COMPUTING THE RESOLVENT

THEOREM: The coefficients

$$a_0, a_1, \ldots, a_{n-1}$$

$$E_0, E_1, \ldots E_{n-1}$$

in the expression

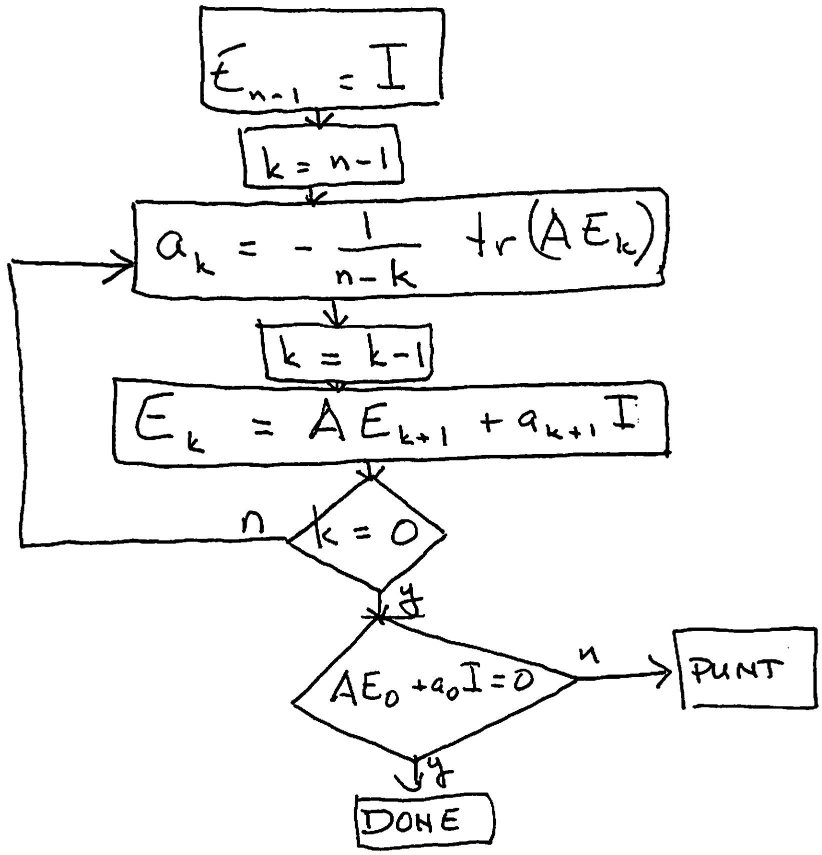
$$(sI - A)^{-1} = \frac{E_{n-1}s^{n-1} + \ldots + E_1 s + E_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

may be determined successively by means of the recursive formulas

$$(1) \qquad \begin{cases} E_{n-1} = I \\ E_k = AE_{k+1} + a_{k+1} I & k = n-2, n-3, \ldots, 0 \end{cases}$$

$$(2) \qquad a_k = -\frac{1}{n-k}\, tr(AE_k) \qquad k = n-1, n-3, \ldots, 0$$

REMARK: The algorithm based on this theorem proceeds as follows.



Proof of theorem: It is really only required, since we already have seen (1) is valid, to show (2). First note

$$AE_{n-1} = A \cdot I = A$$

$$AE_{n-2} = A^2 E_{n-1} + a_{n-1} A$$

$$= A^2 + a_{n-1} A$$

$$AE_{n-3} = A^2 E_{n-2} + a_{n-2} A$$

$$= A^3 + a_{n-1} A^2 + a_{n-2} A$$

$$\vdots$$

$$AE_k = A^{n-k} + a_{n-1} A^{n-k-1} + \cdots + a_{k+1} A$$

$$\vdots$$

$$\therefore \quad tr(A E_k) = tr(A^{n-k}) + a_{n-1}\, tr(A^{n-k-1}) + \cdots + a_{k+1}\, tr(A)$$

Claim 1: $\quad tr\, A^j = \sum_{i=1}^{n} \lambda_i^j$

Proof: For any matrix B, $tr\, B = \sum \lambda_i(B)$.

If λ is an eigenvalue of A, then λ^j is an eigenvalue of A^j

for

$$A x = \lambda x$$

$$\Rightarrow A^2 x = \lambda A x = \lambda^2 x$$

$$\vdots$$

$$\boxed{\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}}$$

q.e.d. for claim 1.

Hence

$$tr\, A E_k = S_{n-k} + a_{n-1} S_{n-k-1} + \cdots + a_{k+1} S_1$$

where $$s_j = \sum_{i=1}^{n} \lambda_i^j .$$

The theorem now follows from Newton's formula

$$s_{n-k} + a_{n-1} s_{n-k-1} + \cdots + a_{k+1} s_1 = -(n-k) a_k .$$

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FADDEEV, V.H. <u>Computational Aspects of Linear Algebra</u>
Dover, N.Y. 1959.

