# Dynamic Systems - State Space Control - Lecture 3 *

September 11, 2012

Recall some basic facts about finite dimensional vector spaces.

Let v be a vector space whose field of scalars is $\mathbb{R}$ or $\mathbb{C}$.

There are two laws of combination,

## 1 Vector Addition

• For $v_1, v_2, v_3 \in$ V, the laws of vector addition are associative;

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

• For $v_1, v_2 \in$ V, the laws of vector addition are commutative;

$$v_1 + v_2 = v_2 + v_1$$

• There exists a zero vector $0 \in$ V, such that;

$$v + 0 = v \text{ for all } v \in V$$

• For every $v \in V, -v \in V$ as well.

*This work is being done by various members of the class of 2012

# 2 Scalar Multiplication

- For any $\alpha \in \mathbb{R}(\mathbb{C})$ and $v \in V$,

$$\alpha v \in V$$

- For $\alpha, \beta \in \mathbb{R}(\mathbb{C})$

$$\alpha(\beta v) = (\alpha\beta)v \qquad \text{(scalar multiplication is associative)}$$

- For $\alpha \in \mathbb{R}(\mathbb{C})$ and $v, w \in V$,

$$\alpha(v + w) = \alpha v + \alpha w \qquad \text{(Distributive over vector spaces)}$$

- For $\alpha, \beta \in \mathbb{R}(\mathbb{C}), v\ inV$

$$(\alpha + \beta)v = \alpha v + \beta v \qquad \text{(Distributive over scalar addition)}$$

A basis for a vector space $V$ is a linearly independent set of vectors: $v_1, ..., v_n$ that spans $V$. Saying that $v_1, ..., v_n$ spans $V$ means that for every $w \in V$ there are scalars $\alpha_1, ..., \alpha_n$ such that,

$$W = \alpha_1 v_1, ..., \alpha_n v_n$$

$1, s, s^2, ..., s^n-$ These monomials are a basis of the vector space of polynomials p(s) having degree $\leq n$ A vestor space $V$ is finite dimensional of dimension n if it has a basis with n elements.

A finite dimensional vector space over $\mathbb{R}^n$(resp $\mathbb{C}^n$) is isomorphic to
$\mathbb{R}^n$(resp $\mathbb{C}^n$).

**THEOREM:** Any square matrix with distinct eigenvalues can be put into a diagonal form by a change of basis.

**PROOF:** List the n eigenvalues corresponding to distinct eigenvalues. Prove this is a basis (DO THIS!). With respect to this basis, the matrix has diagonal form.

### Matrices whose eigenvalues are not distinct

Examples:

1.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2.

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

**Jordan Canonical Form:**

Recall that the eigenvalues of a matrix $A$ are the roots of the characteristic polynomial,

$$p(\lambda) = \det(\lambda I - A)$$

Suppose p($\lambda$) can be factored

$$p(\lambda) = (\lambda - \lambda_1)^{r_1} ... (\lambda - \lambda_p)^{r_p}$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$ (to be sure that such factorisation exists, we must allow for complex eigenvalues)

For each $\lambda_j$ we are interested in the associated generalized eigenspace,

$$M^k = \ker(A - \lambda_j I)^k = \{v \in V : (A - \lambda_j I)^k v = 0\}$$

What are these?

$$M^0 = \ker(A - \lambda_j I)^0 = \ker(I) = \{0\}$$

$M^1 = \ker(A - \lambda_j I) =$ eigenspace of $\lambda_j$

For $v \in M^k$,

$$(A - \lambda_j I)^{k+1} v = (A - \lambda_j I)(A - \lambda_j I)^k v = (A - \lambda_j I)0 = 0$$

Therefore,$M^k \in M^{k+1}$ we have an increasing chain of subspaces $M^0 \subset M^1 \subset$

Since $M^k \subset \mathbb{C}$ for all k, there is some value $t$ such that $M^k = M^t$ for all $k \geq t$, let $t$ be the smallest integer such that $M^k = M^t$ for all $k \geq t$ (call $M^t = M_{\lambda_j}$)

