

1 ELEMENTS OF LINEAR SYSTEM THEORY

1.1 INTRODUCTION

This book deals with the analysis and design of linear control systems. A prerequisite for studying linear control systems is a knowledge of linear system theory. We therefore devote this first chapter to a review of the most important ingredients of linear system theory. The introduction of control problems is postponed until Chapter 2.

The main purpose of this chapter is to establish a conceptual framework, introduce notational conventions, and give a survey of the basic facts of linear system theory. The starting point is the state space description of linear systems. We then proceed to discussions of the solution of linear state differential equations, the stability of linear systems, and the transform analysis of such systems. The topics next dealt with are of a more advanced nature; they concern controllability, reconstructibility, duality, and phase-variable canonical forms of linear systems. The chapter concludes with a discussion of vector stochastic processes and the response of linear systems to white noise. These topics play an important role in the development of the theory.

Since the reader of this chapter is assumed to have had an introduction to linear system theory, the proofs of several well-known theorems are omitted. References to relevant textbooks are provided, however. Some topics are treated in sections marked with an asterisk, notably controllability, reconstructibility, duality and phase-variable canonical forms. The asterisk indicates that these notions are of a more advanced nature, and needed only in the sections similarly marked in the remainder of the book.

1.2 STATE DESCRIPTION OF LINEAR SYSTEMS

1.2.1 State Description of Nonlinear and Linear Differential Systems

Many systems can be described by a set of simultaneous differential equations of the form

$$\dot{x}(t) = f[x(t), u(t), t]. \quad 1-1$$

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Here t is the time variable, $x(t)$ is a real n -dimensional time-varying column vector which denotes the *state* of the system, and $u(t)$ is a real k -dimensional column vector which indicates the *input variable* or *control variable*. The function f is real and vector-valued. For many systems the choice of the state follows naturally from the physical structure, and 1-1, which will be called the *state differential equation*, usually follows directly from the elementary physical laws that govern the system.

Let $y(t)$ be a real l -dimensional system variable that can be observed or through which the system influences its environment. Such a variable we call an *output variable* of the system. It can often be expressed as

$$y(t) = g[x(t), u(t), t]. \quad 1-2$$

This equation we call the *output equation* of the system.

We call a system that is described by 1-1 and 1-2 a *finite-dimensional differential system* or, for short, a *differential system*. Equations 1-1 and 1-2 together are called the *system equations*. If the vector-valued function g contains u explicitly, we say that the system has a *direct link*.

In this book we are mainly concerned with the case where f and g are linear functions. We then speak of a (*finite-dimensional*) *linear differential system*. Its state differential equation has the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad 1-3$$

where $A(t)$ and $B(t)$ are time-varying matrices of appropriate dimensions. We call the dimension n of x the *dimension* of the system. The output equation for such a system takes the form

$$y(t) = C(t)x(t) + D(t)u(t). \quad 1-4$$

If the matrices A , B , C , and D are constant, the system is *time-invariant*.

1.2.2 Linearization

It is the purpose of this section to show that if $u_0(t)$ is a given input to a system described by the state differential equation 1-1, and $x_0(t)$ is a known solution of the state differential equation, we can find approximations to neighboring solutions, for small deviations in the initial state and in the input, from a linear state differential equation. Suppose that $x_0(t)$ satisfies

$$\dot{x}_0(t) = f[x_0(t), u_0(t), t], \quad t_0 \leq t \leq t_1. \quad 1-5$$

We refer to u_0 as a *nominal input* and to x_0 as a *nominal trajectory*. Often we can assume that the system is operated close to nominal conditions, which means that u and x deviate only slightly from u_0 and x_0 . Let us therefore write

$$\begin{aligned} u(t) &= u_0(t) + \tilde{u}(t), & t_0 \leq t \leq t_1, \\ x(t_0) &= x_0(t_0) + \tilde{x}(t_0), \end{aligned} \quad 1-6$$

where $\tilde{u}(t)$ and $\tilde{x}(t_0)$ are small perturbations. Correspondingly, let us introduce $\tilde{x}(t)$ by

$$x(t) = x_0(t) + \tilde{x}(t), \quad t_0 \leq t \leq t_1. \quad 1-7$$

Let us now substitute x and u into the state differential equation and make a Taylor expansion. It follows that

$$\begin{aligned} \dot{\tilde{x}}(t) + \tilde{x}(t) &= f[x_0(t), u_0(t), t] + J_x[x_0(t), u_0(t), t]\tilde{x}(t) \\ &+ J_u[x_0(t), u_0(t), t]\tilde{u}(t) + h(t), \quad t_0 \leq t \leq t_1. \end{aligned} \quad 1-8$$

Here J_x and J_u are the Jacobian matrices of f with respect to x and u , respectively, that is, J_x is a matrix the (i, j) -th element of which is

$$(J_x)_{i,j} = \frac{\partial f_i}{\partial \xi_j}, \quad 1-9$$

where f_i is the i -th component of f and ξ_j the j -th component of x . J_u is similarly defined. The term $h(t)$ is an expression that is supposed to be "small" with respect to \tilde{x} and \tilde{u} . Neglecting h , we see that \tilde{x} and \tilde{u} approximately satisfy the *linear* equation

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t) + B(t)\tilde{u}(t), \quad t_0 \leq t \leq t_1, \quad 1-10$$

where $A(t) = J_x[x_0(t), u_0(t), t]$ and $B(t) = J_u[x_0(t), u_0(t), t]$. We call 1-10 the *linearized state differential equation*. The initial condition of 1-10 is $\tilde{x}(t_0)$.

The linearization procedure outlined here is very common practice in the solution of control problems. Often it is more convenient to linearize the system differential equations before arranging them in the form of state differential equations. This leads to the same results, of course (see the examples of Section 1.2.3).

It can be inferred from texts on differential equations (see, e.g., Roseau, 1966) that the approximation to $x(t)$ obtained in this manner can be made arbitrarily accurate, provided the function f possesses partial derivatives with respect to the components of x and u near the nominal values x_0, u_0 , the interval $[t_0, t_1]$ is finite, and the initial deviation $\tilde{x}(t_0)$ and the deviation of the input \tilde{u} are chosen sufficiently small.

In Section 1.4.4 we present further justification of the extensive use of linearization in control engineering.

1.2.3 Examples

In this section several examples are given which serve to show how physical equations are converted into state differential equations and how linearization is performed. We discuss these examples at some length because later they are extensively used to illustrate the theory that is given.

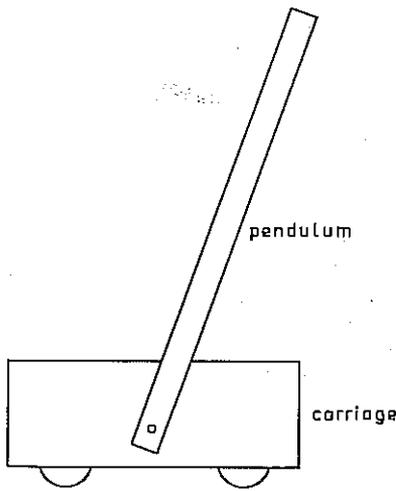


Fig. 1.1. An inverted pendulum positioning system.

Example 1.1. *Inverted pendulum positioning system.*

Consider the inverted pendulum of Figure 1.1 (see also, for this example, Cannon, 1967; Elgerd, 1967). The pivot of the pendulum is mounted on a carriage which can move in a horizontal direction. The carriage is driven by a small motor that at time t exerts a force $\mu(t)$ on the carriage. This force is the input variable to the system.

Figure 1.2 indicates the forces and the displacements. The displacement of the pivot at time t is $s(t)$, while the angular rotation at time t of the pendulum is $\phi(t)$. The mass of the pendulum is m , the distance from the pivot to the center of gravity L , and the moment of inertia with respect to the center of gravity J . The carriage has mass M . The forces exerted on the pendulum are

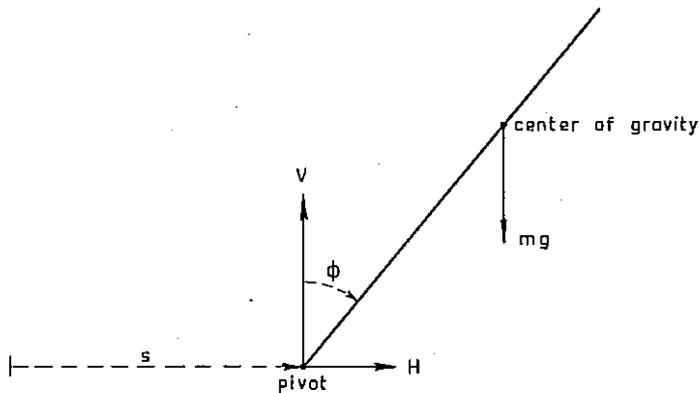


Fig. 1.2. Inverted pendulum: forces and displacements.

the force mg in the center of gravity, a horizontal reaction force $H(t)$, and a vertical reaction force $V(t)$ in the pivot. Here g is the gravitational acceleration. The following equations hold for the system:

$$m \frac{d^2}{dt^2} [s(t) + L \sin \phi(t)] = H(t), \quad 1-11$$

$$m \frac{d^2}{dt^2} [L \cos \phi(t)] = V(t) - mg, \quad 1-12$$

$$J \frac{d^2 \phi(t)}{dt^2} = LV(t) \sin \phi(t) - LH(t) \cos \phi(t), \quad 1-13$$

$$M \frac{d^2 s(t)}{dt^2} = \mu(t) - H(t) - F \frac{ds(t)}{dt}. \quad 1-14$$

Friction is accounted for only in the motion of the carriage and not at the pivot; in 1-14, F represents the friction coefficient. Performing the differentiations indicated in 1-11 and 1-12, we obtain

$$m\ddot{s}(t) + mL\ddot{\phi}(t) \cos \phi(t) - mL\dot{\phi}^2(t) \sin \phi(t) = H(t), \quad 1-15$$

$$-mL\ddot{\phi}(t) \sin \phi(t) - mL\dot{\phi}^2(t) \cos \phi(t) = V(t) - mg, \quad 1-16$$

$$J\ddot{\phi}(t) = LV(t) \sin \phi(t) - LH(t) \cos \phi(t), \quad 1-17$$

$$M\ddot{s}(t) = \mu(t) - H(t) - F\dot{s}(t). \quad 1-18$$

To simplify the equations we assume that m is small with respect to M and therefore neglect the horizontal reaction force $H(t)$ on the motion of the carriage. This allows us to replace 1-18 with

$$M\ddot{s}(t) = \mu(t) - F\dot{s}(t). \quad 1-19$$

Elimination of $H(t)$ and $V(t)$ from 1-15, 1-16, and 1-17 yields

$$(J + mL^2)\ddot{\phi}(t) - mgL \sin \phi(t) + mL\dot{s}(t) \cos \phi(t) = 0. \quad 1-20$$

Division of this equation by $J + mL^2$ yields

$$\ddot{\phi}(t) - \frac{g}{L'} \sin \phi(t) + \frac{1}{L'} \dot{s}(t) \cos \phi(t) = 0, \quad 1-21$$

where

$$L' = \frac{J + mL^2}{mL}. \quad 1-22$$

This quantity has the significance of "effective pendulum length" since a mathematical pendulum of length L' would also yield 1-21.

Let us choose as the nominal solution $\mu(t) \equiv 0$, $s(t) \equiv 0$, $\phi(t) \equiv 0$. Linearization can easily be performed by using Taylor series expansions for $\sin \phi(t)$ and $\cos \phi(t)$ in 1-21 and retaining only the first term of the series. This yields the linearized version of 1-21:

$$\ddot{\phi}(t) - \frac{g}{L'} \phi(t) + \frac{1}{L'} \dot{s}(t) = 0. \quad 1-23$$

We choose the components of the state $x(t)$ as

$$\begin{aligned} \xi_1(t) &= s(t), \\ \xi_2(t) &= \dot{s}(t), \\ \xi_3(t) &= s(t) + L' \phi(t), \\ \xi_4(t) &= \dot{s}(t) + L' \dot{\phi}(t). \end{aligned} \quad 1-24$$

The third component of the state represents a linearized approximation to the displacement of a point of the pendulum at a distance L' from the pivot. We refer to $\xi_3(t)$ as the displacement of the pendulum. With these definitions we find from 1-19 and 1-23 the linearized state differential equation

$$\begin{aligned} \dot{\xi}_1(t) &= \xi_2(t), \\ \dot{\xi}_2(t) &= \frac{1}{M} \mu(t) - \frac{F}{M} \xi_2(t), \\ \dot{\xi}_3(t) &= \xi_4(t), \\ \dot{\xi}_4(t) &= g\phi(t) = \frac{g}{L'} [\xi_3(t) - \xi_1(t)]. \end{aligned} \quad 1-25$$

In vector notation we write

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{g}{L'} & 0 & \frac{g}{L'} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ 0 \end{pmatrix} \mu(t), \quad 1-26$$

where $x(t) = \text{col} [\xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t)]$.

Later the following numerical values are used:

$$\frac{F}{M} = 1 \text{ s}^{-1},$$

$$\frac{1}{M} = 1 \text{ kg}^{-1},$$

1-27

$$\frac{g}{L} = 11.65 \text{ s}^{-2},$$

$$L = 0.842 \text{ m}.$$

Example 1.2. *A stirred tank.*

As a further example we treat a system that is to some extent typical of process control systems. Consider the stirred tank of Fig. 1.3. The tank is fed

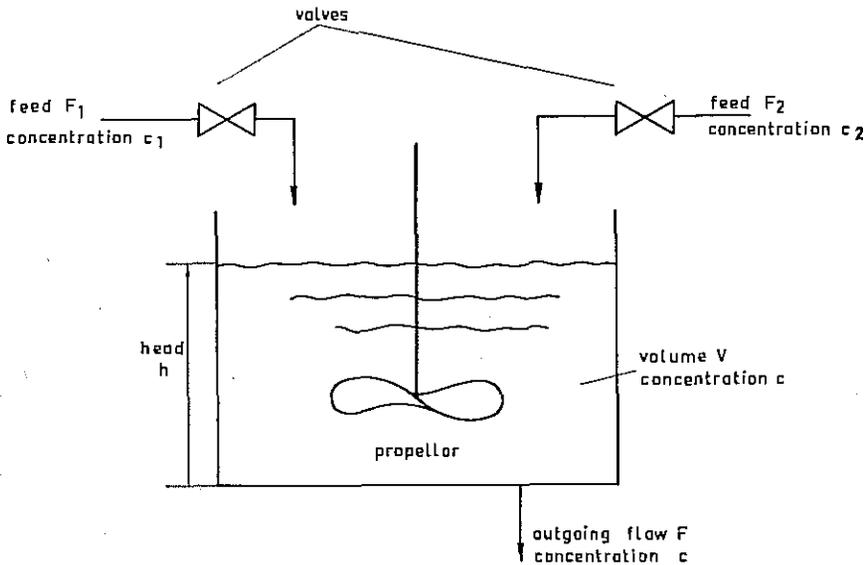


Fig. 1.3. A stirred tank.

with two incoming flows with time-varying flow rates $F_1(t)$ and $F_2(t)$. Both feeds contain dissolved material with constant concentrations c_1 and c_2 , respectively. The outgoing flow has a flow rate $F(t)$. It is assumed that the tank is stirred well so that the concentration of the outgoing flow equals the concentration $c(t)$ in the tank.

The mass balance equations are

$$\frac{dV(t)}{dt} = F_1(t) + F_2(t) - F(t), \quad 1-28$$

$$\frac{d}{dt} [c(t)V(t)] = c_1F_1(t) + c_2F_2(t) - c(t)F(t), \quad 1-29$$

where $V(t)$ is the volume of the fluid in the tank. The outgoing flow rate $F(t)$ depends upon the head $h(t)$ as follows

$$F(t) = k\sqrt{h(t)}, \quad 1-30$$

where k is an experimental constant. If the tank has constant cross-sectional area S , we can write

$$F(t) = k\sqrt{\frac{V(t)}{S}}, \quad 1-31$$

so that the mass balance equations are

$$\frac{dV(t)}{dt} = F_1(t) + F_2(t) - k\sqrt{\frac{V(t)}{S}}, \quad 1-32$$

$$\frac{d}{dt} [c(t)V(t)] = c_1F_1(t) + c_2F_2(t) - c(t)k\sqrt{\frac{V(t)}{S}}. \quad 1-33$$

Let us first consider a steady-state situation where all quantities are constant, say F_{10} , F_{20} , and F_0 for the flow rates, V_0 for the volume, and c_0 for the concentration in the tank. Then the following relations hold:

$$0 = F_{10} + F_{20} - F_0, \quad 1-34$$

$$0 = c_1F_{10} + c_2F_{20} - c_0F_0, \quad 1-35$$

$$F_0 = k\sqrt{\frac{V_0}{S}}. \quad 1-36$$

For given F_{10} and F_{20} , these equations can be solved for F_0 , V_0 , and c_0 . Let us now assume that only small deviations from steady-state conditions occur. We write

$$\begin{aligned} F_1(t) &= F_{10} + \mu_1(t), \\ F_2(t) &= F_{20} + \mu_2(t), \\ V(t) &= V_0 + \xi_1(t), \\ c(t) &= c_0 + \xi_2(t), \end{aligned} \quad 1-37$$

where we consider μ_1 and μ_2 input variables and ξ_1 and ξ_2 state variables. By assuming that these four quantities are small, linearization of 1-32 and 1-33 gives

$$\dot{\xi}_1(t) = \mu_1(t) + \mu_2(t) - \frac{k}{2V_0} \sqrt{\frac{V_0}{S}} \xi_1(t), \quad 1-38$$

$$\dot{\xi}_2(t)V_0 + c_0\dot{\xi}_1(t) = c_1\mu_1(t) + c_2\mu_2(t) - c_0\frac{k}{2V_0} \sqrt{\frac{V_0}{S}} \xi_1(t) - k\sqrt{\frac{V_0}{S}} \xi_2(t). \quad 1-39$$

Substitution of 1-36 into these equations yields

$$\dot{\xi}_1(t) = \mu_1(t) + \mu_2(t) - \frac{1}{2} \frac{F_0}{V_0} \xi_1(t), \quad 1-40$$

$$\dot{\xi}_2(t)V_0 + c_0\dot{\xi}_1(t) = c_1\mu_1(t) + c_2\mu_2(t) - \frac{1}{2} c_0 \frac{F_0}{V_0} \xi_1(t) - F_0\xi_2(t). \quad 1-41$$

We define

$$\frac{V_0}{F_0} = \theta, \quad 1-42$$

and refer to θ as the *holdup time* of the tank. Elimination of $\dot{\xi}_1$ from 1-41 results in the linearized state differential equation

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{2\theta} & 0 \\ 0 & -\frac{1}{\theta} \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ \frac{c_1 - c_0}{V_0} & \frac{c_2 - c_0}{V_0} \end{pmatrix} u(t), \quad 1-43$$

where $x(t) = \text{col} [\xi_1(t), \xi_2(t)]$ and $u(t) = \text{col} [\mu_1(t), \mu_2(t)]$. If we moreover define the output variables

$$\eta_1(t) = F(t) - F_0 \simeq \frac{1}{2} \frac{F_0}{V_0} \xi_1(t) = \frac{1}{2\theta} \xi_1(t), \quad 1-44$$

$$\eta_2(t) = c(t) - c_0 = \xi_2(t),$$

we can complement 1-43 with the linearized output equation

$$y(t) = \begin{pmatrix} \frac{1}{2\theta} & 0 \\ 0 & 1 \end{pmatrix} x(t), \quad 1-45$$

where $y(t) = \text{col} [\eta_1(t), \eta_2(t)]$. We use the following numerical values:

$$\begin{aligned}
 F_{10} &= 0.015 \text{ m}^3/\text{s}, \\
 F_{20} &= 0.005 \text{ m}^3/\text{s}, \\
 F_0 &= 0.02 \text{ m}^3/\text{s}, \\
 c_1 &= 1 \text{ kmol/m}^3, \\
 c_2 &= 2 \text{ kmol/m}^3, \\
 c_0 &= 1.25 \text{ kmol/m}^3, \\
 V_0 &= 1 \text{ m}^3, \\
 \theta &= 50 \text{ s}.
 \end{aligned}
 \tag{1-46}$$

This results in the linearized system equations

$$\begin{aligned}
 \dot{x}(t) &= \begin{pmatrix} -0.01 & 0 \\ 0 & -0.02 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ -0.25 & 0.75 \end{pmatrix} u(t), \\
 y(t) &= \begin{pmatrix} 0.01 & 0 \\ 0 & 1 \end{pmatrix} x(t).
 \end{aligned}
 \tag{1-47}$$

1.2.4 State Transformations

As we shall see, it is sometimes useful to employ a transformed representation of the state. In this section we briefly review linear state transformations for time-invariant linear differential systems. Consider the linear time-invariant system

$$\begin{aligned}
 \dot{x}(t) &= Ax(t) + Bu(t), \\
 y(t) &= Cx(t).
 \end{aligned}
 \tag{1-48}$$

Let us define a transformed state variable

$$x'(t) = Tx(t), \tag{1-49}$$

where T is a constant, nonsingular transformation matrix. Substitution of $x(t) = T^{-1}x'(t)$ into 1-48 yields

$$\begin{aligned}
 T^{-1}\dot{x}'(t) &= AT^{-1}x'(t) + Bu(t), \\
 y(t) &= CT^{-1}x'(t),
 \end{aligned}
 \tag{1-50}$$

or

$$\begin{aligned}
 \dot{x}'(t) &= TAT^{-1}x'(t) + TBu(t), \\
 y(t) &= CT^{-1}x'(t).
 \end{aligned}
 \tag{1-51}$$

These are the state differential equation and the output equation of the system in terms of the new state $x'(t)$. It is clear that the transformed representation is completely equivalent to the original system, since we can always reconstruct the behavior of the system in terms of the original state by the relation $x(t) = T^{-1}x'(t)$. This derivation shows that the choice of the state is to some extent arbitrary and therefore can be adapted to suit various purposes. Many properties of linear, time-invariant systems remain unchanged under a state transformation (Problems 1.3, 1.6, 1.7).

1.3 SOLUTION OF THE STATE DIFFERENTIAL EQUATION OF LINEAR SYSTEMS

1.3.1 The Transition Matrix and the Impulse Response Matrix

In this section we discuss the solution of the linear state differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t). \quad 1-52$$

We first have the following result (Zadeh and Desoer, 1963; Desoer, 1970).

Theorem 1.1. *Consider the homogeneous equation*

$$\dot{x}(t) = A(t)x(t). \quad 1-53$$

Then if $A(t)$ is continuous for all t , 1-53 always has a solution which can be expressed as

$$x(t) = \Phi(t, t_0)x(t_0), \quad \text{for all } t. \quad 1-54$$

The transition matrix $\Phi(t, t_0)$ is the solution of the matrix differential equation

$$\frac{d}{dt}\Phi(t, t_0) = A(t)\Phi(t, t_0), \quad \text{for all } t, \quad 1-55$$

$$\Phi(t_0, t_0) = I,$$

where I is the unit matrix.

For a general time-varying system, the transition matrix rarely can be obtained in terms of standard functions, so that one must resort to numerical integration techniques. For time-invariant systems of low dimensions or of a simple structure, the transition matrix can be computed by any of the methods discussed in Sections 1.3.2, 1.3.3, and 1.5.1. For complicated time-invariant problems, one must employ numerical methods such as described in Section 1.3.2.

The transition matrix can be shown to possess the following properties (Zadeh and Desoer, 1963).

Theorem 1.2. *The transition matrix $\Phi(t, t_0)$ of a linear differential system has the following properties:*

$$(a) \quad \Phi(t_2, t_1)\Phi(t_1, t_0) = \Phi(t_2, t_0) \quad \text{for all } t_0, t_1, t_2; \quad 1-56$$

$$(b) \quad \Phi(t, t_0) \quad \text{is nonsingular for all } t, t_0; \quad 1-57$$

$$(c) \quad \Phi^{-1}(t, t_0) = \Phi(t_0, t) \quad \text{for all } t, t_0; \quad 1-58$$

$$(d) \quad \frac{d}{dt} \Phi^T(t_0, t) = -A^T(t)\Phi^T(t_0, t) \quad \text{for all } t, t_0, \quad 1-59$$

where the superscript T denotes the transpose.

Property (d) shows that the system $\dot{x}(t) = -A^T(t)x(t)$ has the transition matrix $\Phi^T(t_0, t)$. This can be proved by differentiating the identity $\Phi(t, t_0)\Phi(t_0, t) = I$.

Once the transition matrix has been found, it is easy to obtain solutions to the state differential equation 1-52.

Theorem 1.3. *Consider the linear state differential equation*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t). \quad 1-60$$

Then if $A(t)$ is continuous and $B(t)$ and $u(t)$ are piecewise continuous for all t , the solution of 1-60 is

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau) d\tau \quad 1-61$$

for all t .

This result is easily verified by direct substitution into the state differential equation (Zadeh and Desoer, 1963).

Consider now a system with the state differential equation 1-60 and the output equation

$$y(t) = C(t)x(t). \quad 1-62$$

For the output variable we write

$$y(t) = C(t)\Phi(t, t_0)x(t_0) + C(t) \int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau) d\tau. \quad 1-63$$

If the system is initially in the zero state, that is, $x(t_0) = 0$, the response of the output variable is given by

$$y(t) = \int_{t_0}^t K(t, \tau)u(\tau) d\tau, \quad t \geq t_0, \quad 1-64$$

where

$$K(t, \tau) = C(t)\Phi(t, \tau)B(\tau), \quad t \geq \tau. \quad 1-65$$

The matrix $K(t, \tau)$ is called the *impulse response matrix* of the system because the (i, j) -th element of this matrix is the response at time t of the i -th component of the output variable to an impulse applied at the j -th component of the input at time $\tau > t_0$ while all other components of the input are zero and the initial state is zero. The *step response matrix* $S(t, \tau)$ is defined as

$$S(t, \tau) = \int_{\tau}^t K(t, \tau') d\tau', \quad t \geq \tau. \quad 1-66$$

The (i, j) -th element of the step response matrix is the response at time t of the i -th component of the output when the j -th component of the input is a step function applied at time $\tau > t_0$ while all other components of the input are zero and the initial state is the zero state.

1.3.2 The Transition Matrix of a Time-Invariant System

For a time-invariant system, the transition matrix can be given in an explicit form (Zadeh and Desoer, 1963; Desoer, 1970; Polak and Wong, 1970).

Theorem 1.4. *The time-invariant system*

$$\dot{x}(t) = Ax(t) \quad 1-67$$

has the transition matrix

$$\Phi(t, t_0) = e^{A(t-t_0)}, \quad 1-68$$

where the exponential of a square matrix M is defined as

$$e^{M} = I + M + \frac{1}{2!} M^2 + \frac{1}{3!} M^3 + \dots \quad 1-69$$

This series converges for all M .

For small dimensions or simple structures of the matrix A , this result can be used to write down the transition matrix explicitly in terms of elementary functions (see Example 1.3). For high dimensions of the matrix A , Theorem 1.4 is quite useful for the computation of the transition matrix by a digital computer since the repeated multiplications and additions are easily programmed and performed. Such programs must include a stopping rule to truncate the infinite series after a finite number of terms. A usual stopping rule is to truncate when the addition of a new term changes each of the elements of the partial sum by less than a specified fraction. Numerical difficulties may occur when M is too large; this means that $t - t_0$ in 1-68 cannot be chosen too large (see Kalman, 1966; Kalman and Englar, 1966). Having a program for computing matrix exponentials is essential for anyone who wishes to simulate linear time-invariant systems. There are numerous references on the computation of matrix exponentials and simulating linear

systems; some of these are: Everling (1967), Liou (1966a,b, 1967, 1968), Whitney (1966a-c), Bickart (1968), Fath (1968), Plant (1969), Wallach (1969), Levis (1969), Rohrer (1970), Mastascusa and Simes (1970), and Krouse and Ward (1970). Melsa (1970) gives listings of FORTRAN computer programs.

By using 1-68 the time-invariant version of 1-63 becomes

$$y(t) = Ce^{A(t-t_0)}x(t_0) + C \int_{t_0}^t e^{A(t-\tau)}Bu(\tau) d\tau. \quad 1-70$$

Comparing 1-64 and 1-70 we see that the impulse response matrix of a time-invariant linear differential system depends on $t - \tau$ only and can be expressed as

$$K(t - \tau) = Ce^{A(t-\tau)}B, \quad t \geq \tau. \quad 1-71$$

Example 1.3. *Stirred tank.*

The homogeneous part of the linearized state differential equation of the stirred tank of Example 1.2 is given by

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{2\theta} & 0 \\ 0 & -\frac{1}{\theta} \end{pmatrix} x(t). \quad 1-72$$

It is easily found that its transition matrix is given by

$$\Phi(t, t_0) = e^{A(t-t_0)}, \quad 1-73$$

where

$$\begin{aligned} e^{At} &= \begin{pmatrix} 1 - \frac{t}{2\theta} + \frac{1}{2!} \left(\frac{t}{2\theta}\right)^2 - \frac{1}{3!} \left(\frac{t}{2\theta}\right)^3 + \dots & 0 \\ 0 & 1 - \frac{t}{\theta} + \frac{1}{2!} \left(\frac{t}{\theta}\right)^2 - \frac{1}{3!} \left(\frac{t}{\theta}\right)^3 + \dots \end{pmatrix} \\ &= \begin{pmatrix} e^{-t/2\theta} & 0 \\ 0 & e^{-t/\theta} \end{pmatrix}. \end{aligned} \quad 1-74$$

The impulse response matrix of the system is

$$K(t - \tau) = \begin{pmatrix} \frac{1}{2\theta} e^{-(t-\tau)/2\theta} & \frac{1}{2\theta} e^{-(t-\tau)/2\theta} \\ \frac{c_1 - c_0}{V_0} e^{-(t-\tau)/\theta} & \frac{c_2 - c_0}{V_0} e^{-(t-\tau)/\theta} \end{pmatrix}. \quad 1-75$$

We find for the step response matrix of the stirred tank:

$$S(t - \tau) = \begin{pmatrix} 1 - e^{-(t-\tau)/20} & 1 - e^{-(t-\tau)/20} \\ \frac{c_1 - c_0}{F_0} (1 - e^{-(t-\tau)/\theta}) & \frac{c_2 - c_0}{F_0} (1 - e^{-(t-\tau)/\theta}) \end{pmatrix}. \quad 1-76$$

In Fig. 1.4 the step responses are sketched for the numerical data of Example 1.2.

