Boston University
ME/SE/EC501

Lecture 2

Course Logistics

- Starting Monday, Sept. 11, all lectures for both sections in PHO 203, 8:00-9:45 AM
- Course website: http://people.bu.edu/johnb/ME501.html

Observations

- In our investigation/study of “systems theory” we see that areas from “mathematics” play a fundamental role. These areas include: Calculus, Differential Equations, Algebra, Linear Algebra, . . .

- We begin today by recalling a number of concepts from Linear Algebra and more specifically “Finite Dimensional Vector Spaces”

A field $\mathcal{F}$ is a non-empty set with two operations (labelled "+" for addition and "·" for multiplication) that obey certain properties. For $\alpha, \beta, \gamma \in \mathcal{F}$ we have:

1. $\alpha + \beta \in \mathcal{F}$ (closure under addition)
   $\alpha \cdot \beta \in \mathcal{F}$ (closure under multiplication)

2. $\alpha + \beta = \beta + \alpha$ (addition is commutative)
   $\alpha \cdot \beta = \beta \cdot \alpha$ (multiplication is commutative)

3. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ (addition is associative)
   $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ (multiplication is associative)

4. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$ (multiplication is distributive with respect to addition)

5. $\mathcal{F}$ contains an element denoted as "0" such that $\alpha + 0 = \alpha$ for every element in $\mathcal{F}$ (0 is the additive identity)
   $\mathcal{F}$ contains an element denoted as "1" such that $\alpha \cdot 1 = \alpha$ for every element in $\mathcal{F}$ (1 is the multiplicative identity)

6. To every element $\alpha \in \mathcal{F}$ there exists an element β such that: $\alpha + \beta = 0$, $\beta = -\alpha$ "additive inverse"
   To every element $\alpha \in \mathcal{F}, \alpha \neq 0$ there exists an element γ such that: $\alpha \cdot \gamma = 1$, $\gamma = \alpha^{-1}$ "multiplicative inverse"

Examples

1. Real numbers $\mathbb{R}$ under the usual addition and multiplication

2. Complex numbers $\mathbb{C}$ under the usual addition and multiplication

Vector Spaces

A nonempty set $\mathbb{X}$, is said to be a vector space over some field $\mathcal{F}$ if there exists an operation "+" defined on $\mathbb{X}$ with the following properties for $x, y, z \in \mathbb{X}$

1. $x + y \in \mathbb{X}$
2. $x + y = y + x$
3. $(x + y) + z = x + (y + z)$
4. There exists an element in $\mathbb{X}$ denoted by "0" such that: $x + 0 = x$ for all $x \in \mathbb{X}$
5. To every element $x \in \mathbb{X}$ there exists an element $y \in \mathbb{X}$ such that $x + y = 0$

and there is an operation "$\cdot$" multiplication between elements of $\mathcal{F}$ and $\mathbb{X}$ subject to:

6. for any $\alpha \in \mathcal{F}$ and $x \in \mathbb{X}$, $\alpha \cdot x \in \mathbb{X}$

7. $\alpha \in \mathcal{F}, x, y \in \mathbb{X}, \qquad \alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$

8. $\alpha, \beta \in \mathcal{F}, x \in \mathbb{X}, \qquad (\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$

9. $\alpha, \beta \in \mathcal{F}, x \in \mathbb{X}, \qquad \alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$

10. $1 \cdot x = x$

Notation $(\mathbb{X}, \mathcal{F})$

Examples

1. $(\mathbb{R}^n, \mathbb{R})$ n-tuples of real numbers $\mathbb{R}^n$ over the reals $\mathbb{R}$ with the usual operations of vector addition and scalar-vector multiplication

2. $(\mathbb{C}^n, \mathbb{C})$ n-tuples of complex numbers $\mathbb{C}^n$ over the complex numbers $\mathbb{C}$ with the usual operations of vector addition and scalar-vector multiplication

Consider a map (function) $\mathcal{L} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ that maps n-tuples to n-tuples with the properties:

1. $\mathcal{L}(x + y) = \mathcal{L}(x) + \mathcal{L}(y)$

2. $\mathcal{L}(\alpha \cdot x) = \alpha \cdot \mathcal{L}(x)$

This map is called a Linear Transformation

Definition

A vector space $\mathbb{X}$ is said to be finite dimensional (over its field $\mathcal{F}$) if there is a finite set of elements $\{x_1, x_2, \cdots x_n\}$ such that every element x of $\mathbb{X}$, is a linear combination of $\{x_1, x_2, \cdots x_n\}$ (i.e., $x = \alpha_1 x_1, +\alpha_2 x_2, + \cdots + \alpha_n x_n$)

Example $(\mathbb{R}^n, \mathbb{R})$,

$$(\mathbb{R}^n, \mathbb{R}), \qquad \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

Definition

Let $\mathbb{X}$ be a vector space over $\mathcal{F}$, with $\{x_1,\ x_2,\ \cdots\ x_n\} \in \mathbb{X}$. We say that $\{x_1,\ x_2,\ \cdots\ x_n\}$ are linearly dependent over $\mathcal{F}$ if there exist elements $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathcal{F}$ not all zero such that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$.

If $\{x_1,\ x_2,\ \cdots\ x_n\}$ are NOT linearly dependent they are said to be linearly independent.

Lemma 1: If $\{x_1,\ x_2,\ \cdots\ x_n\} \in \mathbb{X}$ are linearly independent, every element in their span can be UNIQUELY represented in the form

$$x = \lambda_1 x_1 + \lambda_2 x_2, + \cdots + \lambda_n x_n, \qquad \lambda_i \in \mathcal{F}$$

Definition A subset $\{x_1, x_2, \cdots x_n\}$ of some vector space $\mathbb{X}$ is called a basis of $\mathbb{X}$ if $\{x_1, x_2, \cdots x_n\}$ are linearly independent and any $x \in \mathbb{X}$ can be written as:

$$x = \lambda_1 x_1 + \lambda_2 x_2, + \cdots + \lambda_n x_n, \qquad \lambda_i \in \mathcal{F}$$

Let us focus on $(\mathbb{C}^n, \mathbb{C})$ the vector space of n-tuples of complex numbers over the field of complex numbers.
If we choose a basis for $\mathbb{C}^n$, $\{v_1, v_2, \cdots, v_n\}$ we can represent each $x \in \mathbb{C}^n$ uniquely as:

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, \qquad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

If we choose a different basis $\{w_1, w_2, \cdots, w_n\}$ then the same x can be represented as:

$$x = \bar{\alpha}_1 w_1 + \bar{\alpha}_2 w_2 + \cdots + \bar{\alpha}_n w_n, \qquad \bar{\alpha} = \begin{pmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \vdots \\ \bar{\alpha}_n \end{pmatrix}$$

What is the relationship between α and $\bar{\alpha}$?

For each j, $1 \le j \le n$ we can write:

$$(\star) \qquad w_j = \sum_{i=1}^{n} p_{ij} v_i$$

Now

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \bar{\alpha}_1 w_1 + \bar{\alpha}_2 w_2 + \cdots + \bar{\alpha}_n w_n$$

$$= \bar{\alpha}_1(p_{11}v_1 + p_{21}v_2 + \cdots + p_{n1}v_n) + \bar{\alpha}_2(p_{12}v_1 + p_{22}v_2 + \cdots + p_{n2}v_n) + \cdots$$
$$+\bar{\alpha}_n(p_{1n}v_1 + p_{2n}v_2 + \cdots + p_{nn}v_n)$$

$$= \underbrace{(\bar{\alpha}_1 p_{11} + \bar{\alpha}_2 p_{12} + \cdots + \bar{\alpha}_n p_{1n})}_{\alpha_1} v_1 + \underbrace{(\bar{\alpha}_1 p_{21} + \bar{\alpha}_2 p_{22} + \cdots + \bar{\alpha}_n p_{2n})}_{\alpha_2} v_2 + \cdots$$
$$+ \underbrace{(\bar{\alpha}_1 p_{n1} + \bar{\alpha}_2 p_{n2} + \cdots + \bar{\alpha}_n p_{nn})}_{\alpha_n} v_n$$

since the representation is unique!

