Lecture 19 Observability Canonical Form and the Theory of Observers *

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Observability Canonical Form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Suppose this is observable:

Let $S = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$ Assume scalar output.

Let$z = Sx$. Then

$$\begin{aligned} \dot{z} &= SAS^{-1}z + SBu \\ y &= CS^{-1}z \end{aligned}$$

Let $S^{-1} = \begin{pmatrix} S1 & \vdots & \cdots & \vdots & Sn \end{pmatrix}$

$CA^kS_j = \begin{cases} 1 & k = j-1 \\ -a_{j-1} & k = n \\ 0 & otherwise \end{cases}$

Hence,

$$SAS^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}$$

$$CS^{-1} = \begin{pmatrix} 1, & 0, & 0, & \cdots & ,0 \end{pmatrix}$$

*This work is being done by various members of the class of 2012

$$SB = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix}$$

Summary: For Single Input/Single Output systems

$$\dot{x} = Ax + bu$$

$$y = Cx$$

The system is said to be in controllability canonical form if:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

It is in observability canonical form if

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}, \; c = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Observers

Suppose we are given

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

and we wish to estimate x_0. "After a while", the state is given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma$$

If A is a stable matrix (i.e. eigenvalues in the l.h.p), then the dependence of $x(t)$ on x_0 diminishes over time. Thus, eventually depends only on the second term. If we run another system

$$\dot{z} = Az + Bu; \; z(0) = z_0$$

in parallel, then the over time $e = z - x$ decreases to 0.

Proof. $e = z - x$ satisfies the homogeneous linear ordinary differential equation $\dot{e} = Ae$.

Identity Observer

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

We propose an observer of the form

$$\dot{z} = Az + E(y - Cz) + Bu,\ z(0) = z_0$$

where E is a coefficient matrix to be specified.

Theorem. *If the system (1) is observable, the coefficients of the characteristic polynomial of $A - EC$ may be selected arbitrarily by appropriate choice of E.*

Proof. The result follows from the eigenvalue placement problem follow the controllable pair $(A^T,\ C^T)$. Hence, we can find a K, such that $A^T + C^T K$ has any chosen characteristic polynomial. Let $E = -K^T$. Then $A - EC$ has the same characteristic polynomial. □

Note that if $e = z - x$

$$\begin{aligned} \dot{e} &= \dot{z} - \dot{x} \\ &= Az + E(y - Cz) + Bu - Ax - Bu \\ &= Az + E(Cx - Cz) - Ax \\ &= (A - EC)e \end{aligned}$$

If the eigenvalue of $A - EC$ are in the l.h.p, then $e(t) \to 0$ asymptotically. □

Reduced Order Observers

We don't need to propagate an n-dimensional reconstruction of the state since we directly observe p-dimensions in the form of the output y.

Thus, we seek to reconstruct $n - p$ observer states. We are given the system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

We assume that the $p \times n$ matrix C has full rank p. Let v be any $(n-p) \times n$ matrix such that $p = \begin{pmatrix} v \\ c \end{pmatrix}$ is non-singular.

The set of independent vectors that are the row of C can be completed by an additional $n - p$ independent row vectors to yield a basis for $\mathbb{R}^n$. These rows are grouped together to gain the $(n-p) \times n$ V, making $p = \begin{pmatrix} v \\ c \end{pmatrix}$ invertible.

Let $\bar{x} = P_x$, and write $\bar{x} = \begin{pmatrix} w \\ y \end{pmatrix}$. In terms of $\bar{x}$, the system dynamics are of the form:

$$\begin{pmatrix} \dot{w} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$

It is possible to extract from this system a system of order $n - p$ which has as inputs the known quantities $u,\ y$.

Let E be $(n-p) \times p$ matrix, and subtract $E \cdot$ (*bottom part of the system*) from the top.

$$\begin{aligned} \dot{w} - E\dot{y} &= (A_{11} - EA_{21})w + (A_{12} - EA_{22})y + (B_1 - EB_2)u \\ &= (A_{11} - EA_{21})(w - Ey) + (A_{11}E - EA_{21}E + A_{12} - EA_{22})y + (B_1 - EB_2)u \end{aligned}$$

Letting $v = w - Ey$, this reduced order system takes the form:

$$\dot{v} = (A_{11} - EA_{21})v + (A_{11}E - EA_{21}E + A_{12} - EA_{22})y + (B_1 - EB_2)u$$

where $y,\ u$ are known inputs. Form an observer by copying this system

$$\dot{z} = (A_{11} - EA_{21})z + (A_{11}E - EA_{21}E + A_{12} - EA_{22})y + (B_1 - EB_2)u$$

The error in the z estimate of z propagates according to

$$\begin{aligned} \dot{e} &= \dot{z} - \dot{v} \\ &= (A_{11} - EA_{21})e \end{aligned}$$

the rate at which $e \to 0$ depends on the eigenvalues of $(A_{11} - EA_{21})$. This can be placed arbitrarily provided that $(A_{11},\ A_{21})$ is an observable pair.

FACT. *If the original system* $(A,\ B,\ C)$ *is observable, then* $(A_{11},\ A_{21})$ *must be an observable pair.*

Proof. Left to students as homework. □

Reconstructing the original state: Since $v = w - Ey,\ z \sim v,\ \hat{w} = z + Ey$ is my observer estimate of w. The state estimate is thus:

$$\hat{x} = p^{-1} \begin{pmatrix} \hat{w} \\ y \end{pmatrix} = p^{-1} \begin{pmatrix} z + Ey \\ y \end{pmatrix}$$

Observers in Feedback Loops (the Separation Principle)

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$\dot{z} = Az + E(y - Cz) + Bu$$

We would like to choose a gain matrix K, defining a feedback $u = Kx$ such that the resulting closed loop system is stable. Unfortunately, x is not available. For this purpose, assuming the observer works, we can by feeding back $u = Kz$, the resulting closed loop system is of the form:

$$\dot{x} = Ax + BKz$$

$$\dot{z} = (A - EC)z + E\underbrace{Cx}_{\text{y}} + BKz$$

Let $e = z - x$, then the above may be rewritten as:

$$\dot{x} = (A + BK)x + BKe$$
$$\dot{e} = (A - EC)e$$

or

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A + BK & BK \\ 0 & A - EC \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}$$

FACT. *$det \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} = det(X)det(Z)$, where X and Z are square blocks.*

Proof.

$$\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} X & Y \\ 0 & I \end{pmatrix}$$

$$det \begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix} = \mathbf{1} det \begin{pmatrix} I_{n-1} & 0 \\ 0 & Z \end{pmatrix} = \cdots = det(Z)$$

Similarly,

$$det \begin{pmatrix} X & Y \\ 0 & I \end{pmatrix} = det(X)$$

We know that

$$det \left[\begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} X & Y \\ 0 & I \end{pmatrix} \right] = det \begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix} det \begin{pmatrix} X & Y \\ 0 & I \end{pmatrix}$$

and the result (fact) follows. From this, it is easy to see that the eigenvalues of $\begin{pmatrix} A + BK & BK \\ 0 & A - EC \end{pmatrix}$ are the eigenvalues of $A + BK$ together with the eigenvalues of $A - EC$. □