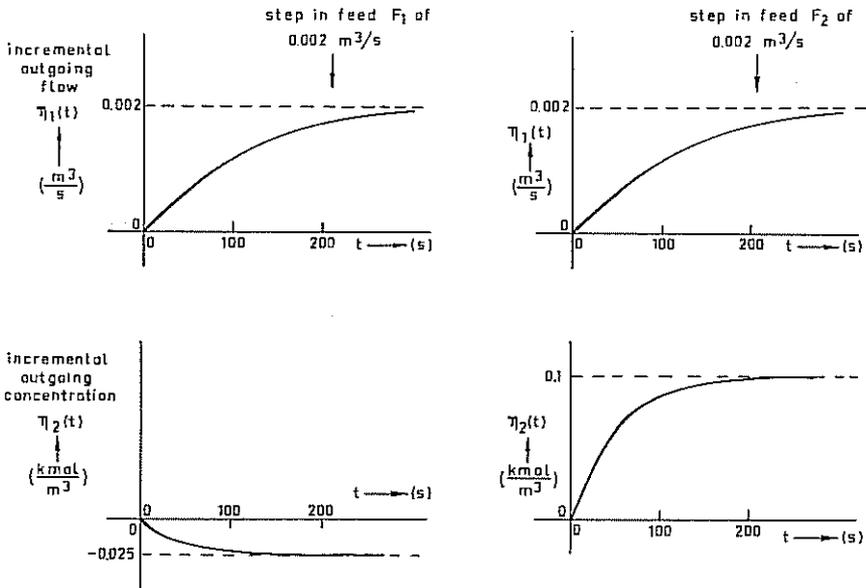


Fig. 1.4. Response of the stirred tank to a step of $0.002 \text{ m}^3/\text{s}$ in the feed F_1 (left column) and to a step of $0.002 \text{ m}^3/\text{s}$ in the feed F_2 (right column).

1.3.3 Diagonalization

An explicit form of the transition matrix of a time-invariant system can be obtained by diagonalization of the matrix A . The following result is available (Noble, 1969).

Theorem 1.5. Suppose that the constant $n \times n$ matrix A has n distinct characteristic values $\lambda_1, \lambda_2, \dots, \lambda_n$. Let the corresponding characteristic vectors be e_1, e_2, \dots, e_n . Define the $n \times n$ matrices

$$T = (e_1, e_2, \dots, e_n), \quad 1-77a$$

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n). \quad 1-77b$$

Then T is nonsingular and A can be represented as

$$A = T\Lambda T^{-1}. \quad 1-78$$

Here the notation **1-77a** implies that the vectors e_1, e_2, \dots, e_n are the columns of the matrix T , and **1-77b** means that Λ is a diagonal matrix with $\lambda_1, \lambda_2, \dots, \lambda_n$ as diagonal elements. It is said that T diagonalizes A .

The following fact is easily verified.

Theorem 1.6. Consider the matrix A that satisfies the assumptions of Theorem 1.5. Then

$$(a) \quad e^{At} = T e^{\Lambda t} T^{-1}, \quad 1-79$$

$$(b) \quad e^{\Lambda t} = \text{diag} (e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}). \quad 1-80$$

This result makes it simple to compute $\exp (At)$ once A is diagonalized. It is instructive to present the same result in a different form.

Theorem 1.7. Consider the time-invariant system

$$\dot{x}(t) = Ax(t), \quad 1-81$$

where A satisfies the assumptions of Theorem 1.5. Write the matrix T^{-1} in the form

$$T^{-1} = \begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ f_n \end{pmatrix}, \quad 1-82$$

that is, the row vectors f_1, f_2, \dots, f_n are the rows of T^{-1} . Then the solution of **1-81** can be written as

$$x(t) = \sum_{i=1}^n e^{\lambda_i t} e_i f_i x(0). \quad 1-83$$

This is easily shown by expanding $x(t) = T \exp (\Lambda t) T^{-1} x(0)$ in terms of e_i, f_i , and $\exp (\lambda_i t)$, $i = 1, 2, \dots, n$. We write **1-83** in the form

$$x(t) = \sum_{i=1}^n \mu_i e^{\lambda_i t} e_i, \quad 1-84$$

where the μ_i are the scalars $f_i x(0)$, $i = 1, 2, \dots, n$. This clearly shows that the response of the system **1-81** is a composition of motions along the characteristic vectors of the matrix A . We call such a motion a *mode* of the system. A particular mode is *excited* by choosing the initial state to have a component along the corresponding characteristic vector.

It is clear that the characteristic values $\lambda_i, i = 1, 2, \dots, n$, to a considerable extent determine the dynamic behavior of the system. We often refer to these numbers as the *poles of the system*.

Even if the matrix A has multiple characteristic values, it can be diagonalized provided that the number of linearly independent characteristic vectors for each characteristic value equals the multiplicity of the characteristic value. The more complicated case, where the matrix A cannot be diagonalized, is discussed in Section 1.3.4.

Example 1.4. *Inverted pendulum.*

The homogeneous part of the state differential equation of the inverted pendulum balancing system of Example 1.1 is

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{g}{L'} & 0 & \frac{g}{L'} & 0 \end{pmatrix} x(t). \quad 1-85$$

The characteristic values and characteristic vectors of the matrix A can be found to be

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{F}{M}, \quad \lambda_3 = \sqrt{\frac{g}{L'}}, \quad \lambda_4 = -\sqrt{\frac{g}{L'}},$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 \\ -\frac{F}{M} \\ \alpha \\ -\alpha \frac{F}{M} \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \sqrt{\frac{g}{L'}} \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\sqrt{\frac{g}{L'}} \end{pmatrix}, \quad 1-86$$

where

$$\alpha = \frac{\frac{g}{L'}}{\frac{g}{L'} - \frac{F^2}{M^2}}, \quad 1-87$$

and where we assume that the denominator of α differs from zero. The matrix T and its inverse are

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 1 & \alpha & 1 & 1 \\ 0 & -\alpha\frac{F}{M} & \sqrt{\frac{g}{L'}} & -\sqrt{\frac{g}{L'}} \end{pmatrix},$$

$$T^{-1} = \begin{pmatrix} 1 & \frac{M}{F} & 0 & 0 \\ 0 & -\frac{M}{F} & 0 & 0 \\ -\frac{1}{2} & -\frac{\frac{1}{2}}{\frac{F}{M} + \sqrt{\frac{g}{L'}}}} & \frac{1}{2} & \frac{1}{2}\sqrt{\frac{L'}{g}} \\ -\frac{1}{2} & -\frac{\frac{1}{2}}{-\frac{F}{M} + \sqrt{\frac{g}{L'}}}} & \frac{1}{2} & -\frac{1}{2}\sqrt{\frac{L'}{g}} \end{pmatrix}. \quad 1-88$$

The modes of the system are

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -\frac{F}{M} \\ \alpha \\ -\alpha\frac{F}{M} \end{pmatrix} e^{-(F/M)t}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ \sqrt{\frac{g}{L'}} \end{pmatrix} e^{t\sqrt{g/L'}}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\sqrt{\frac{g}{L'}} \end{pmatrix} e^{-t\sqrt{g/L'}}.$$

1-89

The first mode represents the indifference of the system with respect to horizontal translations, while the third mode exhibits the unstable character of the inverted pendulum.

1.3.4* The Jordan Form

In the preceding section we saw that the representation of the transition matrix can be facilitated by diagonalizing the matrix A . This diagonalization is not possible if the $n \times n$ matrix A does not have n linearly independent characteristic vectors. In this case, however, it is possible to bring A into the so-called Jordan normal form which is almost diagonal and from which the transition matrix can easily be obtained.

We first recall a few facts from linear algebra. If M is a matrix, the *null space* of M is defined as

$$\mathcal{N}(M) = \{x: x \in \mathcal{C}^n, Mx = 0\}, \quad 1-90$$

where \mathcal{C}^n is the n -dimensional complex vector space. Furthermore, if \mathcal{M}_1 and \mathcal{M}_2 are two linear subspaces of an n -dimensional space, a linear subspace \mathcal{M}_3 is said to be the *direct sum* of \mathcal{M}_1 and \mathcal{M}_2 , written as

$$\mathcal{M}_3 = \mathcal{M}_1 \oplus \mathcal{M}_2, \quad 1-91$$

if any vector $x_3 \in \mathcal{M}_3$ can be written in one and only one way as $x_3 = x_1 + x_2$, where $x_1 \in \mathcal{M}_1$ and $x_2 \in \mathcal{M}_2$.

We have the following result (Zadeh and Desoer, 1963).

Theorem 1.8. *Suppose that the $n \times n$ matrix A has k distinct characteristic values λ_i , $i = 1, 2, \dots, k$. Let the multiplicity of each characteristic value λ_i in the characteristic polynomial of A be given by m_i . Define*

$$M_i = (A - \lambda_i I)^{m_i}, \quad 1-92$$

and let

$$\mathcal{N}_i = \mathcal{N}(M_i). \quad 1-93$$

Then

- (a) *The dimension of the linear subspace \mathcal{N}_i is m_i , $i = 1, 2, \dots, k$;*
- (b) *The whole n -dimensional complex space \mathcal{C}^n is the direct sum of the null spaces \mathcal{N}_i , $i = 1, 2, \dots, k$, that is,*

$$\mathcal{C}^n = \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \dots \oplus \mathcal{N}_k. \quad 1-94$$

When the matrix A has n distinct characteristic values, the null spaces \mathcal{N}_i reduce to one-dimensional subspaces each of which is spanned by a characteristic vector of A .

We have the following fact (Noble, 1969).

Theorem 1.9. *Consider the matrix A with the same notation as in Theorem 1.8. Then it is always possible to find a nonsingular transformation matrix T*

* See the Preface for the significance of the sections marked with an asterisk.

which can be partitioned as

$$T = (T_1, T_2, \dots, T_k), \tag{1-95}$$

such that

$$A = TJT^{-1}, \tag{1-96}$$

where

$$J = \text{diag} (J_1, J_2, \dots, J_k). \tag{1-97}$$

The block J_i has dimensions $m_i \times m_i$, $i = 1, 2, \dots, k$, and the partitioning of T matches that of J . The columns of T_i form a specially chosen basis for the null space \mathcal{N}_i , $i = 1, 2, \dots, k$. The blocks J_i can be subpartitioned as

$$J_i = \text{diag} (J_{i1}, J_{i2}, \dots, J_{i\ell_i}), \tag{1-98}$$

where each subblock J_{ij} is of the form

$$J_{ij} = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & \dots \\ 0 & \lambda_i & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \lambda_i & 1 \\ 0 & \dots & 0 & 0 & \lambda_i \end{pmatrix}. \tag{1-99}$$

J is called the *Jordan normal form* of A .

Expression 1-96 suggests the following practical method of computing the transformation matrix T (Noble, 1969). From 1-96 it follows

$$AT = TJ. \tag{1-100}$$

Let us denote the columns of T as q_1, q_2, \dots, q_n . Then from the form of J , it follows with 1-100 that

$$Aq_i = \lambda q_i + \gamma_i q_{i-1}, \tag{1-101}$$

where γ_i is either 0 or 1, depending on J , and where λ is a characteristic value of A . Let us subpartition the block T_i of T corresponding to the subpartitioning 1-98 of J_i as $T_{i1}, T_{i2}, \dots, T_{i\ell_i}$. Then the number γ_i is zero whenever the corresponding column q_i of T is the first column of a subblock. Since if $\gamma_i = 0$ the vector q_i is a characteristic vector of A , we see that we can find the first columns of each subblock as the characteristic vectors of A . The remaining columns of each subblock then follow from 1-101 with $\gamma_i = 1$. Those remaining columns are known as *generalized* characteristic vectors of the matrix A . We stop this process when 1-101 fails to have a solution. Example 1.5 at the end of this section illustrates the procedure.

Once the matrix A has been brought into Jordan normal form, the exponential of A is easily found.

Theorem 1.10. Consider the matrix A with the same notation as in Theorems 1.8 and 1.9. Then

$$(a) e^{At} = Te^{Jt}T^{-1}, \tag{1-102}$$

$$(b) e^{Jt} = \text{diag}(e^{J_{11}t}, e^{J_{22}t}, \dots, e^{J_{kk}t}), \tag{1-103}$$

$$(c) e^{J_{ij}t} = \text{diag}(e^{J_{i1}t}, e^{J_{i2}t}, \dots, e^{J_{in_i}t}), \tag{1-104}$$

$$(d) e^{J_{ij}t} = e^{\lambda_{ij}t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} \dots \frac{t^{n_{ij}-1}}{(n_{ij}-1)!} \\ 0 & 1 & t \dots \frac{t^{n_{ij}-2}}{(n_{ij}-2)!} \\ \dots & \dots & \dots \\ 0 & \dots & \dots & 1 \end{pmatrix}, \tag{1-105}$$

where n_{ij} is the dimension of J_{ij} .

It is seen from this theorem that the response of the system

$$\dot{x}(t) = Ax(t) \tag{1-106}$$

may contain besides purely exponential terms of the form $\exp(\lambda_i t)$ also terms of the form $t \exp(\lambda_i t)$, $t^2 \exp(\lambda_i t)$, and so on.

Completely in analogy with Section 1.3.3, we have the following fact (Zadeh and Desoer, 1963).

Theorem 1.11. Consider the time-invariant linear system

$$\dot{x}(t) = Ax(t). \tag{1-107}$$

Express the initial state $x(0)$ as

$$x(0) = \sum_{i=1}^k v_i \quad \text{with } v_i \in \mathcal{N}_i, \quad i = 1, 2, \dots, k. \tag{1-108}$$

Write

$$T^{-1} = \begin{pmatrix} U_1 \\ U_2 \\ \cdot \\ \cdot \\ U_k \end{pmatrix}, \tag{1-109}$$

where the partitioning corresponds to that of T in Theorem 1.9. Then the response of the system can be expressed as

$$x(t) = \sum_{i=1}^k T_i \exp(J_i t) U_i v_i. \quad 1-110$$

From this theorem we see that if the initial state is within one of the null spaces \mathcal{N}_i , the nature of the response of the system to this initial state is completely determined by the corresponding characteristic value. In analogy with the simple case of Section 1.3.3, we call the response of the system to any initial state within one of the null spaces a *mode* of the system.

Example 1.5. *Inverted pendulum.*

Consider the inverted pendulum of Example 1.1, but suppose that we neglect the friction of the carriage so that $F = 0$. The homogeneous part of the linearized state differential equation is now given by $\dot{x}(t) = Ax(t)$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{g}{L'} & 0 & \frac{g}{L'} & 0 \end{pmatrix}. \quad 1-111$$

The characteristic values of A can be found to be

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{\frac{g}{L'}}, \quad \lambda_4 = -\sqrt{\frac{g}{L'}}. \quad 1-112$$

It is easily found that corresponding to the double characteristic value 0 there is only one characteristic vector, given by

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \quad 1-113$$

To λ_3 and λ_4 correspond the characteristic vectors

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ \sqrt{\frac{g}{L'}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\sqrt{\frac{g}{L'}} \end{pmatrix}. \quad 1-114$$

Since the characteristic values λ_3 and λ_4 are single, the corresponding null spaces have dimension one and are spanned by the corresponding characteristic vectors. Since zero is a double characteristic value, the corresponding null space is two-dimensional. The fact that there do not exist two linearly independent characteristic vectors gives rise to one subblock in the Jordan form of size 2×2 . Let the characteristic vector 1-113 be the first column q_1 of the transformation matrix T . Then the second column q_2 must follow from

$$Aq_2 = 0 \cdot q_2 + q_1. \quad 1-115$$

It is easily found that the general solution to this equation is

$$q_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad 1-116$$

where β is an arbitrary constant. We take $\beta = 0$. Since q_3 and q_4 have to be the characteristic vectors given by 1-114, we find for the transformation matrix T ,

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & \sqrt{\frac{g}{L}} & -\sqrt{\frac{g}{L}} \end{pmatrix}. \quad 1-117$$

The corresponding Jordan normal form of A is

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{g}{L}} & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{g}{L}} \end{pmatrix}. \quad 1-118$$

The exponential of A can now easily be found from 1-102, 1-117, and 1-118.

1.4 STABILITY

1.4.1 Definitions of Stability

In this section we are interested in the overall time behavior of differential systems. Consider the general nonlinear state differential equation

$$\dot{x}(t) = f[x(t), u(t), t]. \quad 1-119$$

An important property of the system is whether or not the solutions of the state differential equation tend to grow indefinitely as $t \rightarrow \infty$. In order to simplify this question, we assume that we are dealing with an autonomous system, that is, a system without an input u or, equivalently, a system where u is a fixed time function. Thus we reduce our attention to the system

$$\dot{x}(t) = f[x(t), t]. \quad 1-120$$

Just as in Section 1.2.2 on linearization, we introduce a *nominal solution* $x_0(t)$ which satisfies the state differential equation:

$$\dot{x}_0(t) = f[x_0(t), t]. \quad 1-121$$

A case of special interest occurs when $x_0(t)$ is a constant vector x_a ; in this case we say that x_a is an *equilibrium state* of the system.

We now discuss the stability of *solutions* of state differential equations. First we have the following definition (for the whole sequence of definitions that follows, see also Kalman and Bertram, 1960; Zadeh and Desoer, 1963; Brockett, 1970).

Definition 1.1. Consider the state differential equation

$$\dot{x}(t) = f[x(t), t] \quad 1-122$$

with the nominal solution $x_0(t)$. Then the nominal solution is **stable in the sense of Lyapunov** if for any t_0 and any $\varepsilon > 0$ there exists a $\delta(\varepsilon, t_0) > 0$ (depending upon ε and possibly upon t_0) such that $\|x(t_0) - x_0(t_0)\| \leq \delta$ implies $\|x(t) - x_0(t)\| < \varepsilon$ for all $t \geq t_0$.

Here $\|x\|$ denotes the norm of a vector x ; the Euclidean norm

$$\|x\| = \sqrt{\sum_{i=1}^n \xi_i^2}, \quad 1-123$$

where the ξ_i , $i = 1, 2, \dots, n$, are the components of x , can be used. Other norms are also possible.

Stability in the sense of Lyapunov guarantees that the state can be prevented from departing too far from the nominal solution by choosing the initial state close enough to the nominal solution. Stability in the sense of

Lyapunov is a rather weak form of stability. We therefore extend our concept of stability.

Definition 1.2. *The nominal solution $x_0(t)$ of the state differential equation*

$$\dot{x}(t) = f[x(t), t] \quad 1-124$$

is asymptotically stable if

- (a) *It is stable in the sense of Lyapunov;*
 (b) *For all t_0 there exists a $\rho(t_0) > 0$ (possibly depending upon t_0) such that $\|x(t_0) - x_0(t_0)\| < \rho$ implies*

$$\|x(t) - x_0(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Thus asymptotic stability implies, in addition to stability in the sense of Lyapunov, that the solution always approaches the nominal solution, provided the initial deviation is within the region defined by

$$\|x(t_0) - x_0(t_0)\| < \rho.$$

Asymptotic stability does not always give information for large initial deviations from the nominal solution. The following definition refers to the case of arbitrary initial deviations.

Definition 1.3. *The nominal solution $x_0(t)$ of the state differential equation*

$$\dot{x}(t) = f[x(t), t] \quad 1-125$$

is asymptotically stable in the large if

- (a) *It is stable in the sense of Lyapunov;*
 (b) *For any $x(t_0)$ and any t_0*

$$\|x(t) - x_0(t)\| \rightarrow 0 \quad 1-126$$

as $t \rightarrow \infty$.

A solution that is asymptotically stable in the large has therefore the property that all other solutions eventually approach it.

So far we have discussed only the stability of *solutions*. For nonlinear systems this is necessary because of the complex phenomena that may occur. In the case of linear systems, however, the situation is simpler, and we find it convenient to speak of the *stability of systems* rather than that of solutions. To make this point clear, let $x_0(t)$ be any nominal solution of the linear differential system

$$\dot{x}(t) = A(t)x(t), \quad 1-127$$

and denote by $x(t)$ any other solution of 1-127. Since both $x_0(t)$ and $x(t)$ are solutions of the linear state differential equation 1-127 $x(t) - x_0(t)$ is also a

solution, that is,

$$\frac{d}{dt} [x(t) - x_0(t)] = A(t)[x(t) - x_0(t)]. \quad 1-128$$

This shows that in order to study the stability of the nominal solution $x_0(t)$, we may as well study the stability of the zero solution, that is, the solution $x(t) \equiv 0$. If the zero solution is stable in any sense (of Lyapunov, asymptotically or asymptotically in the large), any other solution will also be stable in that sense. We therefore introduce the following terminology.

Definition 1.4. *The linear differential system*

$$\dot{x}(t) = A(t)x(t) \quad 1-129$$

is stable in a certain sense (of Lyapunov, asymptotically or asymptotically in the large), if the zero solution $x_0(t) \equiv 0$ is stable in that sense.

In addition to the fact that all nominal solutions of a linear differential system exhibit the same stability properties, for linear systems there is no need to make a distinction between asymptotic stability and asymptotic stability in the large as stated in the following theorem.

Theorem 1.12. *The linear differential system*

$$\dot{x}(t) = A(t)x(t) \quad 1-130$$

is asymptotically stable if and only if it is asymptotically stable in the large.

This theorem follows from the fact that for linear systems solutions may be scaled up or down without changing their behavior.

We conclude this section by introducing another form of stability, which we define only for linear systems (Brockett, 1970).

Definition 1.5. *The linear time-varying differential system*

$$\dot{x}(t) = A(t)x(t) \quad 1-131$$

is exponentially stable if there exist positive constants α and β such that

$$\|x(t)\| \leq \alpha e^{-\beta(t-t_0)} \|x(t_0)\|, \quad t \geq t_0, \quad 1-132$$

for every initial state $x(t_0)$.

A system that is exponentially stable has the property that the state converges exponentially to the zero state irrespective of the initial state.

We clarify the concepts introduced in this section by some examples.

Example 1.6. *Inverted pendulum.*

The equilibrium position $s(t) \equiv 0$, $\phi(t) \equiv 0$, $\mu(t) \equiv 0$ of the inverted pendulum of Example 1.1 (Section 1.2.3) obviously is not stable in any sense.

Example 1.7. *Suspended pendulum.*

Consider the pendulum discussed in Example 1.1 (Section 1.2.3). Suppose that $\mu(t) \equiv 0$. From physical considerations it is clear that the solution $s(t) \equiv 0$, $\phi(t) \equiv \pi$ (corresponding to a suspended pendulum) is stable in the sense of Lyapunov; by choosing sufficiently small initial offsets and velocities, the motions of the system can be made to remain arbitrarily small. The system is not asymptotically stable, however, since no friction is assumed for the pendulum; once it is in motion, it remains in motion. Moreover, if the carriage has an initial displacement, it will not return to the zero position without an external force.

Example 1.8. *Stirred tank.*

Consider the stirred tank of Example 1.2 (Section 1.2.3). For $u(t) \equiv 0$ the linearized system is described by

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{2\theta} & 0 \\ 0 & -\frac{1}{\theta} \end{pmatrix} x(t), \quad \text{1-133}$$

which has the solution

$$\begin{aligned} \xi_1(t) &= e^{-t/2\theta} \xi_1(0), & t \geq 0, \\ \xi_2(t) &= e^{-t/\theta} \xi_2(0), & t \geq 0. \end{aligned} \quad \text{1-134}$$

Obviously $\xi_1(t)$ and $\xi_2(t)$ always approach the value zero as t increases since $\theta > 0$. As a result, the linearized system is asymptotically stable. Moreover, since the convergence to the equilibrium state is exponential, the system is exponentially stable.

In Section 1.4.4 it is seen that if a linearized system is asymptotically stable then the equilibrium state about which the linearization is performed is asymptotically stable but not necessarily asymptotically stable in the large. Physical considerations, however, lead us to expect that in the present case the system is also asymptotically stable in the large.

1.4.2 Stability of Time-Invariant Linear Systems

In this section we establish under what conditions time-invariant linear systems possess any of the forms of stability we have discussed. Consider the system

$$\dot{x}(t) = Ax(t), \quad \text{1-135}$$

where A is a constant $n \times n$ matrix. In Section 1.3.3 we have seen that if A has n distinct characteristic values $\lambda_1, \lambda_2, \dots, \lambda_n$ and corresponding characteristic vectors e_1, e_2, \dots, e_n , the response of the system to any initial state

can be represented as

$$x(t) = \sum_{i=1}^n \mu_i e^{\lambda_i t} e_i, \quad 1-136$$

where the scalars μ_i , $i = 1, 2, \dots, n$ follow from the initial state $x(0)$. For systems with nondiagonalizable A , this expression contains additional terms of the form $t^k \exp(\lambda_i t)$ (Section 1.3.4). Clearly, the stability of the system in both cases is determined by the characteristic values λ_i . We have the following result.

Theorem 1.13. *The time-invariant linear system*

$$\dot{x}(t) = Ax(t) \quad 1-137$$

is stable in the sense of Lyapunov if and only if

- (a) *all of the characteristic values of A have nonpositive real parts, and*
- (b) *to any characteristic value on the imaginary axis with multiplicity m there correspond exactly m characteristic vectors of the matrix A .*

Condition (b) is necessary to prevent terms that grow as t^k (see Section 1.3.4). This condition is always satisfied if A has no multiple characteristic values on the imaginary axis. For asymptotic stability we need slightly stronger conditions.

Theorem 1.14. *The time-invariant system*

$$\dot{x}(t) = Ax(t) \quad 1-138$$

is asymptotically stable if and only if all of the characteristic values of A have strictly negative real parts.

This result is also easily recognized to be valid. We furthermore see that if a time-invariant linear system is asymptotically stable the convergence of the state to the zero state is exponential. This results in the following theorem.

Theorem 1.15. *The time-invariant system*

$$\dot{x}(t) = Ax(t) \quad 1-139$$

is exponentially stable if and only if it is asymptotically stable.

Since it is really the matrix A that determines whether a time-invariant system is asymptotically stable, it is convenient to use the following terminology.

Definition 1.6. *The $n \times n$ constant matrix A is asymptotically stable if all its characteristic values have strictly negative real parts.*

The characteristic values of A are the roots of the characteristic polynomial $\det(\lambda I - A)$. Through the well-known Routh-Hurwitz criterion (see, e.g.,

Schwarz and Friedland, 1965) the stability of A can be tested directly from the coefficients of the characteristic polynomial without explicitly evaluating the roots. With systems that are not asymptotically stable, we find it convenient to refer to those characteristic values of A that have strictly negative real parts as the *stable poles* of the system, and to the remaining ones as the *unstable poles*.

We conclude this section with a simple example. An additional example is given in Section 1.5.1.

Example 1.9. *Stirred tank.*

The matrix A of the linearized state differential equation of the stirred tank of Example 1.2 has the characteristic values $-(1/2\theta)$ and $-(1/\theta)$. As we concluded before (Example 1.8), the linearized system is asymptotically stable since $\theta > 0$.

1.4.3* Stable and Unstable Subspaces for Time-Invariant Linear Systems

In this section we show how the state space of a linear time-invariant differential system can be decomposed into two subspaces, such that the response of the system from an initial state in the first subspace always converges to the zero state while the response from a nonzero initial state in the other subspace never converges.

Let us consider the time-invariant system

$$\dot{x}(t) = Ax(t) \tag{1-140}$$

and assume that the matrix A has distinct characteristic values (the more general case is discussed later in this section). Then we know from Section 1.3.3 that the response of this system can be written as

$$x(t) = \sum_{i=1}^n \mu_i e^{\lambda_i t} e_i, \tag{1-141}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the characteristic values of A , and e_1, \dots, e_n are the corresponding characteristic vectors. The numbers $\mu_1, \mu_2, \dots, \mu_n$ are the coefficients that express how the initial state $x(0)$ is decomposed along the vectors e_1, e_2, \dots, e_n .

Let us now suppose that the system is not asymptotically stable, which means that some of the characteristic values λ_i have nonnegative real parts. Then it is clear that the state will converge to the zero state only if the initial state has components only along those characteristic vectors that correspond to stable poles.

If the initial state has components only along the characteristic vectors that correspond to unstable poles, the response of the state will be composed of nondecreasing exponentials. This leads to the following decomposition of the state space.

Definition 1.7. Consider the n -dimensional system $\dot{x}(t) = Ax(t)$ with A a constant matrix. Suppose that A has n distinct characteristic values. Then we define the **stable subspace** for this system as the real linear subspace spanned by those characteristic vectors of A that correspond to characteristic values with strictly negative real parts. The **unstable subspace** for this system is the real subspace spanned by those characteristic vectors of A that correspond to characteristic values with nonnegative real parts.

We now extend this concept to more general time-invariant systems. In Section 1.3.4 we saw that the response of the system can be written as

$$x(t) = \sum_{i=1}^k T_i \exp(J_i t) U_i v_i, \quad 1-142$$

where the v_i are in the null spaces \mathcal{N}_i , $i = 1, 2, \dots, k$. The behavior of the factor $\exp(J_i t)$ is determined by the characteristic value λ_i ; only if λ_i has a strictly negative real part does the corresponding component of the state approach the zero state. This leads us in analogy with the simple case of Definition 1.7 to the following decomposition:

Definition 1.8. Consider the n -dimensional linear time-invariant system $\dot{x}(t) = Ax(t)$. Then we define the **stable subspace** for this system as the real subspace of the direct sum of those null spaces \mathcal{N}_i that correspond to characteristic values of A with strictly negative real parts. Similarly, we define the **unstable subspace** of A as the real subspace of the direct sum of those null spaces \mathcal{N}_i that correspond to characteristic values of A with nonnegative real parts.

As a result of this definition the whole real n -dimensional space \mathcal{R}^n is the direct sum of the stable and the unstable subspace.

Example 1.10. *Inverted pendulum.*

In Example 1.4 (Section 1.3.3), we saw that the matrix A of the linearized state differential equation of the inverted pendulum has the characteristic values and vectors:

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{F}{M}, \quad \lambda_3 = \sqrt{\frac{g}{L}}, \quad \lambda_4 = -\sqrt{\frac{g}{L}}, \quad 1-143$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 \\ -\frac{F}{M} \\ \alpha \\ -\alpha \frac{F}{M} \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \sqrt{\frac{g}{L}} \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\sqrt{\frac{g}{L}} \end{pmatrix}.$$

Apparently, the stable subspace of this system is spanned by the vectors e_2 and e_4 , while the unstable subspace is spanned by e_1 and e_3 .