In matrix form this relationship can be expressed as:

$$\underbrace{\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ & \vdots & & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \vdots \\ \bar{\alpha}_n \end{pmatrix}}_{\bar{\alpha}} = \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}}_{\alpha}, \qquad P\bar{\alpha} = \alpha$$

Repating the process we can represent each v_j in terms of $\{w_1, w_2, \cdots, w_n\}$

$$v_j = \sum_{i=1}^{n} q_{ij} w_i$$

and obtain

$$\underbrace{\begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ & \vdots & & \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}}_{\alpha} = \underbrace{\begin{pmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \vdots \\ \bar{\alpha}_n \end{pmatrix}}_{\bar{\alpha}}, \qquad Q\alpha = \bar{\alpha}$$

P and Q are invertible and in fact, $P^{-1} = Q, \;\; Q^{-1} = P$

Continuing to be focused on $(\mathbb{C}^n, \mathbb{C})$, we now consider some Linear Transformation

$$\mathcal{L} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

If we choose a basis for $\mathbb{C}^n$, $\{v_1, v_2, \cdots, v_n\}$ then the image of each basis element is uniquely expressed:

$$\mathcal{L}v_j = \sum_{i=1}^{n} \alpha_{ij} v_i, \qquad 1 \leq j \leq n$$

and the n^2 elements α_{ij} generate a matrix A:

$$\begin{aligned}\mathcal{L}x = \mathcal{L}(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) &= \alpha_1 \mathcal{L}v_1 + \alpha_2 \mathcal{L}v_2 + \cdots + \alpha_n \mathcal{L}v_n \\ &= \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n\end{aligned}$$

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ & \vdots & & \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}, \qquad A \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}}_{\alpha} = \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

that represents $\mathcal{L}$ (j^{th} column of A is the representation of $\mathcal{L}v_j$ in the $\{v_1, v_2, \cdots, v_n\}$ basis)

Suppose that we choose a different basis for $\mathbb{C}^n$, $\{w_1, w_2, \cdots, w_n\}$, we will have

$$(\star\star) \qquad \mathcal{L}w_j = \sum_{i=1}^{n} \beta_{ij} w_i, \qquad 1 \le j \le n$$

and we will generate a matrix B

$$\mathcal{L}x = \mathcal{L}(\bar{\alpha}_1 w_1 + \bar{\alpha}_2 w_2 + \cdots + \bar{\alpha}_n w_n) = \bar{\alpha}_1 \mathcal{L}w_1 + \bar{\alpha}_2 \mathcal{L}w_2 + \cdots + \bar{\alpha}_n \mathcal{L}w_n$$
$$= \bar{\beta}_1 w_1 + \bar{\beta}_2 w_2 + \cdots + \bar{\beta}_n w_n$$

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ & \vdots & & \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{pmatrix}, \qquad B \underbrace{\begin{pmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \vdots \\ \bar{\alpha}_n \end{pmatrix}}_{\bar{\alpha}} = \bar{\beta} = \begin{pmatrix} \bar{\beta}_1 \\ \bar{\beta}_2 \\ \vdots \\ \bar{\beta}_n \end{pmatrix}$$

that represents $\mathcal{L}$ (j^{th} column of B is the representation of $\mathcal{L}w_j$ in the $\{w_1, w_2, \cdots, w_n\}$ basis)

Question: What is the relationship between A and B?
Recall that $(\star)$

$$w_j = \sum_{\ell=1}^{n} p_{\ell j} v_\ell \qquad 1 \leq j \leq n$$

Apply $\mathcal{L}$ to both sides:

$$\mathcal{L} w_j = \mathcal{L} \sum_{\ell=1}^{n} p_{\ell j} v_\ell \qquad 1 \leq j \leq n$$

and since $\mathcal{L}$ is a linear transformation we have:

$$\mathcal{L} w_j = \sum_{\ell=1}^{n} p_{\ell j} \mathcal{L} v_\ell \qquad 1 \leq j \leq n$$