NEWTON'S FORMULA

Theorem: Consider the polynomial

$$s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = p(s).$$

Let $\lambda_1, \dots, \lambda_n$ be the (not nec. distinct) roots of this polynomial, and let

$$s_k = \sum_{i=1}^{n} \lambda_i^k .$$

Then Newton's formulas hold:

$$s_k + a_{n-1} s_{k-1} + a_{n-2} s_{k-2} + \dots + a_{n-k+1} s_1 + k a_{n-k} = 0$$

$$k = 1, \dots, n$$

Proof: Associate to each root λ_i a monic polynomial of degree $n-1$ which has roots

$$\lambda_1, \dots, \widehat{\lambda_i}, \dots \lambda_n$$

where "$\wedge$" means the i-th root is omitted from the list. Denote this polynomial by

$$s^{n-1} + a^{i}_{n-2}\, s^{n-2} + \cdots + a^{i}_{0} = {}_{i}P(s)$$

Now

$$s^{n-1} + a^{i}_{n-2}\, s^{n-2} + \cdots + a_0^{i} = \left(s^{n} + a_{n-1} s^{n-1} + \cdots + a_0\right) / (s - \lambda_i)$$

Exercise: verify this.

$$\begin{aligned}
&= s^{n-1} + (a_{n-1} + \lambda_i)\, s^{n-2} \\
&\quad + (a_{n-2} + a_{n-1}\lambda_i + \lambda_i^{2})\, s^{n-3} \\
&\quad + \cdots + \\
&\quad + (a_2 + a_3\lambda_i + \cdots + a_{n-1}\lambda_i^{n-3} + \lambda_i^{n-2})\, s \\
&\quad + (a_1 + a_2\lambda_i + \cdots + a_{n-1}\lambda_i^{n-2} + \lambda_i^{n-1}) \\
&\quad + (a_0 + a_1\lambda_i + \cdots + a_{n-1}\lambda_i^{n-1} + \lambda_i^{n})\, s^{-1} \\
&\quad + \cdots
\end{aligned}$$

But, of course, there are no terms with negative exponents. Equating coefficients

$$a_{n-2}^{i} = a_{n-1} + \lambda_i$$

$$a_{n-3}^{i} = a_{n-2} + a_{n-1}\lambda_i + \lambda_i^{2}$$

$$\vdots$$

$$a_{1}^{i} = a_2 + a_3\lambda_i + \cdots + a_{n-1}\lambda_i^{n-3} + \lambda_i^{n-2}$$

$$a_{0}^{i} = a_1 + a_2\lambda_i + \cdots + a_{n-1}\lambda_i^{n-2} + \lambda_i^{n-1}$$

$$\lambda_1 a_k^1 + \lambda_2 a_k^2 + \cdots + \lambda_n a_k^n$$

$$= \lambda_1\left(a_{k+1} + a_{k+2}\lambda_1 + \cdots + a_{n-1}\lambda_1^{n-k-2} + \lambda_1^{n-k-1}\right)$$

$$+ \lambda_2\left(a_{k+1} + a_{k+2}\lambda_2 + \cdots + a_{n-1}\lambda_2^{n-k-2} + \lambda_2^{n-k-1}\right)$$

$$+ \cdots +$$

$$+ \lambda_n\left(a_{k+1} + a_{k+2}\lambda_n + \cdots + a_{n-1}\lambda_n^{n-k-2} + \lambda_n^{n-k-1}\right)$$

$$= a_{k+1}S_1 + a_{k+2}S_2 + \cdots a_{n-1}S_{n-k-1} + S_{n-k}$$

Now we know each a_k^i is the sum of products of the roots $\lambda_1, \ldots, \lambda_n$:

Eg:

$$s^{n-1} + a^i_{n-2} s^{n-2} + \cdots + a^i_1 s + a^i_0$$

$$= (s-\lambda_1) \cdots \widehat{(s-\lambda_i)} \cdots (s-\lambda_n)$$

and $\quad a^i_0 = (-\lambda_1)(-\lambda_2)\cdots\widehat{(-\lambda_i)}\cdots(-\lambda_n)$

$$a^i_{n-2} = -\lambda_1 - \lambda_2 - \cdots - \widehat{\lambda_i} \cdots - \lambda_n$$

$$a^i_{n-3} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n$$

no prod. inv. λ_i appears

The coeff. a^i_k involves sums of products of roots taken $n-k-1$ at a time. From this, we may explicitly write down an expression for $\lambda_1 a^1_k + \lambda_2 a^2_k + \cdots + \lambda_n a^n_k$:

Eg:

$$\lambda_1 a^1_{n-2} + \lambda_2 a^2_{n-2} + \cdots + \lambda_n a^n_{n-2}$$

$$= \lambda_1(-\lambda_2 - \lambda_3 \cdots - \lambda_n)$$

$$+\lambda_2(-\lambda_1 - \lambda_3 - \cdots - \lambda_n)$$

$$+ \cdots +$$

$$+ \lambda_n(-\lambda_1 - \cdots - \lambda_{n-1})$$

$$= -2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n)$$

$$= -2\, a_{n-2}$$

and

$$\lambda_1 a_{n-3}^1 + \cdots + \lambda_n a_{n-3}^n$$

$$= \lambda_1(\lambda_2\lambda_3 + \lambda_2\lambda_4 + \cdots + \lambda_{n-1}\lambda_n)$$

$$+ \lambda_2(\lambda_1\lambda_3 + \lambda_1\lambda_4 + \cdots + \lambda_{n-1}\lambda_n)$$

$$+ \cdots +$$

$$+\lambda_n(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-2}\lambda_{n-1})$$

$$= 3(\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \cdots + \lambda_{n-2}\lambda_{n-1}\lambda_n)$$

$$= -3 a_{n-3}$$

In general

$$\lambda_1 a^1_{n-k} + \cdots + \lambda_n a^n_{n-k} = -k\, a_{n-k}$$

and this proves the theorem that

$$\boxed{S_k + a_{n-1} S_{k-1} + \cdots + a_{n-k+1} S_1 + k a_{n-k} = 0}$$

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