MIMO linear systems controller form and pole placement a la Jack Rugh

$$\dot{x} = Ax + Bu$$

where A is $n \times n$ and B is $n \times m$. Assume *controllable* so that

$$[B \,\vdots\, AB \,\vdots\, \ldots A^{n-1}B]$$

has full rank n. Write this out column-wise

$$[B_1, B_2, \ldots, B_m, AB_1, \ldots, AB_m, \ldots, A^{n-1}B_1, \ldots, A^{n-1}B_m] \quad (\star)$$

Definition: For $j = 1,\ldots,m,$ the $j-th$ $\underline{controllability\ index}$ ρ_j is the smallest integer such that the column $A^{\rho_j}B_j$ is linearly dependent on columns to the left of it in the rendering ($\star$).

By virtue of this construction,

$$\{B_1, AB_1, \ldots, A^{\rho_1 - 1}B_1, \ldots, B_m, \ldots, A^{\rho_m - 1}B_m\}$$

is a linearly independent set, and $\rho_1, \ldots, \rho_m \geq 1$ and $\rho_1 + \cdots + \rho_m = n$.

Exercise: Show that the controllability indices are invariant under a linear change of state variables. $\square$

In terms of the above, we define an $n \times n$ invertible matrix M by

$$M^{-1} = [B_1, AB_1, \ldots, A^{\rho_1 - 1}B_1, \ldots, B_m, \ldots, A^{\rho_m - 1}B_m],$$

and partition M by rows.

$$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix}.$$

The change of variables to be used is constructed from rows in this matrix as follows. We form blocks

$$P = \begin{bmatrix} P_1 \\ \cdots \\ \vdots \\ \cdots \\ P_m \end{bmatrix} \quad \text{where the } i\text{-th block is} \quad P_i = \begin{bmatrix} M_{\rho_1+\cdots+\rho_i} \\ M_{\rho_1+\cdots+\rho_i} A \\ \vdots \\ M_{\rho_1+\cdots+\rho_i} A^{\rho_i - 1} \end{bmatrix}$$

Remark: We've see the special case $m = 1$ already: $P = \begin{bmatrix} M_n \\ M_n A \\ \vdots \\ M_n A^{n-1} \end{bmatrix}$.

Lemma: The matrix P is invertible.

<u>Proof</u>: Omitted.

Definition: Given a set of k positive integers $\alpha_1, \ldots, \alpha_k = n$, the corresponding *integrator coefficient matrices* are defined by

$$A_0 = \text{block diagonal} \left[\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{\alpha_i \times \alpha_i}, \quad i = 1, \ldots, k \right], \text{ and}$$

$$B_0 = \text{block diagonal} \left[\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{\alpha_i \times 1}, \quad i = 1, \ldots, k \right].$$

Theorem: Suppose the time-invariant linear system

$$\dot{x} = Ax + Bu$$

is controllable, A is $n \times n$, B is $n \times m$, *with* $m \leq n$, and B having full rank m. Suppose the controllability indices are $\rho_1, \ldots, \rho_m$. With P defined as on slide (page) 3, the change of variables $z(t) = Px(t)$ yields the controller form state equation

$$\dot{z} = (A_0 + B_0 U P^{-1})z + B_0 R u(t)$$

where A_0, B_0 are integrator coefficient matrices corresponding to $\rho_1, \ldots, \rho_m$, and where the $m \times n$ coefficient matrix U and the $m \times m$ invertible matrix R are given by

$$U = \begin{bmatrix} M_{\rho_1} A^{\rho_1} \\ M_{\rho_1+\rho_2} A^{\rho_2} \\ \vdots \\ M_n A^{\rho_m} \end{bmatrix}, \quad R = \begin{bmatrix} M_{\rho_1} A^{\rho_1 - 1} B \\ M_{\rho_1+\rho_2} A^{\rho_2 - 1} B \\ \vdots \\ M_n A^{\rho_m - 1} B \end{bmatrix}.$$

Example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

$$\begin{pmatrix} B, & AB \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$ has rank = 3

$\Rightarrow$ **system controllable**

According to our definitions, this system is in controller form. It is not difficult to show that to assign coefficients such that the characteristic polynomial of the closed loop system is $s^3 + a_2 s^2 + a_1 s + a_0$, we choose the feedback law

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = Kx = \begin{pmatrix} 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For details, see W.J. Rugh, *Linear System Theory*, Second Ed., Prentice Hall, 1996, esp., p. 248.