \\
\eta(t) &= (1, 0)x(t) + \nu_m(t),
\end{aligned} \qquad \textbf{5-268}$$

where τ_d and ν_m are described as white noise processes with intensities V_d and V_m, respectively. As in Example 5.3, we choose the criterion to be minimized as

$$\lim_{t_0 \to -\infty} E\{\zeta^2(t) + \rho\mu^2(t)\}, \qquad \textbf{5-269}$$

where $\zeta(t) = (1, 0)x(t)$ is the controlled variable. As we saw in Example 5.3, the optimal controller without limitations on its dimension is of dimension two. The only possible controller without a direct link of smaller dimension is a first-order controller, described by the scalar equations

$$\dot{q}(t) = \delta q(t) + \eta(t),$$
$$\mu(t) = -\varepsilon q(t). \tag{5-270}$$

Here we have taken the coefficient of $\eta(t)$ equal to 1, without loss of generality. The problem to be solved thus is: Find the constants δ and ε that minimize the criterion **5-269**.

In Example 5.3 we used the following numerical values:

$$\kappa = 0.787 \text{ rad/(V s}^2), \qquad \alpha = 4.6 \text{ s}^{-1}, \qquad \gamma = 0.1 \text{ kg}^{-1}\text{ m}^{-2},$$
$$V_d = 10 \text{ N}^2\text{ m}^2\text{ s}, \qquad V_m = 10^{-7} \text{ rad}^2\text{ s}. \tag{5-271}$$

For $\rho = 0.00002 \text{ rad}^2/\text{V}^2$ we found an optimal controller characterized by the data in the first column of Table 5.2.

Table 5.2 A Comparison of the Performances of the Position Control System with Controllers of Dimensions One and Two

	Second-order optimal controller with $\rho = 0.00002$	First-order optimal controller with $\rho = 0.00002$	First-order optimal controller with rms input 1.5 V
Rms input voltage (V)	1.5	1.77	1.5
Rms regulating error (rad)	0.00674	0.00947	0.0106
$E\{\zeta^2(t)\} + \rho E\{\mu^2(t)\}$ (rad^2)	$9.08 + 10^{-5}$	15.2×10^{-5}	15.8×10^{-5}
Closed-loop poles (s^{-1})	$-9.66 \pm j9.09$ $-22.48 \pm j22.24$	-400 $-2.13 \pm j11.3$	-350 $-2.15 \pm j9.92$

It is not difficult to find the parameters of the first-order controller **5-270** that minimize the criterion **5-269**. In the present case explicit expressions for the rms regulating error and input voltage can be found. Numerical or analytical evaluation of the optimal parameter values for $\rho = 0.00002$ rad^2/V^2 leads to

$$\delta = -400 \text{ s}^{-1}, \qquad \varepsilon = 6.75 \times 10^4 \text{ V/(rad s)}. \tag{5-272}$$

The performance of the resulting controller is listed in the second column of Table 5.2. It is observed that this controller results in an rms input voltage that is larger than that for the second-order optimal controller. By slightly increasing ρ a first-order controller is obtained with the same rms input voltage as the second-order controller. The third column of Table 5.2 gives the performance of this controller. It is characterized by the parameters

$$\delta = -350\ \mathrm{s}^{-1}, \qquad \varepsilon = 4.65 \times 10^4\ \mathrm{V/(rad\ s)}. \tag{5-273}$$

A comparison of the data of Table 5.2 shows that the first-order optimal controller has an rms regulating error that is about 1.5 times that of the second-order controller. Whether or not this is acceptable depends on the system specifications. We note that the locations of the dominating closed-loop poles at $-2.15 \pm j9.92$ of the reduced-order control system are not at all close to the locations of the dominant poles at $-9.66 \pm j9.09$ of the second-order system. Finally, we observe that the first-order controller transfer function is

$$G(s) = \frac{\varepsilon}{s - \delta} = \frac{4.65 \times 10^4}{s + 350}\ \mathrm{V/rad}. \tag{5-274}$$

This controller has a very large bandwidth. Unless the bandwidth of the observation noise (which we approximated as white noise but in practice has a limited bandwidth) is larger than the bandwidth of the controller, the controller may as well be replaced with a constant gain of

$$\frac{4.65 \times 10^4}{350} \simeq 133\ \mathrm{V/rad}. \tag{5-275}$$

This suggests, however, that the optimization procedure probably should be repeated, representing the observation noise with its proper bandwidth, and searching for a zero-order controller (consisting of a constant gain).