Now each $\mathcal{L} v_\ell = \sum_{k=1}^{n} \alpha_{k\ell} v_k$ so,

$$\begin{aligned} \mathcal{L} w_j &= \sum_{\ell=1}^{n} p_{\ell j} (\sum_{k=1}^{n} \alpha_{k\ell} v_k) \\ &= \sum_{\ell=1}^{n} \sum_{k=1}^{n} p_{\ell j} \alpha_{k\ell} v_k \\ &= \sum_{k=1}^{n} \sum_{\ell=1}^{n} \alpha_{k\ell} p_{\ell j} v_k \end{aligned}$$

But we also have the images $\mathcal{L}w_j$ directly expressed from $(\star\star)$

$$\mathcal{L}w_j = \sum_{i=1}^{n} \beta_{ij} w_i$$

now each w_i is expressed in terms of the $\{v_1, v_2, \cdots, v_n\}$ basis as (from $(\star)$)

$$w_i = \sum_{k=1}^{n} p_{ki} v_k$$

So

$$\begin{aligned}
\mathcal{L}w_j &= \sum_{i=1}^{n} \beta_{ij} w_i \\
&= \sum_{i=1}^{n} \beta_{ij} (\sum_{k=1}^{n} p_{ki} v_k) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} \beta_{ij} p_{ki} v_k \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} p_{ki} \beta_{ij} v_k
\end{aligned}$$

Since $\{v_1, v_2, \cdots, v_n\}$ is a basis the representation of each $\mathcal{L}w_j$ is unique. This means that for each $k, 1 \leq k \leq n$:

$$\sum_{\ell=1}^{n} \alpha_{k\ell} p_{\ell j} = \sum_{i=1}^{n} p_{ki} \beta_{ij}$$

$$\underbrace{(\alpha_{k1}\ \alpha_{k2}\ \cdots\ \alpha_{kn})}_{k^{th}\text{row of } A} \underbrace{\begin{pmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{nj} \end{pmatrix}}_{j^{th}\text{column of } P} = \underbrace{(p_{k1}\ p_{k2}\ \cdots\ p_{kn})}_{k^{th}\text{row of } P} \underbrace{\begin{pmatrix} \beta_{1j} \\ \beta_{2j} \\ \vdots \\ \beta_{nj} \end{pmatrix}}_{j^{th}\text{column of } B}$$

This is true for $1 \leq j \leq n, \quad 1 \leq k \leq n$ which implies:

$$AP = PB$$

Recall that P is invertible so we can write:

$$P^{-1}AP = B$$

Theorem Two square matrices A, B represent the same linear transformation if and only if there exists a non-singular matrix P such that $B = P^{-1}AP$

proof

$\Longrightarrow$ We just proved that if A, B are the matrix representations of some linear transformation then there exists a P, invertible, such that $B = P^{-1}AP$.

