Lecture 17 Discrete Time Theorem and Control Canonical Form*

November 6, 2012

1 Discrete Time Theorem

$$
\begin{aligned}
x(k+1) &= A(k)x(k) + B(k)u(k) \qquad (1) \\
x(k+1) &= \Phi(k,1)x(1) + \sum_{j=1}^{k} \Phi(k,j+1)B(j)u(j) \\
\Phi(k,k+1) &= I \qquad \text{for } k \geq 1 \\
\Phi(k,j) &= A(k)A(k-1)\cdots A(j)
\end{aligned}
$$

<u>Discrete Time Theorem:</u>

If for some positive integer N and pair of points $x_1, x_2 \in R^n$,

$$x_2 - \Phi(N,1)x_1$$

belongs to the range space of

$$W_N = \sum_{j=1}^{N} \Phi(N,j+1)B(j)B^T(j)\Phi(N,j+1)^T$$

then the control sequence

$$u(j) = B^T(j)\Phi(N,j+1)^T\eta \qquad (j = 1, \cdots, N)$$

where $\eta \in R^n$ satisfies.

$$x_2 = \Phi(N,1)x_1 = W_N \cdot \eta$$

steers the state of (1) from $x(1) = x_1$ to $x(N+1) = x_2$.

*This work is being done by various members of the class of 2012

<u>Definition:</u>

The discrete time system (1) is said to be <u>controllable</u> if for any $x_1, x_2 \in R^n$, there exists a positive integer N and an input sequence $u(1), u(2), \cdots, u(N)$, such that

$$x_2 = \Phi(N,1)x_1 + \sum_{j=1}^{N} \Phi(N, j+1)B(j)u(j)$$

<u>Theorem:</u>

A constant coefficient (i.e. time invariant) discrete time system with scalar input

$$x(k+1) = Ax(k) + bu(k)$$
$$A \in R^{n\times n} \qquad b \in R^{n\times 1}$$

is controllable $\Leftrightarrow$ rank $(b, Ab, \cdots, A^{n-1}b) = n$.

<u>Corollary:</u>
For the time invariant system

$$x(k+1) = Ax(k) + Bu(k)$$

in which B is $n \times m$ with $m \geq 2$, the condition

rank $(B, AB, \cdots, A^{n-1}B) = n$ is sufficient but not necessary for controllability.

$$\dot{x} = Ax + Bu$$

$$x(k+1) = e^A x(k) + \int_0^1 e^{A(1-s)} B ds\, u(k)$$

This is the sampled system at unit intervals.

<u>Example:</u>

$$A = \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad e^A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \Phi$$

$$\begin{aligned}\Gamma &= \int_0^1 e^{A(t-s)}Bds \\ &= \int_0^1 \begin{pmatrix} cos\pi(1-s) & sin\pi(1-s) \\ -sin\pi(1-s) & cos\pi(1-s)\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} ds \\ &= \begin{pmatrix}\frac{2}{\pi}\\0\end{pmatrix}\end{aligned}$$

$$(\Gamma, \Phi\Gamma) = \begin{pmatrix}\frac{3}{\pi} & -\frac{2}{\pi} \\ 0 & 0\end{pmatrix}$$

$\rightarrow$ rank=1. This system is not controllable.

Invertibility:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \qquad (2)$$

This system is observable $\Leftrightarrow$

$$\text{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1}\end{pmatrix} = n$$

Definition:

The control system (2) is said to be invertible if wherever there are two input/output pairs (y_1, u_1), (y_2, u_2) such that $y_1 \equiv y_2$, then $u_1 \equiv u_2$
.

Theorem:

The control system

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= Cx + du\end{aligned}$$

is convertible $\Longleftrightarrow$

$$\text{rank}\begin{pmatrix} d & 0 & 0 & \cdots & 0 \\ cb & d & 0 & \cdots & 0 \\ cAb & cb & d & \cdots & 0 \\ \vdots & \cdots & & \ddots & \vdots \\ cA^{n-2}b & \cdots & \cdots & \cdots & d \end{pmatrix} \geq n$$

(As usual, the use of linear case b, c means that we are considering single input-single output [SISO] system)

Given

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx + du \end{aligned} \qquad (s)$$

where does there exist a system

$$\begin{aligned} \dot{z} &= Fz + gy \\ u &= hz + ky \end{aligned} \qquad (I)$$

such that the control $u(\cdot)$ which produces a given $y(\cdot)$

from (s) is generated by the output of (I)?