5.8 CONCLUSIONS

In this final chapter on the design of continuous-time optimal linear feedback systems, we have seen how the results of the preceding chapters can be combined to yield optimal output feedback control systems. We have also analyzed the properties of such systems. Table 5.3 summarizes the main properties and characteristics of linear optimal output feedback control system designs of full order. Almost all of the items listed can be considered favorable features except the last two.

We first discuss the aspects of digital computation. Linear optimal control system design usually requires the use of a digital computer, but this hardly constitutes an objection because of the widespread availability of computing facilities. In fact, the need for digital computation can be converted into an

Table 5.3 Characteristics of Linear Optimal Output Feedback Control System Designs

Design characteristic	Characteristic judged favorable (+), indifferent (□), or unfavorable (−)
Stability is guaranteed	+
A good response from initial conditions and to a reference variable can be obtained	+
Information about the closed-loop poles is available	+
The input amplitude or, equivalently, the loop gain, is easily controlled	+
Good protection against disturbances can be obtained	+
Adequate protection against observation noise can be obtained	+
The control system offers protection against plant variations	+
Digital computation is usually necessary for control system design	□
The control system may turn out to be rather complex	−

advantage, since it is possible to develop computer programs that largely automate the control system design procedure and at the same time produce a great deal of detailed information about the proposed design. Table 5.4 lists several subroutines that could be contained in a computer program package for the design and analysis of time-invariant, continuous-time linear optimal control systems. Apart from the subroutines listed, such a package should contain programs for coordinating the subroutines and handling the data.

The last item in the list of Table 5.3, concerning the complexity of linear output feedback controllers, raises a substantial objection. In Section 5.7 we discussed methods for obtaining controllers of reduced complexity. At present, too little experience with such design methods is available, however, to conclude that this approach solves the complexity problem.

Altogether, the perspective that linear optimal control theory offers for the solution of real, everyday, complex linear control problems is very favorable. It truly appears that this theory is a worthy successor to traditional control theory.

Table 5.4 Computer Subroutines for a Linear Optimal Control System Design and Analysis Package

Subroutine task	For discussion and references see
Computation of the exponential of a matrix	Section 1.3.2
Simulation of a time-invariant linear system	Section 1.3.2
Computation of the transfer matrix and characteristic values of a linear time-invariant system	Section 1.5.1
Computation of the zeroes of a square transfer matrix	Section 1.5.3
Simulation of a linear time-invariant system driven by white noise	Section 1.11.2
Solution of the linear matrix equation $M_1X + XM_2^T = M_3$	Section 1.11.3
Solution of the algebraic Riccati equation and computation of the corresponding closed-loop regulator or observer poles	Section 3.5
Numerical determination of an optimal controller of reduced dimension	Section 5.7.3

5.9 PROBLEMS

5.1. *Angular velocity regulation system*
Consider the angular velocity system described by the state differential equation

$$\dot{\xi}(t) = -\alpha\xi(t) + \kappa\mu(t) + w_1(t). \qquad \textbf{5-276}$$

Here ξ is the angular velocity, μ the driving voltage, and the disturbance w_1 is represented as white noise with intensity N. The controlled variable is the angular velocity:

$$\zeta(t) = \xi(t). \qquad \textbf{5-277}$$

The observed variable is also the angular velocity:

$$\eta(t) = \xi(t) + w_2(t), \qquad \textbf{5-278}$$

where w_2 is represented as white noise with intensity M. The following numerical values are assumed:

$$\begin{aligned} \alpha &= 0.5 \text{ s}^{-1}, \\ \kappa &= 150 \text{ rad}/(\text{Vs}^2), \\ N &= 600 \text{ rad}^2/\text{s}^2, \\ M &= 0.5 \text{ rad}^2/\text{s}. \end{aligned} \qquad \text{5-279}$$

Suppose that the angular velocity system is to be made into a regulator system, which keeps the angular velocity at a constant value. Determine the optimal output feedback regulator such that the rms input is 10 V. Compute the rms regulating error and compare this to the rms regulating error when no control is applied.

5.2. *Angular velocity tracking system*

Suppose that the system of Problem 5.1 is to be made into an angular velocity tracking system. For the reference variable we assume exponentially correlated noise with time constant θ and rms value σ. Furthermore, we assume that the reference variable is measured with additive white noise with intensity M_r. Compute the optimal tracking system. Assume the numerical values

$$\begin{aligned} \theta &= 1 \text{ s}, \\ \sigma &= 30 \text{ rad/s}, \\ M_r &= 0.8 \text{ rad}^2/\text{s}. \end{aligned} \qquad \text{5-280}$$

Determine the optimal tracking system such that the total rms input is 10 V. Compute the total rms tracking error and compare this to the rms value of the reference variable.