Example 1.11. *Inverted pendulum without friction.*

In Example 1.5 (Section 1.3.4), we discussed the Jordan normal form of the A matrix of the inverted pendulum with negligible friction. There we found a double characteristic value 0 and the single characteristic values $\sqrt{(g/L')}$ and $-\sqrt{(g/L')}$. The null space corresponding to the characteristic value 0 is spanned by the first two columns of the transformation matrix T , that is, by

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \quad 1-145$$

These two column vectors, together with the characteristic vector corresponding to $\sqrt{(g/L')}$, that is,

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ \sqrt{\frac{g}{L'}} \end{pmatrix}, \quad 1-146$$

span the unstable subspace of the system. The stable subspace is spanned by the remaining characteristic vector

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ -\sqrt{\frac{g}{L'}} \end{pmatrix}. \quad 1-147$$

1.4.4* Investigation of the Stability of Nonlinear Systems through Linearization

Most of the material of this book is concerned with the design of linear control systems. One major goal in the design of such systems is stability. In

later chapters very powerful techniques for finding stable linear feedback control systems are developed. As we have seen, however, actual systems are never linear, and the linear models used are obtained by linearization.

This means that we design systems whose linearized models possess good properties. The question now is: What remains of these properties when the actual nonlinear system is implemented? Here the following result is helpful.

Theorem 1.16. *Consider the time-invariant system with state differential equation*

$$\dot{x}(t) = f[x(t)]. \quad \mathbf{1-148}$$

Suppose that the system has an equilibrium state x_0 and that the function f possesses partial derivatives with respect to the components of x at x_0 . Suppose that the linearized state differential equation about x_0 is

$$\dot{\tilde{x}}(t) = A\tilde{x}(t), \quad \mathbf{1-149}$$

where the constant matrix A is the Jacobian of f at x_0 . Then if A is asymptotically stable, the solution $x(t) = x_0$ is an asymptotically stable solution of 1-148.

For a proof we refer the reader to Roseau (1966). Note that of course we cannot conclude anything about stability in the large from the linearized state differential equation.

This theorem leads to a reassuring conclusion. Suppose that we are confronted with an initially unstable system, and that we use linearized equations to find a controller that makes the linearized system stable. Then it can be shown from the theorem that the actual nonlinear system with this controller will at least be asymptotically stable for small deviations from the equilibrium state.

Note, however, that the theorem is reassuring only when the system contains "smooth" nonlinearities. If discontinuous elements occur (dead zones, stiction) this theory is of no help.

We conclude by noting that if some of the characteristic values of A have zero real parts while all the other characteristic values have strictly negative real parts no conclusions about the stability of x_0 can be drawn from the linearized analysis. If A has some characteristic values with positive real parts, however, x_0 is not stable in any sense (Roseau, 1966).

An example of the application of this theorem is given in Chapter 2 (Example 2.6, Section 2.4).

1.5 TRANSFORM ANALYSIS OF TIME-INVARIANT SYSTEMS

1.5.1 Solution of the State Differential Equation through Laplace Transformation

Often it is helpful to analyze time-invariant linear systems through Laplace transformation. We define the Laplace transform of a time-varying vector $z(t)$ as follows

$$\mathbf{Z}(s) = \mathcal{L}[z(t)] = \int_0^{\infty} e^{-st} z(t) dt, \quad 1-150$$

where s is a complex variable. A boldface capital indicates the Laplace transform of the corresponding lowercase time function. The Laplace transform is defined for those values of s for which 1-150 converges. We see that the Laplace transform of a time-varying vector $z(t)$ is simply a vector whose components are the Laplace transforms of the components of $z(t)$.

Let us first consider the homogeneous state differential equation

$$\dot{x}(t) = Ax(t), \quad 1-151$$

where A is a constant matrix. Laplace transformation yields

$$s\mathbf{X}(s) - x(0) = A\mathbf{X}(s), \quad 1-152$$

since all the usual rules of Laplace transformations for scalar expressions carry over to the vector case (Polak and Wong, 1970). Solution for $\mathbf{X}(s)$ yields

$$\mathbf{X}(s) = (sI - A)^{-1}x(0). \quad 1-153$$

This is the equivalent of the time domain expression

$$x(t) = e^{At}x(0). \quad 1-154$$

We conclude the following.

Theorem 1.17. *Let A be a constant $n \times n$ matrix. Then $(sI - A)^{-1} = \mathcal{L}[e^{At}]$, or, equivalently, $e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$.*

The Laplace transform of a time-varying matrix is obtained by transforming each of its elements. Theorem 1.17 is particularly convenient for obtaining the explicit form of the transition matrix as long as n is not too large, irrespective of whether or not A is diagonalizable.

The matrix function $(sI - A)^{-1}$ is called the *resolvent* of A . The following result is useful (Zadeh and Desoer, 1963; Bass and Gura, 1965).

Theorem 1.18. Consider the constant $n \times n$ matrix A with characteristic polynomial

$$\det (sI - A) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1s + \alpha_0. \quad 1-155$$

Then the resolvent of A can be written as

$$(sI - A)^{-1} = \frac{1}{\det (sI - A)} \sum_{i=1}^n s^{i-1} R_i, \quad 1-156$$

where the matrices R_i are given by

$$R_i = \sum_{j=i}^n \alpha_j A^{j-i}, \quad i = 1, 2, \cdots, n, \quad 1-157$$

with $\alpha_n = 1$. The coefficients α_i and the matrices R_i , $i = 1, 2, \cdots, n$ can be obtained through the following algorithm. Set

$$\alpha_n = 1, \quad R_n = I. \quad 1-158$$

Then

$$\alpha_{n-k} = -\frac{1}{k} \operatorname{tr} (AR_{n-k+1}), \quad 1-159$$

$$R_{n-k} = \alpha_{n-k} I + AR_{n-k+1}, \quad 1-160$$

for $k = 1, 2, \cdots, n$. For $k = n$ we have

$$R_0 = 0. \quad 1-161$$

Here we have employed the notation

$$\operatorname{tr} (M) = \sum_{i=1}^n M_{ii}, \quad 1-162$$

if M is an $n \times n$ matrix with diagonal elements M_{ii} , $i = 1, 2, \cdots, n$. We refer to the algorithm of the theorem as *Leverrier's algorithm* (Bass and Gura, 1965). It is also known as *Souriau's method* or *Faddeeva's method* (Zadeh and Desoer, 1963). The fact that $R_0 = 0$ can be used as a numerical check. The algorithm is very convenient for a digital computer. It must be pointed out, however, that the algorithm is relatively sensitive to round-off errors (Forsythe and Strauss, 1955), and double precision is usually employed in the computations. Melsa (1970) gives a listing of a FORTRAN computer program.

Let us now consider the inhomogeneous equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad 1-163$$

where A and B are constant. Laplace transformation yields

$$sX(s) - x(0) = AX(s) + BU(s), \quad 1-164$$

which can be solved for $\mathbf{X}(s)$. We find

$$\mathbf{X}(s) = (sI - A)^{-1}\mathbf{x}(0) + (sI - A)^{-1}B\mathbf{U}(s). \quad 1-165$$

Let the output equation of the system be given by

$$y(t) = C\mathbf{x}(t), \quad 1-166$$

where C is constant. Laplace transformation and substitution of 1-165 yields

$$\mathbf{Y}(s) = C\mathbf{X}(s) = C(sI - A)^{-1}\mathbf{x}(0) + C(sI - A)^{-1}B\mathbf{U}(s), \quad 1-167$$

which is the equivalent in the Laplace transform domain of the time domain expression 1-70 with $t_0 = 0$:

$$y(t) = Ce^{At}\mathbf{x}(0) + C \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau) d\tau. \quad 1-168$$

For $\mathbf{x}(0) = 0$ the expression 1-167 reduces to

$$\mathbf{Y}(s) = H(s)\mathbf{U}(s), \quad 1-169$$

where

$$H(s) = C(sI - A)^{-1}B. \quad 1-170$$

The matrix $H(s)$ is called the *transfer matrix* of the system. If $H(s)$ and $\mathbf{U}(s)$ are known, the zero initial state response of the system can be found by inverse Laplace transformation of 1-169.

By Theorem 1.17 it follows immediately from 1-170 that the transfer matrix $H(s)$ is the Laplace transform of the matrix function $K(t) = C \exp (At)B$, $t \geq 0$. It is seen from 1-168 that $K(t - \tau)$, $t \geq \tau$, is precisely the impulse response matrix of the system.

From Theorem 1.18 we note that the transfer matrix can be written in the form

$$H(s) = \frac{1}{\det (sI - A)} P(s), \quad 1-171$$

where $P(s)$ is a matrix whose elements are polynomials in s . The elements of the transfer matrix $H(s)$ are therefore rational functions of s . The common denominator of the elements of $H(s)$ is $\det (sI - A)$, unless cancellation occurs of factors of the form $s - \lambda_i$, where λ_i is a characteristic value of A , in all the elements of $H(s)$.

We call the roots of the common denominator of $H(s)$ the *poles of the transfer matrix* $H(s)$. If no cancellation occurs, the poles of the transfer matrix are precisely the poles of the system, that is, the characteristic values of A .

If the input $u(t)$ and the output variable $y(t)$ are both one-dimensional, the transfer matrix reduces to a scalar *transfer function*. For multiinput multioutput systems, each element $H_{ij}(s)$ of the transfer matrix $H(s)$ is the transfer function from the j -th component of the input to the i -th component of the output.

Example 1.12. *A nondiagonalizable system.*

Consider the system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t). \quad 1-172$$

It is easily verified that this system has a double characteristic value 0 but only a single characteristic vector, so that it is not diagonalizable. We compute its transition matrix by Laplace transformation. The resolvent of the system can be found to be

$$(sI - A)^{-1} = \frac{1}{s^2} \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix} = \begin{pmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{pmatrix}. \quad 1-173$$

Inverse Laplace transformation yields

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \quad 1-174$$

Note that this system is not stable in the sense of Lyapunov.

Example 1.13. *Stirred tank.*

The stirred tank of Example 1.2 is described by the linearized state differential equation

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{2\theta} & 0 \\ 0 & -\frac{1}{\theta} \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ \frac{c_1 - c_0}{V_0} & \frac{c_2 - c_0}{V_0} \end{pmatrix} u(t) \quad 1-175$$

and the output equation

$$y(t) = \begin{pmatrix} \frac{1}{2\theta} & 0 \\ 0 & 1 \end{pmatrix} x(t). \quad 1-176$$

The resolvent of the matrix A is

$$(sI - A)^{-1} = \begin{pmatrix} \frac{1}{s + \frac{1}{2\theta}} & 0 \\ 0 & \frac{1}{s + \frac{1}{\theta}} \end{pmatrix}. \quad 1-177$$

The system has the transfer matrix

$$H(s) = \begin{pmatrix} \frac{1}{2\theta} & \frac{1}{2\theta} \\ s + \frac{1}{2\theta} & s + \frac{1}{2\theta} \\ \frac{c_1 - c_0}{V_0} \frac{1}{s + \frac{1}{\theta}} & \frac{c_2 - c_0}{V_0} \frac{1}{s + \frac{1}{\theta}} \end{pmatrix}. \quad 1-178$$

The impulse response matrix 1-75 of the system follows immediately by inverse Laplace transformation of 1-178.

1.5.2 Frequency Response

In this section we study the frequency response of time-invariant systems, that is, the response to an input of the form

$$u(t) = u_m e^{j\omega t}, \quad t \geq 0, \quad 1-179$$

where u_m is a constant vector. We express the solution of the state differential equation

$$\dot{x}(t) = Ax(t) + Bu(t) \quad 1-180$$

in terms of the solution of the homogeneous equation plus a particular solution. Let us first try to find a particular solution of the form

$$x_p(t) = x_m e^{j\omega t}, \quad 1-181$$

where x_m is a constant vector to be determined. It is easily found that this particular solution is given by

$$x_p(t) = (j\omega I - A)^{-1} Bu_m e^{j\omega t}, \quad t \geq 0. \quad 1-182$$

The general solution of the homogeneous equation $\dot{x}(t) = Ax(t)$ can be written as

$$x_h(t) = e^{At} a, \quad 1-183$$

where a is an arbitrary constant vector. The general solution of the inhomogeneous equation 1-180 is therefore

$$x(t) = e^{At} a + (j\omega I - A)^{-1} Bu_m e^{j\omega t}, \quad t \geq 0. \quad 1-184$$

The constant vector a can be determined from the initial conditions. If the system 1-180 is *asymptotically stable*, the first term of the solution will eventually vanish as t increases, and the second term represents the *steady-state response* of the state to the input 1-179. The corresponding steady-state

response of the output

$$y(t) = Cx(t) \quad 1-185$$

is given by

$$\begin{aligned} y(t) &= C(j\omega I - A)^{-1} B u_m e^{j\omega t} \\ &= H(j\omega) u_m e^{j\omega t}. \end{aligned} \quad 1-186$$

We note that in this expression the transfer matrix $H(s)$ appears with s replaced by $j\omega$. We call $H(j\omega)$ the *frequency response matrix* of the system.

Once we have obtained the response to complex periodic inputs of the type 1-179, the steady-state response to real, sinusoidal inputs is easily found. Suppose that the k -th component $\mu_k(t)$ of the input $u(t)$ is given as follows

$$\mu_k(t) = \hat{\mu}_k \sin(\omega t + \phi_k), \quad t \geq 0. \quad 1-187$$

Assume that all other components of the input are identically zero. Then the steady-state response of the i -th component $\eta_i(t)$ of the output $y(t)$ is given by

$$\eta_i(t) = |H_{ik}(j\omega)| \hat{\mu}_k \sin(\omega t + \phi_k + \psi_{ik}), \quad 1-188$$

where $H_{ik}(j\omega)$ is the (i, k) -th element of $H(j\omega)$ and

$$\psi_{ik} = \arg [H_{ik}(j\omega)]. \quad 1-189$$

A convenient manner of representing scalar frequency response functions is through asymptotic Bode plots (D'Azzo and Houpis, 1966). Melsa (1970) gives a FORTRAN computer program for plotting the modulus and the argument of a scalar frequency response function.

In conclusion, we remark that it follows from the results of this section that the steady-state response of an asymptotically stable system with frequency response matrix $H(j\omega)$ to a *constant* input

$$u(t) = u_m \quad 1-190$$

is given by

$$y(t) = H(0)u_m. \quad 1-191$$

Example 1.14. *Stirred tank.*

The stirred tank of Example 1.2 has the transfer matrix (Example 1.13)

$$H(s) = \begin{pmatrix} \frac{1}{2\theta} & \frac{1}{2\theta} \\ s + \frac{1}{2\theta} & s + \frac{1}{2\theta} \\ \frac{c_1 - c_0}{V_0} \frac{1}{s + \frac{1}{\theta}} & \frac{c_2 - c_0}{V_0} \frac{1}{s + \frac{1}{\theta}} \end{pmatrix}. \quad 1-192$$

The system is asymptotically stable so that it makes sense to consider the frequency response matrix. With the numerical data of Example 1.2, we have

$$H(j\omega) = \begin{pmatrix} \frac{0.01}{j\omega + 0.01} & \frac{0.01}{j\omega + 0.01} \\ \frac{-0.25}{j\omega + 0.02} & \frac{0.75}{j\omega + 0.02} \end{pmatrix}. \quad 1-193$$

1.5.3 Zeroes of Transfer Matrices

Let us consider the single-input single-output system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b\mu(t), \\ \eta(t) &= cx(t), \end{aligned} \quad 1-194$$

where $\mu(t)$ and $\eta(t)$ are the scalar input and output variable, respectively, b is a column vector, and c a row vector. The transfer matrix of this system reduces to a transfer function which is given by

$$H(s) = c(sI - A)^{-1}b. \quad 1-195$$

Denote the characteristic polynomial of A as

$$\det(sI - A) = \phi(s). \quad 1-196$$

Then $H(s)$ can be written as

$$H(s) = \frac{\psi(s)}{\phi(s)}, \quad 1-197$$

where, if A is an $n \times n$ matrix, then $\phi(s)$ is a polynomial of degree n and $\psi(s)$ a polynomial of degree $n - 1$ or less. The roots of $\psi(s)$ we call the *zeroes of the system 1-194*. Note that we determine the zeroes *before* cancelling any common factors of $\psi(s)$ and $\phi(s)$. The zeroes of $H(s)$ that remain *after* cancellation we call the *zeroes of the transfer function*.

In the case of a multiinput multioutput system, $H(s)$ is a matrix. Each entry of $H(s)$ is a transfer function which has its own zeroes. It is not obvious how to define "the zeroes of $H(s)$ " in this case. In the remainder of this section we give a definition that is motivated by the results of Section 3.8. Only square transfer matrices are considered.

First we have the following result (Haley, 1967).

Theorem 1.19. *Consider the system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \quad 1-198$$

where the state x has dimension n and both the input u and the output variable y have dimension m . Let $H(s) = C(sI - A)^{-1}B$ be the transfer-matrix of the system. Then

$$\det [H(s)] = \frac{\psi(s)}{\phi(s)}, \quad \mathbf{1-199}$$

where

$$\phi(s) = \det (sI - A), \quad \mathbf{1-200}$$

and $\psi(s)$ is a polynomial in s of degree $n - m$ or less.

Since this result is not generally known we shall prove it. We first state the following fact from matrix theory.

Lemma 1.1. Let M and N be matrices of dimensions $m \times n$ and $n \times m$, respectively, and let I_m and I_n denote unit matrices of dimensions $m \times m$ and $n \times n$. Then

$$(a) \quad \det (I_m + MN) = \det (I_n + NM). \quad \mathbf{1-201}$$

(b) Suppose $\det (I_m + MN) \neq 0$; then

$$(I_m + MN)^{-1} = I_m - M(I_n + NM)^{-1}N. \quad \mathbf{1-202}$$

The proof of (a) follows from considerations involving the characteristic values of $I_m + MN$ (Plotkin, 1964; Sain, 1966). Part (b) is easily verified. It is not needed until later.

To prove Theorem 1.19 consider the expression

$$\det [\lambda I_m + C(sI_n - A)^{-1}B], \quad \mathbf{1-203}$$

where λ is a nonzero arbitrary scalar which later we let approach zero. Using part (a) of the lemma, we have

$$\begin{aligned} \det [\lambda I_m + C(sI_n - A)^{-1}B] &= \det (\lambda I_m) \det \left[I_m + \frac{1}{\lambda} C(sI_n - A)^{-1}B \right] \\ &= \lambda^m \det \left[I_n + \frac{1}{\lambda} (sI_n - A)^{-1}BC \right] \\ &= \frac{\lambda^m \det \left[sI_n - A + \frac{1}{\lambda} BC \right]}{\det (sI_n - A)}. \end{aligned} \quad \mathbf{1-204}$$

We see that the left-hand and the right-hand side of **1-204** are polynomials in λ that are equal for all nonzero λ ; hence by letting $\lambda \rightarrow 0$ we obtain

$$\det [C(sI - A)^{-1}B] = \frac{\psi(s)}{\phi(s)}, \quad \mathbf{1-205}$$

where

$$\psi(s) = \lim_{\lambda \rightarrow 0} \lambda^m \det \left(sI_n - A + \frac{1}{\lambda} BC \right). \quad 1-206$$

This immediately shows that $\psi(s)$ is a polynomial in s . We now consider the degree of this polynomial. For $|s| \rightarrow \infty$ we see from Theorem 1.18 that

$$\lim_{|s| \rightarrow \infty} s(sI - A)^{-1} = I. \quad 1-207$$

Consequently,

$$\begin{aligned} \lim_{|s| \rightarrow \infty} \frac{s^m \psi(s)}{\phi(s)} &= \lim_{|s| \rightarrow \infty} s^m \det [C(sI - A)^{-1}B] \\ &= \lim_{|s| \rightarrow \infty} \det [Cs(sI - A)^{-1}B] = \det (CB). \end{aligned} \quad 1-208$$

This shows that the degree of $\phi(s)$ is greater than that of $\psi(s)$ by at least m , hence $\psi(s)$ has degree $n - m$ or less. If $\det (CB) \neq 0$, the degree of $\psi(s)$ is exactly $n - m$. This terminates the proof of Theorem 1.19.

We now introduce the following definition.

Definition 1.9. *The zeroes of the system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \quad 1-209$$

where the state x has dimension n and both the input u and the output y have dimension m , are the zeroes of the polynomial $\psi(s)$, where

$$\det [H(s)] = \frac{\psi(s)}{\phi(s)}. \quad 1-210$$

Here $H(s) = C(sI - A)^{-1}B$ is the transfer matrix and $\phi(s) = \det (sI - A)$ the characteristic polynomial of the system.

An n -dimensional system with m -dimensional input and output thus has at most $n - m$ zeroes. Note that for single-input single-output systems our definition of the zeroes of the system reduces to the conventional definition as described in the beginning of this section. In this case the system has at most $n - 1$ zeroes.

The numerical computation of the numerator polynomial for a system of some complexity presents problems. One possible way of going about this is to write the numerator polynomial as

$$\psi(s) = \phi(s) \det [H(s)], \quad 1-211$$

where $\phi(s)$ is the characteristic polynomial of the system. The coefficients of $\psi(s)$ can then be found by substituting $n - m + 1$ suitable values for s into

the right-hand side of 1-211 and solving the resulting linear equations. Another, probably more practical, approach results from using the fact that from 1-206 we have

$$\psi(s) = \lim_{\lambda \rightarrow 0} \psi(s, \lambda), \quad 1-212$$

where

$$\psi(s, \lambda) = \lambda^m \det \left(sI - A + \frac{1}{\lambda} BC \right). \quad 1-213$$

Inspection shows that we can write

$$\psi(s, \lambda) = \sum_{i=0}^m \lambda^i \alpha_i(s), \quad 1-214$$

where $\alpha_i(s)$, $i = 0, 1, \dots, m$, are polynomials in s . These polynomials can be computed by calculating $\psi(s, \lambda)$ for m different values of λ . The desired polynomial $\psi(s)$ is precisely $\alpha_0(s)$.

We illustrate the results of this section by the following example.

Example 1.15. *Stirred tank.*

The stirred tank of Example 1.2 (Section 1.2.3) has the transfer matrix

$$H(s) = \begin{pmatrix} \frac{1}{2\theta} & \frac{1}{2\theta} \\ s + \frac{1}{2\theta} & s + \frac{1}{2\theta} \\ \frac{c_1 - c_0}{V_0} \frac{1}{s + \frac{1}{\theta}} & \frac{c_2 - c_0}{V_0} \frac{1}{s + \frac{1}{\theta}} \end{pmatrix}. \quad 1-215$$

The characteristic polynomial of the system is

$$\phi(s) = \left(s + \frac{1}{2\theta} \right) \left(s + \frac{1}{\theta} \right). \quad 1-216$$

We find for the determinant of the transfer matrix

$$\det [H(s)] = \frac{\frac{1}{2\theta} \frac{c_2 - c_1}{V_0}}{\left(s + \frac{1}{2\theta} \right) \left(s + \frac{1}{\theta} \right)}. \quad 1-217$$

Apparently, the transfer matrix has no zeroes. This is according to expectation, since in this case $n - m = 0$ so that the degree of $\psi(s)$ is zero.

1.5.4 Interconnections of Linear Systems

In this section we discuss interconnections of linear systems. Two important examples of interconnected systems that we frequently encounter are the *series connection* of Fig. 1.5 and the *feedback configuration* or *closed-loop system* of Fig. 1.6.

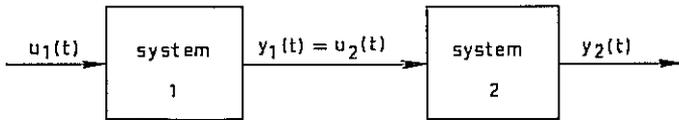


Fig. 1.5. Series connection.

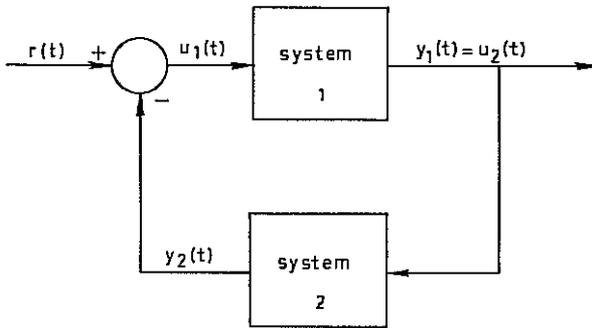


Fig. 1.6. Feedback connection.

We often describe interconnections of systems by the *state augmentation technique*. In the series connection of Fig. 1.5, let the individual systems be described by the state differential and output equations

$$\left. \begin{aligned} \dot{x}_1(t) &= A_1(t)x_1(t) + B_1(t)u_1(t) \\ y_1(t) &= C_1(t)x_1(t) + D_1(t)u_1(t) \end{aligned} \right\} \text{system 1,} \quad \mathbf{1-218}$$

$$\left. \begin{aligned} \dot{x}_2(t) &= A_2(t)x_2(t) + B_2(t)u_2(t) \\ y_2(t) &= C_2(t)x_2(t) + D_2(t)u_2(t) \end{aligned} \right\} \text{system 2.}$$

Defining the *augmented state*

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad \mathbf{1-219}$$

the interconnected system is described by the state differential equation

$$\dot{x}(t) = \begin{pmatrix} A_1(t) & 0 \\ B_2(t)C_1(t) & A_2(t) \end{pmatrix} x(t) + \begin{pmatrix} B_1(t) \\ B_2(t)D_1(t) \end{pmatrix} u_1(t), \quad 1-220$$

where we have used the relation $u_2(t) = y_1(t)$. Taking $y_2(t)$ as the output of the interconnected system, we obtain for the output equation

$$y_2(t) = [D_2(t)C_1(t), C_2(t)]x(t) + D_2(t)D_1(t)u_1(t). \quad 1-221$$

In the case of time-invariant systems, it is sometimes convenient to describe an interconnection in terms of transfer matrices. Suppose that the individual transfer matrices of the systems 1 and 2 are given by $H_1(s)$ and $H_2(s)$, respectively. Then the overall transfer matrix is $H_2(s)H_1(s)$, as can be seen from

$$Y_2(s) = H_2(s)U_2(s) = H_2(s)H_1(s)U_1(s). \quad 1-222$$

Note that the order of H_2 and H_1 generally cannot be interchanged.

In the feedback configuration of Fig. 1.6, $r(t)$ is the input to the overall system. Suppose that the individual systems are described by the state differential and output equations

$$\left. \begin{aligned} \dot{x}_1(t) &= A_1(t)x_1(t) + B_1(t)u_1(t) \\ y_1(t) &= C_1(t)x_1(t) \end{aligned} \right\} \text{system 1,} \\ \left. \begin{aligned} \dot{x}_2(t) &= A_2(t)x_2(t) + B_2(t)u_2(t) \\ y_2(t) &= C_2(t)x_2(t) + D_2(t)u_2(t) \end{aligned} \right\} \text{system 2.} \quad 1-223$$

Note that we have taken system 1 without a direct link. This is to avoid implicit algebraic equations. In terms of the augmented state $x(t) = \text{col}[x_1(t), x_2(t)]$, the feedback connection can be described by the state differential equation

$$\dot{\bar{x}}(t) = \begin{pmatrix} A_1(t) - B_1(t)D_2(t)C_1(t) & -B_1(t)C_2(t) \\ B_2(t)C_1(t) & A_2(t) \end{pmatrix} x(t) + \begin{pmatrix} B_1(t) \\ 0 \end{pmatrix} r(t), \quad 1-224$$

where we have used the relations $u_2(t) = y_1(t)$ and $u_1(t) = r(t) - y_2(t)$. If $y_1(t)$ is the overall output of the system, we have for the output equation

$$y_1(t) = [C_1(t), 0]x(t). \quad 1-225$$

Consider now the time-invariant case. Then we can write in terms of transfer matrices

$$Y_1(s) = H_1(s)[R(s) - H_2(s)Y_1(s)], \quad 1-226$$

where $H_1(s)$ and $H_2(s)$ are the transfer matrices of the individual systems. Solving for $Y_1(s)$, we find

$$Y_1(s) = [I + H_1(s)H_2(s)]^{-1}H_1(s)R(s). \quad 1-227$$

It is convenient to give the expression $I + H_1(s)H_2(s)$ a special name:

Definition 1.10. Consider the feedback configuration of Fig. 1.6. and let the systems 1 and 2 be time-invariant systems with transfer matrices $H_1(s)$ and $H_2(s)$, respectively. Then the matrix function

$$J(s) = I + H_1(s)H_2(s) \quad 1-228$$

is called the return difference matrix. The matrix function

$$L(s) = H_1(s)H_2(s) \quad 1-229$$

is called the loop gain matrix.

The term “return difference” can be clarified by Fig. 1.7. Here the loop is cut at the point indicated, and an external input variable $u_2(t)$ is connected.

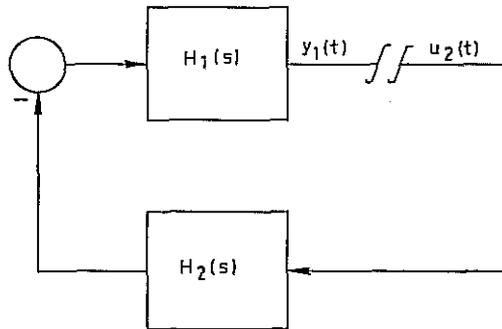


Fig. 1.7. Illustration of return difference.

This yields (putting $r(t) \equiv 0$)

$$Y_1(s) = -H_1(s)H_2(s)U_2(s). \quad 1-230$$

The difference between the “returned variable” $y_1(t)$ and the “injected variable” $u_2(t)$ is

$$\begin{aligned} U_2(s) - Y_1(s) &= [I + H_1(s)H_2(s)]U_2(s) \\ &= J(s)U_2(s). \end{aligned} \quad 1-231$$

Note that the loop can also be cut elsewhere, which will result in a different return difference matrix. We strictly adhere to the definition given above, however. The term “loop gain matrix” is self-explanatory.

A matter of great interest in control engineering is the stability of interconnections of systems. For series connections we have the following result, which immediately follows from a consideration of the characteristic polynomial of the augmented state differential equation 1-220.

Theorem 1.20. *Consider the series connection of Fig. 1.5, where the systems 1 and 2 are time-invariant systems with characteristic polynomials $\phi_1(s)$ and $\phi_2(s)$, respectively. Then the interconnection has the characteristic polynomial $\phi_1(s)\phi_2(s)$. Hence the interconnected system is asymptotically stable if and only if both system 1 and system 2 are asymptotically stable.*

In terms of transfer matrices, the stability of the feedback configuration of Fig. 1.6 can be investigated through the following result (Chen, 1968a; Hsu and Chen, 1968).

Theorem 1.21. *Consider the feedback configuration of Fig. 1.6 in which the systems 1 and 2 are time-invariant linear systems with transfer matrices $H_1(s)$ and $H_2(s)$ and characteristic polynomials $\phi_1(s)$ and $\phi_2(s)$, respectively, and where system 1 does not have a direct link. Then the characteristic polynomial of the interconnected system is*

$$\phi_1(s)\phi_2(s) \det [I + H_1(s)H_2(s)]. \quad 1-232$$

Hence the interconnected system is stable if and only if the polynomial 1-232 has zeroes with strictly negative real parts only.