$\Longleftarrow$ Suppose P invertible exists and that $B = P^{-1}AP$

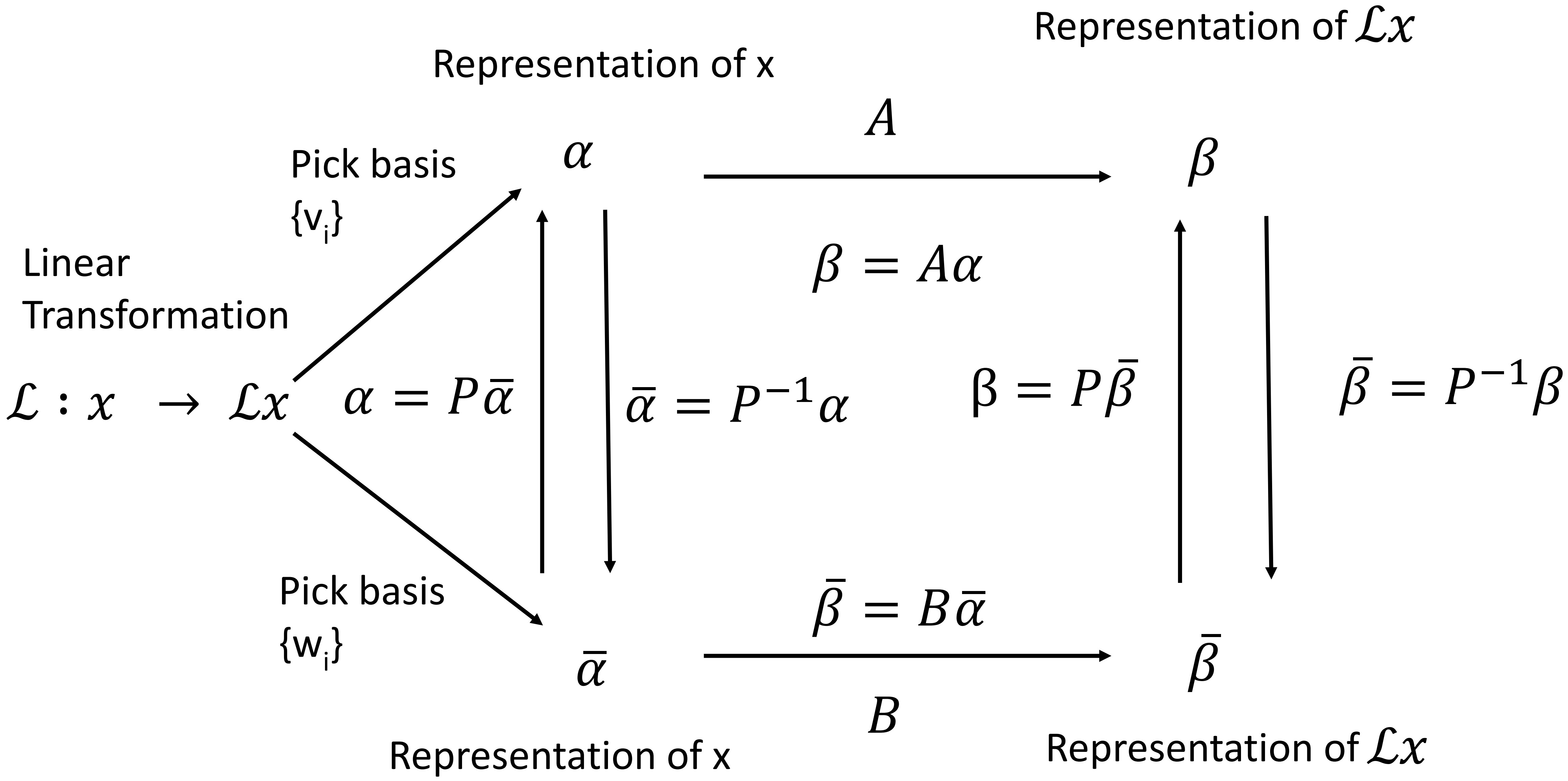
Let $\mathcal{L}_1, \mathcal{L}_2$ be two linear transformations

$\mathcal{L}_1 : x \longrightarrow Ax$ (working with basis $\{v_i\}$ where A represents $\mathcal{L}_1$ in this basis)
$\mathcal{L}_2 : \bar{x} \longrightarrow B\bar{x}$ (working with basis $\{w_i\}$ where B represents $\mathcal{L}_2$ in this basis)

where P relates the two bases $P\bar{x} = x$, $\bar{x} = P^{-1}x$. Show that $\mathcal{L}_1 = \mathcal{L}_2$ i.e., that in either basis $\mathcal{L}_1 x = \mathcal{L}_2 x \ \ \forall x$

$$\mathcal{L}_2\bar{x} = \underbrace{B\bar{x}}_{\text{basis}\{w_i\}} = BP^{-1} \underbrace{x}_{\text{basis}\{v_i\}}$$

multiplying on the left by P will translate this into the $\{v_i\}$ basis, $PBP^{-1}x$. But this is equal to $Ax = \mathcal{L}_1 x$


