Solutions

ENG EC/ME/SC 501:

Exercises (Set 8)  (Study questions—answers posted December 7)

1. Given that \( x(0) = 1 \), find \( x(\cdot) \) on the interval \( 0 \leq t \leq T \) such that

\[
J = \int_0^T \dot{x}^2 + x^2 \, dt
\]

is minimized. (Hint: Convert this into a control problem by setting \( \dot{x} = u \).)

2. Suppose that the partitioned system

\[
\begin{pmatrix}
\dot{w}(t) \\
\dot{y}(t)
\end{pmatrix} =
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\begin{pmatrix}
w(t) \\
y(t)
\end{pmatrix}
\]

with output \( y(t) \) is observable. Show that \( \{A_{11}, A_{21}\} \) is an observable pair.

3. Design an observer for the system shown in the figure. The observer should be of second order with both eigenvalues equal to \(-3\).

![Block diagram](image)

4. Consider the linear system

\[
\begin{pmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix} +
\begin{pmatrix}
0 \\
u(t)
\end{pmatrix}.
\]

(1)

For \( T > 0 \), find the control input that steers the state of (1) from \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) to \( \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \) in \( T \) units of time so as to minimize the performance metric

\[
\eta = \int_0^T u(t)^2 \, dt.
\]

(2)
1

Allow \( \dot{x} = u \). Note that this represents a controllable system. Then

\[
J = \int_0^T x^2 + \dot{x}^2 \, dt = \int_0^T u^2 + x^2 \, dt.
\]

This is equivalent to

\[
J = \int_0^T x^T Q x + u^T R u \, dt,
\]

where \( Q = R = 1 \). Note that both \( Q \) and \( R \) are positive definite.

The associated Ricatti equation is

\[
\dot{m} = (m + 1)(m - 1),
\]

which is separable, simplifying to the equation of integrals

\[
\int \frac{dm}{(m + 1)(m - 1)} = \int dt.
\]

The integral on the right hand side of (2) produces \( t + c \). The integral on the left hand side of (2), after consulting an integration table, evaluates to

\[
\frac{1}{2} \ln \left( \frac{m - 1}{m + 1} \right).
\]

Thus, equation (2) becomes

\[
\frac{1}{2} \ln \left( \frac{m - 1}{m + 1} \right) = t + c,
\]

which, after some algebra, produces the solution to (1) as

\[
m(t) = \frac{1 + ce^{2t}}{1 - ce^{2t}}.
\]

In order for \( m(\cdot) \) to pass through 0 at time \( t = T \), \( c \) must be \( -e^{-2T} \). Accordingly, the solution in (3) becomes

\[
m(t) = \frac{1 - e^{2(t-T)}}{1 + e^{2(t-T)}} = -\tanh(t - T).
\]

Since \( m(t) \geq 0 \) for all \( t \in [0, T] \), \( m(t) \) is positive semidefinite on the interval \([0, T]\). Therefore, the control that minimizes \( J \) is given by

\[
u(t) = x(t) \tanh(t - T).
\]

Thus \( x(\cdot) \) satisfies

\[
\dot{x}(t) = x(t) \tanh(t - T).
\]

Solving for \( x(\cdot) \),

\[
\begin{align*}
\frac{dx}{dt} &= x \tanh(t - T) \\
\int \frac{dx}{x} &= \int \tanh(t - T) \, dt \\
\ln x &= \ln \cosh(t - T) + c \\
x(t) &= c \cosh(t - T).
\end{align*}
\]
Employing the initial condition produces

\[ x(0) = 1 = c \cosh(-T) = c \cosh(T) \]
\[ c = \text{sech}(T). \]

And thus,

\[ x(t) = \text{sech}(T) \cosh(t - T). \]

\[ 2 \]

We have

\[ \dot{w} = A_{11} w + A_{12} y \]
\[ \dot{y} = A_{21} w + A_{22} y. \]

Allow \( z = \dot{y} - A_{22} y \). Since \( y \) is observable, \( z \) is a known quantity.

The solution for \( w \) is obtained by

\[ w(t) = e^{A_{11} t} w_0 + f(y) \]

where \( f \) is some function of observable \( y \) and therefore evaluates to a known quantity. This gives

\[ z(t) = A_{21} e^{A_{11} t} w_0 + f'(y), \]

where \( f' \) is some other function of observable \( y \) and therefore evaluates to a known quantity. Differentiating \( z \) produces

\[ \dot{z}(t) = A_{21} A_{11} e^{A_{11} t} w_0 + f''(y), \]

where again \( f'' \) is some function that evaluates to a known quantity. Eventually, after repeated differentiation and evaluation, we obtain

\[
\begin{bmatrix}
  z(0) \\
  \dot{z}(0) \\
  \vdots \\
  z^{(n)}(0)
\end{bmatrix}
= \begin{bmatrix}
  A_{21} \\
  A_{21} A_{11} \\
  \vdots \\
  A_{21} A_{11}^{n-1}
\end{bmatrix} w_0 + F(y),
\]

where \( F \) is some vector that we can assume is known. Since \( F, w_0, \) and \( [z(0), \dot{z}(0), \ldots, z^{(n)}(0)]^T \) are all known quantities, the matrix

\[
\begin{bmatrix}
  A_{21} \\
  A_{21} A_{11} \\
  \vdots \\
  A_{21} A_{11}^{n-1}
\end{bmatrix}
\]

is invertible, and therefore full rank. Thus, \((A_{11}, A_{21})\) is an observable pair.

