Solutions

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(11/2/21)

ENG EC/ME/SC 501:

Exercises (Set 8) (Study questions—answers posted December 7)

1. Given that $x(0) = 1$, find $x(\cdot)$ on the interval $0 \le t \le T$ such that

$$J = \int_0^T \dot{x}^2 + x^2 \, dt$$

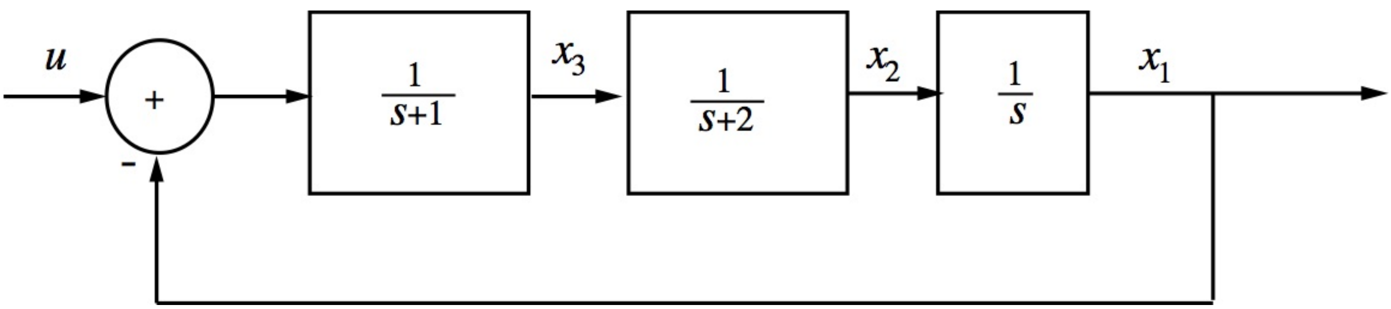
is minimized. (*Hint:* Convert this into a control problem by setting $\dot{x} = u$.)

2. Suppose that the partitioned system

$$\begin{pmatrix} \dot{w}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} w(t) \\ y(t) \end{pmatrix}$$

with output $y(t)$ is observable. Show that $\{A_{11}, A_{21}\}$ is an observable pair.

3. Design an observer for the system shown in the figure. The observer should be of second order with both eigenvalues equal to -3.



4. Consider the linear system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ u(t) \end{pmatrix}. \tag{1}$$

For $T > 0$, find the control input that steers the state of (1) from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ in T units of time so as to minimize the performance metric

$$\eta = \int_0^T u(t)^2 \, dt. \tag{2}$$

Study Questions
Solutions

$\boxed{1}$

Allow $\dot{x} = u$. Note that this represents a controllable system. Then

$$J = \int_0^T \dot{x}^2 + x^2 dt = \int_0^T u^2 + x^2 dt.$$

This is equivalent to

$$J = \int_0^T \mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u} dt,$$

where $\mathbf{Q} = \mathbf{R} = 1$. Note that both $\mathbf{Q}$ and $\mathbf{R}$ are positive definate.

The associated Ricatti equation is

$$\dot{m} = (m+1)(m-1), \tag{1}$$

which is separable, simplifying to the equation of integrals

$$\int \frac{dm}{(m+1)(m-1)} = \int dt. \tag{2}$$

The integral on the right hand side of (2) produces $t + c$. The integral on the left hand side of (2), after consulting an integration table, evaluates to

$$\frac{1}{2} \ln \left(\frac{m-1}{m+1} \right).$$

Thus, equation (2) becomes

$$\frac{1}{2} \ln \left(\frac{m-1}{m+1} \right) = t + c,$$

which, after some algebra, produces the solution to (1) as

$$m(t) = \frac{1 + ce^{2t}}{1 - ce^{2t}}. \tag{3}$$

In order for $m(\cdot)$ to pass through 0 at time $t = T$, c must be $-e^{-2T}$. Accordingly, the solution in (3) becomes

$$m(t) = \frac{1 - e^{2(t-T)}}{1 + e^{2(t-T)}} = -\tanh(t - T). \tag{4}$$

Since $m(t) \geq 0$ for all $t \in [0, T]$, $m(t)$ is positive semidefinate on the interval $[0, T]$. Therefore, the control that minimizes J is given by

$$u(t) = x(t) \tanh(t - T). \tag{5}$$

Thus $x(\cdot)$ satisfies

$$\dot{x}(t) = x(t) \tanh(t - T).$$

Solving for $x(\cdot)$,

$$\begin{aligned} \tfrac{dx}{dt} &= x \tanh(t - T) \\ \int \tfrac{dx}{x} &= \int \tanh(t - T) dt \\ \ln x &= \ln \cosh(t - T) + c \\ x(t) &= c \cosh(t - T). \end{aligned}$$

Employing the initial condition produces

$$\begin{aligned} x(0) = 1 &= c\cosh(-T) = c\cosh(T) \\ c &= \text{sech}(T). \end{aligned}$$

And thus,

$$x(t) = \text{sech}(T)\cosh(t - T).$$

2

We have

$$\begin{aligned} \dot{\mathbf{w}} &= \mathbf{A}_{11}\mathbf{w} + \mathbf{A}_{12}\mathbf{y} \\ \dot{\mathbf{y}} &= \mathbf{A}_{21}\mathbf{w} + \mathbf{A}_{22}\mathbf{y}. \end{aligned}$$

Allow $\mathbf{z} = \dot{\mathbf{y}} - \mathbf{A}_{22}\mathbf{y}$. Since $\mathbf{y}$ is observable, $\mathbf{z}$ is a known quantity.