Before proving this result we remark the following. The expression $\det [I + H_1(s)H_2(s)]$ is a rational function in s . Unless cancellations take place, the denominator of this function is $\phi_1(s)\phi_2(s)$ so that the numerator of $\det [I + H_1(s)H_2(s)]$ is the characteristic polynomial of the interconnected system. We often refer to 1-232 as the *closed-loop characteristic polynomial*.

Theorem 1.21 can be proved as follows. In the time-invariant case, it follows from 1-224 for the state differential equation of the interconnected system

$$\dot{x}(t) = \begin{pmatrix} A_1 - B_1 D_2 C_1 & -B_1 C_2 \\ B_2 C_1 & A_2 \end{pmatrix} x(t) + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} r(t). \quad 1-233$$

We show that the characteristic polynomial of this system is precisely 1-232. For this we need the following result from matrix theory.

Lemma 1.2. *Let M be a square, partitioned matrix of the form*

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}. \quad 1-234$$

Then if $\det(M_1) \neq 0$,

$$\det(M) = \det(M_1) \det(M_4 - M_3 M_1^{-1} M_2). \quad 1-235$$

If $\det(M_4) \neq 0$,

$$\det(M) = \det(M_4) \det(M_1 - M_2 M_4^{-1} M_3). \quad 1-236$$

The lemma is easily proved by elementary row and column operations on M . With the aid of Lemmas 1.2 and 1.1 (Section 1.5.3), the characteristic polynomial of 1-233 can be written as follows.

$$\begin{aligned} \det \begin{pmatrix} sI - A_1 + B_1 D_2 C_1 & B_1 C_2 \\ -B_2 C_1 & sI - A_2 \end{pmatrix} \\ = \det(sI - A_2) \det[sI - A_1 + B_1 D_2 C_1 + B_1 C_2 (sI - A_2)^{-1} B_2 C_1] \\ = \det(sI - A_2) \det(sI - A_1) \\ \quad \cdot \det\{I + B_1 [D_2 + C_2 (sI - A_2)^{-1} B_2] C_1 (sI - A_1)^{-1}\} \\ = \det(sI - A_1) \det(sI - A_2) \\ \quad \cdot \det\{I + C_1 (sI - A_1)^{-1} B_1 [C_2 (sI - A_2)^{-1} B_2 + D_2]\}. \end{aligned} \quad 1-237$$

Since

$$\begin{aligned} \det(sI - A_1) &= \phi_1(s), \\ \det(sI - A_2) &= \phi_2(s), \\ C_1 (sI - A_1)^{-1} B_1 &= H_1(s), \\ C_2 (sI - A_2)^{-1} B_2 + D_2 &= H_2(s), \end{aligned} \quad 1-238$$

1-237 can be rewritten as

$$\phi_1(s) \phi_2(s) \det[I + H_1(s) H_2(s)]. \quad 1-239$$

This shows that 1-232 is the characteristic polynomial of the interconnected system; thus the stability immediately follows from the roots of 1-232.

This method for checking the stability of feedback systems is usually more convenient for single-input single-output systems than for multivariable systems. In the case of single-input single-output systems, we write

$$H_1(s) = \frac{\psi_1(s)}{\phi_1(s)}, \quad H_2(s) = \frac{\psi_2(s)}{\phi_2(s)}, \quad 1-240$$

where $\psi_1(s)$ and $\psi_2(s)$ are the numerator polynomials of the systems. By Theorem 1.21 stability now follows from the roots of the polynomial

$$\phi_1(s) \phi_2(s) \left[1 + \frac{\psi_1(s) \psi_2(s)}{\phi_1(s) \phi_2(s)} \right] = \phi_1(s) \phi_2(s) + \psi_1(s) \psi_2(s). \quad 1-241$$

It often happens in designing linear feedback control systems that either

in the feedback path or in the feedforward path a gain factor is left undetermined until a late stage in the design. Suppose by way of example that

$$H_1(s) = \rho \frac{\psi_1(s)}{\phi_1(s)}, \quad 1-242$$

where ρ is the undetermined gain factor. The characteristic values of the interconnected system are now the roots of

$$\phi_1(s)\phi_2(s) + \rho\psi_1(s)\psi_2(s). \quad 1-243$$

An interesting problem is to construct the loci of the roots of this polynomial as a function of the scalar parameter ρ . This is a special case of the more general problem of finding in the complex plane the loci of the roots of

$$\phi(s) + \rho\psi(s) \quad 1-244$$

as the parameter ρ varies, where $\phi(s)$ and $\psi(s)$ are arbitrary given polynomials. The rules for constructing such loci are reviewed in the next section.

Example 1.16. Inverted pendulum

Consider the inverted pendulum of Example 1.1 (Section 1.2.3) and suppose that we wish to stabilize it. It is clear that if the pendulum starts falling to the right the carriage must also move to the right. We therefore attempt a method of control whereby we apply a force $\mu(t)$ to the carriage which is proportional to the angle $\phi(t)$. This angle can be measured by a potentiometer at the pivot; the force $\mu(t)$ is exerted through a small servomotor. Thus we have

$$\mu(t) = k\phi(t), \quad 1-245$$

where k is a constant. It is easily found that the transfer function from $\mu(t)$ to $\phi(t)$ is given by

$$H_1(s) = \frac{-\frac{1}{LM}s}{\left(s + \frac{F}{M}\right)\left(s^2 - \frac{g}{L}\right)}. \quad 1-246$$

The transfer function of the feedback part of the system follows from 1-245:

$$H_2(s) = -k. \quad 1-247$$

The characteristic polynomial of the pendulum positioning system is

$$\phi_1(s) = s\left(s + \frac{F}{M}\right)\left(s^2 - \frac{g}{L}\right), \quad 1-248$$

while the characteristic polynomial of the feedback part is

$$\phi_2(s) = 1. \quad 1-249$$

It follows from 1-246 and 1-247 that in this case

$$1 + H_1(s)H_2(s) = \frac{s^3 + s^2 \frac{F}{M} + s \left(\frac{k}{LM} - \frac{g}{L} \right) - \frac{Fg}{ML}}{\left(s + \frac{F}{M} \right) \left(s^2 - \frac{g}{L} \right)}, \quad 1-250$$

while from 1-248 and 1-249 we obtain

$$\phi_1(s)\phi_2(s) = s \left(s + \frac{F}{M} \right) \left(s^2 - \frac{g}{L} \right). \quad 1-251$$

We note that in this case the denominator of $1 + H_1(s)H_2(s)$ is not the product of the characteristic polynomials 1-251, but that a factor s has been canceled. Therefore, the numerator of 1-250 is not the closed-loop characteristic polynomial. By multiplication of 1-250 and 1-251, it follows that the characteristic polynomial of the feedback system is

$$s \left(s^3 + s^2 \frac{F}{M} + s \left(\frac{k}{LM} - \frac{g}{L} \right) - \frac{Fg}{ML} \right). \quad 1-252$$

We see that one of the closed-loop characteristic values is zero. Moreover, since the remaining factor contains a term with a negative coefficient, according to the well-known Routh-Hurwitz criterion (Schwarz and Friedland, 1965) there is at least one root with a positive real part. This means that the system cannot be stabilized in this manner. Example 2.6 (Section 2.4) presents a more sophisticated control scheme which succeeds in stabilizing the system.

Example 1.17. Stirred tank

Consider the stirred tank of Example 1.2 (Section 1.2.3). Suppose that it is desired to operate the system such that a constant flow $F(t)$ and a constant concentration $c(t)$ are maintained. One way of doing this is to use the main flow F_1 to regulate the flow F , and the minor flow F_2 to regulate the concentration c . Let us therefore choose μ_1 and μ_2 according to

$$\begin{aligned} \mu_1(t) &= -k_1 \eta_1(t), \\ \mu_2(t) &= -k_2 \eta_2(t). \end{aligned} \quad 1-253$$

This means that the system in the feedback loop has the transfer matrix

$$H_2(s) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}. \quad 1-254$$

It is easily found with the numerical data of Example 1.2 that the transfer

matrix of the system in the forward loop is given by

$$H_1(s) = \begin{pmatrix} \frac{0.01}{s + 0.01} & \frac{0.01}{s + 0.01} \\ \frac{-0.25}{s + 0.02} & \frac{0.75}{s + 0.02} \end{pmatrix}. \quad 1-255$$

With this the return difference matrix is

$$J(s) = I + H_1(s)H_2(s) = \begin{pmatrix} \frac{s + 0.01k_1 + 0.01}{s + 0.01} & \frac{0.01k_2}{s + 0.01} \\ \frac{-0.25k_1}{s + 0.02} & \frac{s + 0.75k_2 + 0.02}{s + 0.02} \end{pmatrix}. \quad 1-256$$

For the characteristic polynomials of the two systems, we have

$$\begin{aligned} \phi_1(s) &= (s + 0.01)(s + 0.02), \\ \phi_2(s) &= 1. \end{aligned} \quad 1-257$$

It follows from 1-256 that

$$\det [J(s)] = \frac{(s + 0.01k_1 + 0.01)(s + 0.75k_2 + 0.02) + 0.0025k_1k_2}{(s + 0.01)(s + 0.02)}. \quad 1-258$$

Since the denominator of this expression is the product $\phi_1(s)\phi_2(s)$, its numerator is the closed-loop characteristic polynomial. Further evaluation yields for the closed-loop characteristic polynomial

$$s^2 + s(0.01k_1 + 0.75k_2 + 0.03) + (0.0002k_1 + 0.0075k_2 + 0.01k_1k_2 + 0.0002). \quad 1-259$$

This expression shows that for positive k_1 and k_2 the feedback system is stable. Let us choose for the gain coefficients $k_1 = 10$ and $k_2 = 0.1$. This gives for the characteristic polynomial

$$s^2 + 0.205s + 0.01295. \quad 1-260$$

The characteristic values are

$$-0.1025 \pm j0.04944. \quad 1-261$$

The effectiveness of such a control scheme 1-253 is investigated in Example 2.8 (Section 2.5.3).

1.5.5* Root Loci

In the preceding section we saw that sometimes it is of interest to find in the complex plane the loci of the roots of an expression of the form

$$\phi(s) + \rho\psi(s), \quad 1-262$$

where $\phi(s)$ and $\psi(s)$ are polynomials in s , as the scalar parameter ρ varies. In this section we give some rules pertaining to these loci, so as to allow us to determine some special points of the loci and, in particular, to determine the asymptotic behavior. These rules make it possible to sketch root loci quite easily for simple problems; for more complicated problems the assistance of a digital computer is usually indispensable. Melsa (1970) gives a FORTRAN computer program for computing root loci.

We shall assume the following forms for the polynomials $\phi(s)$ and $\psi(s)$:

$$\begin{aligned} \phi(s) &= \prod_{i=1}^n (s - \pi_i), \\ \psi(s) &= \prod_{i=1}^m (s - \nu_i). \end{aligned} \quad 1-263$$

We refer to the π_i , $i = 1, 2, \dots, n$, as the *open-loop poles*, and to the $\nu_i = 1, 2, \dots, m$, as the *open-loop zeroes*. The roots of 1-262 will be called the *closed-loop poles*. This terminology stems from the significance that the polynomials $\phi(s)$ and $\psi(s)$ have in Section 1.5.4. We assume that $m \leq n$; this is no restriction since if $m > n$ the roles of $\phi(s)$ and $\psi(s)$ can be reversed by choosing $1/\rho$ as the parameter.

The most important properties of the root loci are the following.

(a) *Number of roots:* The number of roots of 1-262 is n . Each of the roots traces a continuous locus as ρ varies from $-\infty$ to ∞ .

(b) *Origin of loci:* The loci originate for $\rho = 0$ at the poles π_i , $i = 1, 2, \dots, n$. This is obvious, since for $\rho = 0$ the roots of 1-262 are the roots of $\phi(s)$.

(c) *Behavior of loci as $\rho \rightarrow \pm\infty$:* As $\rho \rightarrow \pm\infty$, m of the loci approach the zeroes ν_i , $i = 1, 2, \dots, m$. The remaining $n - m$ loci go to infinity. This follows from the fact that the roots of 1-262 are also the roots of

$$\frac{1}{\rho} \phi(s) + \psi(s). \quad 1-264$$

(d) *Asymptotes of loci:* Those $n - m$ loci that go to infinity approach asymptotically $n - m$ straight lines which make angles

$$\frac{\pi + k2\pi}{n - m}, \quad k = 0, 1, \dots, n - m - 1, \quad 1-265$$

with the positive real axis as $\rho \rightarrow +\infty$, and angles

$$\frac{k2\pi}{n-m}, \quad k = 0, 1, \dots, n-m-1, \quad 1-266$$

as $\rho \rightarrow -\infty$. The $n-m$ asymptotes intersect in one point on the real axis given by

$$\frac{\sum_{i=1}^n \pi_i - \sum_{i=1}^m \nu_i}{n-m}. \quad 1-267$$

These properties can be derived as follows. For large s we approximate 1-262 by

$$s^n + \rho s^m. \quad 1-268$$

The roots of this polynomial are

$$(-\rho)^{1/(n-m)}, \quad 1-269$$

which gives a first approximation for the faraway roots. A more refined analysis shows that a better approximation for the roots is given by

$$\frac{\sum_{i=1}^n \pi_i - \sum_{i=1}^m \nu_i}{n-m} + (-\rho)^{1/(n-m)}. \quad 1-270$$

This proves that the asymptotic behavior is as claimed.

(e) **Portions of root loci on real axis:** If ρ assumes only positive values, any portion of the real axis to the right of which an odd number of poles and zeroes lies on the real axis is part of a root locus. If ρ assumes only negative values, any portion of the real axis to the right of which an even number of poles and zeroes lies on the real axis is part of a root locus. This can be seen as follows. The roots of 1-262 can be found by solving

$$\frac{\phi(s)}{\psi(s)} = -\rho. \quad 1-271$$

If we assume ρ to be positive, 1-271 is equivalent to the real equations

$$\left| \frac{\phi(s)}{\psi(s)} \right| = \rho, \quad 1-272$$

$$\arg \frac{\phi(s)}{\psi(s)} = \pi + 2\pi k, \quad 1-273$$

where k is any integer. If s is real, there always exists a ρ for which 1-272 is satisfied. To satisfy 1-273 as well, there must be an odd number of zeroes and poles to the right of s . For negative ρ a similar argument holds.

Several other properties of root loci can be established (D'Azzo and Houpis, 1966) which are helpful in sketching root locus plots, but the rules listed above are sufficient for our purpose.

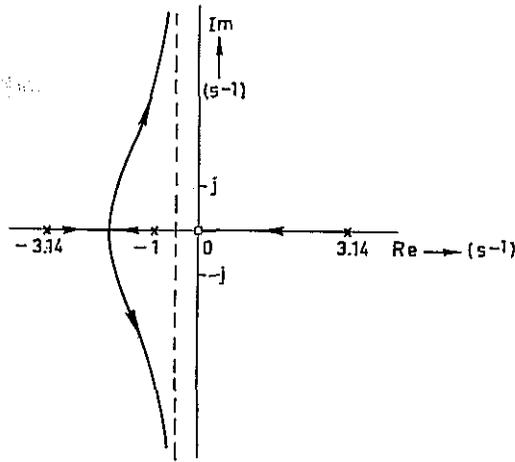


Fig. 1.8. Root locus for inverted pendulum. x, open-loop poles; O, open-loop zero.

Example 1.18. Inverted pendulum

Consider the proposed proportional feedback scheme of Example 1.16 where we found for the closed-loop characteristic polynomial

$$s \left(s + \frac{F}{M} \right) \left(s^2 - \frac{g}{L'} \right) + \frac{k}{L'M} s^3. \tag{1-274}$$

Here k is varied from 0 to ∞ . The poles are at 0, $-F/M$, $\sqrt{g/L'}$, and $-\sqrt{g/L'}$, while there is a double zero at 0. The asymptotes make angles of $\pi/2$ and $-\pi/2$ with the real axis as $k \rightarrow \infty$ since $n - m = 2$. The asymptotes intersect at $-\frac{1}{2}(F/M)$. The portions of the real axis between $\sqrt{g/L'}$ and 0, and between $-F/M$ and $-\sqrt{g/L'}$ belong to a locus. The pole at 0 coincides with a zero; this means that 0 is always one of the closed-loop poles. The loci of the remaining roots are sketched in Fig. 1.8 for the numerical values given in Example 1.1. It is seen that the closed-loop system is not stable for any k , as already concluded in Example 1.16.

1.6* CONTROLLABILITY

1.6.1* Definition of Controllability

For the solution of control problems, it is important to know whether or not a given system has the property that it may be steered from any given state

to any other given state. This leads to the concept of controllability (Kalman, 1960), which is discussed in this section. We give the following definition.

Definition 1.11. *The linear system with state differential equation*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad 1-275$$

is said to be **completely controllable** if the state of the system can be transferred from the zero state at any initial time t_0 to any terminal state $x(t_1) = x_1$ within a finite time $t_1 - t_0$.

Here, when we say that the system can be transferred from one state to another, we mean that there exists a piecewise continuous input $u(t)$, $t_0 \leq t \leq t_1$, which brings the system from one state to the other.

Definition 1.11 seems somewhat limited, since the only requirement is that the system can be transferred from the zero state to any other state. We shall see, however, that the definition implies more. The response from an arbitrary initial state is by 1-61 given by

$$x(t_1) = \Phi(t_1, t_0)x(t_0) + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau) d\tau, \quad 1-276$$

so that

$$x(t_1) - \Phi(t_1, t_0)x(t_0) = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau) d\tau. \quad 1-277$$

This shows that transferring the system from the state $x(t_0) = x_0$ to the state $x(t_1) = x_1$ is achieved by the same input that transfers $x(t_0) = 0$ to the state $x(t_1) = x_1 - \Phi(t_1, t_0)x_0$. This implies the following fact.

Theorem 1.22. *The linear differential system*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad 1-278$$

is **completely controllable** if and only if it can be transferred from any initial state x_0 at any initial time t_0 to any terminal state $x(t_1) = x_1$ within a finite time $t_1 - t_0$.

Example 1.19. *Stirred tank*

Suppose that the feeds F_1 and F_2 of the stirred tank of Example 1.2 (Section 1.2.3) have equal concentrations $c_1 = c_2 = \bar{c}$. Then the steady-state concentration c_0 in the tank is also \bar{c} , and we find for the linearized state differential equation

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{2\theta} & 0 \\ 0 & -\frac{1}{\theta} \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} u(t). \quad 1-279$$

It is clear from this equation that the second component of the state, which is the incremental concentration, cannot be controlled by manipulating the input, whose components are the incremental incoming flows. This is also clear physically, since the incoming feeds are assumed to have equal concentrations.

Therefore, the system obviously is not completely controllable if $c_1 = c_2$. If $c_1 \neq c_2$, the system is completely controllable, as we shall see in Example 1.21.

1.6.2* Controllability of Linear Time-Invariant Systems

In this section the controllability of linear time-invariant systems is studied. We first state the main result.

Theorem 1.23. *The n -dimensional linear time-invariant system*

$$\dot{x}(t) = Ax(t) + Bu(t) \quad 1-280$$

is completely controllable if and only if the column vectors of the controllability matrix

$$P = (B, AB, A^2B, \dots, A^{n-1}B) \quad 1-281$$

span the n -dimensional space.

This result can be proved formally as follows. We write for the state at t_1 , when at time t_0 the system is in the zero state,

$$x(t_1) = \int_{t_0}^{t_1} e^{A(t_1-\tau)} Bu(\tau) d\tau. \quad 1-282$$

The exponential may be represented in terms of its Taylor series; doing this we find

$$\begin{aligned} x(t_1) = & B \int_{t_0}^{t_1} u(\tau) d\tau + AB \int_{t_0}^{t_1} (t_1 - \tau) u(\tau) d\tau \\ & + A^2B \int_{t_0}^{t_1} \frac{(t_1 - \tau)^2}{2!} u(\tau) d\tau + \dots \end{aligned} \quad 1-283$$

We see that the terminal state is in the linear subspace spanned by the column vectors of the infinite sequence of matrices B, AB, A^2B, \dots . In this sequence there must eventually be a matrix, say $A^l B$, the column vectors of which are all linearly dependent upon the combined column vectors of the preceding matrices $B, AB, \dots, A^{l-1}B$. There must be such a matrix since there cannot be more than n linearly independent vectors in n -dimensional space. This also implies that $l \leq n$.

Let us now consider $A^{l+1}B = A(A^lB)$. Since the column vectors of A^lB depend linearly upon the combined column vectors of $B, AB, \dots, A^{l-1}B$, we can write

$$A^lB = B\Lambda_0 + AB\Lambda_1 + \dots + A^{l-1}B\Lambda_{l-1}, \quad 1-284$$

where the Λ_i , $i = 0, 1, \dots, l-1$ are matrices which provide the correct coefficients to express each of the column vectors of A^lB in terms of the column vectors of $B, AB, \dots, A^{l-1}B$. Consequently, we write

$$A^{l+1}B = ABA\Lambda_0 + A^2B\Lambda_1 + \dots + A^lB\Lambda_{l-1}, \quad 1-285$$

which very clearly shows that the columns of $A^{l+1}B$ also depend linearly upon the column vectors of $B, AB, \dots, A^{l-1}B$. Similarly, it follows that the column vectors of all matrices $A^k B$ for $k \geq l$ depend linearly upon the column vectors of $B, AB, \dots, A^{l-1}B$.

Returning now to **1-283**, we see that the terminal state $x(t_1)$ is in the linear subspace spanned by the column vectors of $B, AB, \dots, A^{l-1}B$. Since $l \leq n$ we can just as well say that $x(t_1)$ is in the subspace spanned by the column vectors of $B, AB, \dots, A^{n-1}B$. Now if these column vectors do *not* span the n -dimensional space, clearly only states in a linear subspace that is of smaller dimension than the entire n -dimensional space can be reached, hence the system is not completely controllable. This proves that if the system is completely controllable the column vectors of the controllability matrix P span the n -dimensional space.

To prove the other direction of the theorem, assume that the columns of P span the n -dimensional space. Then by a suitable choice of the input $u(\tau)$, $t_0 \leq \tau \leq t_1$ (e.g., involving orthogonal polynomials), the coefficient vectors

$$\int_{t_0}^{t_1} \frac{(t_1 - \tau)^i}{i!} u(\tau) d\tau \quad 1-286$$

in **1-283** can always be chosen so that the right-hand side of **1-283** equals any given vector in the space spanned by the columns of P . Since by assumption the columns of P span the entire n -dimensional space, this means that any terminal state can be reached, hence that the system is completely controllable. This terminates the proof of Theorem 1.23.

The controllability of the system **1-280** is of course completely determined by the matrices A and B . It is therefore convenient to introduce the following terminology.

Definition 1.12. Let A be an $n \times n$ and B an $n \times k$ matrix. Then we say that the pair $\{A, B\}$ is completely controllable if the system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad 1-287$$

is completely controllable.

Example 1.20. *Inverted pendulum*

The inverted pendulum of Example 1.1 (Section 1.2.3) is a single-input system which is described by the state differential equation

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{g}{L} & 0 & \frac{g}{L} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ 0 \end{pmatrix} \mu(t). \quad 1-288$$

The controllability matrix of the system is

$$P = \begin{pmatrix} 0 & \frac{1}{M} & -\frac{F}{M} \frac{1}{M} & \left(\frac{F}{M}\right)^2 \frac{1}{M} \\ \frac{1}{M} & -\frac{F}{M} \frac{1}{M} & \left(\frac{F}{M}\right)^2 \frac{1}{M} & -\left(\frac{F}{M}\right)^3 \frac{1}{M} \\ 0 & 0 & 0 & -\frac{g}{L} \frac{1}{M} \\ 0 & 0 & -\frac{g}{L} \frac{1}{M} & \frac{g}{L} \frac{F}{M} \frac{1}{M} \end{pmatrix}. \quad 1-289$$

It is easily seen that P has rank four for all values of the parameters, hence that the system is completely controllable.

1.6.3* The Controllable Subspace

In this section we analyze in some detail the structure of linear time-invariant systems that are not completely controllable. If a system is not completely controllable, clearly it is of interest to know what part of the state space can be reached. This motivates the following definition.

Definition 1.13. *The controllable subspace of the linear time-invariant system*

$$\dot{x}(t) = Ax(t) + Bu(t) \quad 1-290$$

is the linear subspace consisting of the states that can be reached from the zero state within a finite time.

In view of the role that the controllability matrix P plays, the following result is not surprising.

Theorem 1.24. *The controllable subspace of the n -dimensional linear time-invariant system*

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \mathbf{1-291}$$

is the linear subspace spanned by the columns of the controllability matrix

$$P = (B, AB, \dots, A^{n-1}B). \quad \mathbf{1-292}$$

This theorem immediately follows from the proof of Theorem 1.23 where we showed that any state that can be reached from the zero state is spanned by the columns of P , and any state not spanned by the columns of P cannot be reached. The controllable subspace possesses the following property.

Lemma 1.3. *The controllable subspace of the system $\dot{x}(t) = Ax(t) + Bu(t)$ is invariant under A , that is, if a vector x is in the controllable subspace, Ax is also in this subspace.*

The proof of this lemma follows along the lines of the proof of Theorem 1.23. The controllable subspace is spanned by the column vectors of $B, AB, \dots, A^{n-1}B$. Thus the vector Ax , where x is in the controllable subspace, is in the linear subspace spanned by the column vectors of AB, A^2B, \dots, A^nB . The column vectors of A^nB , however, depend linearly upon the column vectors of $B, AB, \dots, A^{n-1}B$; therefore Ax is in the subspace spanned by the column vectors of $B, AB, \dots, A^{n-1}B$, which means that Ax is in the controllable subspace. The controllable subspace is therefore invariant under A .

The concept of a controllable subspace can be further clarified by the following fact.

Theorem 1.25. *Consider the linear time-invariant system $\dot{x}(t) = Ax(t) + Bu(t)$. Then any initial state x_0 in the controllable subspace can be transferred to any terminal state x_1 in the controllable subspace within a finite time.*

We prove this result by writing for the state of the system at time t_1 :

$$x(t_1) = e^{A(t_1-t_0)}x_0 + \int_{t_0}^{t_1} e^{A(t_1-\tau)}Bu(\tau) d\tau. \quad \mathbf{1-293}$$

Now if x_0 is in the controllable subspace, $\exp [A(t_1 - t_0)]x_0$ is also in the controllable subspace, since the controllable subspace is invariant under A and $\exp [A(t_1 - t_0)] = I + A(t_1 - t_0) + \frac{1}{2}A^2(t_1 - t_0)^2 + \dots$. Therefore, if x_1 is in the controllable subspace, $x_1 - \exp [A(t_1 - t_0)]x_0$ is also in the controllable subspace. Expression 1-293 shows that any input that transfers the zero state to the state $x_1 - \exp [A(t_1 - t_0)]x_0$ also transfers x_0 to x_1 . Since $x_1 - \exp [A(t_1 - t_0)]x_0$ is in the controllable subspace, such an input exists; Theorem 1.25 is thus proved.

We now find a state transformation that represents the system in a canonical form, which very clearly exhibits the controllability properties of the system. Let us suppose that P has rank $m \leq n$, that is, P possesses m linearly independent column vectors. This means that the controllable subspace of the system 1-290 has dimension m . Let us choose a basis e_1, e_2, \dots, e_m for the controllable subspace. Furthermore, let $e_{m+1}, e_{m+2}, \dots, e_n$ be $n - m$ linearly independent vectors which together with e_1, e_2, \dots, e_m span the whole n -dimensional space. We now form the nonsingular transformation matrix

$$T = (T_1, T_2), \quad 1-294$$

where

$$T_1 = (e_1, e_2, \dots, e_m), \quad 1-295$$

and

$$T_2 = (e_{m+1}, e_{m+2}, \dots, e_n). \quad 1-296$$

Finally, we introduce a transformed state variable $x'(t)$ defined by

$$Tx'(t) = x(t). \quad 1-297$$

Substituting this into the state differential equation 1-290, we obtain

$$Tx'(t) = ATx'(t) + Bu(t) \quad 1-298$$

or

$$\dot{x}'(t) = T^{-1}ATx'(t) + T^{-1}Bu(t). \quad 1-299$$

We partition T^{-1} as follows

$$T^{-1} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \quad 1-300$$

where the partitioning corresponds to that of T in the sense that U_1 has m rows and U_2 has $n - m$ rows. With this partitioning it follows

$$T^{-1}T = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} (T_1, T_2) = \begin{pmatrix} U_1 T_1 & U_1 T_2 \\ U_2 T_1 & U_2 T_2 \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_{n-m} \end{pmatrix}. \quad 1-301$$

From this we conclude that

$$U_2 T_1 = 0. \quad 1-302$$

T_1 is composed of the vectors e_1, e_2, \dots, e_m which span the controllable subspace. This means that 1-302 implies that

$$U_2 x = 0 \quad 1-303$$

for any vector x in the controllable subspace.

With the partitionings 1-294 and 1-300, we write

$$T^{-1}AT = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} A(T_1, T_2) = \begin{pmatrix} U_1AT_1 & U_1AT_2 \\ U_2AT_1 & U_2AT_2 \end{pmatrix} \quad 1-304$$

and

$$T^{-1}B = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} B = \begin{pmatrix} U_1B \\ U_2B \end{pmatrix}. \quad 1-305$$

All the columns of T_1 are in the controllable subspace. This means that all the columns of AT_1 are also in the controllable subspace, since the controllable subspace is invariant under A (Lemma 1.3). However, then 1-303 implies that

$$U_2AT_1 = 0. \quad 1-306$$

The columns of B are obviously all in the controllable subspace, since B is part of the controllability matrix. Therefore, we also have

$$U_2B = 0. \quad 1-307$$

Our findings can be summarized as follows.

Theorem 1.26. Consider the n -dimensional time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t). \quad 1-308$$

Form a nonsingular transformation matrix $T = (T_1, T_2)$ where the columns of T_1 form a basis for the m -dimensional ($m \leq n$) controllable subspace of 1-308 and the column vectors of T_2 together with those of T_1 form a basis for the whole n -dimensional space. Define the transformed state

$$x'(t) = T^{-1}x(t). \quad 1-309$$

Then the state differential equation 1-308 is transformed into the controllability canonical form

$$\dot{x}'(t) = \begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix} x'(t) + \begin{pmatrix} B'_1 \\ 0 \end{pmatrix} u(t). \quad 1-310$$

Here A'_{11} is an $m \times m$ matrix, and the pair $\{A'_{11}, B'_1\}$ is completely controllable.

