Dynamic Systems - State Space Control - Lecture 5 *

September 17, 2012

1 Example of Jordan Normal Form

$$A = \begin{pmatrix} 5 & -1 & -3 & 2 & -5 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & -2 \\ 0 & -1 & 0 & 3 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

Begin by computing eigenvalues

$$A - \lambda I = \begin{pmatrix} 5-\lambda & -1 & -3 & 2 & -5 \\ 0 & 2-\lambda & 0 & 0 & 0 \\ 1 & 0 & 1-\lambda & 1 & -2 \\ 0 & -1 & 0 & 3-\lambda & 1 \\ 1 & -1 & -1 & 1 & 1-\lambda \end{pmatrix}$$

Compute the determinant by expanding using the 2nd row

$$= (2-\lambda) \begin{pmatrix} 5-\lambda & -3 & 2 & -5 \\ 1 & 1-\lambda & 1 & -2 \\ 0 & 0 & 3-\lambda & 1 \\ 1 & -1 & 1 & 1-\lambda \end{pmatrix}$$

Now expand using the 3rd row

$$= (2-\lambda) \left[(3-\lambda) \begin{pmatrix} 5-\lambda & -3 & -5 \\ 1 & 1-\lambda & -2 \\ 1 & -1 & 1-\lambda \end{pmatrix} - (1) \begin{pmatrix} 5-\lambda & -3 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & -1 & 1 \end{pmatrix} \right]$$

*This work is being done by various members of the class of 2012

$$= (2-\lambda)[(3-\lambda)[14-17\lambda+7\lambda^2-\lambda^3]-(3-\lambda)(2-\lambda)]$$
$$= (2-\lambda)(3-\lambda)(12-16\lambda+7\lambda^2-\lambda^3)$$
$$= (2-\lambda)^3(3-\lambda)^2$$
$$\lambda 1 = 2$$
$$\lambda 2 = 3$$

STEP 1: Find the generalized eigenspaces.

$$\text{For}\lambda = 2$$

$$A - 2I = \begin{pmatrix} 3 & -1 & -3 & 2 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -2 \\ 0 & -1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 \end{pmatrix}$$

To find the kernel of this matrix we put the matrix into reduced row echelon form

First, swap the second and fifth rows
Second, swap the third and first rows
Then, use the new first row to zero out other entries in the first column

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 2 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Multiply the second row by -1
Add row 2 to row 3
Subtract row 2 from row 4

$$\begin{pmatrix} 1 & 0 & -1 & 1 & -2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Multiply row 3 by 1/2
Subtract row 3 from row 4

$$\begin{pmatrix} 1 & 0 & -1 & 1 & -2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Subtract row 3 from rows 1 and 2

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Non trivial solutions to$(A - 2I)X = 0$ are

$$x_4 = 0, x_2 = x_5, x_1 = x_3 + 2x_5$$

The general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} s + 2t \\ t \\ s \\ 0 \\ t \end{pmatrix} = s \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Next, consider the kernel of

$$(A-\lambda I)^2 = \begin{pmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & -1 & 2 & 0 \\ 1 & -1 & -1 & 1 & -1 \end{pmatrix}$$

The first few steps towards rendering this matrix in reduced row echelon form yield the following matrix

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & -2 & 0 & 2 & 2 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Which directly leads to

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The solutions to $(A-2I)^2X = 0$ take the form

$$x_2 = x_4 + x_5, x_1 = x_3 + 2x_5$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} s+2t \\ u+t \\ s \\ u \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + u\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

These vectors span the generalized eigenspace. The new vector corresponding to the coefficient u is the generalized eigenvector.

For $\lambda = 3$,

$$A-3I = \begin{pmatrix} 2 & -1 & -3 & 2 & -5 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & -2 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 1 & -2 \end{pmatrix}$$

To put into reduced row echelon form, swap row 3 and row 1, and zero out other values in the 1st column

$$\begin{pmatrix} 1 & 0 & -2 & 1 & -2 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

Multiply row 4 by -1 and swap with row 2
Zero out other values in the 2nd column

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{The solutions to}(A - 3I)X = 0 \text{ take the form}$$

$$x_2 = x_5 = x_3 = 0, x_1 = -x_4$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ 0 \\ s \\ 0 \end{pmatrix} = s \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$(A - 3I)^2 = \begin{pmatrix} -4 & 2 & 5 & -4 & 8 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 3 & -2 & 4 \\ 1 & 0 & -1 & 1 & -2 \\ -1 & 1 & 1 & -1 & 2 \end{pmatrix}$$

Put this matrix into reduced row echelon form

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & -12 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The solutions to$(A - 3I)^2X = 0$ has solutions

$$x_2 = x_3 = 0, x_1 = -x_4 + 2x_5$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -s+2t \\ 0 \\ 0 \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$A \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$A \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

With respect to the basis

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Matrix A takes the form

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

2 Dynamic Systems

Dynamic systems have inputs, outputs, disturbances, etc ..

State Space Control Theory is concerned with systems whose state evolution may be described recursively.

A state describes everything knowable about the system at a given instant in time, but it is not typically known a priori how the system will be changing. State evolution is typically described by an equation

$$x(k+1) = x(k) + f(x(k), u(k), d(k))$$

Typically the state is modeled but not observed. We only get to measure some function of it.

$$y(k) = h(x(k), u(k), e(k))$$

All quantities may be vector quantities (multi-dimensional).
Implicit in this model is a quantization of time. There is some time interval Δt between ticks of the system clock. The real equation of evolution may be rendered.

$$x((k+1)\Delta t) = x(k\Delta t) + f(x(k\Delta t), u(k\Delta t), d(k\Delta t))$$

If we let $k \to \infty$ and $\Delta t \to 0$ then in the limit,

$$x(t+\Delta t) = x(t) + f(x(t), u(t), d(t))\Delta t or \dot{x} = f(x(t), u(t), d(t))$$

or

$$\dot{x} = f(x(t), u(t), d(t)).$$

Examples

1 - Metabolic Processes

Many drugs effect the time it takes blood to clot. The clotting time may be taken to be directly proportional to the amount of drug in the body. This amount after a single dose typically decays with time.

μ is the dose introduced at time t = 0.

If a dose is administered according to the regimen $\mu(k)$ on the k-th day, the amount in the body on day k+1 is :

$$x(k+1) = \alpha x(k) + \mu(k+1),$$

where

$$\alpha = e^{-r} \text{ is decay over the course of one day.}$$

This may be written as

$$x(k+1) - \alpha x(k) = \mu(k+1).$$

2 - Moving Average Models

More generally we are interested in moving average models.

$$x(k+n) + a_n - 1(k)x(k+n-1) + ... + a_0(k)x(k) = \mu(k)$$

Both of these examples represent systems with inputs. We frequently want to write models of this form as first order systems.

$$x(k+1) = \alpha x(k) + \mu(k+1)$$

$$\begin{gathered} x_1(k) = x(k) \\ x_2(k) = x(k+1) \\ ... \\ x_n(k) = x(k+n-1) \end{gathered}$$

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ ... \\ ... \\ x_n(k+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ ... & & & & \\ 0 & 0 & 0 & ... & 1 \\ -a_0(k) & -a_1(k) & ... & ... & -a_n(k) \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ ... \\ x_{n-1}(k) \\ x_n(k) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \mu(k).$$