<u>Idea</u>

$$\begin{pmatrix} y(0) \\ y'(0) \\ y''(0) \\ \vdots \end{pmatrix} = \begin{pmatrix} d & 0 & 0 & \cdots & 0 \\ cb & d & 0 & \cdots & 0 \\ cAb & cb & d & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ cA^{n-2}b & \cdots & \cdots & \cdots & d \end{pmatrix} \begin{pmatrix} u(0) \\ \dot{u}(0) \\ \ddot{u}(0) \\ \vdots \end{pmatrix}$$

In order to have invertibility and to realize the system (I), we probably read

$$k = \frac{1}{d}$$

and we may further conjecture that

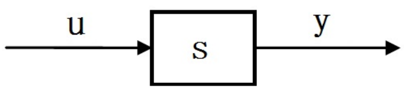
$$h = -\frac{c}{d}, \quad g = \frac{b}{d}, \quad F = A - \frac{bc}{d}$$

<u>Theorem:</u>

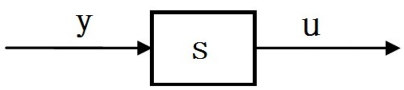
Given (s), we find (I) given by the above values, and the frequency domain statement of invertibility is

$$\begin{aligned} c(Is - A)^{-1}b + d \qquad & (s) \\ h(Is - F)^{-1}g + k \qquad & (I) \end{aligned}$$

where the system



has the inverse



2 Canonical forms and realizations

$$\dot{x} = Ax + bu$$

The system is <u>controllable</u> $\Leftrightarrow$

$$M = (b, Ab, \cdots, A^{n-1}b)$$

is invertible. Suppose this is the case

$$M^{-1} = \begin{pmatrix} e_1^T \\ \vdots \\ e_n^T \end{pmatrix} \qquad M^{-1}M = \begin{pmatrix} e_1^T b & e_1^T Ab & \cdots & e_1^T A^{n-1}b \\ e_2^T b & e_2^T Ab & \cdots & e_2^T A^{n-1}b \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

we claim that $e_n^T, e_n^T A, e_n^T A^{n-1}$ are linearly independant.
<u>Proof:</u>

Suppose

$$\alpha_1 e_n^T + \alpha_2 e_n^T A + \cdots + \alpha_2 e_n^T A^{n-1} = 0$$

Multiply both sides of this equation on the right by b,

$$\alpha_1 e_n^T b + \alpha_2 e_n^T Ab + \cdots + \alpha_2 e_n^T A^{n-1} b = 0$$

Hence we have

$$\alpha_1 e_n^T + \alpha_2 e_n^T A + \cdots + \alpha_2 e_n^T A^{n-2} = 0$$

Multiply on the right side by Ab as before, this implies $\alpha_{n-1} = 0$. Continuing this way, we establish our claim.

Let

$$P = \begin{pmatrix} e_n^T \\ e_n^T A \\ \vdots \\ e_n^T A^{n-1} \end{pmatrix}$$

By our calculation, P is invertible.

Let $z(t) = Px(t)$, then $\dot{z} = PAP^{-1}z(t) + Pbu(t)$.

Note that

$$Pb = \begin{pmatrix} e_n^T b \\ e_n^T Ab \\ \vdots \\ e_n^T A^{n-1} b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

To see what PAP^{-1} is, denote $P^{-1} = (P_1 \vdots \cdots \vdots P_n)$

Then,

$$e_n^T A^k P_j \begin{cases} 1 & \text{if } k = j-1 \\ -a_{j-1} & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$PAP^{-1} = \begin{pmatrix} e_n^T \\ e_n^T A \\ \vdots \\ e_n^T A^{n-1} \end{pmatrix} A(P_1 \vdots \cdots \vdots P_n)$$

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{pmatrix}$$

$$det(\bar{A} - \lambda I) = (-1)^n a_0 + (-1)^n a_1 \lambda + \cdots + (-1)^n a_{n-1} \lambda^{n-1} + (-1)^n \lambda^n$$

The $a_j's$ in the bottom row are the coefficients of the characteristic polynomial.(called companion form)

The pair

$$\bar{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_n - 1 \end{pmatrix} \qquad \bar{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Is said to be control canonical form.