Partitioning

$$x'(t) = \begin{pmatrix} x'_1(t) \\ x'_2(t) \end{pmatrix}, \quad 1-311$$

where x'_1 has dimension m and x'_2 dimension $n - m$, we see from Theorem 1.26 that the transformed system can be represented as in Fig. 1.9. We note that

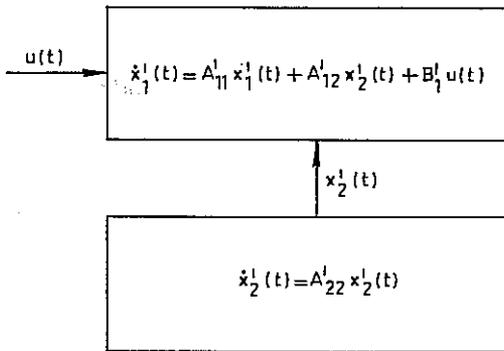


Fig. 1.9. The controllability canonical form of a linear time-invariant differential system.

x_2' behaves completely independently, while x_1' is influenced both by x_2' and the input u . The fact that $\{A_{11}', B_1'\}$ is completely controllable follows from the fact that any state of the form $\text{col}(x_0', 0)$ is in the controllable subspace of the system 1-310. The proof is left as an exercise.

It should be noted that the controllability canonical form is not at all unique, since both T_1 and T_2 can to some extent be freely chosen. It is easily verified, however, that no matter how the transformation T is chosen the characteristic values of both A_{11}' and A_{22}' are always the same (Problem 1.5). Quite naturally, this leads us to refer to the characteristic values of A_{11}' as the *controllable poles* of the system, and to the characteristic values of A_{22}' as the *uncontrollable poles*. Let us now assume that all the characteristic values of the system 1-310 are distinct (this is not an essential restriction). Then it is not difficult to recognize (Problem 1.5) that *the controllable subspace of the system 1-310 is spanned by the characteristic vectors corresponding to the controllable poles of the system*. This statement is also true for the original representation 1-308 of the system. Then a natural definition for the *uncontrollable subspace* of the system, which we have so far avoided, is *the subspace spanned by the characteristic vectors corresponding to the uncontrollable poles of the system*.

Example 1.21. *Stirred tank*

The stirred tank of Example 1.2 (Section 1.2.3) is described by the state differential equation

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{2\theta} & 0 \\ 0 & -\frac{1}{\theta} \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ \frac{c_1 - c_0}{V_0} & \frac{c_2 - c_0}{V_0} \end{pmatrix} u(t). \quad 1-312$$

The controllability matrix is

$$P = \begin{pmatrix} 1 & 1 & -\frac{1}{2\theta} & -\frac{1}{2\theta} \\ \frac{c_1 - c_0}{V_0} & \frac{c_2 - c_0}{V_0} & -\frac{1}{\theta} \frac{c_1 - c_0}{V_0} & -\frac{1}{\theta} \frac{c_2 - c_0}{V_0} \end{pmatrix}. \quad 1-313$$

P has rank two provided $c_1 \neq c_2$. The system is therefore completely controllable if $c_1 \neq c_2$.

If $c_1 = c_2 = \bar{c}$, then $c_0 = \bar{c}$ also and the controllability matrix takes the form

$$P = \begin{pmatrix} 1 & 1 & -\frac{1}{2\theta} & -\frac{1}{2\theta} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad 1-314$$

The controllable subspace is therefore spanned by the vector $\text{col}(1, 0)$. This means, as we saw in Example 1.19, that only the volume of fluid in the tank can be controlled but not the concentration.

We finally remark that if $c_1 = c_2 = c_0 = \bar{c}$ the state differential equation 1-312 takes the form 1-279, which is already in controllability canonical form. The controllable pole of the system is $-1/(2\theta)$; the uncontrollable pole is $-1/\theta$.

1.6.4* Stabilizability

In this section we develop the notion of stabilizability (Galperin and Krasovski, 1963; Wonham 1968a). The terminology will be motivated in Section 3.2. In Section 1.4.3 we defined the stable and unstable subspaces for a time-invariant system. Any initial state $x(0)$ can be uniquely written as

$$x(0) = x_s(0) + x_u(0), \quad 1-315$$

where $x_s(0)$ is in the stable subspace and $x_u(0)$ in the unstable subspace. Clearly, in order to control the system properly, we must require that the unstable component can be completely controlled. This is the case if the unstable component $x_u(0)$ is in the controllable subspace. We thus state.

Definition 1.14. *The linear time-invariant system*

$$\dot{x}(t) = Ax(t) + Bu(t) \quad 1-316$$

is *stabilizable* if its unstable subspace is contained in its controllable subspace, that is, any vector x in the unstable subspace is also in the controllable subspace.

It is sometimes convenient to employ the following abbreviated terminology.

Definition 1.15. The pair $\{A, B\}$ is stabilizable if the system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad 1-317$$

is stabilizable.

Obviously, we have the following result.

Theorem 1.27. Any asymptotically stable time-invariant system is stabilizable. Any completely controllable system is stabilizable.

The stabilizability of a system can conveniently be checked when the state differential equation is in controllability canonical form. This follows from the following fact.

Theorem 1.28. Consider the time-invariant linear system

$$\dot{x}(t) = Ax(t) + Bu(t). \quad 1-318$$

Suppose that it is transformed according to Theorem 1-26 into the controllability canonical form

$$\dot{x}'(t) = \begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix} x'(t) + \begin{pmatrix} B'_1 \\ 0 \end{pmatrix} u(t), \quad 1-319$$

where the pair $\{A'_{11}, B'_1\}$ is completely controllable. Then the system 1-318 is stabilizable if and only if the matrix A'_{22} is asymptotically stable.

This theorem can be summarized by stating that a system is stabilizable if and only if its uncontrollable poles are stable. We prove the theorem as follows.

(a) *Stabilizability implies A'_{22} asymptotically stable.* Suppose that the system 1-318 is stabilizable. Then the transformed system 1-319 is also stabilizable (Problem 1.6). Let us partition

$$x'(t) = \begin{pmatrix} x'_1(t) \\ x'_2(t) \end{pmatrix}, \quad 1-320$$

where the dimension m of $x'_1(t)$ is the dimension of the controllable subspace of the original system 1-318. Suppose that A'_{22} is not stable. Choose an $(n - m)$ -dimensional vector x'_2 in the unstable subspace of A'_{22} . Then obviously, the n -dimensional column vector $\text{col}(0, x'_2)$ is in the unstable subspace of 1-319. This vector, however, is clearly not in the controllable subspace of 1-319. This means that there is a vector that is in the unstable subspace of 1-319 but not in the controllable subspace. This contradicts the assumption of stabilizability. This proves that if the system 1-318 is stabilizable A'_{22} must be stable.

(b) *A'_{22} stable implies stabilizability:* Assume that A'_{22} is stable. Then any vector that is in the unstable subspace of 1-319 must be of the form

col $(x'_1, 0)$. However, since the pair $\{A'_{11}, B'_1\}$ is completely controllable, this vector is also in the controllable subspace of **1-319**. This shows that any vector in the unstable subspace of **1-319** is also in the controllable subspace, hence that **1-319** is stabilizable. Consequently (Problem 1.6), the original system **1-318** is also stabilizable.

Example 1.22. Stirred tank

The stirred tank of Example 1.2 (Section 1.2.3) is described by the state differential equation

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{2\theta} & 0 \\ 0 & -\frac{1}{\theta} \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} u(t), \quad \mathbf{1-321}$$

if we assume that $c_1 = c_2 = c_0 = \bar{c}$. As we have seen before, this system is not completely controllable. The state differential equation is already in the decomposed form for controllability. We see that the matrix A'_{22} has the characteristic value $-1/\theta$, which implies that the system is stabilizable. This means that even if the incremental concentration $\xi_2(t)$ initially has an incorrect value it will eventually approach zero.

1.6.5* Controllability of Time-Varying Linear Systems

The simple test for controllability of Theorem 1.24 does not apply to time-varying linear systems. For such systems we have the following result, which we shall not prove.

Theorem 1.29. Consider the linear time-varying system with state differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t). \quad \mathbf{1-322}$$

Define the nonnegative-definite symmetric matrix function

$$W(t_0, t) = \int_{t_0}^t \Phi(t, \tau) B(\tau) B^T(\tau) \Phi^T(t, \tau) d\tau, \quad \mathbf{1-323}$$

where $\Phi(t, t_0)$ is the transition matrix of the system. Then the system is completely controllable if and only if there exists for all t_0 a t_1 with $t_0 < t_1 < \infty$ such that $W(t_0, t_1)$ is nonsingular.

For a proof of this theorem, the reader is referred to Kalman, Falb, and Arbib (1969).

The matrix $W^{-1}(t_0, t_1)$ is related to the minimal "control energy" needed to transfer the system from one state to another when the "control energy"

is measured as

$$\int_{t_0}^{t_1} u^T(t)u(t) dt. \quad 1-324$$

A stronger form of controllability results if certain additional conditions are imposed upon the matrix $W(t_0, t)$ (Kalman, 1960):

Definition 1.16. *The time-varying system 1-322 is uniformly completely controllable if there exist positive constants σ , α_0 , α_1 , β_0 , and β_1 such that*

$$(a) \alpha_0 I \leq W(t_0, t_0 + \sigma) \leq \alpha_1 I \quad \text{for all } t_0; \quad 1-325$$

$$(b) \beta_0 I \leq \Phi(t_0, t_0 + \sigma)W(t_0, t_0 + \sigma)\Phi^T(t_0, t_0 + \sigma) \leq \beta_1 I \quad \text{for all } t_0, \quad 1-326$$

where $W(t_0, t)$ is the matrix 1-323 and $\Phi(t, t_0)$ is the transition matrix of the system.

Uniform controllability implies not only that the system can be brought from any state to any other state but also that the control energy involved in this transfer and the transfer time are roughly independent of the initial time. In view of this remark, the following result for time-invariant systems is not surprising.

Theorem 1.30. *The time-invariant linear system*

$$\dot{x}(t) = Ax(t) + Bu(t) \quad 1-327$$

is uniformly completely controllable if and only if it is completely controllable.

1.7* RECONSTRUCTIBILITY

1.7.1* Definition of Reconstructibility

In Chapter 4 we discuss the problem of reconstructing the behavior of the state of the system from incomplete and possibly inaccurate observations. Before studying such problems it is important to know whether or not a given system has the property that it is at all possible to determine from the behavior of the output what the behavior of the state is. This leads to the concept of *reconstructibility* (Kalman, Falb, and Arbib, 1969), which is the subject of this section.

We first consider the following definition.

Definition 1.17. *Let $y(t; t_0, x_0, u)$ denote the response of the output variable $y(t)$ of the linear differential system*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t), \end{aligned} \quad 1-328$$

to the initial state $x(t_0) = x_0$. Then the system is called **completely reconstructible** if for all t_1 there exists a t_0 with $-\infty < t_0 < t_1$ such that

$$y(t; t_0, x_0, u) = y(t; t_0, x'_0, u), \quad t_0 \leq t \leq t_1, \quad 1-329$$

for all $u(t)$, $t_0 \leq t \leq t_1$, implies $x_0 = x'_0$.

The definition implies that if a system is completely reconstructible, and the output variable is observed up to any time t_1 , there always exists a time $t_0 < t_1$ at which the state of the system can be uniquely determined. If $x(t_0)$ is known, of course $x(t_1)$ can also be determined.

The following result shows that in order to study the reconstructibility of the system 1-328 we can confine ourselves to considering a simpler situation.

Theorem 1.31. *The system 1-328 is completely reconstructible if and only if for all t_1 there exists a t_0 with $-\infty < t_0 < t_1$ such that*

$$y(t; t_0, x_0, 0) = 0, \quad t_0 \leq t \leq t_1, \quad 1-330$$

implies that $x_0 = 0$.

This result is not difficult to prove. Of course if the system 1-328 is completely reconstructible, it follows immediately from the definition that if 1-330 holds then $x_0 = 0$. This proves one direction of the theorem. However, since

$$y(t; t_0, x_0, u) = C(t) \left[\Phi(t, t_0)x_0 + \int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau) d\tau \right], \quad 1-331$$

the fact that

$$y(t; t_0, x_0, u) = y(t; t_0, x'_0, u) \quad \text{for } t_0 \leq t \leq t_1 \quad 1-332$$

implies and is implied by

$$C(t)\Phi(t, t_0)x_0 = C(t)\Phi(t, t_0)x'_0 \quad \text{for } t_0 \leq t \leq t_1. \quad 1-333$$

This in turn is equivalent to

$$C(t)\Phi(t, t_0)(x_0 - x'_0) = 0 \quad \text{for } t_0 \leq t \leq t_1. \quad 1-334$$

Evidently if 1-334 implies that $x_0 - x'_0 = 0$, that is, $x_0 = x'_0$, the system is completely reconstructible. This finishes the proof of the other direction of Theorem 1.31.

The definition of reconstructibility is due to Kalman (Kalman, Falb, and Arbib, 1969). It should be pointed out that reconstructibility is complementary to *observability*. A system of the form 1-328 is said to be completely observable if for all t_0 there exists a $t_1 < \infty$ such that

$$y(t; t_0, x_0, u) = y(t; t_0, x'_0, u), \quad t_0 \leq t \leq t_1, \quad 1-335$$

for all $u(t)$, $t_0 \leq t \leq t_1$, implies that $x_0 = x'_0$. We note that observability

means that it is possible to determine the state at time t_0 from the *future* output. In control and filtering problems, however, usually only past output values are available. It is therefore much more natural to consider reconstructibility, which regards the problem of determining the present state from *past* observations. It is easy to recognize that for time-invariant systems complete reconstructibility implies and is implied by complete observability.

Example 1.23. *Inverted pendulum*

Consider the inverted pendulum of Example 1.1 (Section 1.2.3) and take as the output variable the angle $\phi(t)$. Let us compare the states

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} d_0 \\ 0 \\ d_0 \\ 0 \end{pmatrix} \quad 1-336$$

The second state differs from the zero state in that both carriage and pendulum are displaced over a distance d_0 ; otherwise, the system is at rest. If an input identical to zero is applied, the system stays in these positions, and $\phi(t) \equiv 0$ in both cases. It is clear that if only the angle $\phi(t)$ is observed it is impossible to decide at a given time whether the system is in one state or the other; as a result, the system is not completely reconstructible.

1.7.2* Reconstructibility of Linear Time-Invariant Systems

In this section the reconstructibility of linear time-invariant systems is discussed. The main result is the following.

Theorem 1.32. *The n -dimensional linear time-invariant system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \quad 1-337$$

is completely reconstructible if and only if the row vectors of the reconstructibility matrix

$$Q = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \quad 1-338$$

span the n -dimensional space.

This can be proved as follows. Let us first assume that the system 1-337 is completely reconstructible. Then it follows from Theorem 1.31 that for all t_1 there exists a t_0 such that

$$Ce^{A(t-t_0)}x_0 = 0, \quad t_0 \leq t \leq t_1, \quad 1-339$$

implies that $x_0 = 0$. Expanding $\exp [A(t - t_0)]$ in terms of its Taylor series, 1-339 is equivalent to

$$\left[C + CA(t - t_0) + CA^2 \frac{(t - t_0)^2}{2!} + CA^3 \frac{(t - t_0)^3}{3!} + \cdots \right] x_0 = 0, \quad t_0 \leq t \leq t_1. \quad 1-340$$

Now if the reconstructibility matrix Q does not have full rank, there exists a nonzero x_0 such that

$$Cx_0 = 0, \quad CAx_0 = 0, \quad \cdots, \quad CA^{n-1}x_0 = 0. \quad 1-341$$

By using the Cayley-Hamilton theorem, it is not difficult to see also that $CA^l x_0 = 0$ for $l \geq n$. Thus if Q does not have full rank there exists a nonzero x_0 such that 1-340 holds. Clearly, in this case 1-339 does not imply $x_0 = 0$, and the system is not completely reconstructible. This contradicts our assumption, which proves that Q must have full rank.

We now prove the other direction of Theorem 1.32. Assume that Q has full rank. Suppose that

$$y(t) = Ce^{A(t-t_0)}x_0 = 0 \quad \text{for } t_0 \leq t \leq t_1. \quad 1-342$$

It follows by repeated differentiation of $y(t)$ that

$$\begin{aligned} y(t_0) &= Cx_0 = 0, \\ y'(t_0) &= CAx_0 = 0, \\ y''(t_0) &= CA^2x_0 = 0, \\ &\vdots \\ &\vdots \\ y^{(n-1)}(t_0) &= CA^{n-1}x_0 = 0, \end{aligned} \quad 1-343$$

or

$$Qx_0 = 0. \quad 1-344$$

Since Q has full rank, 1-344 implies that $x_0 = 0$. Hence by Theorem 1.31 the system is completely reconstructible. This terminates the proof of Theorem 1.32.

Since the reconstructibility of the system 1-337 depends only on the matrices A and C , it is convenient to employ the following terminology.

Definition 1.18. Let A be an $n \times n$ and C an $l \times n$ matrix. Then we call the pair $\{A, C\}$ completely reconstructible if the system

$$\dot{x}(t) = Ax(t), \quad 1-345$$

$$y(t) = Cx(t), \quad 1-346$$

is completely reconstructible.

Example 1.24. *Inverted pendulum*

The inverted pendulum of Example 1.1 (Section 1.2.3) is described by the state differential equation

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{g}{L} & 0 & \frac{g}{L} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ 0 \end{pmatrix} \mu(t). \quad 1-347$$

If we take as the output variable $\eta(t)$ the angle $\phi(t)$, we have

$$\eta(t) = \left(-\frac{1}{L}, 0, \frac{1}{L}, 0 \right) x(t). \quad 1-348$$

The reconstructibility matrix is

$$Q = \begin{pmatrix} -\frac{1}{L} & 0 & \frac{1}{L} & 0 \\ 0 & -\frac{1}{L} & 0 & \frac{1}{L} \\ -\frac{g}{L} \frac{1}{L} & \frac{F}{M} \frac{1}{L} & \frac{g}{L} \frac{1}{L} & 0 \\ 0 & -\frac{g}{L} \frac{1}{L} - \left(\frac{F}{M} \right)^2 \frac{1}{L} & 0 & \frac{g}{L} \frac{1}{L} \end{pmatrix}. \quad 1-349$$

This matrix has rank three; the system is therefore not completely reconstructible. This confirms the conclusion of Example 1.23. If we add as a second component of the output variable the displacement $s(t)$ of the carriage, we have

$$y(t) = \begin{pmatrix} -\frac{1}{L} & 0 & \frac{1}{L} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} x(t). \quad 1-350$$

This yields for the reconstructibility matrix

$$Q = \begin{pmatrix} -\frac{1}{L} & 0 & \frac{1}{L} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{L} & 0 & \frac{1}{L} \\ 0 & 1 & 0 & 0 \\ -\frac{g}{L} & \frac{F}{M} \frac{1}{L} & \frac{g}{L} \frac{1}{L} & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & -\frac{g}{L} \frac{1}{L} - \left(\frac{F}{M}\right)^2 \frac{1}{L} & 0 & \frac{g}{L} \frac{1}{L} \\ 0 & \left(\frac{F}{M}\right)^2 & 0 & 0 \end{pmatrix} \quad 1-351$$

With this output the system is completely reconstructible, since Q has rank four.

1.7.3* The Unreconstructible Subspace

In this section we analyze in some detail the structure of systems that are not completely reconstructible. If a system is not completely reconstructible, it is never possible to establish uniquely from the output what the state of the system is. Clearly, it is of interest to know exactly what uncertainty remains. This introduces the following definition.

Definition 1.19. *The unreconstructible subspace of the linear time-invariant system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \quad 1-352$$

is the linear subspace consisting of the states x_0 for which

$$y(t; x_0, t_0, 0) = 0, \quad t \geq t_0. \quad 1-353$$

The following theorem characterizes the unreconstructible subspace.

Theorem 1.33. *The unreconstructible subspace of the n -dimensional linear time-invariant system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \quad 1-354$$

is the null space of the reconstructibility matrix

$$Q = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ \vdots \\ CA^{n-1} \end{pmatrix}. \quad 1-355$$

The proof of this theorem immediately follows from the proof of Theorem 1.32 where we showed that any initial state in the null space of Q produces an output that is identical to zero in response to a zero input. Any initial state not in the null space of Q produces a nonzero response, which proves that the null space of Q is the unreconstructible subspace. The unreconstructible subspace possesses the following property.

Lemma 1.4. *The unreconstructible subspace of the system $\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$ is invariant under A .*

We leave the proof of this lemma as an exercise.

The concept of unreconstructible subspace can be clarified by the following fact.

Theorem 1.34. *Consider the time-invariant system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \quad 1-356$$

Suppose that the output $y(t)$ and the input $u(t)$ are known over an interval $t_0 \leq t \leq t_1$. Then the initial state of the system at time t_0 is determined within the addition of an arbitrary vector in the unreconstructible subspace. As a result, also the terminal state at time t_1 is determined within the addition of an arbitrary vector in the unreconstructible subspace.

To prove the first part of the theorem, we must show that if two initial states $x(t_0) = x_0$ and $x(t_0) = x'_0$ produce the same output $y(t)$, $t_0 \leq t \leq t_1$, for any input $u(t)$, $t_0 \leq t \leq t_1$, then $x_0 - x'_0$ lies in the unreconstructible subspace. This is obviously true since by the linearity of the system,

$$y(t; t_0, x_0, u) = y(t; t_0, x'_0, u), \quad t_0 \leq t \leq t_1, \quad 1-357$$

is equivalent to

$$y(t; t_0, x_0 - x'_0, 0) = 0, \quad t_0 \leq t \leq t_1, \quad 1-358$$

which shows that $x_0 - x'_0$ is in the unreconstructible subspace.

The second part of the theorem is proved as follows. The addition of an arbitrary vector x''_0 in the unreconstructible subspace to x_0 results in the

addition of $\exp [A(t_1 - t_0)]x_0''$ to the terminal state. Since $\exp [A(t_1 - t_0)]$ can be expanded in powers of A , and the unreconstructible subspace is invariant under A , $\exp [A(t_1 - t_0)]x_0''$ is also in the unreconstructible subspace. Moreover, since $\exp [A(t_1 - t_0)]$ is nonsingular, this proves that also the terminal state is determined within the addition of an arbitrary vector in the unreconstructible subspace.

We now discuss a state transformation that represents the system in a *canonical form*, which clearly exhibits the reconstructibility properties of the system. Let us suppose that Q has rank $m \leq n$, that is, Q possesses m linearly independent row vectors. This means that the null space of Q , hence the unreconstructible subspace of the system, has dimension $n - m$. The row vectors of Q span an m -dimensional linear subspace; let the row vectors f_1, f_2, \dots, f_m be a basis for this subspace. An obvious choice for this basis is a set of m independent row vectors from Q . Furthermore, let $f_{m+1}, f_{m+2}, \dots, f_n$ be $n - m$ linearly independent row vectors which together with f_1, \dots, f_m span the whole n -dimensional space. Now form the nonsingular transformation matrix

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \quad 1-359$$

where

$$U_1 = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix} \quad \text{and} \quad U_2 = \begin{pmatrix} f_{m+1} \\ f_{m+2} \\ \vdots \\ f_n \end{pmatrix}. \quad 1-360$$

Finally, introduce a transformed state variable $x'(t)$ as

$$x'(t) = Ux(t). \quad 1-361$$

Substitution into 1-356 yields

$$\begin{aligned} U^{-1}\dot{x}'(t) &= AU^{-1}x'(t) + Bu(t), \\ y(t) &= CU^{-1}x'(t), \end{aligned} \quad 1-362$$

or

$$\begin{aligned} \dot{x}'(t) &= UAU^{-1}x'(t) + UB u(t), \\ y(t) &= CU^{-1}x'(t). \end{aligned} \quad 1-363$$

We partition U^{-1} as follows

$$U^{-1} = (T_1, T_2), \quad 1-364$$

where the partitioning corresponds to that of U so that T_1 has m and T_2 $n - m$ columns. We have

$$UU^{-1} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} (T_1, T_2) = \begin{pmatrix} U_1 T_1 & U_1 T_2 \\ U_2 T_1 & U_2 T_2 \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_{n-m} \end{pmatrix}, \quad 1-365$$

from which we conclude that

$$U_1 T_2 = 0. \quad 1-366$$

The rows of U_1 are made up of linear combinations of the linearly independent rows of the reconstructibility matrix Q . This means that any vector x that satisfies $U_1 x = 0$ also satisfies $Qx = 0$, hence is in the unreconstructible subspace. Since

$$U_1 T_2 = 0, \quad 1-367$$

all column vectors of T_2 must be in the unreconstructible subspace. Because T_2 has $n - m$ linearly independent column vectors, and the unreconstructible subspace has dimension $n - m$, the column vectors of T_2 form a basis for the subspace. With this it follows from 1-367 that $U_1 x = 0$ for any x in the subspace.

With the partitionings 1-359 and 1-364, we have

$$UAU^{-1} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} A(T_1, T_2) = \begin{pmatrix} U_1 A T_1 & U_1 A T_2 \\ U_2 A T_1 & U_2 A T_2 \end{pmatrix} \quad 1-368$$

and

$$CU^{-1} = (CT_1, CT_2). \quad 1-369$$

All column vectors of T_2 are in the unreconstructible subspace; because the subspace is invariant under A (Lemma 1.4), the columns of AT_2 are also in the subspace, and we have from 1-367

$$U_1 A T_2 = 0. \quad 1-370$$

Since the rows of C are rows of the reconstructibility matrix Q , and the columns of T_2 are in the unreconstructible subspace, hence in the null space of Q , we must also have

$$CT_2 = 0. \quad 1-371$$

We summarize our results as follows.

Theorem 1.35. Consider the n -th order time-invariant linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \quad 1-372$$

Form a nonsingular transformation matrix

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \quad 1-373$$

where the m rows of U_1 form a basis for the m -dimensional ($m \leq n$) subspace spanned by the rows of the reconstructibility matrix of the system. The $n - m$ rows of U_2 form together with the m rows of U_1 a basis for the whole n -dimensional space. Define a transformed state variable $x'(t)$ by

$$x'(t) = Ux(t). \quad 1-374$$

Then in terms of the transformed state variable the system is represented in the reconstructibility canonical form

$$\begin{aligned} \dot{x}'(t) &= \begin{pmatrix} A'_{11} & 0 \\ A'_{21} & A'_{22} \end{pmatrix} x'(t) + \begin{pmatrix} B'_1 \\ B'_2 \end{pmatrix} u(t), \\ y(t) &= (C'_1, 0) x'(t). \end{aligned} \quad 1-375$$

Here A'_{11} is an $m \times m$ matrix, and the pair $\{A'_{11}, C'_1\}$ is completely reconstructible.

Partitioning

$$x'(t) = \begin{pmatrix} x'_1(t) \\ x'_2(t) \end{pmatrix}, \quad 1-376$$

where x'_1 has dimension m and x'_2 dimension $n - m$, we see from Theorem 1.35 that the system can be represented as in Fig. 1.10. We note that nothing about x'_2 can be inferred from observing the output y . The fact that the pair $\{A'_{11}, C'_1\}$ is completely reconstructible follows from the fact that if an initial

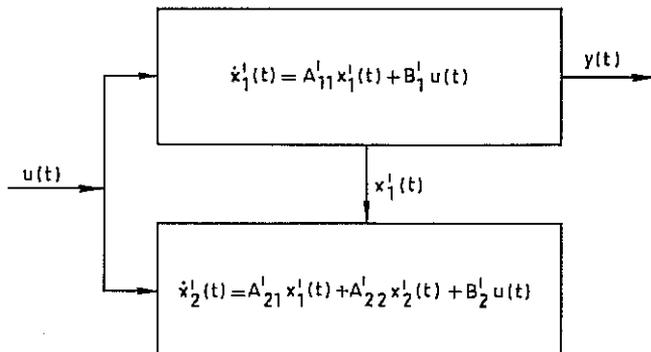


Fig. 1.10. Reconstructibility canonical form of a time-invariant linear differential system.

state $x'(t_0)$ produces a zero input response identical to zero, it must be of the form $x'(t_0) = \text{col}(0, x'_{20})$. The complete proof is left as an exercise.

We finally note that the reconstructibility canonical form is not unique because both U_1 and U_2 can to some extent be arbitrarily chosen. No matter how the transformation is performed, however, the characteristic values of A'_{11} and A'_{22} can be shown to be always the same. This leads us to refer to the characteristic values of A'_{11} as the *reconstructible poles*, and the characteristic values of A'_{22} as the *unreconstructible poles* of the system 1-372. Let us assume for simplicity that all characteristic values of the system are distinct. Then it can be proved that *the unreconstructible subspace of the system is spanned by those characteristic vectors of the system that correspond to the unreconstructible poles*. This is true both for the transformed version 1-375 and the original representation 1-372 of the system. Quite naturally, we now define the *reconstructible subspace* of the system 1-372 as the *subspace spanned by the characteristic vectors of the system corresponding to the reconstructible poles*.

Example 1.25. Inverted pendulum

In Example 1.24 we saw that the inverted pendulum is not completely reconstructible if the angle $\phi(t)$ is chosen as the observed variable. We now determine the unreconstructible subspace and the reconstructibility canonical form. It is easy to see that the rows of the reconstructibility matrix Q as given by 1-349 are spanned by the row vectors

$$(-1, 0, 1, 0), (0, -1, 0, 1), \quad \text{and} \quad (0, 1, 0, 0). \quad 1-377$$

Any vector $x = \text{col}(\xi_1, \xi_2, \xi_3, \xi_4)$ in the null space of Q must therefore satisfy

$$\begin{aligned} -\xi_1 + \xi_3 &= 0, \\ -\xi_2 + \xi_4 &= 0, \\ \xi_2 &= 0. \end{aligned} \quad 1-378$$

This means that the unreconstructible subspace of the system is spanned by

$$\text{col}(1, 0, 1, 0). \quad 1-379$$

Any initial state proportional to this vector is indistinguishable from the zero state, as shown in Example 1.23.