Another (this time decreasing) subspace chain which is of interest is,

$$(A - \lambda_j I)^k \mathbb{C}^n = W^k$$

Here we have $\mathbb{C}^n = W^0 \subset W^1 \subset \dots$

Let $m_{\lambda_j} = \dim M_{(\lambda_j)}$ The $\dim W^t = n - m_{(\lambda_j)}$ (Since the dimensions of the range space and the null spce of $(A - \lambda_j I)^t$ must add up to r) We must have,

$W^k = W^t$ for $k \geq t$

And we denote $W^t$ by $W_{\lambda_j}$

**Proposition 1:** $\mathbb{C}^n$ is the direct sum of $M_{(\lambda_j)}$ and $W_{(\lambda_j)}$ (i.e any $v \in \mathbb{C}^n$ maybe writen uniquely as $W = v_m + w_m$ where $v_m \in M_{(\lambda_j)}, w_m \in W_{(\lambda_j)}$)

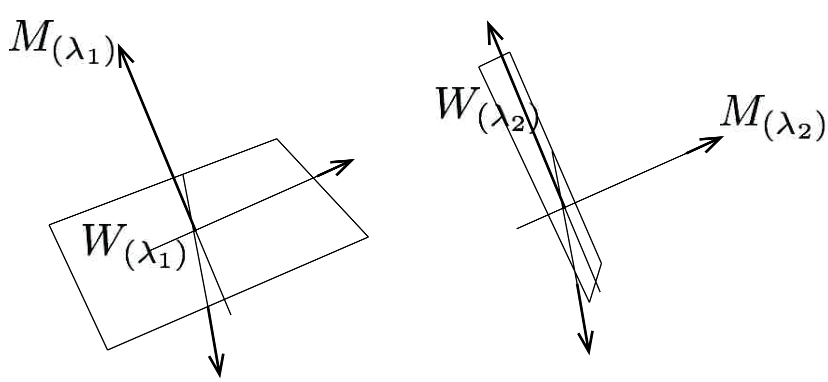
**Proof:** Since $(A - \lambda_j I)W_{(\lambda_j)} = (A - \lambda_j I)^{t+1}\mathbb{C}^n = (A - \lambda_j I)^t \mathbb{C}^n = W_{\lambda_j}$
We see that $A - \lambda_j I$ is nonsingular on $W_{(\lambda_j)}$. Let $v$ be any vector in $\mathbb{C}^n$. Then $(A - \lambda_j I)^t v = 0 \in W_{(\lambda_j)}$. Because $(A - \lambda_j I)^t$ is nonsingular on $W_{(\lambda_j)}$. There is a unique $w_m = W_{(\lambda_j)}$ such that $(A - \lambda_j I)^t W_m = \gamma$. Let $v_m = V - w_m$. It is easily seen $(A - \lambda_j I)^t V_m = 0$, so that $V_m \in M_{(\lambda_j)}$.

Hence $\mathbb{C}^n = M_{(\lambda_j)} + W_{(\lambda_j)}$ and remains only to show that the expression $W = v_m + w_m$ is unique. This follows since $\dim M_{(\lambda_j)} + \dim W_{(\lambda_j)} = n$, and any pair of respective basis.

For $M_{(\lambda_j)}$ and $W_{(\lambda_j)} \{v_1, \dots, v_n\}, \{w_1, \dots, w_n\}$ together define a basis for $\mathbb{C}^n$.

$$\mathbb{C}^n = M_{(\lambda_j)} \oplus W_{(\lambda_j)}$$

Where, $\oplus$ is the notation for direct sum.



**Proposition 2:** For any $i, j, A - \lambda I$ maps $M_{(\lambda_j)}, W_{(\lambda_j)}$ onto themselves.

**Proof:** In the case $i = j$ we've shown that $(A - \lambda_j I)M^{k+1} \subset M^k \subset M^{k+1}$ and $(A - \lambda_j I)W^k = W^{k+1} \subset W^k$. This implies $W_{(\lambda_j)}$ and $M_{(\lambda_j)}$ are invarient under $(A - \lambda_i I)$
But from this it follows that these subspaces are invariant under any polynomial in $A - \lambda_i I$. In particular, under $A - \lambda_i I = A - \lambda_i I + (\lambda_i - \lambda_j)I$.