The solution for $\mathbf{w}$ is obtained by

$$\mathbf{w}(t) = e^{\mathbf{A}_{11}t}\mathbf{w}_0 + f(\mathbf{y})$$

where f is some function of observable $\mathbf{y}$ and therefore evaluates to a known quantity. This gives

$$\mathbf{z}(t) = \mathbf{A}_{21}e^{\mathbf{A}_{11}t}\mathbf{w}_0 + f'(\mathbf{y}),$$

where f' is some other function of observable $\mathbf{y}$ and therefore evaluates to a known quantity. Differentiating $\mathbf{z}$ produces

$$\dot{\mathbf{z}}(t) = \mathbf{A}_{21}\mathbf{A}_{11}e^{\mathbf{A}_{11}t}\mathbf{w}_0 + f''(\mathbf{y}),$$

where again f'' is some function that evaluates to a known quantity. Eventually, after repeated differentiation and evaluation, we obtain

$$\begin{bmatrix} \mathbf{z}(0) \\ \dot{\mathbf{z}}(0) \\ \vdots \\ \mathbf{z}^{(n)}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{21} \\ \mathbf{A}_{21}\mathbf{A}_{11} \\ \vdots \\ \mathbf{A}_{21}\mathbf{A}_{11}^{n-1} \end{bmatrix} \mathbf{w}_0 + \mathbf{F}(\mathbf{y}),$$

where $\mathbf{F}$ is some vector that we can assume is known. Since $\mathbf{F}$, $\mathbf{w}_0$, and $[\mathbf{z}(0), \dot{\mathbf{z}}(0), \ldots, \mathbf{z}^{(n)}(0)]^T$ are all known quantities, the matrix

$$\begin{bmatrix} \mathbf{A}_{21} \\ \mathbf{A}_{21}\mathbf{A}_{11} \\ \vdots \\ \mathbf{A}_{21}\mathbf{A}_{11}^{n-1} \end{bmatrix}$$

is invertible, and therefore full rank. Thus, $(\mathbf{A}_{11}, \mathbf{A}_{21})$ is an observable pair.

3

The system diagram produces the equations

$$\begin{aligned} x_1 &= \tfrac{1}{s}x_2 \\ x_2 &= \tfrac{1}{s+2}x_3 \\ x_3 &= \tfrac{1}{s+1}(-x_1 + u) \\ y &= x_1. \end{aligned}$$

After some rearrangement, this becomes

$$\begin{aligned} sx_1 &= x_2 \\ sx_2 &= -2x_2 + x_3 \\ sx_3 &= -x_1 - x_3 + u \\ y &= x_1, \end{aligned}$$

which produces

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

Further rearranging the above produces

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

with partitions

$$\mathbf{A}_{11} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \quad \mathbf{A}_{12} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \mathbf{A}_{21} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \mathbf{A}_{22} = \begin{bmatrix} 0 \end{bmatrix}.$$

Hence,

$$\mathbf{A}_{11} - \mathbf{E}\mathbf{A}_{21} = \begin{bmatrix} -2 - e_1 & 1 \\ -e_2 & -1 \end{bmatrix},$$

with characteristic polynomial

$$|\mathbf{A}_{11} - \mathbf{E}\mathbf{A}_{21} - \mathbf{I}\lambda| = \begin{vmatrix} -2 - e_1 - \lambda & 1 \\ -e_2 & -1 - \lambda \end{vmatrix} = \lambda^2 + (3 + e_1)\lambda + e_1 + e_2 + 2.$$

For both eigenvalues of $\mathbf{A}_{11} - \mathbf{E}\mathbf{A}_{21}$ to be -3 it must be the case that

$$3 + e_1 = 6 \Rightarrow e_1 = 3$$

and

$$e_1 + e_2 + 2 = 9 \Rightarrow e_2 = 4.$$

The observer equation is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -11 \\ -17 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

The estimate $\hat{\mathbf{x}}$ of $\mathbf{x}$ is then reconstructed as

$$\begin{aligned} \mathbf{P}^{-1} \begin{bmatrix} w_1 \\ w_2 \\ y \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 + e_1 y \\ z_2 + e_2 y \\ y \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 + 3y \\ z_2 + 4y \\ y \end{bmatrix} \\ &= \begin{bmatrix} y \\ z_1 + 3y \\ z_2 + 4y \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} y. \end{aligned}$$

4.

$$W(0,T) = \int_0^T \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0,1) \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} ds$$

$$= \begin{pmatrix} \frac{T^3}{3} & -\frac{T^2}{2} \\ -\frac{T^2}{2} & T \end{pmatrix}$$

$$W(0,T)^{-1} = \begin{pmatrix} \frac{12}{T^3} & \frac{6}{T^2} \\ \frac{6}{T^2} & \frac{4}{T} \end{pmatrix}$$

$$u(t) = (0,1) \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} W(0,T)^{-1} \begin{pmatrix} 1 & -T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

From this it is easy to see that

$$\eta = \int_0^T u(s)^2 \, ds$$

$$= \frac{12\cos^2(\theta)}{T^3} - \frac{12\sin(\theta)\cos(\theta)}{T^2} + \frac{4\sin^2(\theta)}{T}.$$

Hence

$$\eta(0) = \eta(\pi) = \frac{12}{T^3}, \quad \text{and} \quad \eta(\pi/2) = \eta(3\pi/2) = \frac{4}{T}.$$

Note that a general form of the optimal cost of steering

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$$

from x_0 to x_1 so as to minimize

$$\eta = \int_0^T \|u(t)\|^2 \, dt.$$

is

$$\eta_0 = \left[x_0 - e^{-AT})x_1\right]^{\mathrm{T}} W(0,T)^{-1} \left[x_0 - e^{-AT}x_1\right].$$

The values above can be read off directly from this formula.