To bring the system equations into reconstructibility canonical form, let us choose the row vectors 1-377 as the first three rows of the transformation matrix U . For the fourth row we select, rather arbitrarily, the row vector

$$(1, 0, 0, 0). \quad 1-380$$

With this we find for the transformation matrix U and its inverse

$$U = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \quad 1-381$$

It follows for the transformed representation

$$\dot{x}'(t) = \begin{pmatrix} 0 & 1 & 0 & | & 0 \\ \frac{g}{L} & 0 & \frac{F}{M} & | & 0 \\ 0 & 0 & -\frac{F}{M} & | & 0 \\ \hline 0 & 0 & 1 & | & 0 \end{pmatrix} x'(t) + \begin{pmatrix} 0 \\ -\frac{1}{M} \\ \frac{1}{M} \\ 0 \end{pmatrix} \mu(t), \quad 1-382$$

$$\eta(t) = \left(\frac{1}{L}, \quad 0, \quad 0, \quad | \quad 0 \right) x'(t).$$

The components of the transformed state are, from 1-24,

$$\begin{aligned} \xi_1'(t) &= -\xi_1(t) + \xi_3(t) = L\dot{\phi}(t), \\ \xi_2'(t) &= -\xi_2(t) + \xi_4(t) = L\dot{\phi}(t), \\ \xi_3'(t) &= \xi_3(t) = \dot{s}(t), \\ \xi_4'(t) &= \xi_4(t) = s(t). \end{aligned} \quad 1-383$$

In this representation the position and velocity of the pendulum relative to the carriage, as well as the velocity of the carriage, can be reconstructed from the observed variable, but not the position of the carriage.

It is easily seen that the reconstructible poles of the system are $-F/M$ and $\pm\sqrt{g/L}$. The unreconstructible pole is 0.

1.7.4* Detectability

In the preceding section it was found that if the output variable of a not completely reconstructible system is observed there is always an uncertainty about the actual state of the system since to any possible state we can always add an arbitrary vector in the unreconstructible subspace (Theorem 1.34). The best we can hope for in such a situation is that any state in the unreconstructible subspace has the property that the zero input response of the system to this

state converges to zero. This is the case when any state in the unreconstructible subspace is also in the stable subspace of the system. Then, whatever we guess for the unreconstructible component of the state, the error will never grow indefinitely. A system with this property will be called detectable (Wonham, 1968a). We define this property as follows.

Definition 1.20. *The linear time-invariant system*

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t),\end{aligned}\tag{1-384}$$

is detectable if its unreconstructible subspace is contained in its stable subspace.

It is convenient to employ the following abbreviated terminology.

Definition 1.21. *The pair $\{A, C\}$ is detectable if the system*

$$\begin{aligned}\dot{\tilde{x}}(t) &= A\tilde{x}(t), \\ y(t) &= C\tilde{x}(t),\end{aligned}\tag{1-385}$$

is detectable.

The following result is an immediate consequence of the definition:

Theorem 1.36. *Any asymptotically stable system of the form 1-384 is detectable. Any completely reconstructible system of the form 1-384 is detectable.*

Detectable systems possess the following property.

Theorem 1.37. *Consider the linear time-invariant system*

$$\begin{aligned}\dot{x}(t) &= Ax(t), \\ y(t) &= Cx(t).\end{aligned}\tag{1-386}$$

Suppose that it is transformed according to Theorem 1.35 into the form

$$\begin{aligned}\dot{x}'(t) &= \begin{pmatrix} A'_{11} & 0 \\ A'_{21} & A'_{22} \end{pmatrix} x'(t), \\ y(t) &= (C'_1, 0)x'(t),\end{aligned}\tag{1-387}$$

where the pair $\{A'_{11}, C'_1\}$ is completely reconstructible. Then the system is detectable if and only if the matrix A'_{22} is asymptotically stable.

This theorem can be summarized by stating that a system is detectable if and only if its unreconstructible poles are stable. We prove the theorem as follows.

(a) *Detectability implies A'_{22} asymptotically stable*: Let us partition the transformed state variable as

$$x'(t) = \begin{pmatrix} x'_1(t) \\ x'_2(t) \end{pmatrix}, \quad 1-388$$

where the dimension m of $x'_1(t)$ is equal to the rank m of the reconstructibility matrix. The fact that the system is detectable implies that any initial state in the unreconstructible subspace gives a response that converges to zero. Any initial state in the unreconstructible subspace has in the transformed representation the form

$$x'(0) = \begin{pmatrix} 0 \\ x'_2(0) \end{pmatrix}. \quad 1-389$$

The response of the transformed state to this initial state is given by

$$x'(t) = \begin{pmatrix} 0 \\ e^{-A'_{22}t} x'_2(0) \end{pmatrix}. \quad 1-390$$

Since this must give a response that converges to zero, A'_{22} must be stable.

(b) *A'_{22} asymptotically stable implies detectability*: Any initial state $x(0)$ in the unreconstructible subspace must in the transformed representation have the form

$$x'(0) = \begin{pmatrix} 0 \\ x'_2(0) \end{pmatrix}. \quad 1-391$$

The response to this initial state is

$$x'(t) = \begin{pmatrix} 0 \\ e^{-A'_{22}t} x'_2(0) \end{pmatrix}. \quad 1-392$$

Since A'_{22} is stable, this response converges to zero, which shows that $x(0)$, which was assumed to be in the unreconstructible subspace, is also in the stable subspace. This implies that the system is detectable.

Example 1.26. *Inverted pendulum*

Consider the inverted pendulum in the transformed representation of Example 1.25. The matrix A'_{22} has the characteristic value 0, which implies that the system is not detectable. This means that if initially there is an uncertainty about the position of the carriage, the error made in guessing it will remain constant in time.

1.7.5* Reconstructibility of Time-Varying Linear Systems

The reconstructibility of time-varying linear systems can be ascertained by the following test.

Theorem 1.38. Consider the linear time-varying system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t).\end{aligned}\tag{1-393}$$

Define the nonnegative-definite matrix function

$$M(t, t_1) = \int_t^{t_1} \Phi^T(\tau, t) C^T(\tau) C(\tau) \Phi(\tau, t) d\tau,\tag{1-394}$$

where $\Phi(t, t_0)$ is the transition matrix of the system. Then the system is completely reconstructible if and only if for all t_1 there exists a t_0 with $-\infty < t_0 < t_1$ such that $M(t_0, t_1)$ is nonsingular.

For a proof we refer the reader to Bucy and Joseph (1968) and Kalman, Falb, and Arbib (1969). A stronger form of reconstructibility results by imposing further conditions on the matrix M (Kalman, 1960):

Definition 1.22. The time-varying system 1-393 is uniformly completely reconstructible if there exist positive constants σ , α_0 , α_1 , β_0 , and β_1 such that

$$(a) \quad \alpha_0 I \leq M(t_1 - \sigma, t_1) \leq \alpha_1 I \quad \text{for all } t_1;\tag{1-395}$$

$$(b) \quad \beta_0 I \leq \Phi^T(t_1 - \sigma, t_1) M(t_1 - \sigma, t_1) \Phi(t_1 - \sigma, t_1) \leq \alpha_1 I \quad \text{for all } t_1,\tag{1-396}$$

where $M(t, t_1)$ is the matrix function 1-394.

Uniform reconstructibility guarantees that identification of the state is always possible within roughly the same time. For time-invariant systems the following holds.

Theorem 1.39. The time-invariant linear system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t),\end{aligned}\tag{1-397}$$

is uniformly completely reconstructible if and only if it is completely reconstructible.

1.8* DUALITY OF LINEAR SYSTEMS

In the discussion of controllability and reconstructibility, we have seen that there is a striking symmetry between these properties. This symmetry can be made explicit by introducing the idea of duality (Kalman, 1960; Kalman, Falb, and Arbib, 1969).

Definition 1.23. Consider the linear time-varying system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t),\end{aligned}\tag{1-398}$$

and also the system

$$\begin{aligned}\dot{x}^*(t) &= A^T(t^* - t)x^*(t) + C^T(t^* - t)u^*(t), \\ y^*(t) &= B^T(t^* - t)x^*(t),\end{aligned}\tag{1-399}$$

where t^* is an arbitrary fixed time. Then 1-399 is called the *dual* of the system 1-398 with respect to the time t^* .

The purpose of introducing the dual system becomes apparent in Chapter 4 when we discuss the duality of linear optimal control problems and linear optimal observer problems. The following result is immediate.

Theorem 1.40. The dual of the system 1-399 with respect to the time t^* is the original system 1-398.

There is a close connection between the reconstructibility and controllability of a system and its dual.

Theorem 1.41. Consider the system 1-398 and its dual 1-399 where t^* is arbitrary.

- The system 1-398 is (uniformly) completely controllable if and only if its dual is (uniformly) completely reconstructible.
- The system 1-398 is (uniformly) completely reconstructible if and only if its dual is (uniformly) completely controllable.
- Assume that 1-398 is time-invariant. Then 1-398 is stabilizable if and only if its dual is detectable.
- Assume that 1-398 is time-invariant. Then 1-398 is detectable if and only if its dual is stabilizable.

We give the proof only for time-invariant systems. The reconstructibility matrix of the dual system is given by

$$Q^* = \begin{pmatrix} B^T \\ B^T(A^T) \\ \vdots \\ B^T(A^T)^{n-1} \end{pmatrix} = P^T,\tag{1-400}$$

where P is the controllability matrix of the original system. This immediately proves (a).

Part (b) of the theorem follows similarly. The controllability matrix of the dual system is given by

$$P^* = (C^T, A^T C^T, \dots, (A^T)^{n-1} C^T) = Q^T, \quad 1-401$$

where Q is the reconstructibility matrix of the original system. This implies the validity of (b).

Part (c) can be proved as follows. The original system can be transformed by a transformation $x' = T^{-1}x$ according to Theorem 1.26 (Section 1.6.3) into the controllability canonical form

$$\dot{x}'(t) = \begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix} x'(t) + \begin{pmatrix} B'_1 \\ 0 \end{pmatrix} u(t), \quad 1-402$$

$$y(t) = (C'_1, C'_2) x'(t).$$

If 1-398 is stabilizable, the pair $\{A'_{11}, B'_1\}$ is completely controllable and A'_{22} is stable. The dual of the transformed system is

$$\dot{x}'^*(t) = \begin{pmatrix} A'^T_{11} & 0 \\ A'^T_{12} & A'^T_{22} \end{pmatrix} x'^*(t) + \begin{pmatrix} C'^T_1 \\ C'^T_2 \end{pmatrix} u^*(t), \quad 1-403$$

$$y^*(t) = (B'^T_1, 0) x'^*(t).$$

Since $\{A'_{11}, B'_1\}$ is completely controllable, $\{A'^T_{11}, B'^T_1\}$ is completely reconstructible [part (a)]. Since A'_{22} is stable, A'^T_{22} is also stable. This implies that the system 1-403 is detectable. By the transformation $T^T x^* = x'^*$ (see Problem 1.8), the system 1-403 is transformed into the dual of the original system. Therefore, since 1-403 is detectable, the dual of the original system is also detectable. By reversing the steps of the proof, the converse of Theorem 1.41(c) can also be proved. Part (d) can be proved completely analogously. The proofs of (a) and (b) for the time-varying case are left as an exercise for the reader.

We conclude this section with the following fact, relating the stability of a system and its dual.

Theorem 1.42. *The system 1-398 is exponentially stable if and only if its dual 1-399 is exponentially stable.*

This result is easily proved by first verifying that if the system 1-398 has the transition matrix $\Phi(t, t_0)$ its dual 1-399 has the transition matrix $\Phi^T(t^* - t_0, t^* - t)$, and then verifying Definition 1.5 (Section 1.4.1).

1.9* PHASE-VARIABLE CANONICAL FORMS

For single-input time-invariant linear systems, it is sometimes convenient to employ the so-called phase-variable canonical form.

Definition 1.24. *A single-input time-invariant linear system is in phase-variable canonical form if its system equations have the form*

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & \cdots & -\alpha_{n-1} & \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix} \mu(t) \quad , \quad \mathbf{1-404}$$

$$y(t) = Cx(t).$$

Note that no special form is imposed upon the matrix C in this definition. It is not difficult to see that the numbers $\alpha_i, i = 0, \dots, n - 1$ are the coefficients of the characteristic polynomial

$$\sum_{i=0}^n \alpha_i s^i \quad \mathbf{1-405}$$

of the system, where $\alpha_n = 1$.

It is easily verified that the system **1-404** is always completely controllable. In fact, any completely controllable single-input system can be transformed into phase-variable canonical form.

Theorem 1.43. *Consider the completely controllable single-input time-invariant linear system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b\mu(t), \\ y(t) &= Cx(t), \end{aligned} \quad \mathbf{1-406}$$

where b is a column vector. Let P be the controllability matrix of the system,

$$P = (b, Ab, A^2b, \dots, A^{n-1}b), \quad \mathbf{1-407}$$

and let

$$\det (sI - A) = \sum_{i=0}^n \alpha_i s^i, \quad \mathbf{1-408}$$

where $\alpha_n = 1$, be the characteristic polynomial of the matrix A . Then the system is transformed into phase-variable canonical form by a transformation

$x(t) = Tx'(t)$. T is the nonsingular transformation matrix

$$T = PM,$$

where

$$M = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_2 & \alpha_3 & \cdots & \alpha_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-1} & \alpha_n & 0 & \cdots & 0 \\ \alpha_n & 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{1-409}$$

If the system 1-406 is not completely controllable, no such transformation exists.

This result can be proved as follows (Anderson and Luenberger, 1967). That the transformation matrix T is nonsingular is easily shown: P is nonsingular due to the assumption of complete controllability, and $\det(M) = \chi'(-1)^n$ because $\alpha_n = 1$. We now prove that T transforms the system into phase-variable canonical form. By postmultiplying P by M , it is easily seen that T can be written as

$$T = (t_1, t_2, \cdots, t_n), \tag{1-410}$$

where the column vectors t_i of T are given by

$$\begin{aligned} t_1 &= \alpha_1 b + \alpha_2 Ab + \alpha_3 A^2 b + \cdots + \alpha_n A^{n-1} b, \\ t_2 &= \alpha_2 b + \alpha_3 Ab + \cdots + \alpha_n A^{n-2} b, \\ &\dots \\ t_{n-1} &= \alpha_{n-1} b + \alpha_n Ab, \\ t_n &= \alpha_n b. \end{aligned} \tag{1-411}$$

It is seen from 1-411 that

$$At_i = t_{i-1} - \alpha_{i-1} t_n, \quad i = 2, 3, \cdots, n, \tag{1-412}$$

since $b = t_n$.

Now in terms of the new state variable, the state differential equation of the system is given by

$$\dot{x}'(t) = T^{-1}ATx'(t) + T^{-1}b\mu(t). \tag{1-413}$$

Let us consider the matrix $T^{-1}AT$. To this end denote the rows of T^{-1} by r_i , $i = 1, 2, \cdots, n$. Then for $i = 1, 2, \cdots, n$ and $j = 2, 3, \cdots, n$, the (i, j) -th entry of $T^{-1}AT$ is given by

$$(T^{-1}AT)_{ij} = r_i(At_j) = r_i(t_{j-1} - \alpha_{j-1} t_n) = \begin{cases} 1 & \text{if } i = j - 1, \\ -\alpha_{j-1} & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases} \tag{1-414}$$

This proves that the last $n - 1$ columns of $T^{-1}AT$ have the form as required in the phase-variable canonical form. To determine the first column, we observe from 1-411 that

$$At_1 = (\alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_n A^n)b = -\alpha_0 b = -\alpha_0 t_n, \quad \mathbf{1-415}$$

since according to the Cayley-Hamilton theorem

$$\alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_n A^n = 0. \quad \mathbf{1-416}$$

Thus we have for $i = 1, 2, \dots, n$,

$$(T^{-1}AT)_{i1} = r_i(At_1) = -\alpha_0 r_i t_n = \begin{cases} -\alpha_0 & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases} \quad \mathbf{1-417}$$

Similarly, we can show that $T^{-1}b$ is in the form required, which terminates the proof of the first part of Theorem 1.43. The last statement of Theorem 1.43 is easily verified: if the system 1-406 is not completely controllable, no nonsingular transformation can bring the system into phase-variable canonical form, since nonsingular transformations preserve controllability properties (see Problem 1.6). An alternate method of finding the phase-variable canonical form is given by Ramaswami and Ramar (1968). Computational rules are described by Tuel (1966), Rane (1966), and Johnson and Wonham (1966).

For single-input systems represented in phase-variable canonical form, certain linear optimal control problems are much easier to solve than if the system is given in its general form (see, e.g., Section 3.2). Similarly, certain filtering problems involving the reconstruction of the state from observations of the output variable are more easily solved when the system is in the dual phase-variable canonical form.

Definition 1.25. *A single-output linear time-invariant system is in dual phase-variable canonical form if it is represented as follows:*

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & 0 & \cdots & 0 & -\alpha_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & \cdots & 0 & 1 & -\alpha_{n-1} \end{pmatrix} x(t) + Bu(t), \\ \eta(t) &= (0 \ 0 \ 0 \ \cdots \ 0 \ 1)x(t). \end{aligned} \quad \mathbf{1-418}$$

It is noted that the definition imposes no special form on the matrix B . By "dualizing" Theorem 1.43, it is not difficult to establish a transformation to transform completely reconstructible systems into dual canonical form.

Related canonical forms can be derived for multiinput and multioutput systems (Anderson and Luenberger, 1967; Luenberger, 1967; Johnson, 1971a; Wolovich and Falb, 1969).

1.10 VECTOR STOCHASTIC PROCESSES

1.10.1 Definitions

In later chapters of this book we use stochastic processes as mathematical models for disturbances and noise phenomena. Often several disturbances and noise phenomena simultaneously influence a given system. This makes it necessary to introduce vector-valued stochastic processes, which constitute the topic of this section.

A stochastic process can be thought of as a family of time functions. Each time function we call a *realization* of the process. Suppose that $v_1(t), v_2(t), \dots, v_n(t)$ are n scalar stochastic processes which are possibly mutually dependent. Then we call

$$v(t) = \text{col } [v_1(t), v_2(t), \dots, v_n(t)] \tag{1-419}$$

a *vector stochastic process*. We always assume that each of the components of $v(t)$ takes real values, and that $t \geq t_0$, with t_0 given.

A stochastic process can be characterized by specifying the joint probability distributions

$$P\{v(t_1) \leq v_1, v(t_2) \leq v_2, \dots, v(t_m) \leq v_m\} \tag{1-420}$$

for all real v_1, v_2, \dots, v_m , for all $t_1, t_2, \dots, t_m \geq t_0$ and for every natural number m . Here the vector inequality $v(t_i) \leq v_i$ is by definition satisfied if the inequalities

$$v_j(t_i) \leq v_{ij}, \quad j = 1, 2, \dots, n, \tag{1-421}$$

are simultaneously satisfied. The v_{ij} are the components of v_i , that is, $v_i = \text{col } (v_{i1}, v_{i2}, \dots, v_{in})$.

A special class of stochastic processes consists of those processes the statistical properties of which do not change with time. We define more precisely.

Definition 1.26. A stochastic process $v(t)$ is *stationary* if

$$\begin{aligned} P\{v(t_1) \leq v_1, \dots, v(t_m) \leq v_m\} \\ = P\{v(t_1 + \theta) \leq v_1, \dots, v(t_m + \theta) \leq v_m\} \end{aligned} \tag{1-422}$$

for all t_1, t_2, \dots, t_m , for all v_1, \dots, v_m , for every natural number m , and for all θ .

The joint probability distributions that characterize a stationary stochastic process are thus invariant with respect to a shift in the time origin.

Here the notation $M \geq 0$, where M is a square symmetric real matrix, means that M is nonnegative-definite, that is,

$$x^T M x \geq 0 \quad \text{for all real } x. \quad \text{1-432}$$

The theorem is easily proved from the definitions of $R_v(t_1, t_2)$ and $C_v(t_1, t_2)$. Since the second-order properties of the stochastic process are equally well characterized by the covariance matrix as by the joint moment matrix, we usually consider only the covariance matrix.

For stationary processes we have the following result.

Theorem 1.45. *Suppose that $v(t)$ is a stationary stochastic process. Then its mean $m(t)$ is constant and its covariance matrix $R_v(t_1, t_2)$ depends on $t_1 - t_2$ only.*

This is easily shown from the definition of stationarity.

It sometimes happens that a stochastic process has a constant mean and a covariance matrix that depends on $t_1 - t_2$ only, while its other statistical properties are not those of a stationary process. Since frequently we are interested only in the first- and second-order properties of a stochastic process, we introduce the following notion.

Definition 1.28. *The stochastic process $v(t)$ is called wide-sense stationary if its second-order moment matrix $C_v(t, t)$ is finite for all t , its mean $m(t)$ is constant, and its covariance matrix $R_v(t_1, t_2)$ depends on $t_1 - t_2$ only.*

Obviously, any stationary process with finite second-order moment matrix is also wide-sense stationary.

Let $v_1(t)$ and $v_2(t)$ be two vector stochastic processes. Then v_1 and v_2 are called *independent* processes if $\{v_1(t_1), v_1(t_2), \dots, v_1(t_l)\}$ and $\{v_2(t'_1), v_2(t'_2), \dots, v_2(t'_m)\}$ are independent sets of stochastic variables for all $t_1, t_2, \dots, t_l, t'_1, t'_2, \dots, t'_m \geq t_0$ and for all natural numbers m and l . Furthermore, v_1 and v_2 are called *uncorrelated* stochastic processes if $v_1(t_1)$ and $v_2(t_2)$ are uncorrelated vector stochastic variables for all $t_1, t_2 \geq t_0$, that is,

$$E\{[v_1(t_1) - m_1(t_1)][v_2(t_2) - m_2(t_2)]^T\} = 0$$

for all t_1 and t_2 , where m_1 is the mean of v_1 and m_2 that of v_2 .

Example 1.27. *Gaussian stochastic process*

A Gaussian stochastic process v is a stochastic process where for each set of instants of time $t_1, t_2, \dots, t_m \geq t_0$ the n -dimensional vector stochastic variables $v(t_1), v(t_2), \dots, v(t_m)$ have a Gaussian joint probability distribution.

2. For any sequence of instants $t_1, t_2, t_3,$ and $t_4,$ with $t_0 \leq t_1 \leq t_2 \leq t_3 \leq t_4,$ the increments $v(t_2) - v(t_1)$ and $v(t_4) - v(t_3)$ have zero means and are uncorrelated, that is,

$$\begin{aligned} E\{v(t_2) - v(t_1)\} &= E\{v(t_4) - v(t_3)\} = 0, \\ E\{[v(t_2) - v(t_1)][v(t_4) - v(t_3)]^T\} &= 0. \end{aligned} \tag{1-438}$$

The mean of such a process is easily determined:

$$\begin{aligned} m(t) &= E\{v(t)\} = E\{v(t) - v(t_0)\} \\ &= 0, \quad t \geq t_0. \end{aligned} \tag{1-439}$$

Suppose for the moment that $t_2 \geq t_1.$ Then we have for the covariance matrix

$$\begin{aligned} R_v(t_1, t_2) &= E\{v(t_1)v^T(t_2)\} \\ &= E\{[v(t_1) - v(t_0)][v(t_2) - v(t_1) + v(t_1) - v(t_0)]^T\} \\ &= E\{[v(t_1) - v(t_0)][v(t_1) - v(t_0)]^T\} \\ &= E\{v(t_1)v^T(t_1)\} \\ &= Q(t_1), \quad t_2 \geq t_1 \geq t_0, \end{aligned} \tag{1-440}$$

where

$$Q(t) = E\{v(t)v^T(t)\} \tag{1-441}$$

is the variance matrix of the process. Similarly,

$$R_v(t_1, t_2) = Q(t_2) \quad \text{for } t_1 \geq t_2 \geq t_0. \tag{1-442}$$

Clearly, a process with uncorrelated increments cannot be stationary or wide-sense stationary, except in the trivial case in which $Q(t) = 0, t \geq t_0.$

Let us now consider the variance matrix of the process. We can write for $t_2 \geq t_1 \geq t_0:$

$$\begin{aligned} Q(t_2) &= E\{v(t_2)v^T(t_2)\} \\ &= E\{[v(t_2) - v(t_1) + v(t_1) - v(t_0)][v(t_2) - v(t_1) + v(t_1) - v(t_0)]^T\} \\ &= E\{[v(t_2) - v(t_1)][v(t_2) - v(t_1)]^T\} + Q(t_1). \end{aligned} \tag{1-443}$$

Obviously, $Q(t)$ is a monotonically nondecreasing matrix function of t in the sense that

$$Q(t_2) \geq Q(t_1) \quad \text{for all } t_2 \geq t_1 \geq t_0. \tag{1-444}$$

Here, if A and B are two symmetric real matrices, the notation

$$A \geq B \tag{1-445}$$

implies that the matrix $A - B$ is nonnegative-definite. Let us now assume that

the matrix function $Q(t)$ is absolutely continuous, that is, we can write

$$Q(t) = \int_{t_0}^t V(\tau) d\tau, \quad 1-446$$

where $V(t)$ is a nonnegative-definite symmetric matrix function. It then follows from 1-443 that the variance matrix of the increment $v(t_2) - v(t_1)$ is given by

$$\begin{aligned} E\{[v(t_2) - v(t_1)][v(t_2) - v(t_1)]^T\} &= Q(t_2) - Q(t_1) \\ &= \int_{t_1}^{t_2} V(\tau) d\tau. \end{aligned} \quad 1-447$$

Combining 1-440 and 1-442, we see that if 1-446 holds the covariance matrix of the process can be expressed as

$$R_v(t_1, t_2) = \int_{t_0}^{\min(t_1, t_2)} V(\tau) d\tau. \quad 1-448$$

One of the best-known processes with uncorrelated increments is the *Brownian motion process*, also known as the *Wiener process* or the *Wiener-Lévy process*. This is a process with uncorrelated increments where each of the increments $v(t_2) - v(t_1)$ is a Gaussian stochastic vector with zero mean and variance matrix $(t_2 - t_1)I$, where I is the unit matrix. A generalization of this process is obtained when it is assumed that each increment $v(t_2) - v(t_1)$ is a Gaussian stochastic vector with zero mean and variance matrix given in the form 1-447. Since in the Brownian motion process the increments are uncorrelated and Gaussian, they are independent. Obviously, Brownian motion is a Gaussian process. It is an important tool in the theory of stochastic processes.

1.10.2 Power Spectral Density Matrices

For scalar wide-sense stationary stochastic processes, the power spectral density function is defined as the Fourier transform of the covariance function. Similarly, we define for vector stochastic processes:

Definition 1.29. *The power spectral density matrix $\Sigma_v(\omega)$ of a wide-sense stationary vector stochastic process is defined as the Fourier transform, if it exists, of the covariance matrix $R_v(t_1 - t_2)$ of the process, that is,*

$$\Sigma_v(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} R_v(\tau) d\tau. \quad 1-449$$

Note that we have allowed a slight inconsistency in the notation of the covariance matrix by replacing the two variables t_1 and t_2 by the single variable $t_1 - t_2$. The power spectral density matrix has the following properties.

Theorem 1.46. *Suppose that $\Sigma_v(\omega)$ is the spectral density matrix of a wide-sense stationary process $v(t)$. Then $\Sigma_v(\omega)$ is a complex matrix that has the properties:*

$$(a) \Sigma_v(-\omega) = \Sigma_v^T(\omega) \quad \text{for all } \omega; \quad 1-450$$

$$(b) \Sigma_v^*(\omega) = \Sigma_v(\omega) \quad \text{for all } \omega; \quad 1-451$$

$$(c) \Sigma_v(\omega) \geq 0 \quad \text{for all } \omega. \quad 1-452$$

Here the asterisk denotes the complex conjugate transpose, while $M \geq 0$, where M is a complex matrix, indicates that M is a nonnegative-definite matrix, that is, $x^*Mx \geq 0$ for all complex x .

The proofs of parts (a) and (b) follow in a straightforward manner from the definition of $\Sigma_v(\omega)$ and Theorem 1.44. In order to prove part (c), one can extend the proof given by Davenport and Root (1958, Chapter 6) to the vector case. The reason for the term power spectral density matrix becomes apparent in Section 1.10.4.

Example 1.30. *Exponentially correlated noise*

In Example 1.28 we considered exponentially correlated noise, a scalar wide-sense stationary process $v(t)$ with covariance function

$$R_v(t_1 - t_2) = \sigma^2 \exp\left(-\frac{|t_1 - t_2|}{\theta}\right). \quad 1-453$$

By Fourier transformation it easily follows that $v(t)$ has the power spectral density function

$$\Sigma_v(\omega) = \frac{2\sigma^2\theta}{1 + \omega^2\theta^2}, \quad 1-454$$

provided $\theta > 0$.

1.10.3 The Response of Linear Systems to Stochastic Inputs

In this section we study the statistical properties of the response of a linear system if the input is a realization of a stochastic process. We have the following result.

Theorem 1.47. *Consider a linear system with impulse response matrix $K(t, \tau)$ which at time t_0 is in the zero state. Suppose that the input to the system is a realization of a stochastic process $u(t)$ with mean $m_u(t)$ and covariance matrix $R_u(t_1, t_2)$. Then the output of the system is a realization of a stochastic process $y(t)$ with mean*

$$m_y(t) = \int_{t_0}^t K(t, \tau) m_u(\tau) d\tau, \quad 1-455$$

and covariance matrix

$$R_y(t_1, t_2) = \int_{t_0}^{t_1} d\tau_1 \int_{t_0}^{t_2} K(t_1, \tau_1) R_u(\tau_1, \tau_2) K^T(t_2, \tau_2) d\tau_2, \quad \mathbf{1-456}$$

provided the integrals exist.

We present a formal proof of these results. The output y , which is a stochastic process, is given by

$$y(t) = \int_{t_0}^t K(t, \tau) u(\tau) d\tau. \quad \mathbf{1-457}$$

Taking the expectation of both sides of **1-457**, interchanging the order of the integration and the expectation, one obtains **1-455**.

Similarly, we can write (assuming for simplicity that $m_u(t) = 0$)

$$\begin{aligned} R_y(t_1, t_2) &= E\{y(t_1)y^T(t_2)\} \\ &= E\left\{\left[\int_{t_0}^{t_1} K(t_1, \tau_1)u(\tau_1) d\tau_1\right]\left[\int_{t_0}^{t_2} K(t_2, \tau_2)u(\tau_2) d\tau_2\right]^T\right\} \\ &= E\left\{\int_{t_0}^{t_1} d\tau_1 \int_{t_0}^{t_2} d\tau_2 K(t_1, \tau_1)u(\tau_1)u^T(\tau_2)K^T(t_2, \tau_2)\right\} \\ &= \int_{t_0}^{t_1} d\tau_1 \int_{t_0}^{t_2} d\tau_2 K(t_1, \tau_1)E\{u(\tau_1)u^T(\tau_2)\}K^T(t_2, \tau_2) \\ &= \int_{t_0}^{t_1} d\tau_1 \int_{t_0}^{t_2} d\tau_2 K(t_1, \tau_1)R_u(\tau_1, \tau_2)K^T(t_2, \tau_2). \end{aligned} \quad \mathbf{1-458}$$

For a time-invariant system and a wide-sense stationary input process, we have the following result.