**Proposition 3:** For $\lambda_i \neq \lambda_j, M_{(\lambda_i)} \subset W_{(\lambda_j)}$

**Proof:** Let $v \in M_{(\lambda_j)}$. I claim that $v$ cannot be in kernal of $A - \lambda_j I$. For $j \neq i$ suppose to the contrary, $V \in \ker(A - \lambda_j I)$. Then,

$$(\lambda_j - \lambda_i)^{t_i} v = \{A - \lambda_i I - (A - \lambda_j I)\}^{t_i} v$$

$$= (A - \lambda_i I)^{t_i} v + \sum_{k=1}^{t_i} (-1)^k \binom{t_i}{k} (A - \lambda_{iI})^{t_i - k} (A - \lambda_{jI})^k v$$

$= 0$ (the first term $= 0$ because $v \in M_{(\lambda_j)}$), the remaining terms would need to be zero under the assumption $v \in \ker(A - \lambda_j I)$

Since $\lambda_i - \lambda_j \neq 0$, this implies $v = 0$. This in turn implies $(A - \lambda_j I)|_{M_{\lambda_i}}$ (The linear transformation restricted to subspace $M_{(\lambda_i)}$) is non-singular. This means that $A - \lambda_j I$ maps $M_{(\lambda_i)}$ onto it-
self and hence, $M_{(\lambda_i)}$ is contained in the range of $(A - \lambda_j I)^{t_j}$ which is just $W_{(\lambda_j)}$.

**Proposition 4:** $\mathbb{C}^n = M_{(\lambda_1)} \oplus M_{(\lambda_2)} \oplus \cdots \oplus M_{(\lambda_p)}$
*Remark:* The direct sum notation means that any vector $v \in \mathbb{C}^n$ can be uniquely expressed as a sum, $V = v_1 + \cdots + v_p, v_j \in M_{(\lambda_j)}$

**Proof:** $\mathbb{C}^n = M_{(\lambda_1)} \oplus W_{(\lambda_1)}$ and since $W_{(\lambda_1)} \supset M_{(\lambda_2)}$,
We can write,

$$\begin{aligned} \mathbb{C}^n \quad &= M_{(\lambda_1)} \oplus (W_{(\lambda_1)} \cap [M_{(\lambda_2)} \oplus M_{(\lambda_1)}]) \\ &= M_{(\lambda_1)} \oplus ([W_{(\lambda_1)} \cap M_{(\lambda_2)}] \oplus (W_{(\lambda_1)} \cap W_{(\lambda_2)})) \\ &= M_{(\lambda_1)} \oplus M_{(\lambda_2)} \oplus (W_{(\lambda_1)} \cap W_{(\lambda_2)}) \end{aligned}$$

Similiarly we obtain,

$$\mathbb{C}^n = M_{(\lambda_1)} \oplus M_{(\lambda_2)} \oplus M_{(\lambda_3)} \oplus (W_{(\lambda_1)} \cap W_{(\lambda_2)} \cap W_{(\lambda_3)})$$

Eventually we obtain,

$$\mathbb{C}^n = M_{(\lambda_1)} \oplus M_{(\lambda_2)} \cdots \oplus M_{(\lambda_p)} \oplus (W_{(\lambda_1)} \cap \cdots \cap W_{(\lambda_p)})$$

To complete the proof, we show that

$$W_{(\lambda_1)} \cap \cdots \cap W_{(\lambda_p)} = \{0\}$$

Since each $W_{(\lambda_i)}$ is invarient under all $A - \lambda_j I, W_{(\lambda_1)} \cap \cdots \cap W_{(\lambda_p)}$ is invarient as well.
$A - \lambda_j I$ is non-singular on $W_{(\lambda_1)} \cap \cdots \cap W_{(\lambda_p)}$.Otherwise it would be singular on all $W_{(\lambda_i)}$s and $W_{(\lambda_j)}$ in particular. Hence,

$$(A - \lambda_1 I)^{r_1} (A - \lambda_2 I)^{r_2} \ldots (A - \lambda_p I)^{r_p}$$

is nonsingular on $W_{(\lambda_1)} \cap \cdots \cap W_{(\lambda_p)}$. But since this operator maps all vectors in $\mathbb{C}^n$ to zero, the only way it could be nonsingular on $W_{(\lambda_1)} \cap \cdots \cap W_{(\lambda_p)}$ is if this intersection is the zero subspace.