Representation of $\mathcal{L}x$
Representation of x
A
Pick basis
$\{v_i\}$
α
β
$\beta = A\alpha$
Linear
Transformation
$\mathcal{L} : x \rightarrow \mathcal{L}x$
$\alpha = P\bar{\alpha}$
$\bar{\alpha} = P^{-1}\alpha$
$\beta = P\bar{\beta}$
$\bar{\beta} = P^{-1}\beta$
Pick basis
$\{w_i\}$
$\bar{\beta} = B\bar{\alpha}$
$\bar{\alpha}$
$\bar{\beta}$
B
Representation of x
Representation of $\mathcal{L}x$

Terminology The operation $A \longrightarrow P^{-1}AP$ is called a similarity transformation. Two matrices represent the same transformation if and only if they are similar.

Observation/Question

By choosing different bases we obtain different representations of some linear transformation. Well, are there "special" bases that make the matrix representations "simple" i.e., plainly reveal special characteristics of the structure of the linear transformation?

Example

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

and let

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Clearly P is invertible (det P $= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 1$) so form $P^{-1}AP$ (the representation in the second basis)

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

Observation One can check and see that 2 and 4 are the two eigenvalues of A $\det(sI - A) = (s-2)(s-4)$.

Theorem

Any square $n \times n$ matrix with n distinct eigenvalues can be put in diagonal form by a change of basis

Proof: Let $\{e_1, e_2, \cdots, e_n\}$ be any set of eigenvectors corresponding to eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ i.e.,

$$Ae_j = \lambda_j e_j$$

Let A be the representation for some basis $\{v_i\}$ and pick a second basis $\{e_i\}$ be the basis formed by the eigenvectors. Form M

$$M = (e_1 \; e_2 \; \cdots \; e_n)$$

Here M plays the role of the "P" matrix in our earlier discussion. Note that M is invertible because $\{e_i\}$ are linearly independent. Now the j^{th} column of M is the representation of e_j in terms of the $\{v_i\}$.

Let Λ be the representation of A in the new basis so we have:

$$\begin{aligned}\Lambda &= M^{-1}AM \\ &= M^{-1}\begin{pmatrix}\underbrace{Ae_1}_{\lambda_1 e_1} & \underbrace{Ae_2}_{\lambda_2 e_2} & \cdots & \underbrace{Ae_n}_{\lambda_n e_n}\end{pmatrix} \\ &= \begin{pmatrix}\lambda_1 M^{-1}e_1 & \lambda_2 M^{-1}e_2 & \cdots & \lambda_n M^{-1}e_n\end{pmatrix}\end{aligned}$$

Now note that since $M^{-1}M = I$ we can write:

$$M^{-1}e_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \quad M^{-1}e_2 = \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \quad \cdots \quad M^{-1}e_n = \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}$$

$$\Lambda = \begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ & \vdots & & \\ 0 & 0 & \cdots & \lambda_n\end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix}$$

Eigenvalues are 1 and a:

$$\det(sI - A) = \begin{pmatrix} s-1 & -1 \\ 0 & s-a \end{pmatrix} = (s - 1)(s - a)$$

they are distinct as long as $a \neq 1$. We now need to find corresponding eigenvectors. We can see that for 1:

$$\begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and for a

$$\begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 \\ a-1 \end{pmatrix} = a \begin{pmatrix} 1 \\ a-1 \end{pmatrix}$$

this means that:

$$M = \begin{pmatrix} 1 & 1 \\ 0 & a-1 \end{pmatrix}, \quad M^{-1} = \frac{1}{a-1} \begin{pmatrix} a-1 & -1 \\ 0 & 1 \end{pmatrix}$$

and that:

$$\frac{1}{a-1} \begin{pmatrix} a-1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & a-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$$

Observation As a approaches 1 the matrix M tends to become singular. When $a = 1$ we have multiple eigenvalues!