Theorem 1.48. *Suppose that the linear system of Theorem 1.47 is an asymptotically stable time-invariant system with impulse response matrix $K(t - \tau)$, and that the input stochastic process $u(t)$ is wide-sense stationary with covariance matrix $R_u(t_1 - t_2)$. Then if the input to the system is a realization of the process $u(t)$, which is applied from time $-\infty$ on, the output is a realization of a wide-sense stationary stochastic process $y(t)$ with covariance matrix*

$$R_y(t_1 - t_2) = \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 K(\tau_1)R_u(t_1 - t_2 + \tau_2 - \tau_1)K^T(\tau_2). \quad \mathbf{1-459}$$

Note that we have introduced a slight inconsistency in the notation of the impulse response matrix K and the covariance matrix R_u . It is in Section 1.3.2 that we saw that the impulse response matrix of a time-invariant system

depends on $t - \tau$ only. The result 1-459 can be found from 1-456 by letting $t_0 \rightarrow -\infty$ and making some simple substitutions.

For wide-sense stationary processes, it is of interest to consider the power density matrix.

Theorem 1.49. *Consider an asymptotically stable time-invariant linear system with transfer matrix $H(s)$. Suppose that the input is a realization of a wide-sense stationary stochastic process $u(t)$ with power spectral density matrix $\Sigma_u(\omega)$ which is applied from time $-\infty$ on. Then the output is a realization of a wide-sense stationary stochastic process $y(t)$ with power spectral density matrix*

$$\Sigma_y(\omega) = H(j\omega)\Sigma_u(\omega)H^T(-j\omega). \tag{1-460}$$

This result follows easily by Fourier transforming 1-459 after replacing $t_1 - t_2$ with a variable τ , using the fact that $H(s)$ is the Laplace transform of $K(\tau)$.

Example 1.31. *Stirred tank*

Consider the stirred tank of Example 1.2 (Section 1.2.3) and assume that fluctuations occur in the concentrations c_1 and c_2 of the feeds. Let us therefore write

$$\begin{aligned} c_1(t) &= c_{10} + v_1(t), \\ c_2(t) &= c_{20} + v_2(t), \end{aligned} \tag{1-461}$$

where c_{10} and c_{20} are the average concentrations and $v_1(t)$ and $v_2(t)$ fluctuations about the average. It is not difficult to show that the linearized system equations must be modified to the following:

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -\frac{1}{2\theta} & 0 \\ 0 & -\frac{1}{\theta} \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ \frac{c_{10} - c_0}{V_0} & \frac{c_{20} - c_0}{V_0} \end{pmatrix} u(t) \\ &+ \begin{pmatrix} 0 & 0 \\ \frac{F_{10}}{V_0} & \frac{F_{20}}{V_0} \end{pmatrix} \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}, \\ y(t) &= \begin{pmatrix} \frac{1}{2\theta} & 0 \\ 0 & 1 \end{pmatrix} x(t). \end{aligned} \tag{1-462}$$

If we take the input $u(t) \equiv 0$, the transfer matrix from the disturbances

$v(t) = \text{col } [v_1(t), v_2(t)]$ to the output variable $y(t)$ can be found to be

$$\begin{pmatrix} 0 & 0 \\ \frac{F_{10}/V_0}{s + \frac{1}{\theta}} & \frac{F_{20}/V_0}{s + \frac{1}{\theta}} \end{pmatrix}. \quad 1-463$$

Obviously, the disturbances affect only the second component of the output variable $\eta_2(t) = \xi_2(t)$. Let us assume that $v_1(t)$ and $v_2(t)$ are two independent exponentially correlated noise processes, so that we can write for the covariance matrix of $v(t)$

$$R_v(t_1 - t_2) = \begin{pmatrix} \sigma_1^2 \exp\left(-\frac{|t_1 - t_2|}{\theta_1}\right) & 0 \\ 0 & \sigma_2^2 \exp\left(-\frac{|t_1 - t_2|}{\theta_2}\right) \end{pmatrix}. \quad 1-464$$

With this we find for the power spectral density matrix of $v(t)$

$$\Sigma_v(\omega) = \begin{pmatrix} \frac{2\sigma_1^2\theta_1}{1 + \omega^2\theta_1^2} & 0 \\ 0 & \frac{2\sigma_2^2\theta_2}{1 + \omega^2\theta_2^2} \end{pmatrix}. \quad 1-465$$

It follows from 1-460 for the power spectral density matrix of the contribution of the disturbances $v(t)$ to the output variable $y(t)$

$$\Sigma_y(\omega) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\omega^2 + \frac{1}{\theta^2}} \left[\frac{(F_{10}/V_0)^2 2\sigma_1^2\theta_1}{1 + \omega^2\theta_1^2} + \frac{(F_{20}/V_0)^2 2\sigma_2^2\theta_2}{1 + \omega^2\theta_2^2} \right] \end{pmatrix}. \quad 1-466$$

1.10.4 Quadratic Expressions

In later chapters of this book it will be convenient to use a measure for the mean square value of a stochastic process. For vector stochastic processes we introduce to this end quadratic expressions of the form

$$E\{v^T(t)W(t)v(t)\}, \quad 1-467$$

where $W(t)$ is a symmetric weighting matrix. If $v(t) = \text{col } [v_1(t), \dots, v_n(t)]$

and W has elements W_{ij} , $i, j = 1, 2, \dots, n$, 1-467 can be written as

$$E\{v^T(t)W(t)v(t)\} = E\left\{\sum_{i=1}^n \sum_{j=1}^n v_i(t)W_{ij}(t)v_j(t)\right\}, \quad 1-468$$

which is the expectation of a quadratic expression in the components $v_i(t)$ of $v(t)$. Usually, $W(t)$ is chosen to be nonnegative-definite so that the expression assumes nonnegative values only.

It is helpful to develop expressions for quadratic expressions of this type in terms of the covariance matrix and power spectral density matrix of $v(t)$. We have the following result.

Theorem 1.50. *Let $v(t)$ be a vector-valued stochastic process. Then if $W(t)$ is a symmetric matrix,*

$$E\{v^T(t)W(t)v(t)\} = \text{tr} [W(t)C_v(t, t)], \quad 1-469$$

where $C_v(t_1, t_2)$ is the second-order joint moment matrix of $v(t)$. If $v(t)$ is wide-sense stationary with zero mean and covariance matrix $R_v(t_1 - t_2)$, and W is constant,

$$E\{v^T(t)Wv(t)\} = \text{tr} [WR_v(0)]. \quad 1-470$$

If $v(t)$ has zero mean and the power spectral density matrix $\Sigma_v(\omega)$,

$$E\{v^T(t)Wv(t)\} = \text{tr} \left[\int_{-\infty}^{\infty} W\Sigma_v(\omega) df \right], \quad 1-471$$

where

$$f = \omega/2\pi. \quad 1-472$$

Furthermore,

$$R_v(0) = \int_{-\infty}^{\infty} \Sigma_v(\omega) df. \quad 1-473$$

By $\text{tr}(A)$ we mean the trace of the matrix A , that is,

$$\text{tr}(A) = \sum_{i=1}^n \alpha_{ii}, \quad 1-474$$

where α_{ii} , $i = 1, \dots, n$ are the diagonal elements of the matrix. The first result of the theorem follows in an elementary manner:

$$\begin{aligned} E\{v^T(t)W(t)v(t)\} &= E\left\{\sum_{i,j=1}^n v_i(t)W_{ij}(t)v_j(t)\right\} \\ &= \sum_{i,j=1}^n W_{ij}(t)E\{v_i(t)v_j(t)\} \\ &= \sum_{i,j=1}^n W_{ij}(t)C_{v,ij}(t, t) \\ &= \text{tr} [W(t)C_v(t, t)], \end{aligned} \quad 1-475$$

where $C_{v,ij}(t, t)$ is the (i, j) -th element of $C_v(t, t)$. The second result, 1-470, is immediate since under the assumptions stated $C_v(t, t) = R_v(0)$. The third result can be shown by recalling that the power spectral density matrix $\Sigma_v(\omega)$ is the Fourier transform of $R_v(\tau)$, and that consequently $R_v(\tau)$ is the inverse transform of $\Sigma_v(\omega)$:

$$R_v(\tau) = \int_{-\infty}^{\infty} \Sigma_v(\omega) e^{j\omega\tau} df. \quad 1-476$$

For $\tau = 0$ we immediately obtain 1-471 and 1-473.

Equation 1-471 gives an interpretation of the term power spectral density matrix. Apparently, the total "power" $E\{v^T(t)Wv(t)\}$ of a zero-mean wide-sense stationary process $v(t)$ is obtained by integrating $\text{tr}[W\Sigma_v(\omega)]$ over all frequencies. Thus $\text{tr}[W\Sigma_v(\omega)]$ can be considered as a measure for the power "density" at the frequency ω . The weighting matrix W determines the contributions of the various components of $v(t)$ to the power.

Example 1.32. Stirred tank

We continue Example 1.31 where we computed the spectral density matrix of the output $y(t)$ due to disturbances $v(t)$ in the concentrations of the feeds of the stirred tank. Suppose we want to compute the mean square value of the fluctuations $\eta_2(t)$ in the concentration of the outgoing flow. This mean square value can be written as

$$E\{\eta_2^2(t)\} = E\{y^T(t)W y(t)\}, \quad 1-477$$

where the weighting matrix W has the simple form

$$W = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad 1-478$$

Thus we find for the mean square error

$$\begin{aligned} E\{y^T(t)W y(t)\} &= \int_{-\infty}^{\infty} \text{tr}[W\Sigma_v(\omega)] df \\ &= \int_{-\infty}^{\infty} \frac{1}{\omega^2 + \frac{1}{\theta^2}} \left[\frac{(F_{10}/V_0)^2 2\sigma_1^2 \theta_1}{1 + \omega^2 \theta_1^2} + \frac{(F_{20}/V_0)^2 2\sigma_2^2 \theta_2}{1 + \omega^2 \theta_2^2} \right] df \\ &= \frac{(F_{10}/V_0)^2 \sigma_1^2 \theta_1 \theta^2}{\theta + \theta_1} + \frac{(F_{20}/V_0)^2 \sigma_2^2 \theta_2 \theta^2}{\theta + \theta_2}. \end{aligned} \quad 1-479$$

Integrals of rational functions of the type appearing in 1-479 frequently occur in the computation of quadratic expressions as considered in this section. Tables of such integrals can be found in Newton, Gould, and Kaiser (1957, Appendix E) and Seifert and Steeg (1960, Appendix).

1.11 THE RESPONSE OF LINEAR DIFFERENTIAL SYSTEMS TO WHITE NOISE

1.11.1 White Noise

One frequently encounters in practice zero-mean scalar stochastic processes w with the property that $w(t_1)$ and $w(t_2)$ are uncorrelated even for values of $|t_2 - t_1|$ that are quite small, that is,

$$R_w(t_2, t_1) \simeq 0 \quad \text{for } |t_2 - t_1| > \varepsilon, \quad 1-480$$

where ε is a "small" number. The covariance function of such stochastic processes can be idealized as follows.

$$R_w(t_2, t_1) = V(t_1) \delta(t_2 - t_1), \quad V(t_1) \geq 0. \quad 1-481$$

Here $\delta(t_2 - t_1)$ is a delta function and $V(t)$ is referred to as the *intensity* of the process at time t . Such processes are called *white noise processes* for reasons explained later. We can of course extend the notion of a white noise process to vector-valued processes:

Definition 1.30. Let $w(t)$ be a zero mean vector-valued stochastic process with covariance matrix

$$R_w(t_2, t_1) = V(t_1) \delta(t_2 - t_1), \quad 1-482$$

where $V(t) \geq 0$. The process $w(t)$ is then said to be a *white noise stochastic process with intensity* $V(t)$.

In the case in which the intensity of the white noise process is constant, the process is wide-sense stationary and we can introduce its power spectral density matrix. Formally, taking the Fourier transform of $V\delta(\tau)$, we see that wide-sense stationary white noise has the power spectral density matrix

$$\Sigma_w(\omega) = V. \quad 1-483$$

This shows that a wide-sense stationary white noise process has equal power density at *all* frequencies. This is why, in analogy with light, such processes are called white noise processes. This result also agrees with our physical intuition. A process with little correlation between two nearby values $w(t_1)$ and $w(t_2)$ is very irregular and thus contains power at quite high frequencies.

Unfortunately, when one computes the total power of a white noise process using Eq. 1-470 or 1-471, one obtains an infinite value, which immediately points out that although white noise processes may be convenient to work with, they do not exist in the physical world. Also, from a strict mathematical viewpoint, white noise processes are not really well-defined. As we shall see

in Example 1.33, white noise is the “derivative” of a process with uncorrelated increments; however, such a process can be shown to have no derivative. Once the white noise has passed at least one integration, however, we are again on a firm mathematical ground and the following integration rules, which are needed extensively, can be proved.

Theorem 1.51. *Let $w(t)$ be a vector-valued white noise process with intensity $V(t)$. Also, let $A_1(t)$, $A_2(t)$, and $A(t)$ be given time-varying matrices. Then*

$$(a) \quad E \left\{ \int_{t_1}^{t_2} A(t) w(t) dt \right\} = 0; \quad \mathbf{1-484}$$

$$(b) \quad E \left\{ \left[\int_{t_1}^{t_2} A_1(t) w(t) dt \right]^T W \left[\int_{t_3}^{t_4} A_2(t') w(t') dt' \right] \right\} \\ = \int_I \text{tr} [V(t) A_1^T(t) W A_2(t)] dt, \quad \mathbf{1-485}$$

where I is the intersection of $[t_1, t_2]$ and $[t_3, t_4]$ and W is any weighting matrix;

$$(c) \quad E \left\{ \left[\int_{t_1}^{t_2} A_1(t) w(t) dt \right] \left[\int_{t_3}^{t_4} A_2(t') w(t') dt' \right]^T \right\} \\ = \int_I A_1(t) V(t) A_2^T(t) dt, \quad \mathbf{1-486}$$

where I is as defined before.

Formally, one can prove (a) by using the fact that $w(t)$ is a zero-mean process, while (b) can be made plausible as follows.

$$E \left\{ \left[\int_{t_1}^{t_2} A_1(t) w(t) dt \right]^T W \left[\int_{t_3}^{t_4} A_2(t') w(t') dt' \right] \right\}$$

$$= E \left\{ \int_{t_1}^{t_2} dt \int_{t_3}^{t_4} dt' w^T(t) A_1^T(t) W A_2(t') w(t') \right\} \quad \mathbf{1-487a}$$

$$= E \left\{ \int_{t_1}^{t_2} dt \int_{t_3}^{t_4} \text{tr} [w(t') w^T(t) A_1^T(t) W A_2(t')] dt' \right\} \quad \mathbf{1-487b}$$

$$= \int_{t_1}^{t_2} dt \int_{t_3}^{t_4} \text{tr} [E\{w(t') w^T(t)\} A_1^T(t) W A_2(t')] dt' \quad \mathbf{1-487c}$$

$$= \int_{t_1}^{t_2} dt \int_{t_3}^{t_4} \text{tr} [V(t) A_1^T(t) W A_2(t')] \delta(t - t') dt' \quad \mathbf{1-487d}$$

$$= \int_I \text{tr} [V(t) A_1^T(t) W A_2(t)] dt. \quad \mathbf{1-487e}$$

The transition from 1-487c to 1-487d uses 1-482, and the transition from 1-487d to 1-487e follows from the properties of the delta function. We have

also used the fact that $\text{tr}(AB) = \text{tr}(BA)$ for any two matrices A and B of compatible dimensions.

The proof of (c) is similar to that of (b).

Example 1.33. *White noise as the derivative of a process with uncorrelated increments*

In Example 1.29 (Section 1.10.1) we considered processes $v(t)$, $t \geq t_0$, with uncorrelated increments, which we showed to be processes with zero means and covariance matrices of the form

$$R_v(t_1, t_2) = \begin{cases} Q(t_1) & \text{for } t_2 \geq t_1 \geq t_0, \\ Q(t_2) & \text{for } t_1 \geq t_2 \geq t_0, \end{cases} \quad 1-488$$

Proceeding completely formally, let us show that the covariance matrix of the derivative process

$$\dot{v}(t) = \frac{dv(t)}{dt}, \quad t \geq t_0, \quad 1-489$$

consists of a delta function. For the mean of the derivative process, we have

$$E\{\dot{v}(t)\} = \frac{d}{dt} E\{v(t)\} = 0, \quad t \geq t_0. \quad 1-490$$

For the covariance matrix of the derivative process we write, completely formally,

$$\begin{aligned} R_{\dot{v}}(t_1, t_2) &= E\{\dot{v}(t_1)\dot{v}^T(t_2)\} \\ &= \frac{\partial^2}{\partial t_1 \partial t_2} E\{v(t_1)v^T(t_2)\} \\ &= \frac{\partial^2}{\partial t_1 \partial t_2} R_v(t_1, t_2), \quad t_1, t_2 \geq t_0. \end{aligned} \quad 1-491$$

Now, successively carrying out the partial differentiations, we obtain

$$R_{\dot{v}}(t_1, t_2) = \dot{Q}(t_1) \delta(t_1 - t_2), \quad t_1, t_2 \geq t_0, \quad 1-492$$

where

$$\dot{Q}(t) = \frac{dQ(t)}{dt}. \quad 1-493$$

This shows that the derivative of a process with uncorrelated increments is a white noise process. When each increment $v(t_2) - v(t_1)$ of the process has a variance matrix that may be written in the form

$$\int_{t_1}^{t_2} V(t) dt, \quad 1-494$$

the intensity of the white noise process that derives from the process with uncorrelated increments is $V(t)$, since (see Example 1.29)

$$Q(t) = \int_{t_0}^t V(\tau) d\tau. \quad 1-495$$

A special case that is of considerable interest occurs when the process $v(t)$ from which the white noise process derives is Brownian motion (see Example 1.29). The white noise process then obtained is often referred to as *Gaussian white noise*.

In the rigorous theory of white noise, the white noise process is never defined. Instead, the theory is developed in terms of increments of processes with uncorrelated increments. In particular, integrals of the type appearing in Theorem 1.51 are redefined in terms of such processes. Let us consider the integral

$$\int_{t_1}^{t_2} A(t)w(t) dt. \quad 1-496$$

This is replaced with

$$\int_{t_1}^{t_2} A(t) dv(t) = \lim_{\varepsilon \rightarrow 0} \sum_0^{n-1} A(\tau_i)[v(\tau_{i+1}) - v(\tau_i)], \quad 1-497$$

where $v(t)$ is the process with uncorrelated increments from which the white noise process $w(t)$ derives and where $t_1 = \tau_0 < \tau_1 < \dots < \tau_n = t_2$, with

$$\varepsilon = \max_i |\tau_{i+1} - \tau_i|, \quad 1-498$$

is a partitioning of the interval $[t_1, t_2]$. The limit in 1-497 can be so defined that it is a proper stochastic variable, satisfying the properties of Theorem 1.51. For detailed treatments we refer the reader to Doob (1953), Gikhman and Skorokhod (1969), Åström (1970), and Kushner (1971). For an extensive and rigorous discussion of white noise, one should consult Hida (1970).

The material in this example is offered only for background. For our purposes, in the context of linear systems, it is sufficient to have Theorem 1.51 available.

1.11.2 Linear Differential Systems Driven by White Noise

It will turn out that a linear differential system driven by white noise is a very convenient model for formulating and solving linear control problems that involve disturbances and noise. In this section we obtain some of the statistical properties of the state of a linear differential system with a white noise process as input. In particular, we compute the mean, the covariance, joint moment, variance, and moment matrices of the state x .

Theorem 1.52. Suppose that $x(t)$ is the solution of

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)w(t), \\ x(t_0) &= x_0, \end{aligned} \tag{1-499}$$

where $w(t)$ is white noise with intensity $V(t)$ and x_0 is a stochastic variable independent of $w(t)$, with mean m_0 and $Q_0 = E\{(x_0 - m_0)(x_0 - m_0)^T\}$ as its variance matrix. Then $x(t)$ has mean

$$m_x(t) = \Phi(t, t_0)m_0, \tag{1-500}$$

where $\Phi(t, t_0)$ is the transition matrix of the system 1-499. The covariance matrix of $x(t)$ is

$$\begin{aligned} R_x(t_1, t_2) &= \Phi(t_1, t_0)Q_0\Phi^T(t_2, t_0) \\ &\quad + \int_{t_0}^{\min(t_1, t_2)} \Phi(t_1, \tau)B(\tau)V(\tau)B^T(\tau)\Phi^T(t_2, \tau) d\tau. \end{aligned} \tag{1-501}$$

The variance matrix $Q(t) = R_x(t, t)$ satisfies the matrix differential equation

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A^T(t) + B(t)V(t)B^T(t), \tag{1-502}$$

$$Q(t_0) = Q_0.$$

Furthermore,

$$R_x(t_1, t_2) = \begin{cases} Q(t_1)\Phi^T(t_2, t_1), & t_2 \geq t_1, \\ \Phi(t_1, t_2)Q(t_2), & t_1 \geq t_2. \end{cases} \tag{1-503}$$

The second-order joint moment matrix of $x(t)$ is

$$\begin{aligned} C_x(t_1, t_2) &= E\{x(t_1)x^T(t_2)\} \\ &= \Phi(t_1, t_0)C_x(t_0, t_0)\Phi^T(t_2, t_0) \\ &\quad + \int_{t_0}^{\min(t_1, t_2)} \Phi(t_1, \tau)B(\tau)V(\tau)B^T(\tau)\Phi^T(t_2, \tau) d\tau. \end{aligned} \tag{1-504}$$

The moment matrix $Q'(t) = C_x(t, t)$ satisfies the matrix differential equation

$$\dot{Q}'(t) = A(t)Q'(t) + Q'(t)A^T(t) + B(t)V(t)B^T(t), \tag{1-505}$$

$$Q'(t_0) = E\{x_0x_0^T\}. \tag{1-506}$$

Finally,

$$C_x(t_1, t_2) = \begin{cases} Q'(t_1)\Phi^T(t_2, t_1), & t_2 \geq t_1, \\ \Phi(t_1, t_2)Q'(t_2), & t_1 \geq t_2. \end{cases} \tag{1-507}$$

These results are easily proved by using the integration rules given in Theorem 1.51. Since

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^t \Phi(t, \tau)B(\tau)w(\tau) d\tau, \tag{1-508}$$

it follows by 1-484 that $m_w(t)$ is given by 1-500. To find the covariance and joint moment matrices, consider

$$\begin{aligned} E\{x(t_1)x^T(t_2)\} &= \Phi(t_1, t_0)E\{x_0x_0^T\}\Phi^T(t_2, t_0) \\ &+ E\left\{[\Phi(t_1, t_0)x_0]\left[\int_{t_0}^{t_2}\Phi(t_2, \tau)B(\tau)w(\tau) d\tau\right]^T\right\} \\ &+ E\left\{\left[\int_{t_0}^{t_1}\Phi(t_1, \tau)B(\tau)w(\tau) d\tau\right][\Phi(t_2, t_0)x_0]^T\right\} \\ &+ E\left\{\left[\int_{t_0}^{t_1}\Phi(t_1, \tau)B(\tau)w(\tau) d\tau\right]\left[\int_{t_0}^{t_2}\Phi(t_2, \tau)B(\tau)w(\tau) d\tau\right]^T\right\}. \quad 1-509 \end{aligned}$$

Because of the independence of x_0 and $w(t)$ and the fact that $w(t)$ has zero mean, the second and third terms of the right-hand side of 1-509 are zero. The fourth term is simplified by applying 1-486 so that 1-509 reduces to 1-504. Similarly, 1-501 can be obtained. The variance $Q(t)$ is obtained by setting $t_1 = t_2 = t$ in 1-501:

$$Q(t) = \Phi(t, t_0)Q_0\Phi^T(t, t_0) + \int_{t_0}^t \Phi(t, \tau)B(\tau)V(\tau)B^T(\tau)\Phi^T(t, \tau) d\tau. \quad 1-510$$

The differential equation 1-502 is found by differentiating $Q(t)$ in 1-510 with respect to t . The initial condition 1-502 is obtained by setting $t = t_0$. The differential equation for $C_w(t, t) = Q'(t)$ follows similarly. Finally, 1-503 and 1-507 follow directly from 1-501 and 1-504, respectively.

In passing, we remark that if x_0 is a Gaussian stochastic variable and the white noise $w(t)$ is Gaussian (see Example 1.33), then $x(t)$ is a Gaussian stochastic process. We finally note that in the analysis of linear systems it is often helpful to have a computer program available for the simulation of a linear differential system driven by white noise (see, e.g., Mehra, 1969).

Example 1.34. *A first-order differential system driven by white noise*

Consider the first-order stochastic differential equation

$$\dot{\xi}(t) = -\frac{1}{\theta}\xi(t) + \omega(t), \quad 1-511$$

where $\omega(t)$ is scalar white noise with constant intensity μ . Let us suppose that $\xi(0) = \xi_0$, where ξ_0 is a scalar stochastic variable with mean zero and variance $E(\xi_0^2) = \sigma^2$. It is easily found that $\xi(t)$ has the covariance function

$$R_{\xi}(t_1, t_2) = \left(\sigma^2 - \frac{\mu\theta}{2}\right)e^{-(t_1+t_2)/\theta} + \frac{\mu\theta}{2}e^{-|t_1-t_2|/\theta}, \quad t_1, t_2 \geq 0. \quad 1-512$$

The variance of the process is

$$Q(t) = \left(\sigma^2 - \frac{\mu\theta}{2} \right) e^{-2t/\theta} + \frac{\mu\theta}{2}, \quad t \geq 0. \quad 1-513$$

1.11.3 The Steady-State Variance Matrix for the Time-Invariant Case

In the preceding section we found an expression [Eq. 1-510] for the variance matrix of the state of a differential linear system driven by white noise. In this section we are interested in the asymptotic behavior of the variance matrix in the time-invariant case, that is, when A , B , and V are constant matrices. In this case 1-510 can be written as

$$Q(t) = e^{A(t-t_0)} Q_0 e^{A^T(t-t_0)} + \int_{t_0}^t e^{A(t-r)} B V B^T e^{A^T(t-r)} dr. \quad 1-514$$

It is not difficult to see that if, and only if, A is asymptotically stable, $Q(t)$ has the following limit for arbitrary Q_0 :

$$\lim_{t \rightarrow \infty} Q(t) = \lim_{t_0 \rightarrow -\infty} Q(t) = \bar{Q} = \int_0^{\infty} e^{A^T r} B V B^T e^{A r} dr. \quad 1-515$$

Since $Q(t)$ is the solution of the differential equation 1-502, its limit \bar{Q} must also satisfy that differential equation, so that

$$A\bar{Q} + \bar{Q}A^T + BVB^T = 0. \quad 1-516$$

It is quite helpful to realize that this algebraic matrix equation in \bar{Q} has a unique solution, which must then necessarily be given by 1-515. This follows from the following result from matrix theory (Frame, 1964).

Lemma 1.5. *Let M_1 , M_2 , and M_3 be real $n \times n$, $m \times m$, and $n \times m$ matrices. Let λ_i , $i = 1, 2, \dots, n$, and μ_j , $j = 1, 2, \dots, m$ denote the characteristic values of M_1 and M_2 , respectively. Then the matrix equation*

$$M_1 X + X M_2^T = M_3 \quad 1-517$$

has a unique $n \times m$ solution X if and only if for all i, j

$$\lambda_i + \mu_j \neq 0. \quad 1-518$$

In applying this lemma to 1-516, we let $M_1 = A$, $M_2 = A^T$. It follows that $m = n$ and $\mu_j = \lambda_j$, $j = 1, 2, \dots, m$. Since by assumption A is asymptotically stable, all characteristic values have strictly negative real parts, and necessarily

$$\lambda_i + \lambda_j \neq 0 \quad 1-519$$

for all i, j . Thus 1-516 has a unique solution.

We summarize as follows.

Theorem 1.53. Consider the stochastic differential equation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bw(t), \\ x(t_0) &= x_0,\end{aligned}\tag{1-520}$$

where A and B are constant and $w(t)$ is white noise with constant intensity V . Then if A is asymptotically stable and $t_0 \rightarrow -\infty$ or $t \rightarrow \infty$, the variance matrix of $x(t)$ tends to the constant nonnegative-definite matrix

$$\bar{Q} = \int_0^{\infty} e^{-At} B V B^T e^{-A^T t} dt,\tag{1-521}$$

which is the unique solution of the matrix equation

$$0 = A\bar{Q} + \bar{Q}A^T + BVB^T.\tag{1-522}$$

The matrix \bar{Q} can thus be found as the limit of the solution of the differential equation 1-502, with an arbitrary positive-semidefinite Q_0 as initial condition, from the integral 1-521 or from the algebraic equation 1-522.

Matrix equations of the form 1-522 are also encountered in stability theory and are sometimes known as *Lyapunov equations*. Although the matrix equation 1-522 is linear in \bar{Q} , its solution cannot be directly obtained by simple matrix inversion. MacFarlane (1963) and Chen and Shieh (1968a) give useful suggestions for setting up linear equations from which \bar{Q} can be solved. Barnett and Storey (1967), Davison and Man (1968), Smith (1968), Jameson (1968), Rome (1969), Kleinman (1970a), Müller (1970), Lu (1971), and Smith (1971) give alternative approaches. Hagander (1972) has made a comparison of various methods of solution, but his conclusions do not recommend one particular method. Also Barnett and Storey (1970) and Rothschild and Jameson (1970) review several methods of solution.

We remark that if A is asymptotically stable and $t_0 = -\infty$, the output of the differential system 1-499 is a wide-sense stationary process. The power spectral density of the state x is

$$\Sigma_x(\omega) = (j\omega I - A)^{-1} B V B^T (-j\omega I - A^T)^{-1}.\tag{1-523}$$

Thus using 1-473 one can obtain yet another expression for \bar{Q} ,

$$\bar{Q} = \int_{-\infty}^{\infty} (j\omega I - A)^{-1} B V B^T (-j\omega I - A^T)^{-1} df.\tag{1-524}$$

The steady-state variance matrix \bar{Q} has thus far been found in this section as the asymptotic solution of the variance differential equation for $t_0 \rightarrow -\infty$ or $t \rightarrow \infty$. Suppose now that we choose the steady-state variance matrix

\bar{Q} as the initial variance at time t_0 , that is, we set

$$Q_0 = \bar{Q}. \quad 1-525$$

By 1-502 this leads to

$$Q(t) = \bar{Q}, \quad t \geq t_0. \quad 1-526$$

The process $x(t)$ thus obtained has all the properties of a wide-sense stationary process.