5.3. *Nonzero set point angular velocity control system*

The tracking system of Problem 5.2 does not have the property that a constant value of the reference variable causes a zero steady-state tracking error. To obtain such a controller, design a nonzero set point controller as suggested in Section 5.5.1. For the state feedback law, choose the one obtained in Problem 5.1. Choose the prefilter such that a step of 30 rad/s in the reference variable causes a peak input voltage of 10 V or less. Compare the resulting design to that of Problem 5.2.

5.4.* *Integral control of the angular velocity regulating system*

Consider the angular velocity control system as described in Problem 5.1. Suppose that in addition to the time-varying disturbance represented by $w_1(t)$ there is also a constant disturbance $v_0(t)$ operating upon the dc motor, so that the state differential equation takes the form

$$\dot{\xi}(t) = -\alpha\xi(t) + \kappa\mu(t) + w_1(t) + v_0(t). \qquad \text{5-281}$$

The observed variable is given by **5-278**, while the numerical values **5-279** are assumed. The controlled variable is given by **5-277**. Design for the present situation a zero-steady-state-error controller as described in Section 5.5.2. To this end, assume that $v_0(t)$ is represented as integrated white noise and choose the intensity of this white noise as 250 rad^2/s^3. Compute the response of the resulting integral control system to a step of 50 rad/s^2 in the constant disturbance v_0 from steady-state conditions and comment on this response. What is the effect of increasing or decreasing the assumed white noise intensity of 250 rad^2/s^3?

5.5.* *Adjoint matrix differential equations*

Consider the matrix differential equation

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A^T(t) + R(t), \qquad Q(t_0) = Q_0, \qquad \textbf{5-282}$$

together with the linear functional

$$\mathrm{tr}\left[\int_{t_0}^{t_1} Q(t)S(t)\,dt + Q(t_1)P_1\right]. \qquad \textbf{5-283}$$

Prove that **5-283** equals

$$\mathrm{tr}\left[\int_{t_0}^{t_1} P(t)R(t)\,dt + P(t_0)Q_0\right], \qquad \textbf{5-284}$$

where $P(t)$ is the solution of the adjoint matrix differential equation

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + S(t), \qquad P(t_1) = P_1. \qquad \textbf{5-285}$$

5.6.* *A property of scalar sensitivity functions*

In Section 5.6 we remarked that optimal linear output feedback systems generally do not possess the property that disturbances are attenuated at all frequencies as compared to the equivalent open-loop system. For single-input single-output systems this follows from the following theorem (Bode, 1945; Westcott, 1952).

Consider a single-input single-output linear time-invariant system with transfer function $H(s)$. Let the controller transfer function (see Fig. 5.14) be given by $G(s)$ so that the control system loop gain function is

$$L(s) = H(s)G(s), \qquad \textbf{5-286}$$

and the sensitivity function is

$$S(s) = \frac{1}{1 + L(s)}. \qquad \textbf{5-287}$$

Let ν denote the difference of the degree of the denominator of $L(s)$ and that of its numerator. Assume that the control system is asymptotically stable.

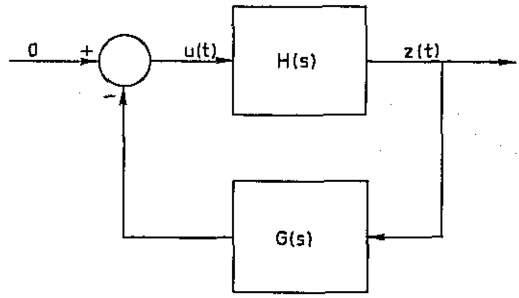

Fig. 5.14. A time-invariant linear feedback system.

Then

$$\int_{-\infty}^{\infty} \ln\,[|S(j\omega)|]\, d\omega = \begin{cases} \pm\infty & \text{for } \nu = 0, \\ -\gamma\dfrac{\pi}{\cancel{2}} & \text{for } \nu = 1, \\ 0 & \text{for } \nu \geq 2, \end{cases} \quad \text{5-288}$$

where

$$\gamma = \lim_{s\to\infty} sL(s). \quad \text{5-289}$$

Prove this result. Conclude that for plants and controllers without direct links the inequality

$$|S(j\omega)| \leq 1 \quad \text{5-290}$$

cannot hold for all ω. *Hint:* Integrate $\ln[S(s)]$ along a contour that consists of part of the imaginary axis closed with a semicircle in the right-half complex s-plane and let the radius of the semicircle go to infinity.