\[ 3 \]

The system diagram produces the equations

\[ x_1 = \frac{1}{2} x_2 \]
\[ x_2 = \frac{1}{x+2} x_3 \]
\[ x_3 = \frac{1}{x+1} (-x_1 + u) \]
\[ y = x_1. \]
After some rearrangement, this becomes
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -2x_2 + x_3 \\
\dot{x}_3 &= -x_1 - x_3 + u \\
y &= x_1,
\end{align*}
\]
which produces
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 \\
0 & -2 & 1 \\
-1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u.
\]
Further rearranging the above produces
\[
\begin{bmatrix}
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_1
\end{bmatrix}
= \begin{bmatrix}
-2 & 1 & 0 \\
0 & -1 & -1 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_3 \\
x_1
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix},
\]
with partitions
\[
A_{11} = \begin{bmatrix}
-2 & 1 \\
0 & -1
\end{bmatrix} \\
A_{12} = \begin{bmatrix}
0 \\
-1
\end{bmatrix} \\
A_{21} = \begin{bmatrix}
1 & 0
\end{bmatrix} \\
A_{22} = \begin{bmatrix}
0
\end{bmatrix}.
\]
Hence,
\[
A_{11} - EA_{21} = \begin{bmatrix}
-2 - e_1 & 1 \\
-e_2 & -1
\end{bmatrix},
\]
with characteristic polynomial
\[
|A_{11} - EA_{21} - I\lambda| = \begin{vmatrix}
-2 - e_1 - \lambda & 1 \\
-e_2 & -1 - \lambda
\end{vmatrix} = \lambda^2 + (3 + e_1)\lambda + e_1 + e_2 + 2.
\]
For both eigenvalues of \(A_{11} - EA_{21}\) to be -3 it must be the case that
\[
3 + e_1 = 6 \Rightarrow e_1 = 3
\]
and
\[
e_1 + e_2 + 2 = 9 \Rightarrow e_2 = 4.
\]
The observer equation is
\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix}
= \begin{bmatrix}
-5 & 1 \\
-4 & -1
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
+ \begin{bmatrix}
-11 \\
-17
\end{bmatrix} y + \begin{bmatrix}
0 \\
1
\end{bmatrix} u.
\]
The estimate \(\hat{x}\) of \(x\) is then reconstructed as
\[
P^{-1} \begin{bmatrix}
w_1 \\
w_2 \\
y
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
z_1 + e_1 y \\
z_2 + e_2 y \\
y
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
z_1 + 3y \\
z_2 + 4y \\
y
\end{bmatrix}
= \begin{bmatrix}
z_1 + 3y \\
z_2 + 4y
\end{bmatrix}
+ \begin{bmatrix}
1 \\
3
\end{bmatrix} y.
4.  

\[ W(0, T) = \int_0^T \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} (0, 1) \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} ds \]

\[ = \begin{pmatrix} T^3/3 & -T^2/2 \\ -T^2/2 & T \end{pmatrix} \]

\[ W(0, T)^{-1} = \begin{pmatrix} 12/6 & -6/4 \\ 6/4 & 4/4 \end{pmatrix} \]

\[ u(t) = (0, 1) \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} W(0, T)^{-1} \begin{pmatrix} 1 & -T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \]

From this it is easy to see that

\[ \eta = \int_0^T u(s)^2 ds \]

\[ = \frac{12 \cos^2(\theta)}{T^3} - \frac{12 \sin(\theta) \cos(\theta)}{T^2} + \frac{4 \sin^2(\theta)}{T}. \]

Hence

\[ \eta(0) = \eta(\pi) = \frac{12}{T^3}, \quad \text{and} \quad \eta(\pi/2) = \eta(3\pi/2) = \frac{4}{T}. \]

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Note that a general form of the optimal cost of steering

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

from \( x_0 \) to \( x_1 \) so as to minimize

\[ \eta = \int_0^T \|u(t)\|^2 dt. \]

is

\[ \eta_0 = [x_0 - e^{-AT}x_1]^T W(0, T)^{-1} [x_0 - e^{-AT}x_1]. \]

The values above can be read off directly from this formula.