Lecture 17 Feedback in Control Design*

November 13, 2012

1 Review

Recall: There is a set of special coordinate to

-The standard controllable realization

-The standard observable realization

-Controllability canonical form

-Observability canonical form

Example: Given $A = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Show that this is a controllable pair $\Leftrightarrow S_1 \neq S_2$

Find the corresponding system matrices in controllability canonical form.

2 THE ROLE OF FEEDBACK IN CONTROL DESIGN

Example: $\dot{x} = u$

Suppose we wish to steer the system from x_0 to x_1 using the techneuqe developed so far.

$$\begin{aligned} w(t_0, t_1) &= \int_{t_0}^{t_1} \Phi(t_0, t) \cdot 1 \cdot 1 \cdot \Phi(t_0, t)^T dt \\ &= t_1 - t_0 \end{aligned}$$

*This work is being done by various members of the class of 2012

$$x_0 - \Phi(t_0, t_1)x_1 = (x_0 - x_1) = (t_1 - t_0)\eta$$

Let

$$\begin{aligned} u(t) &= -B(t)^T \Phi(t_0, t)^T \eta \\ &= -1 \cdot 1 \cdot \eta \\ &= -1 \frac{x_0 - x_1}{t_1 - t_0} = \frac{x_1 - x_0}{t_1 - t_0} \end{aligned}$$

$$u(t) = \begin{cases} \frac{x_1 - x_0}{t_1 - t_0} & t_0 \le t \le t - 1 \\ 0 & else \end{cases}$$

This control law is said to be open loop. The challenge with applying open loop control is randomness and uncertainty in the physical world.

Source of uncertainty:

-model uncertainty

-measurement noise

-sensor errors

-actuator errors

- ...

An alternative to open loop control is feedback control. For $\alpha > 0$, $u(t) = \alpha(x_1 - x_t)$. To pursue the discussion of uncertainty suppose the actual system is

$$\dot{x} = \epsilon x + (1 + \delta)u$$

where ϵ, δ are small.

$$\begin{aligned} x(t) &= e^{\epsilon t} x_0 + \int_{t_0}^{t} e^{\epsilon(t-s)}(1 + \delta)\frac{x_1 - x_0}{t_1 - t_0} ds \\ &= e^{\epsilon t}(x_0 + \frac{1}{\epsilon}(e^{-\epsilon t} - e^{-\epsilon t_0})(1 + \delta)\frac{x_1 - x_0}{t_1 - t_0}) \\ &= e^{\epsilon t} x_0 - \frac{1}{\epsilon}(e^{-\epsilon t_0} - e^{-\epsilon t})(1 + \delta)\frac{x_1 - x_0}{t_1 - t_0} \end{aligned}$$

As $\epsilon \to 0$, this approaches

$$x_0 + (t_1 - t_0)(1+\delta)\frac{x_1 - x_0}{t_1 - t_0} = x_1 + x_0 + \delta x_1 - (1+\delta)x_0 = x_1 + \delta(x_1 + x_0)$$

If $\epsilon > 0$, this residual (terminal error) will be even larger.

With the feedback control $u(t) = \alpha(x_1 - x(t))$, the closed loop dynamics is given as

$$\begin{aligned} \dot{x} &= \epsilon x + (1+\delta)\alpha(x_1 - x) \\ &= [\epsilon - \alpha(1+\delta)]x + \alpha(1+\delta)x_1 \end{aligned}$$

The equilibrium

$$\alpha = \frac{\alpha(1+\delta)}{\alpha(1+\delta) - \epsilon} x_1$$

If $\epsilon > 0$ is chosen to be sufficiently large, this can be made to be arbitrarily close to the desired x_1. Moreover,the equilibrium can be made arbitrarily fast.

THEOREM:

Let (A, B) be the controllable pair. Then given any real polynomial

$$P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

where n is the size of the square matrix A, there exists a real matrix K such that $A + BK$ is a square matrix having $P(s)$ as its characteristic polynomial.

Remark: If (A, B) is a controllable pair, the feedback control

$$u(t) = Kx(t)$$

makes the closed loop system asymptotically stable at the origin for proper choice of K.

Proof:(Only given in scalar input case–i.e. the case that $B = b \in C^{n\times 1}$)

We can assume that the coordinate system has been chosen s.t.

$$A = \begin{pmatrix} 0 & 1 & 0 & & \cdots \\ & 0 & 1 & & \cdots \\ & & & \ddots & \\ & & & & 1 \\ -a_0 & \cdots & \cdots & & -a_{n-1} \end{pmatrix}$$

$$b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Then the feedback matrix will be of the form

$$\begin{aligned}
K &= (k_1, \cdots, k_n) \\
bK &= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} (k_1, \cdots, k_n) \\
&= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ k_1 & k_2 & \cdots & k_n \end{pmatrix} \\
A + bK &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ k_1 - a_0 & k_2 - a_1 & \cdots & & k_n - a_{n-1} \end{pmatrix} \\
x(k+1) &= Ax(k) + Bu(k)
\end{aligned}$$

For discrete time asymptotic stability, the eigenvalues of A should be less than 1 in modulus. (I.e. the eigenvalues should be inside the unit disk)

Why you can't set velocity gains arbitrarily high with digital control systems.

$J\ddot{\theta} = \tau$ (torque driven relating machine)

First order: $\theta_1 = \theta, \theta_2 = \dot{\theta}$.

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + \frac{1}{J} \begin{pmatrix} 0 \\ \tau \end{pmatrix}$$

Suppose $\tau = -k_v\dot{\theta} - k_p(\theta - \theta_d)$

The closed loop system is

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k_p}{J} & -\frac{k_v}{J} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{k_p\theta_d}{J} \end{pmatrix}$$

$$\begin{pmatrix} \theta_1(t+h)-\theta_1(t) \\ \theta_2(t+h)-\theta_2(t) \end{pmatrix} = h \begin{pmatrix} 0 & 1 \\ -\frac{k_p}{J} & -\frac{k_v}{J} \end{pmatrix} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} + b \begin{pmatrix} 0 \\ \frac{k_p\theta_d}{J} \end{pmatrix}$$

$$\theta_1(t+h) = (2-\frac{hk_v}{J})\theta_1(t) - (1-\frac{hk_v}{J})\theta_1(t-h) + \frac{h^2k_p}{J}(\theta_d - \theta_1(t-h))$$

Continuous that feed back designs could be problematic in discrete time implementations depending on the sampling frequency:

$$u = Kx(t)$$

$\Rightarrow$ closed loop system

$$\dot{x} = (A+BK)x$$

$A+BK$ may have all l.h.p. eigenvalues, but if the system is implemented using digital elements, the feedback k_w in this case will be

$$\begin{gathered} u(t) = Kx(t_j) \\ j \le t \le j+1 \end{gathered}$$

By the variation of constant formula

$$\begin{aligned} x(j+1) &= e^A x(j) + \int_j^{j+1} e^{A(j+1-s)} BKx(j)ds \\ &= e^A(I + \int_0^1 e^{-As} BKds)x(j) \end{aligned}$$

The requirement is discrete time eigenvalues are inside the unit dish.