Example 1.35. *The steady-state covariance and variance functions of a first-order system*

Consider as in Example 1.34 the scalar first-order differential equation driven by white noise,

$$\dot{\xi}(t) = -\frac{1}{\theta} \xi(t) + \omega(t), \quad 1-527$$

where the scalar white noise $\omega(t)$ has intensity μ and $\theta > 0$. Denoting by \bar{Q} the limit of $Q(t)$ as $t \rightarrow \infty$, one sees from 1-513 that

$$\bar{Q} = \frac{\mu\theta}{2}. \quad 1-528$$

The Lyapunov equation 1-522 reduces to

$$-\frac{2}{\theta} \bar{Q} + \mu = 0, \quad 1-529$$

which agrees with 1-528. Also, 1-521 yields the same result:

$$\bar{Q} = \mu \int_0^{\infty} e^{-2t/\theta} dt = \frac{\mu\theta}{2}. \quad 1-530$$

Finally, one can also check that 1-524 yields:

$$\bar{Q} = \int_{-\infty}^{+\infty} \frac{\mu}{\left(\omega^2 + \frac{1}{\theta^2}\right)} df = \frac{\mu\theta}{2}. \quad 1-531$$

Note that the covariance function $R_{\xi}(t_1, t_2)$ given in 1-512 converges to

$$\frac{\mu\theta}{2} \exp\left(-\left|\frac{t_1 - t_2}{\theta}\right|\right) \quad 1-532$$

as $t_1 + t_2 \rightarrow \infty$ with $t_1 - t_2$ finite. $R_{\xi}(t_1, t_2)$ equals this limit at finite t_1 and t_2 if the variance of the initial state is

$$\sigma^2 = \frac{\mu\theta}{2}. \quad 1-533$$

Apparently, 1-527 represents exponentially correlated noise, provided $\xi(t_0)$ is a zero-mean stochastic variable with variance 1-533.

1.11.4 Modeling of Stochastic Processes

In later chapters of this book we make almost exclusive use of linear differential systems driven by white noise to represent stochastic processes. This representation of a stochastic process $v(t)$ usually takes the following form. Suppose that $v(t)$ is given by

$$v(t) = C(t)x(t), \quad 1-534$$

with

$$\dot{x}(t) = A(t)x(t) + B(t)w(t), \quad 1-535$$

where $w(t)$ is white noise. Choosing such a representation for the stochastic process v , we call *modeling* of the stochastic process v . The use of such models can be justified as follows.

(a) Very often practical stochastic phenomena are generated by very fast fluctuations which act upon a much slower differential system. In this case the model of white noise acting upon a differential system is very appropriate. A typical example of this situation is thermal noise in an electronic circuit.

(b) As we shall see, in linear control theory almost always only the mean and covariance of the stochastic processes matter. Through the use of a linear model, it is always possible to approximate any experimentally obtained mean and covariance matrix arbitrarily closely.

(c) Sometimes the stochastic process to be modeled is a stationary process with known power spectral density matrix. Again, one can always generate a stochastic process by a linear differential equation driven by white noise so that its power spectral density matrix approximates arbitrarily closely the power spectral density matrix of the original stochastic process.

Examples 1.36 and 1.37, as well as Problem 1.11, illustrate the technique of modeling.

Example 1.36. First-order differential system

Suppose that the covariance function of a stochastic scalar process v , which is known to be stationary, has been measured and turns out to be the exponential function

$$R_v(t_1, t_2) = \sigma^2 e^{-|t_1 - t_2|/\theta}. \quad 1-536$$

One can model this process for $t \geq t_0$ as the state of a first-order differential system (see Example 1.35):

$$\dot{v}(t) = -\frac{1}{\theta} v(t) + \omega(t), \quad 1-537$$

with $\omega(t)$ white noise with intensity $2\sigma^2/\theta$ and where $v(t_0)$ is a stochastic variable with zero mean and variance σ^2 .

Example 1.37. Stirred tank

Consider the stirred tank of Example 1.31 (Section 1.10.3) and suppose that we wish to compute the variance matrix of the output variable $y(t)$. In Example 1.31 the fluctuations in the concentrations in the feeds were assumed to be exponentially correlated noises and can thus be modeled as the solution of a first-order system driven by white noise. We now extend the state differential equation of the stirred tank with the models for the stochastic processes $v_1(t)$ and $v_2(t)$. Let us write

$$v_1(t) = \xi_3(t), \tag{1-538}$$

where

$$\dot{\xi}_3(t) = -\frac{1}{\theta_1} \xi_3(t) + \omega_1(t). \tag{1-539}$$

Here $\omega_1(t)$ is scalar white noise with intensity μ_1 ; to make the variance of $v_1(t)$ precisely σ_1^2 , we take $\mu_1 = 2\sigma_1^2/\theta_1$. For $v_2(t) = \xi_4(t)$, we use a similar model. Thus we obtain the augmented system equation

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{2\theta} & 0 & 0 & 0 \\ 0 & -\frac{1}{\theta} & \frac{F_{10}}{V_0} & \frac{F_{20}}{V_0} \\ 0 & 0 & -\frac{1}{\theta_1} & 0 \\ 0 & 0 & 0 & -\frac{1}{\theta_2} \end{pmatrix} x(t) + \begin{pmatrix} 1 & 1 \\ \frac{c_{10} - c_0}{V_0} & \frac{c_{20} - c_0}{V_0} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} w(t), \tag{1-540}$$

where $w(t) = \text{col} [\omega_1(t), \omega_2(t)]$. The two-dimensional white noise $w(t)$ has intensity

$$V = \begin{pmatrix} \frac{2\sigma_1^2}{\theta_1} & 0 \\ 0 & \frac{2\sigma_2^2}{\theta_2} \end{pmatrix}. \tag{1-541}$$

Solution of 1-522 for the variance matrix \bar{Q} yields, assuming that $u(t) \equiv 0$ in 1-540,

$$\bar{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q_{22} & q_{23} & q_{24} \\ 0 & q_{23} & \sigma_1^2 & 0 \\ 0 & q_{24} & 0 & \sigma_2^2 \end{pmatrix}, \quad 1-542$$

where

$$q_{22} = \frac{(F_{10}/V_0)^2 \sigma_1^2 \theta^2 \theta_1}{\theta + \theta_1} + \frac{(F_{20}/V_0)^2 \sigma_2^2 \theta^2 \theta_2}{\theta + \theta_2}, \quad 1-543$$

$$q_{23} = \frac{(F_{10}/V_0)^2 \sigma_1^2 \theta \theta_1}{\theta + \theta_1}, \quad 1-544$$

$$q_{24} = \frac{(F_{20}/V_0)^2 \sigma_2^2 \theta \theta_2}{\theta + \theta_2}. \quad 1-545$$

The variance of $\eta_2(t) = \xi_2(t)$ is q_{22} , which is in agreement with the result of Example 1.32 (Section 1.10.4).

1.11.5 Quadratic Integral Expressions

Consider the linear differential system

$$\dot{x}(t) = A(t)x(t) + B(t)w(t), \quad 1-546$$

where $w(t)$ is white noise with intensity $V(t)$ and where the initial state $x(t_0)$ is assumed to be a stochastic variable with second-order moment matrix

$$E\{x(t_0)x^T(t_0)\} = Q_0. \quad 1-547$$

In later chapters of this book we extensively employ quadratic integral expressions of the form

$$E\left\{\int_{t_0}^{t_1} x^T(t)R(t)x(t) dt + x^T(t_1)P_1x(t_1)\right\}, \quad 1-548$$

where $R(t)$ is a symmetric nonnegative-definite weighting matrix for all $t_0 \leq t \leq t_1$ and where P_1 is symmetric and nonnegative-definite. In this section formulas for such expressions are derived. These formulas of course are also applicable to the deterministic case, where $w(t) = 0$, $t \geq t_0$, $x(t_0)$ is a deterministic variable, and the expectation sign does not apply.

For the solution of the linear differential equation 1-546, we write

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, \tau)B(\tau)w(\tau) d\tau, \quad 1-549$$

so that

$$\begin{aligned}
 & \int_{t_0}^{t_1} x^T(t)R(t)x(t) dt + x^T(t_1)P_1x(t_1) \\
 &= \int_{t_0}^{t_1} \left[x^T(t_0)\Phi^T(t, t_0) + \int_{t_0}^t w^T(\tau)B^T(\tau)\Phi^T(t, \tau) d\tau \right] \\
 & \quad \cdot R(t) \left[\Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, \tau)B(\tau)w(\tau) d\tau \right] dt \\
 &+ \left[x^T(t_0)\Phi^T(t_1, t_0) + \int_{t_0}^{t_1} w^T(\tau)B^T(\tau)\Phi^T(t_1, \tau) d\tau \right] \\
 & \quad \cdot P_1 \left[\Phi(t_1, t_0)x(t_0) + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)w(\tau) d\tau \right]. \quad 1-550
 \end{aligned}$$

Taking the expectation of this expression and using the integration rules of Theorem 1.51, we obtain the result

$$\begin{aligned}
 & E \left\{ \int_{t_0}^{t_1} x^T(t)R(t)x(t) dt + x^T(t_1)P_1x(t_1) \right\} \\
 &= \text{tr} \left\{ \left[\int_{t_0}^{t_1} \Phi^T(t, t_0)R(t)\Phi(t, t_0) dt + \Phi^T(t_1, t_0)P_1\Phi(t_1, t_0) \right] Q_0 \right. \\
 & \quad + \int_{t_0}^{t_1} \left[\int_{t_0}^t V(\tau)B^T(\tau)\Phi^T(t, \tau)R(t)\Phi(t, \tau)B(\tau) d\tau \right] dt \\
 & \quad \left. + \int_{t_0}^{t_1} V(\tau)B^T(\tau)\Phi^T(t_1, \tau)P_1\Phi(t_1, \tau)B(\tau) d\tau \right\}. \quad 1-551
 \end{aligned}$$

Now if M and N are arbitrary matrices of compatible dimensions, it is easily shown that $\text{tr}(MN) = \text{tr}(NM)$. Application of this fact to the last two terms of 1-551 and an interchange of the order of integration in the third term yields

$$\begin{aligned}
 & \text{tr} \left\{ \int_{t_0}^{t_1} \left[\int_{t_0}^t V(\tau)B^T(\tau)\Phi^T(t, \tau)R(t)\Phi(t, \tau)B(\tau) d\tau \right] dt \right. \\
 & \quad \left. + \int_{t_0}^{t_1} V(\tau)B^T(\tau)\Phi^T(t_1, \tau)P_1\Phi(t_1, \tau)B(\tau) d\tau \right\} \\
 &= \text{tr} \left\{ \int_{t_0}^{t_1} \left[\int_{t_0}^t B(\tau)V(\tau)B^T(\tau)\Phi^T(t, \tau)R(t)\Phi(t, \tau) d\tau \right] dt \right. \\
 & \quad \left. + \int_{t_0}^{t_1} B(\tau)V(\tau)B^T(\tau)\Phi^T(t_1, \tau)P_1\Phi(t_1, \tau) d\tau \right\} \\
 &= \text{tr} \left\{ \int_{t_0}^{t_1} B(\tau)V(\tau)B^T(\tau) \left[\int_{\tau}^{t_1} \Phi^T(t, \tau)R(t)\Phi(t, \tau) \right. \right. \\
 & \quad \left. \left. + \Phi^T(t_1, \tau)P_1\Phi(t_1, \tau) \right] d\tau \right\}. \quad 1-552
 \end{aligned}$$

Substitution of this into 1-551 shows that we can write

$$E\left\{\int_{t_0}^{t_1} x^T(t)R(t)x(t) dt + x^T(t_1)P_1x(t_1)\right\} \\ = \text{tr}\left\{P(t_0)Q_0 + \int_{t_0}^{t_1} B(t)V(t)B^T(t)P(t) dt\right\}, \quad 1-553$$

where the symmetric matrix $P(t)$ is given by

$$P(t) = \int_t^{t_1} \Phi^T(\tau, t)R(\tau)\Phi(\tau, t) d\tau + \Phi^T(t_1, t)P_1\Phi(t_1, t). \quad 1-554$$

By using Theorem 1.2 (Section 1.3.1), it is easily shown by differentiation that $P(t)$ satisfies the matrix differential equation

$$-\dot{P}(t) = A^T(t)P(t) + P(t)A(t) + R(t). \quad 1-555$$

Setting $t = t_1$ in 1-554 yields the terminal condition

$$P(t_1) = P_1. \quad 1-556$$

We summarize these results as follows.

Theorem 1.54. Consider the linear differential system

$$\dot{x}(t) = A(t)x(t) + B(t)w(t), \quad 1-557$$

where $w(t)$ is white noise with intensity $V(t)$ and where $x(t_0) = x_0$ is a stochastic variable with $E\{x_0x_0^T\} = Q_0$. Let $R(t)$ be symmetric and nonnegative-definite for $t_0 \leq t \leq t_1$, and P_1 constant, symmetric, and nonnegative-definite. Then

$$E\left\{\int_{t_0}^{t_1} x^T(t)R(t)x(t) dt + x^T(t_1)P_1x(t_1)\right\} \\ = \text{tr}\left\{P(t_0)Q_0 + \int_{t_0}^{t_1} B(t)V(t)B^T(t)P(t) dt\right\}, \quad 1-558$$

where $P(t)$ is the symmetric nonnegative-definite matrix

$$P(t) = \int_t^{t_1} \Phi^T(\tau, t)R(\tau)\Phi(\tau, t) d\tau + \Phi^T(t_1, t)P_1\Phi(t_1, t). \quad 1-559$$

$\Phi(t, t_0)$ is the transition matrix of the system 1-557. $P(t)$ satisfies the matrix differential equation

$$-\dot{P}(t) = A^T(t)P(t) + P(t)A(t) + R(t) \quad 1-560$$

with the terminal condition

$$P(t_1) = P_1. \quad 1-561$$

In particular, if the differential system 1-557 reduces to an autonomous differential system:

$$\dot{x}(t) = A(t)x(t), \tag{1-562}$$

that is, $V(t) = 0$ and $x(t_0)$ is deterministic, then

$$\int_{t_0}^{t_1} x^T(t)R(t)x(t) dt + x^T(t_1)P_1x(t_1) = x^T(t_0)P(t_0)x(t_0). \tag{1-563}$$

We conclude this section with a discussion of the asymptotic behavior of the matrix $P(t)$ as the terminal time t_1 goes to infinity. We limit ourselves to the time-invariant case where the matrices A , B , V , and R are constant, so that 1-559 reduces to:

$$P(t) = \int_t^{t_1} e^{A^T(t-\tau)}Re^{A(\tau-t)} d\tau + e^{A^T(t_1-t)}P_1e^{A(t_1-t)}. \tag{1-564}$$

If A is asymptotically stable, we obtain in the limit $t_1 \rightarrow \infty$:

$$P(t) \rightarrow \bar{P} = \int_t^\infty e^{A^T(t-\tau)}Re^{A(\tau-t)} d\tau. \tag{1-565}$$

A change of integration variable shows that \bar{P} can be written as

$$\bar{P} = \int_0^\infty e^{A^T t'}Re^{At'} dt', \tag{1-566}$$

which very clearly shows that \bar{P} is a constant matrix. Since \bar{P} satisfies the matrix differential equation 1-560, we have

$$0 = A^T\bar{P} + \bar{P}A + R. \tag{1-567}$$

Since by assumption A is asymptotically stable, Lemma 1.5 (Section 1.11.3) guarantees that this algebraic equation has a unique solution.

In the time-invariant case, it is not difficult to conjecture from 1-558 that for $t_1 \gg t_0$ we can approximate

$$E\left\{\int_{t_0}^{t_1} x^T(t)Rx(t) dt + x^T(t_1)P_1x(t_1)\right\} \simeq \text{tr} [\bar{P}Q_0 + (t_1 - t_0)BVB^T\bar{P}]. \tag{1-568}$$

This shows that as $t_1 \rightarrow \infty$ the criterion 1-558 asymptotically increases with t_1 at the rate $\text{tr}(BVB^T\bar{P})$.

Example 1.38. Stirred tank

Consider the stirred tank extended with the model for the disturbances of Example 1.37. Assume that $u(t) \equiv 0$ and suppose that we are interested in the integral expression

$$E\left\{\int_{t_0}^{t_1} \xi_2^2(t) dt\right\}. \tag{1-569}$$

This integral gives an indication of the average deviation of the concentration $\xi_2(t)$ from zero, where the average is taken both statistically and over

time. This expression is of the general form 1-548 if we set

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_1 = 0. \quad 1-570$$

Solution of the algebraic equation

$$0 = A^T \bar{P} + \bar{P} A + R \quad 1-571$$

yields the steady-state solution

$$\bar{P} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p_{22} & p_{23} & p_{24} \\ 0 & p_{23} & p_{33} & p_{34} \\ 0 & p_{24} & p_{34} & p_{44} \end{pmatrix}, \quad 1-572$$

where

$$p_{22} = \frac{\theta}{2},$$

$$p_{23} = \frac{\frac{\theta}{2} \frac{F_{10}}{V_0}}{\frac{1}{\theta} + \frac{1}{\theta_1}},$$

$$p_{24} = \frac{\frac{\theta}{2} \frac{F_{20}}{V_0}}{\frac{1}{\theta} + \frac{1}{\theta_2}},$$

$$p_{33} = \frac{\left(\frac{F_{10}}{V_0}\right)^2 \frac{\theta \theta_1}{2}}{\frac{1}{\theta} + \frac{1}{\theta_1}}, \quad 1-573$$

$$p_{34} = \frac{\left(\frac{F_{10}}{V_0}\right) \left(\frac{F_{20}}{V_0}\right) \frac{\theta}{2}}{\frac{1}{\theta_1} + \frac{1}{\theta_2}} \left(\frac{1}{\frac{1}{\theta} + \frac{1}{\theta_1}} + \frac{1}{\frac{1}{\theta} + \frac{1}{\theta_2}} \right),$$

$$p_{44} = \frac{\left(\frac{F_{20}}{V_0}\right)^2 \frac{\theta \theta_2}{2}}{\frac{1}{\theta} + \frac{1}{\theta_2}}.$$

If we assume for V the form 1-541, as we did in Example 1.37, we find for the rate at which the integral criterion 1-569 asymptotically increases with t_1 [see 1-568]:

$$\operatorname{tr}(BV B^T \bar{P}) = \frac{\left(\frac{F_{10}}{V_0}\right)^2 \sigma_1^2 \theta}{\frac{1}{\theta} + \frac{1}{\theta_1}} + \frac{\left(\frac{F_{20}}{V_0}\right)^2 \sigma_2^2 \theta}{\frac{1}{\theta} + \frac{1}{\theta_2}}. \quad 1-574$$

Not unexpectedly, this is precisely the steady-state value of $E\{\xi_2^2(t)\}$ computed in Example 1.37.

1.12 PROBLEMS

1.1. Revolving satellite

Consider a satellite that revolves about its axis of symmetry (Fig. 1.11). The angular position of the satellite at time t is $\phi(t)$, while the satellite has a

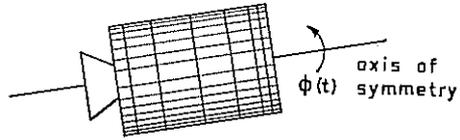


Fig. 1.11. A revolving satellite.

constant moment of inertia J . By means of gas jets, a variable torque $\mu(t)$ can be exerted, which is considered the input variable to the system. The satellite experiences no friction.

(a) Choose as the components of the state the angular position $\phi(t)$ and the angular speed $\dot{\phi}(t)$. Let the output variable be $\eta(t) = \phi(t)$. Show that the state differential equation and the output equation of the system can be represented as

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \beta \end{pmatrix} \mu(t), \\ \eta(t) &= (1, 0)x(t), \end{aligned} \quad 1-575$$

where $\beta = 1/J$.

(b) Compute the transition matrix, the impulse response function, and the step response function of the system. Sketch the impulse response and step response functions.

(c) Is the system stable in the sense of Lyapunov? Is it asymptotically stable?

(d) Determine the transfer function of the system.

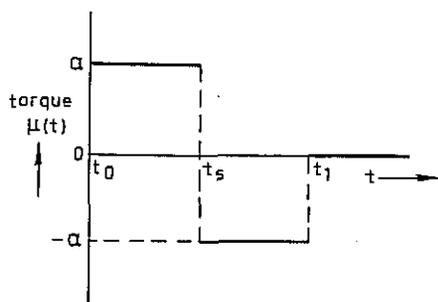


Fig. 1.12. Input torque for satellite repositioning.

(e) Consider the problem of rotating the satellite from one position in which it is at rest to another position, where it is at rest. In terms of the state, this means that the system must be transferred from the state $x(t_0) = \text{col}(\phi_0, 0)$ to the state $x(t_1) = \text{col}(\phi_1, 0)$, where ϕ_0 and ϕ_1 are given angles. Suppose that two gas jets are available; they produce torques in opposite directions such that the input variable assumes only the values $-\alpha$, 0 , and $+\alpha$, where α is a fixed, given number. Show that the satellite can be rotated with an input of the form as sketched in Fig. 1.12. Calculate the switching time t_s and the terminal time t_1 . Sketch the trajectory of the state in the state plane.

1.2. Amplidyne

An amplidyne is an electric machine used to control a large dc power through a small dc voltage. Figure 1.13 gives a simplified representation (D'Azzo and Houpis, 1966). The two armatures are rotated at a constant speed (in fact they are combined on a single shaft). The output voltage of each armature is proportional to the corresponding field current. Let L_1 and R_1 denote the inductance and resistance of the first field windings and L_2 and R_2 those of the first armature windings together with the second field windings.

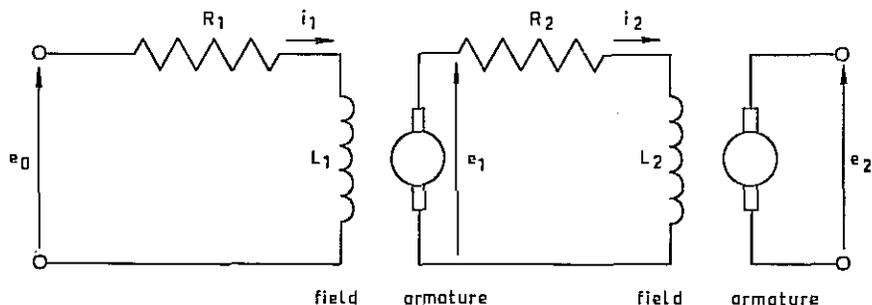


Fig. 1.13. Schematic representation of an amplidyne.

The induced voltages are given by

$$\begin{aligned} e_1 &= k_1 i_1, \\ e_2 &= k_2 i_2. \end{aligned} \quad 1-576$$

The following numerical values are used:

$$\begin{aligned} R_1/L_1 &= 10 \text{ s}^{-1}, & R_2/L_2 &= 1 \text{ s}^{-1}, \\ R_1 &= 5 \Omega, & R_2 &= 10 \Omega, \\ k_1 &= 20 \text{ V/A}, & k_2 &= 50 \text{ V/A}. \end{aligned} \quad 1-577$$

(a) Take as the components of the state $\xi_1(t) = i_1(t)$ and $\xi_2(t) = i_2(t)$ and show that the system equations are

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -\frac{R_1}{L_1} & 0 \\ \frac{k_1}{L_2} & -\frac{R_2}{L_2} \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu(t), \\ \eta(t) &= (0, \quad k_2)x(t), \end{aligned} \quad 1-578$$

where $\mu(t) = e_0(t)$ and $\eta(t) = e_2(t)$.

(b) Compute the transition matrix, the impulse response function, and the step response function of the system. Sketch for the numerical values given the impulse and step response functions.

(c) Is the system stable in the sense of Lyapunov? Is it asymptotically stable?

(d) Determine the transfer function of the system. For the numerical values given, sketch a Bode plot of the frequency response function of the system.

(e) Compute the modes of the system.

1.3. Properties of time-invariant systems under state transformations

Consider the linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \quad 1-579$$

We consider the effects of the state transformation $x' = Tx$.

(a) Show that the transition matrix $\Phi(t, t_0)$ of the system 1-579 and the transition matrix $\Phi'(t_1, t_0)$ of the transformed system are related by

$$\Phi'(t, t_0) = T\Phi(t, t_0)T^{-1}. \quad 1-580$$

(b) Show that the impulse response matrix and the step response matrix of the system do not change under a state transformation.

(c) Show that the characteristic values of the system do not change under a state transformation.

(d) Show that the transformed system is stable in the sense of Lyapunov if and only if the original system 1-579 is stable in the sense of Lyapunov. Similarly, prove that the transformed system is asymptotically stable if and only if the original system 1-579 is asymptotically stable.

(e) Show that the transfer matrix of the system does not change under a state transformation.

1.4. Stability of amplidyne with feedback

In an attempt to improve the performance of the amplidyne of Problem 1.2, the following simple proportional feedback scheme is considered.

$$\mu(t) = \lambda[\eta_r(t) - \eta(t)]. \quad \mathbf{1-581}$$

Here $\eta_r(t)$ is an external reference voltage and λ a gain constant to be determined.

(a) Compute the transfer matrix of the amplidyne interconnected with the feedback scheme 1-581 from the reference voltage $\eta_r(t)$ to the output voltage $\eta(t)$.

(b) Determine the values of the gain constant λ for which the feedback system is asymptotically stable.

1.5*. Structure of the controllable subspace

Consider the controllability canonical form of Theorem 1.26 (Section 1.6.3).

(a) Prove that no matter how the transformation matrix T is chosen the characteristic values of A'_{11} and A'_{22} are always the same.

(b) Define the characteristic values of A'_{11} as the *controllable poles* and the characteristic values of A'_{22} as the *uncontrollable poles* of the system. Prove that the controllable subspace of the system 1-310 is spanned by the characteristic vectors and generalized characteristic vectors of the system that correspond to the controllable poles.

(c) Conclude that in the original representation 1-308 of the system the controllable subspace is similarly spanned by the characteristic vectors and generalized characteristic vectors corresponding to the controllable poles.

1.6*. Controllability and stabilizability of a time-invariant system under a state transformation

Consider the state transformation $x' = Tx$ for the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t). \quad \mathbf{1-582}$$

* See the preface for the significance of the problems marked with an asterisk.

(a) Prove that the transformed system is completely controllable if and only if the original system **1-582** is completely controllable.

(b) Prove directly (without using Theorem 1.26) that the transformed system is stabilizable if and only if the original system **1-582** is stabilizable.

1.7*. *Reconstructibility and detectability of a time-invariant system under a state transformation*

Consider the state transformation $x' = Tx$ for the time-invariant system

$$\dot{x}(t) = Ax(t), \quad y(t) = Cx(t). \quad \mathbf{1-583}$$

(a) Prove that the transformed system is completely reconstructible if and only if the original system **1-583** is completely reconstructible.

(b) Prove directly (without using Theorem 1.35) that the transformed system is detectable if and only if the original system **1-583** is detectable.

1.8*. *Dual of a transformed system*

Consider the time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \quad \mathbf{1-584}$$

Transform this system by defining $x'(t) = Tx(t)$ where T is a nonsingular transformation matrix. Show that the dual of the system **1-584** is transformed into the dual of the transformed system by the transformation $x^*(t) = T^T x'^*(t)$.

1.9. *“Damping” of stirred tank*

Consider the stirred tank with fluctuations in the concentrations c_1 and c_2 as described in Examples 1.31 and 1.32 (Sections 1.10.3 and 1.10.4). Assume that $u(t) \equiv 0$. The presence of the tank has the effect that the fluctuations in the concentrations c_1 and c_2 are reduced. Define the “damping factor” of the tank as the square root of the ratio of the mean square value of the fluctuations in the concentrations $c(t)$ of the outgoing flow and the mean square value of the fluctuations when the incoming feeds are mixed immediately without a tank ($V_0 = 0$). Compute the damping factor as a function of V_0 . Assume $\sigma_1 = \sigma_2$, $\theta_1 = \theta_2 = 10$ s and use the numerical values of Example 1.2 (Section 1.2.3). Sketch a graph of the damping factor as a function of V_0 .

1.10. *State of system driven by Gaussian white noise as a Markov process*

A stochastic process $v(t)$ is a Markov process if

$$P\{v(t_n) \leq v_n \mid v(t_1), v(t_2), \dots, v(t_{n-1})\} = P\{v(t_n) \leq v_n \mid v(t_{n-1})\} \quad \mathbf{1-585}$$

for all n , all t_1, t_2, \dots, t_n with $t_n \geq t_{n-1} \geq t_{n-2} \geq \dots \geq t_1$, and all v_n . Show that the state $x(t)$ of the system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)w(t), \\ x(t_0) &= x_0,\end{aligned}\tag{1-586}$$

where $w(t)$ is Gaussian white noise and x_0 a given stochastic variable, is a Markov process, provided x_0 is independent of $w(t)$, $t \geq t_0$.

1.11. Modeling of second-order stochastic processes

Consider the system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \omega(t).\tag{1-587}$$

For convenience we have chosen the system to be in phase canonical form, but this is not essential. Let $\omega(t)$ be white noise with intensity 1. The output of the system is given by

$$v(t) = (\gamma_1, \gamma_2)x(t).\tag{1-588}$$

(a) Show that if 1-587 is asymptotically stable the power spectral density function of $v(t)$ is given by

$$\Sigma_v(\omega) = \left| \frac{\gamma_1 + (j\omega)\gamma_2}{(j\omega)^2 + \alpha_1(j\omega) + \alpha_0} \right|^2.\tag{1-589}$$

(b) Suppose that a stationary stochastic scalar process is given which has one of two following types of covariance functions:

$$R_v(\tau) = \beta_1 e^{-\sigma_1|\tau|} + \beta_2 e^{-\sigma_2|\tau|},\tag{1-590}$$

or

$$R_v(\tau) = \beta_1 e^{-\sigma_0|\tau|} \cos(\omega_0\tau) + \beta_2 e^{-\sigma_0|\tau|} \cos(\omega_0\tau),\tag{1-591}$$

where $\tau = t_1 - t_2$. Show that 1-587 and 1-588 can be used to model such a process. Express the constants occurring in 1-587 and 1-588 in terms of the constants occurring in 1-590 or 1-591.

(c) Atmospheric turbulence manifests itself in the form of stochastically varying air speeds. The speed fluctuations in a direction perpendicular to the main flow can be represented as a scalar stochastic process with covariance function

$$R(\tau) = \sigma^2 e^{-|\tau|/\theta} \left(1 - \frac{1}{2} \frac{|\tau|}{\theta} \right),\tag{1-592}$$

where $\tau = t_1 - t_2$. Model this process.